REPRODUCING KERNELS OF SPACES OF VECTOR VALUED MONOGENICS

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Abstract. An important class of monogenic functions is that of vector valued monogenic functions, i.e. divergence and rotation free vector fields in arbitrary dimension. These functions are also known as conjugate harmonic functions. In this article we consider the question of reproducing kernels for Hilbert spaces of vector valued monogenics in a domain Ω. We prove the analogues of the classical minimum and symmetry properties of these kernels. Then we consider explicitly the Bergman and the Szegö kernels for the unit ball, this being the case where the corresponding kernels for the module of Clifford valued monogenics are known.

1. Introduction

It is not necessary to use Clifford algebras to study divergence and rotation free vector fields. Under the name of conjugate harmonic functions they have been studied whithout this setting by several authors (see e.g. [7] for an account of such functions in the unit ball). Clifford algebras nevertheless provide a powerful tool, since using these it is possible to consider conjugate harmonic functions as a special class of monogenic functions. For these functions an extensive function theory exists (see [4]), and the results can readily be applied to conjugate harmonic functions. In this introduction we give an overview of Clifford algebras and the theory of monogenic functions.

Let $\mathbb{R}^n$ be Euclidean n-space, endowed with the inner product $\langle \cdot, \cdot \rangle$ and let $e = (e_1, \ldots, e_n)$ be an orthonormal basis for $\mathbb{R}^n$. The Clifford algebra $C\ell(n)$ for the Euclidean space can be constructed: it is generated by the elements of $\mathbb{R}^n$, and the basic multiplication rules are governed by

$$e_j^2 = -1, \qquad j = 1, \ldots, n$$

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and

$$e_i e_j + e_j e_i = 0, \qquad i \neq j.$$

It is natural to identify the multiplicative unit with the real number 1, and so embed $I\!R$ into the Clifford algebra. The real part of a Clifford number a will be denoted as $\mathbf{Re}a$. The main antiinvolution is given by $\overline{e_i} = -e_i$ (and the antiinvolution law $\overline{ab} = \bar{b}\,\bar{a}$), and the main involution by $e'_i = -e_i$ (and of course $(ab)' = a'b'$).

Each vector in $I\!R^n$ can be identified with a Clifford number $\mathbf{x} = x_1 e_1 + \ldots + \mathbf{x}_n e_n$. As the main results of this article deal with spaces which are not modules over a Clifford algebra, only a limited number of properties is needed to express them, the main one being that $\mathbf{Rexy}$ is up to sign the inner product between $\mathbf{x}$ and $\mathbf{y}$. The Spin group of the Clifford algebra can be described as the group of all even products of unit vectors. If α is an element of the Spin group the the mapping $\phi(\alpha) : \mathbf{x} \to \alpha \mathbf{x} \alpha'^{-1}$ is an even orthogonal transformation of the space $I\!R^n$. The Dirac operator in $I\!R^n$ is given by

$$\partial_x = \sum_{j=1}^{n} e_j \partial_j.$$

We have that

$$\partial_x^2 = -\Delta_x,$$

Δ_x being the Laplacian in $I\!R^n$. The Dirac operator has a spherical decomposition

$$\partial_x = \boldsymbol{\xi} \left(\partial_r + \frac{\Gamma_x}{r} \right)$$

where $\mathbf{x} = r\boldsymbol{\xi}$ is the decomposition into spherical coordinates of $\mathbf{x}$ ($\boldsymbol{\xi}$ is a unit vector), and Γ_x is the so-called *spherical Dirac operator* (actually, in terms of operators on manifolds, this is not the Dirac operator on the sphere, but is closely related to it).

Clifford valued functions obeying the equation $\partial_x f = 0$ in a domain Ω are called *(left) monogenic* there. For a vector valued function $f = \sum_i e_i f_i$ the equation $\partial_x f = 0$ can be written as

$$-\sum_{i=1}^{n} \partial_i f_i + \sum_{i<j} e_{ij}(\partial_i f_j - \partial_j f_i) = 0$$

which is equivalent with div $f = 0$ and curl $f = 0$. If f is a monogenic function in a domain Ω then $f^\alpha(\mathbf{x}) = \alpha f(\alpha^{-1} \mathbf{x} \alpha')\alpha'^{-1}$ is a monogenic function in the domain which is the image of Ω under $\phi(\alpha)$.

A function f is called *right monogenic* if $\bar{f}$ is left monogenic. The name right monogenic stems from the fact that instead of writing $\partial_x \bar{f} = 0$ one can write

$$f\partial_x = \sum_{j=1}^{n} (\partial_j f)e_j = 0,$$

i.e. with the coefficients e_j at right. Notice that a vector valued monogenic function is of necessity both left and right monogenic.

An important property of the Dirac operator is given by Stokes' theorem (see e.g. [4]). For a bounded domain C with suitable boundary this is formulated as follows: let f and g be two C^1 functions on the closure of C. Then

$$\int_C \left[-\overline{\partial_x f} g + \bar{f} \partial_x g \right], d\mathbf{x} = \int_{\partial C} \bar{f} \mathbf{n} g dS. \tag{1}$$

Here $\mathbf{n}$ is the outward pointing unit normal of ∂C, and dS is surface measure.

The space of monogenic functions in Ω is denoted by $I\!M(\Omega)$. and it is a right module over $C\ell(n)$ for pointwise multiplication at right. It can be proved (see [1]) that, with very few conditions, a submodule of $I\!M(\Omega)$ which is a Hilbert space can be turned into a Hilbert module, i.e. that a sesquilinear (over $C\ell(n)$) Clifford valued inner product can be constructed having the real inner product as a trace. Important examples of this are the intersections of $I\!M(\Omega)$ with an L_2-space. E.g. the intersection of $I\!M(\Omega)$ with $L_2(\Omega)$ gives the module $ML_2(\Omega)$ with inner product

$$(f,g) = \int_\Omega \overline{f(\mathbf{x})} g(\mathbf{x}) d\mathbf{x}.$$

2. General Properties of Reproducing Kernels

Let M be a right Hilbert module of functions monogenic in a domain Ω with Clifford valued inner product $(\cdot,\cdot)_c$ and the real valued inner product derived from this, $(\cdot,\cdot)$. Suppose moreover that for all $\mathbf{x}$ in Ω the mapping sending f to $f(\mathbf{x})$ is continuous over M. Then there exists a function K such that $K(\mathbf{x},\cdot)$ is in M for all $\mathbf{x}$ in Ω, and K has the reproducing property: $(K(\mathbf{x},\cdot),g)_c = g(\mathbf{x})$ for all g in M. For this reason it is called the reproducing kernel of M. The existence of a reproducing kernel K has been studied in [3]. In most classical cases K is scalar-and-bivector valued. Let V be the subspace of M of vector valued monogenics, i. e. of functions $f \in M$ of the form

$$f = \sum_{i=1}^{n} e_i f_i.$$

where the f_i are real. In the sequel the subindex i will be used for the ith component of the function. As V is a subspace, but not a submodule, it only inherits the scalar valued inner product $(\cdot, \cdot)$ from M. In most classical cases, M will be spanned by V as a module. We have at least the following lemma:

LEMMA 2.1. *Let f be monogenic over Ω. Then there exist vector valued functions g^i, monogenic over Ω and vector valued, and Clifford numbers λ_i, such that*

$$f = \sum_i g^i \lambda_i,$$

where the sum is finite.

Proof.
Since f is C^∞ over Ω there exists a function h such that $f = \partial_x h$. As h is Clifford valued it can be written as $\sum_A h_A e_A$, and $f = \sum_A (\partial_x h_A) e_A$. As each h_A is real valued, $\partial_x h_A$ is vector valued. ∎

This theorem does not state that for $f \in M$ all the elements $\partial_x h_A$ are in V: indeed extra conditions on the inner product may be necessary.

The mapping, for $\mathbf{x}$ fixed, sending f to $f_i(\mathbf{x})$ is obviously continuous over V, so there exist unique functions $K^{(i)}$ in V such that $K^{(i)}(\mathbf{x}, \cdot)$ is in V and $(K^{(i)}(\mathbf{x}, \cdot), f) = f_i(\mathbf{x})$. The function $K^{(i)}$ is the solution of a minimum problem, as is the case with classical reproducing kernels:

THEOREM 2.2. *Take $\mathbf{x}$ fixed such that there exists g in V with $g_i(\mathbf{x}) \neq 0$ and let f be the solution in V to the minimum problem*
(a) $f_i(\mathbf{x}) = 1$
(b) $\|f\|_M$ is minimal.
Then

$$K^{(i)}(\mathbf{x}, \cdot) = \frac{f(\mathbf{y})}{\|f\|_M^2}.$$

If $g_i(\mathbf{x}) = 0$ for all g in V then $K^{(i)}(\mathbf{x}, \cdot) = 0$.

Proof.
That $K^{(i)}(\mathbf{x}, \cdot) = 0$ if $g_i(\mathbf{x}) = 0$ for all g in V is obvious. If evaluation in $\mathbf{x}$ is not identically zero, then $K^{(i)}(\mathbf{x}, \cdot)$ is different from zero, its norm is strictly positive, and so $(K^{(i)}(\mathbf{x}, \cdot), K^{(i)}(\mathbf{x}, \cdot)) = K_i^{(i)}(\mathbf{x}, \mathbf{x})$ is different from zero. The solution of the minimum problem is therefore of the form

$$f = \frac{K^{(i)}(\mathbf{x}, \cdot)}{K_i^{(i)}(\mathbf{x}, \mathbf{x})} + g$$

where $g_i(\mathbf{x}) = 0$. But this implies that $(K^{(i)}(\mathbf{x}, \cdot), g) = 0$, and so $||f||_V^2 = \frac{||K^{(i)}(\mathbf{x}, \cdot)||_V^2}{K^{(i)}(\mathbf{x}, \mathbf{x})^2} + ||g||_V^2$. Since $||f||^2$ is minimal it follows that $g = 0$. The scalar constant $K^{(i)}(\mathbf{x}, \mathbf{x})$ is now found by comparing $(f, f) = ||f||^2 = \left(f, \frac{K^{(i)}(\mathbf{x}, \cdot)}{K^{(i)}(\mathbf{x}, \mathbf{x})} \right)$ and $(f, K^{(i)}(\mathbf{x}, \cdot)) = f_i(\mathbf{x}) = 1$. For the second case notice that $||K^{(i)}(\mathbf{x}, \cdot)||^2 = (K^{(i)}(\mathbf{x}, \cdot), K^{(i)}(\mathbf{x}, \cdot)) = K^{(i)}(\mathbf{x}, \mathbf{x})$ which must be zero since $K^{(i)}(\mathbf{x}, \cdot)$ is an element of V. ∎

The classical reproducing kernel of M is symmetrical with respect to its variables: $K(\mathbf{x}, \mathbf{y}) = (K(\mathbf{y}, \cdot), K(\mathbf{x}, \cdot))_c = \overline{K(\mathbf{y}, \mathbf{x})}$. This symmetry takes a somewhat different form in the case of V: here we have that

$$K_j^{(i)}(\mathbf{x}, \mathbf{y}) = (K^{(j)}(\mathbf{y}, \cdot), K^{(i)}(\mathbf{x}, \cdot)) = K_i^{(j)}(\mathbf{y}, \mathbf{x}).$$

Hence the symmetry property can be written in the following form:

THEOREM 2.3. *The function*

$$T(\mathbf{x}, \mathbf{y}) = \sum_i K^{(i)}(\mathbf{x}, \mathbf{y}) e_i$$

(which is real and bivector valued) is left monogenic in the variable $\mathbf{y}$ *and right monogenic in* $\mathbf{x}$. *Moreover it satisfies the symmetry relation*

$$T(\mathbf{x}, \mathbf{y}) = \overline{T(\mathbf{y}, \mathbf{x})}.$$

Proof.
It follows from 2 and the definition of T that

$$
\begin{aligned}
T(\mathbf{x}, \mathbf{y}) &= \sum_i K^{(i)}(\mathbf{x}, \mathbf{y}) e_i = \sum_{i,j} K_j^{(i)}(\mathbf{x}, \mathbf{y}) e_{ji} \\
&= \sum_{i,j} K_i^{(j)}(\mathbf{y}, \mathbf{x}) e_{ji} = \overline{T(\mathbf{y}, \mathbf{x})}
\end{aligned}
$$

proving the symmetry relation. As a linear combination (with Clifford constants at right) of monogenic functions in $\mathbf{y}$, $T(\mathbf{x}, \mathbf{y})$ itself is left monogenic in $\mathbf{y}$. Symmetry then leads to right monogenicity in $\mathbf{x}$. ∎

When we compare the reproducing kernel K in M and the kernels $K^{(i)}$ in V we see that the latter can not be easily derived from the former.. Indeed, the

solution in M of the minimum problem (a)-(b) is $\frac{K(\mathbf{x},\cdot)e_i}{K(\mathbf{X},\mathbf{X})}$. It is however easy to see that in general for $n > 2$ there is at least one i for which this function is not in V. Indeed, $K(\mathbf{x},\mathbf{y})e_i$ is a vector if and only if $K(\mathbf{x},\mathbf{y})$ is the sum of a real number and a bivector parallel to e_i. But a nonzero bivector can be parallel to at most two coordinate lines, so if $n > 2$ $\frac{K(\mathbf{X},\cdot)e_i}{K(\mathbf{X},\mathbf{X})}$ can only be in V for all i if the bivector part is always zero. Since $K(\mathbf{x},\cdot)$ is monogenic,this implies that K must be constant as a function of its second variable. By the symmetry of K it is independent of its first variable, and as a consequence all functions in M are constant functions.

In order to calculate the kernels $K^{(i)}$ we can start from the reproducing kernel of M. It is clear that in M, for any function $g = \sum_A e_A g_A$, g_i is given by $(K(\mathbf{x},\cdot)e_i, g)$, so we only have to project $K(\mathbf{x},\cdot)e_i$ orthogonally on V (with respect to the scalar valued product) to get $K^{(i)}$. Now $K(\mathbf{x},\cdot)e_i$ is vector-and-trivector valued. So it has the form $K^{(i)} + T^{(i)}$ where the trivector part of $T^{(i)}$ is equal to the trivector part of Ke_i. $T^{(i)}$ is moreover vector-and-trivector valued and monogenic in its second variable. However this method has the drawback of not leading to explicit expressions, i.e. in general we cannot perform the projection explicitly.

3. Monogenic Functions in the Unit Ball

In the case of the unit ball it is possible to arrive at explicit expressions for the kernel functions using spherical harmonic and spherical monogenic functions. In the sequel we shall need two modules of monogenic functions: the module $\mathbf{B}$ of monogenic, Clifford valued L_2-functions in the unit ball $B(1)$ with the inner product

$$(f,g)_B = A \int_{B(1)} \overline{f}g d\mathbf{x}$$

where the normalising constant A is chosen so as to make $(1,1)_B = 1$, and the Hardy space $\mathbf{S}$ of monogenic Clifford valued functions in $B(1)$ with a square integrable boundary value on the sphere, with inner product

$$(f,g)_S = C \int_{S(1)} \overline{f}g dS,$$

(dS is surface measure) where again C is chosen such that $(1,1)_S = 1$. Both modules have a reproducing kernel, denoted by K_B and K_S respectively, satisfying

$$f(\mathbf{x}) = (K_B(\mathbf{x},\cdot), f)_B \qquad g(\mathbf{x}) = (K_S(\mathbf{x},\cdot), f)_S$$

for $f \in \mathbf{B}$ and $g \in \mathbf{S}$ respectively.

The modules $\mathbf{B}$ and $\mathbf{S}$ are intimately linked. Indeed, with the notation $f_t(\mathbf{x}) = f(t\mathbf{x})$, the inequality

$$(f_t, g_t)_S \leq (f, g)_S$$

holds for $t < 1$ and f and g in $\mathbf{S}$. Equality is only reached when f and g are both constant. Hence, for f in $\mathbf{S}$,

$$
\begin{aligned}
\mathbf{Re}(f, f)_B &= \mathbf{Re}\left(A \int_B \overline{f} f d\mathbf{x} \right) \\
&= \mathbf{Re}\left(A \int_0^1 r^{n-1} \frac{1}{B} (f_r, f_r)_S dr \right) \\
&\leq \left(\frac{A}{C} \right) \int_0^1 r^{n-1} dr (f, f)_S.
\end{aligned}
$$

Equality is reached if and only f is constant, which proves that the expression between brackets on the last line is 1, since $(1,1)_S = (1,1)_B$. Hence the embedding mapping $W : \mathbf{B} \to \mathbf{S}$ is continuous, and moreover its image is dense in $\mathbf{S}$. An important operator is the adjoint operator, which has some interesting properties:

LEMMA 3.4. *The adjoint operator W^* of the embedding mapping $W : \mathbf{B} \to \mathbf{S}$ satisfies*
(i) $(W^* f, g)_B = (f, g)_S$ for all g in $\mathbf{B}$ and all f in dom W^*,
(ii) f is in the domain of W^* if and only if

$$\forall h \in \mathbf{B}, |(f, h)_S \leq C\|h\|_{\mathbf{B}}.$$

Now, for a point $\mathbf{x}$ in the open unit ball, the point evaluation map is continuous on $\mathbf{B}$, and so the reproducing kernel $K_S(\mathbf{x}, \cdot)$ certainly satisfies condition (ii) of the lemma above. The reproducing kernels thus are linked by the formula

$$K_B(\mathbf{x}, \cdot) = W^* K_S(\mathbf{x}, \cdot).$$

In order to find an explicit expression for the operator W^* we notice that there exists projection operators P_k onto the modules of homogeneuous monogenic polynomials of degree k such that
(i) Each P_k is selfadjoint both considered as an operator on $\mathbf{B}$ and $\mathbf{S}$.
(ii) $\sum_{k=0}^{+\infty} P_k$ is the identity, both on $\mathbf{B}$ and $\mathbf{S}$.

Clearly for any f and g in $\mathbf{S}$,

$$
\begin{aligned}
(P_k f, P_k g)_b &= \left(\frac{A}{C}\right) \int_0^1 r^{n-1+2k} dr (P_k f, P_k g)_S \\
&= \frac{n}{n+2k} (P_k f, P_k g)_S,
\end{aligned}
$$

which shows that $W^* P_k f = (1 + 2k/n) P_k f$ or

$$
W^* = \left(1 + \frac{2}{n} r \partial r\right).
$$

Explicit expressions for the reproducing kernels are

$$
K_S(\mathbf{x}, \mathbf{y}) = \frac{1 + \mathbf{yx}}{|1 + \mathbf{yx}|^n}
$$

(see [2]) and

$$
\begin{aligned}
K_B(\mathbf{x}, \mathbf{y}) &= \left(1 + \frac{2}{n} r \partial r\right) K_S(\mathbf{x}, \mathbf{y}) \\
&= \frac{(1 + \mathbf{yx})^2}{|1 + \mathbf{yx}|^{n+2}} + \frac{n-2}{n} \frac{\mathbf{yx}}{|1 + \mathbf{yx}|^n}
\end{aligned}
$$

In a similar way we have the vector spaces of harmonic functions B and S, which also have a reproducing kernel. The reproducing kernel of S is called the Poisson kernel and is explicitly given (for the variables $\mathbf{y} = s\eta$ and $\mathbf{x} = r\xi$) by

$$
\begin{aligned}
P(s\eta, r\xi) &= \frac{1 - r^2 s^2}{(1 - 2rs(\eta, \xi) + r^2 s^2)^{n/2}} \\
&= \frac{1 - r^2 s^2}{s^n |\frac{1}{s}\eta - r\xi|^n} \\
&= \frac{1 - r^2 s^2}{r^n |s\eta - \frac{1}{r}\xi|^n} \\
&= \sum_k r^k s^k h_k((\eta, \xi)) \\
&= \sum_k r^k s^k \sum_i F_k^{(i)}(\eta) F_k^{(i)}(\xi)
\end{aligned}
$$

where $h_k((\eta, \xi))$ is a spherical harmonic of degree k both in η and in ξ, the explicit form of which we are not interested in here, and the $F_k^{(i)}$ constitute, for

each k fixed, an orthonormal basis for the harmonics of degree k with respect
to the inner product obtained by integration over the unit sphere. Note that
it is not customary to write down the Poisson kernel for $s \neq 1$. However it
is clear that the expressions above must follow from the $s = 1$ case, as the
development in spherical harmonics for η (which follows from the fact that
$P(s\eta, r\xi)$ is harmonic in the variable $s\eta$) shows that it is sufficient to replace
r in the classical expression by rs.

Notice that the identity

$$P(\mathbf{x}, \mathbf{y}) = K_S(\mathbf{x}, \mathbf{y}) - \mathbf{y} K_S(\mathbf{x}, \mathbf{y}) \mathbf{x}$$

holds, as a straightforward calculation shows.

The space B similarly has as kernel $H(\mathbf{x}, \mathbf{y})$ where

$$H(\mathbf{x}, \mathbf{y}) = K_B(\mathbf{x}, \mathbf{y}) - \mathbf{y} K_B(\mathbf{x}, \mathbf{y}) \mathbf{x}.$$

Both identities follow in a straightforward way if we consider antimonogenic
functions, that is solutions of the equation $\partial_x \mathbf{x}^{-1} f(\mathbf{x}) = 0$, in other words,
functions of the form $f(\mathbf{x}) = \mathbf{x} g(x)$, where g is monogenic. If we call $\mathbf{B}^{\#}$ and
$\mathbf{S}^{\#}$ the modules of antimonogenic functions corresponding to $\mathbf{B}$ and $\mathbf{S}$, their
reproducing kernels are obviously $-\mathbf{y} K_B(\mathbf{x}, \mathbf{y}) \mathbf{x}$ and $-\mathbf{y} K_S(\mathbf{x}, \mathbf{y}) \mathbf{x}$. As B resp.
S are the direct orthogonal sum of $\mathbf{B}$ and $\mathbf{B}^{\#}$ ($\mathbf{S}$ and $\mathbf{S}^{\#}$) these identities
follow directly.

4. Vector Valued Monogenics in the Unit Ball

Let $\mathcal{B}$ be the subspace of $\mathbf{B}$ of vector valued monogenics which inherits the
scalar valued inner product $[\cdot, \cdot]_B = \mathbf{Re}(\cdot, \cdot)_B$ from $\mathbf{B}$. The mapping, for $\mathbf{x}$
fixed, sending f to $f_i(\mathbf{x})$ is obviously continuous, so there exist unique functions
$B_i(\mathbf{x}, \mathbf{y})$ in V such that $B_i(\mathbf{x}, \cdot)$ is in $\mathcal{B}$ and $(B_i(\mathbf{x}, \cdot), f) = f_i$. But of course,
$B_i(\cdot, \mathbf{y})$ is not an element of $\mathcal{B}$ considered as a function of the first variable.

The aim of this paragraph is to find an expression for these 'reproducing
kernels' B_i. Let f be a monogenic function. Then there exists a harmonic
function F such that $f = \partial_x F$. If f is vector valued, we can (at least locally)
take $F = \int f_i dx_i$ and so F is real valued. In the unit ball this holds globally
because of the topological structure of the domain. Notice that if f is a poly-
nomial (homogeneous) of degree n in the variables $x_1, \cdots, x_n$ then F can be
taken a polynomial (homogeneous) of degree $n+1$. Take now two vector valued
monogenics $f = \partial_x F$ and $g = \partial_x G$, where $F = \bar{F}$ because it is real valued, and

$f = -\bar{f}$ because it is vector valued. Then

$$[f,g]_B = [\partial_x F, \partial_x G]_B = \mathbf{Re}\left(\int_B \overline{\partial_x F}(\partial_x G)\right)$$

$$= \mathbf{Re}\left(-\int_B F(\partial_x^2 G) - \int_{S^{n-1}} F\mathbf{x}\partial_x G\, dS\right)$$

$$= \int_{S^{n-1}} F\partial_r G dS$$

For the third transition Stokes' formula 1 was used. Notice that the normal in the point $\mathbf{x}$ of the sphere is equal to $\mathbf{x}$. The fourth transition is valid since $\partial_x^2 G = -\Delta G = 0$, G being harmonic, and $-\mathbf{x}\partial_x = (\partial_r + \Gamma)$ because $r = 1$ on the sphere, where ΓG is bivector valued. Notice moreover that if h is homogeneous of degree t then

$$\int_B h(\mathbf{x})d\mathbf{x} = \int_0^1 r^{t+n-1} \int_S h dS = \frac{1}{t+n}\int_S h dS$$

so we get, if f and g are both homogeneous of degree k,

$$[f,g]_S = (2k+n)[f,g]_B$$

$$= \int_S F(k+1)G dS = (k+1)(2k+n+2)\int_B FG$$

$$= (k+1)[F,G]_S = (k+1)(2k+n+2)[F,G]_B,$$

while $[f,g]_S = [f,g]_B = [F,G]_S = [F,G]_B$ if f and g are homogeneous of different degrees.

Let us now take an orthonormal basis $F_k^{(j)}$ of S, each $F_k^{(j)}$ being homogeneous of degree k.

THEOREM 4.5.   - $\sqrt{2k+n}F_k^{(j)}$ is an orthonormal basis for B.
  - $\frac{1}{\sqrt{k+1}}\partial_x F_{k+1}^{(j)}$ is an orthonormal basis of $\mathcal{S}$.
  - $\sqrt{\frac{2k+n}{k+1}}\partial_x F_{k+1}^{(j)}$ is an orthonormal basis of $\mathcal{B}$.
  - $\sum_{k,j} F_k^{(j)}(\mathbf{y})F_k^{(j)}(\mathbf{x}) = P(\mathbf{x},\mathbf{y})$.
  - $\sum_{k,j}(2k+n+2)F_k^{(j)}(\mathbf{y})F_k^{(j)}(\mathbf{x}) = H(\mathbf{x},\mathbf{y})$.
  - $\sum_{k,j} \frac{1}{k+1}\partial_y F_{k+1}^{(j)}(\mathbf{y})\partial_i F_{k+1}^{(j)}(\mathbf{x}) = B^{(i)}(\mathbf{x},\mathbf{y})$.
  - $\sum_{k,j} \frac{2k+n}{k+1}\partial_x F_{k+1}^{(j)}(\mathbf{y})\partial_i F_{k+1}^{(j)}(\mathbf{x}) = S^{(i)}(\mathbf{x},\mathbf{y})$.

Proof.

The property that, for an orthonormal basis $\phi_k(\mathbf{x})$, the reproducing kernel is given by $\sum_k \phi_k(\mathbf{y})\phi_k(\mathbf{x})$, holds for every Hilbert module. The one before last equality is immediately clear from the fact that $[B^{(i)}(\mathbf{x}, \cdot), \partial_y F_{k+1}^{(j)}]_B = \partial_i F_{k+1}^{(j)}(\mathbf{x})$, thus giving the coefficient of $\partial_y F_{k+1}^{(j)}(\mathbf{y})$ in the series development of $B^{(i)}(\mathbf{x}, \mathbf{y})$; the last equality is proved in a similar way. ∎

In order to calculate the functions explicitly we use the spherical coordinates $\mathbf{x} = r\boldsymbol{\xi}$ and $\mathbf{y} = s\boldsymbol{\eta}$ and first take the function

$$
\begin{aligned}
G(\mathbf{x}, \mathbf{y}) &= \sum_{k,j} \partial_y F_{k+1}^{(j)}(\mathbf{y}) F_{k+1}^{(j)}(\mathbf{x}) \\
&= \partial_y P(\mathbf{x}, \mathbf{y}) \\
&= \partial_y \left(K_S(\mathbf{x}, \mathbf{y}) - \mathbf{y} K_S(\mathbf{x}, \mathbf{y}) \mathbf{x} \right) \\
&= (n - 2\Gamma_y) K_S(\mathbf{x}, \mathbf{y}) \mathbf{x} \\
&= (n + 2s\partial_s) K_S(\mathbf{x}, \mathbf{y}) \mathbf{x}.
\end{aligned}
$$

For these expressions the fact that K_S is monogenic in $\mathbf{y}$ has been used (this implies that $\Gamma_{\mathbf{y}} K_s = -s\partial_s K_s$) as well as the equation $\partial_y(\mathbf{y}f) = -nf + \mathbf{y}\partial_y f + 2\Gamma f$, the normal convergence being clear. Then

$$
B^{(i)}(\mathbf{x}, \mathbf{y}) = \partial_{x_i} \frac{1}{s^2} \int_0^s \rho G(\mathbf{x}, \rho\boldsymbol{\eta}) d\rho
$$

and

$$
S^{(i)}(\mathbf{x}, \mathbf{y}) = 2\partial_{x_i} G(\mathbf{x}, \mathbf{y}) + (n-2) B^{(i)}(\mathbf{x}, \mathbf{y}).
$$

Calculation of the integral gives

$$
\begin{aligned}
\int_0^s \rho G(\mathbf{x}, \rho\boldsymbol{\eta}) d\rho &= \int_0^s \rho(n + 2\rho\partial_\rho) K_S(\mathbf{x}, \rho\boldsymbol{\eta}) \mathbf{x} \, d\rho \\
&= 2s^2 K_S(\mathbf{x}, \mathbf{y}) \mathbf{x} + (n-4) \int_0^s \rho K(\mathbf{x}, \rho\boldsymbol{\eta}) \mathbf{x} \, d\rho,
\end{aligned}
$$

and the last integral can be written as

$$
\begin{aligned}
&\int_0^s \rho K(\mathbf{x}, \rho\boldsymbol{\eta}) \mathbf{x} \, d\rho \\
&= \mathbf{x} \int_0^s \frac{\rho}{(1 - 2\rho\langle\mathbf{x},\eta\rangle + r^2\rho^2)^{n/2}} d\rho - r^2\eta \int_0^s \frac{\rho^2}{(1 - 2\rho\langle\mathbf{x},\eta\rangle + r^2\rho^2)^{n/2}} d\rho.
\end{aligned}
$$

In order to obtain a series decomposition in homogeneous parts of the kernels $B^{(i)}$ and $S^{(i)}$ one can decompose the integrand in homogeneous parts and integrate termwise. It should be noted that, for both kernels, the part which is homogeneous of degree k in $\mathbf{x}$ is also homogeneous of the same degree in $\mathbf{y}$, as is shown immediately by the expression of the kernels in tems of the functions $F_k^{(j)}$. Moreover, since the projection operators P_k are orthogonal we have that

$$[P_k B^{(i)}(\mathbf{x}, \cdot), f]_B = [P_k B^{(i)}(\mathbf{x}, \cdot), P_k f]_B,$$

in other words $P_k B^{(i)}(\mathbf{x}, \mathbf{y})$ is the ith reproducing kernel of the module $P_k \mathcal{B}$ and similarly $P_k S^{(i)}(\mathbf{x}, \mathbf{y})$ is the ith reproducing kernel of the module $P_k \mathcal{S}$. Another way to obtain the series expansion is to start from the series expansion for the Poisson kernel (which can be found e.g. in [6]):

$$\sum_k \frac{2k + n - 2}{n - 2} r^k s^k C_k^{n/2-1}((\boldsymbol{\eta}, \boldsymbol{\xi})).$$

This series expansion can be looked upon as the series expansion for P in terms of the $F_k^{(j)}$, after performing the summations over j. Applying ∂_y to one of the terms given (we use the non-indexed symbol ∂ to denote derivation with respect to the argument):

$$\partial_y r^k s^k C_k^{n/2-1}((\boldsymbol{\eta}, \boldsymbol{\xi}))$$

$$= \sum_i s^{k-1} r^k \left(k \frac{e_i y_i}{s} C_k^{n/2-1}((\boldsymbol{\eta}, \boldsymbol{\xi})) + \left[-\frac{e_i y_i}{s}(\boldsymbol{\eta}, \boldsymbol{\xi}) + \frac{e_i x_i}{r} \right] \partial C_k^{n/2-1}((\boldsymbol{\eta}, \boldsymbol{\xi})) \right)$$

$$= s^{k-1} r^k \left(\frac{\mathbf{y}}{s}(k - (\boldsymbol{\eta}, \boldsymbol{\xi})\partial) C_k^{n/2-1} + \frac{\mathbf{x}}{r} \partial C_k^{n/2-1} \right)$$

$$= (n-2) s^{k-1} r^k \left(\frac{\mathbf{y}}{s} C_{k-2}^{n/2} + \frac{\mathbf{x}}{r} C_{k-1}^{n/2} \right)$$

using equations (23-24) on p. 176 of [5], taking the derivative w.r.t. x_i gives

$$\partial_{x_i}(n-2)\left(-s^{k-2}r^k\mathbf{y}C_{k-2}^{n/2}+s^{k-1}r^{k-1}\mathbf{x}C_{k-1}^{n/2}\right)$$

$$=(n-2)\left(-k\frac{x_i}{r}s^{k-2}r^{k-1}\mathbf{y}C_{k-2}^{n/2}-\mathbf{y}s^{k-2}r^{k-1}\left[-\frac{x_i}{r}(\boldsymbol{\eta},\boldsymbol{\xi})+\frac{y_i}{s}\right]\partial C_{k-2}^{n/2}\right.$$

$$+\left[e_i+(k-1)\frac{x_i\mathbf{x}}{r^2}\right]s^{k-1}r^{k-1}C_{k-1}^{n/2}+$$

$$\left.s^{k-1}r^{k-2}\mathbf{x}\left[-\frac{x_i}{r}(\boldsymbol{\eta},\boldsymbol{\xi})+\frac{y_i}{s}\right]\partial C_{k-1}^{n/2}\right)$$

$$=(n-2)\left(-x_i\mathbf{y}s^{k-2}r^{k-2}\left[(k-(\boldsymbol{\eta},\boldsymbol{\xi})\partial)C_{k-2}^{n/2}\right]-y_i\mathbf{y}s^{k-3}r^{k-1}\partial C_{k-2}^{n/2}\right.$$

$$+e_is^{k-1}r^{k-1}C_{k-1}^{n/2}+s^{k-2}r^{k-2}y_i\mathbf{x}\partial C_{k-1}^{n/2}$$

$$\left.x_i\mathbf{x}s^{k-1}r^{k-3}\left[((k-1)-(\boldsymbol{\eta},\boldsymbol{\xi})\partial)C_{k-1}^{n/2}\right]\right)$$

$$=(n-2)(sr)^{k-3}\left(nrsx_i\mathbf{y}C_{k-4}^{n/2+1}-nr^2y_i\mathbf{y}C_{k-3}^{n/2+1}\right.$$

$$\left.+e_is^2r^2C_{k-1}^{n/2}+nsry_i\mathbf{x}C_{k-2}^{n/2+1}-ns^2x_i\mathbf{x}C_{k-3}^{n/2+1}\right)$$

5. Properties of the Kernels

Let us consider again the functions

$$B(\mathbf{x},\mathbf{y})=\frac{1}{s^2}\int_0^s \rho G(\mathbf{x},\rho\boldsymbol{\eta})d\rho$$

and

$$S(\mathbf{x},\mathbf{y})=2G(\mathbf{x},\mathbf{y})+(n-2)K(\mathbf{x},\mathbf{y})$$

satisfying $B^{(i)}=\partial_{x_i}B$ and $S^{(i)}=\partial_{x_i}S$. These functions can be used to calculate the potential for a vector field in $\mathcal{B}$ or $\mathcal{S}$.

THEOREM 5.6.   - Let f be any function in $\mathcal{B}$. Then the function $F=[B(\mathbf{x},\cdot),f]_B$ is defined in the open unit ball and satisfies the equation $\partial_x F=f$ there.
    - Let f be any function in $\mathcal{S}$. Then the function $F=[S(\mathbf{x},\cdot),f]_S$ is defined in the open unit ball and satisfies the equation $\partial_x F=f$ there.

Moreover we have an invariance relation of the spaces $\mathcal{B}$ and $\mathcal{S}$ under the group $\mathrm{Spin}(n)$. Indeed, for α in the Spin group we have that $f^\alpha(\mathbf{x})$, which is defined as $\alpha f(\alpha^{-1}\mathbf{x}\alpha')\alpha'^{-1}$ is in $\mathcal{B}$ ($\mathcal{S}$) if f is. The mapping $\chi(\alpha) : f \to f^\alpha$ is an isometry, and has as adjoint $(\chi(\alpha))^* = \chi(\alpha^{-1})$. Moreover the Dirac operator commutes with $\chi(\alpha)$. As $F^{\alpha^{-1}}(\mathbf{x}) = F(\alpha\mathbf{x}\alpha'^{-1})$ one has that $F^{\alpha^{-1}}(\mathbf{x}) = [B(\alpha\mathbf{x}\alpha'^{-1},\cdot),f]_B$ while on the other hand $F^\alpha(\mathbf{x}) = [B(\mathbf{x},\cdot),f^\alpha]_B$. Taking the adjoint, this is equal to $[B^{\alpha^{-1}}(\mathbf{x},\cdot),f]_B$, where the index α^{-1} is to be applied to the second variable of B. Combining these two gives
$F(\mathbf{x}) = [B^{\alpha^{-1}}(\alpha\mathbf{x}\alpha'^{-1},\cdot),f]_B$. Since this holds for all $f \in \mathcal{B}$ it follows that $B^{\alpha^{-1}}(\alpha\mathbf{x}\alpha'^{-1},\mathbf{y}) = \alpha^{-1}B^{\alpha^{-1}}(\alpha\mathbf{x}\alpha'^{-1},\alpha\mathbf{y}\alpha'^{-1})\alpha' = B(\mathbf{x},\mathbf{y})$. As a corrolary we have the theorem

THEOREM 5.7. For dimension $n \neq 3$ B has the form

$$B(\mathbf{x},\mathbf{y}) = \mathbf{x}A(\langle\mathbf{x},\mathbf{y}\rangle,|\mathbf{x}|,|\mathbf{y}|) + \mathbf{y}D(\langle\mathbf{x},\mathbf{y}\rangle,|\mathbf{x}|,|\mathbf{y}|).$$

Proof.
Assume first that $\mathbf{x}$ and $\mathbf{y}$ are linearly independent. Let $\mathbf{z}$ be the part of B orthogonal to $\mathbf{x}$ and $\mathbf{y}$ and let α leave $\mathbf{x}$ and $\mathbf{y}$ invariant. Then $\alpha\mathbf{z}\alpha'^{-1} = \mathbf{z}$. Since this holds for all such α $\mathbf{z}$ must be zero (except when the dimension n is 3). Moreover it is clear that the only scalar invariants under $\mathrm{Spin}(n)$ are $\langle\mathbf{x},\mathbf{y}\rangle$, $|\mathbf{x}|$ and $|\mathbf{y}|$, so A and D depend only on these.   ■

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