THE DOWNS-THOMPSON PARADOX IN MULTIMODAL NETWORKS

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Abstract: Users of the transportation networks generally choose their routes in an independent and uncoordinated way in order to minimize their own perceived costs. This non-cooperative behaviour can lead to a suboptimal utilization of the network and, in some situations, increasing the network capacity can make the subutilization even worse. Such phenomenon is described in literature as traffic or network paradoxes. This paper provides a review on two famous network paradoxes, and also introduces a new one.

Keywords: Traffic Paradoxes, Network Paradoxes, Downs-Thomson Paradox, Braess Paradox

1. Introduction

Transportation networks are designed to allow the movement of individuals, the distribution of goods and the execution of service operations. These are essential activities in any society and the efficient management of the transportation networks is vital for its economic development and well-being. The amount of time people spend going to and from work; the logistic costs of goods and services and the pollution level are just some examples of quality of life factors which are very closely related to the efficiency of transportation networks.

In this context, the task of traffic managers is to optimize the flows in a transportation network considering the population's needs and requirements. This means that the network should be designed, controlled and operated in a way to minimize average travel times; avoid traffic jams; reduce air, noise and visual pollution, etc.

The users of the transportation networks, however, may not have exactly the same perception and objectives as the traffic managers. For instance, individuals will choose the routes which minimize their own travel times (rather than the global average); companies will place their facilities in the most economically convenient areas (even if it causes congestion and impacts other users); service providers will schedule their activities in order to maximize their gains (even during rush hours, if the returns compensate). In general, users analyse how the network impact their businesses, not how their businesses impact the network. Not surprisingly, this selfish behaviour can lead to a subutilization of the network capacity and potentially to a net economic loss for the whole society.

Notwithstanding this fact, this lack of coordination is accepted as a natural part of the current socio-political systems of developed economies, which not unjustifiably foster freedom and concurrency as key factors for growth. Given that it cannot – and maybe should not – be easily changed, it is natural to think that the only way to improve the performance of a transportation network is by increasing its capacity. But here a more surprising conclusion has been found in research: in this non-cooperative environment, increasing network capacity (e.g. adding or expanding a road) may actually decrease the overall performance, even when the only and explicit objective of all users is to minimize their travel times.

This counterintuitive fact has been demonstrated by some paradoxes, among which the Downs-Thomson Paradox (Downs, 1962) and the Braess Paradox (Braess, 1968) can be highlighted. Real evidences of these paradoxes have been found and studied, and many experiments have been proposed based on them (Denant-Boèmont et Hammiche, 2010; A. Rapoport et al, 2009). These paradoxes have received growing attention over the recent years, as they seem to occur in other types of networks, such as computer networks and power distribution (Max Planck Institute, 2012).

This paper intends to review the above-mentioned paradoxes and provide some possibly new perspectives from which they could be studied in Operations Research. A third paradox, which is, for the best of our knowledge, still unexplored in literature, is also introduced. It is named *The Downs-Thomson Paradox in Multimodal Networks* and is described in Section 5. Before presenting the paradoxes, we first discuss some theoretical aspects on which our analysis is based.

2. Game Theory Applied To Transportation Networks

As stated in the previous section, the users of a transportation network will generally take decisions in a non-cooperative way in order to maximize their own benefits. Although it would be very difficult to define what each user considers as benefit, it can be assumed that almost every user intends to minimize his travel time, and this is what really matters from the network management point of view.

In the problems discussed in this document it will be considered that a number of individuals travel from one to another point of a network, and that they do it on a regular basis. Each individual wants to minimize his travel time and will therefore choose the route which he perceives as the fastest. Individuals with longer travel times tend to copy the ones with shorter travel times and, eventually, an equilibrium situation is reached, in which the travel time of all users is the same and the flows in each route remain constant. In a game theoretical point of view, such a situation can be seen as a Nash equilibrium.

To illustrate this approach, suppose that 1,000 vehicles/h travel daily from city A to city B during peak-hours using the transportation network depicted in Figure 1.

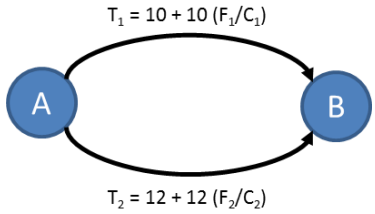

Figure 1
Two separate roads connect two cities

There are two separate roads connecting the cities. The travel time on the first road (T_1) is 10 minutes without traffic, and it rises linearly with the ratio of traffic flow (F_1) to road capacity (C_1). Similarly, the travel time on the second road (T_2) is 12 minutes without traffic, and it rises linearly with the ratio of traffic flow (F_2) to road capacity (C_2).

$$T_1 = 10 + 10\,(F_1/C_1);\ T_2 = 12 + 12\,(F_2/C_2) \tag{1}$$

Let us consider that 600 travellers initially use the first road while the other 400 use the second road, and that the capacities C_1 and C_2 are respectively 400 and 480 vehicles/h. In this situation, the travel times are the following:

$$T_1 = 10 + 10\,(600/400) = 25 \text{ min};\ T_2 = 12 + 12\,(400/480) = 22 \text{ min} \tag{2}$$

Once this information is spread among the travellers, some individuals using the first road will change their route on the following day. It can be expected that, after some time, the travellers will divide themselves in a way that the travel times of both routes are the same or at least very similar.

To determine the equilibrium flows and travel time, the following system of equations can be used:

$$\begin{cases} T_i = T_j \quad \forall i,j\,, i \in \{1 \dots N\}, j \in \{1 \dots N\} \\ \sum_{i=1}^{N} F_i = F_{total} \end{cases} \tag{3}$$

where:

T_i : travel time of route i
F_i : flow of route i
N : number of possible routes from A to B
F_{total} : total flow of individuals from A to B

If the solution of the system contains a negative flow for some route i, it is not valid. It means that the route related to this flow will not be used in equilibrium (F_i will actually be zero). In order to find a valid solution, the equations related to this route should be removed and the system should be solved again.

In the example described in this section, the system becomes:

$$\begin{cases} T_1 = T_2 \Rightarrow 10 + 10(F_1/400) = 12 + 12(F_2/480) \\ F_1 + F_2 = 1000 \end{cases} \tag{4}$$

Solving it, we find that 540 travellers will take the first road and 460 will take the second when the equilibrium is reached. The travel time for all of them will be 23.5 minutes.

$$F_1 = 540;\ F_2 = 460;\ T_1 = 10 + 10\,(540/400) = 23.5 \text{ min};\ T_2 = 12 + 12\,(460/480) = 23.5 \text{ min} \tag{5}$$

The solution is graphically presented in Figure 2.

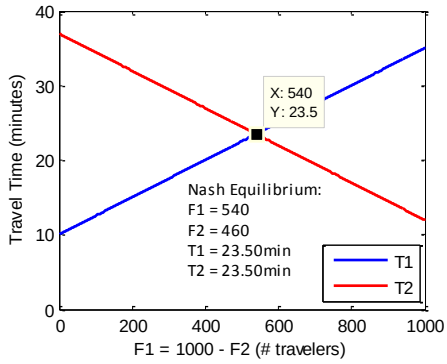

Figure 2
Graphical solution for the Nash Equilibrium

The equilibrium solution of this example is also Pareto optimal, since it is impossible to decrease the travel time of one individual without increasing the travel time of others. Furthermore, an increase of the capacities C_1 or C_2 will certainly decrease the travel time of all network users. There is no paradox in this network topology.

This game theoretical approach can be used, for instance, to minimize the equilibrium travel time in a transportation network given a limited budget and the costs of expanding the capacity of each road. Of course, in realistic problems, many other variables, constraints and objectives would be involved; multiple sources and destinations would exist; multiple types of transportation means would be available; etc. Nonetheless, the game theoretical concepts can undoubtedly provide useful insights and better solutions.

3. The Downs-Thomson Paradox

Suppose that in the previous example the travellers could choose between using the first road and a privately operated train line. The travel time on the road (T_1) is 10 minutes without traffic, and it rises linearly with the ratio of traffic flow (F_1) to road capacity (C_1). The travel time by train (T_2), on the other hand, depends on its frequency, which is adjusted by the train operator accordingly to the demand for the service, i.e. the number of users (F_2). The maximum travel time is 20 minutes, and this can be shortened in the rate of 1 minute per 300 users.

$$T_1 = 10 + 10\,(F_1/C_1);\ T_2 = 20 - (F_2/300) \tag{6}$$

This transportation network is presented in Figure 3.

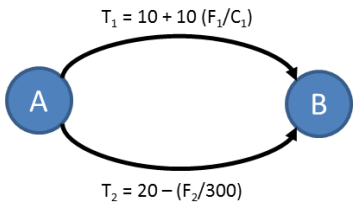

Figure 3 – A road and a train line connect two cities

The equilibrium solution is found by solving the following system of equations:

$$\begin{cases} T_1 = T_2 \Rightarrow 10 + 10(F_1/C_1) = 20 - (F_2/C_2) \\ F_1 + F_2 = 1000 \end{cases} \tag{7}$$

Let us consider that the road capacity C_1 is initially 500 vehicles/h. The equilibrium solution is then:

$$F_1 = 400;\ F_2 = 600;\ T_1 = 10 + 10\,(400/500) = 18\text{min};\ T_2 = 20 - (600/300) = 18\text{min} \tag{8}$$

In equilibrium, 400 travellers will take the road and 600 travellers will take the train. The travel time for everyone is 18 minutes. The graphical solution for this situation is presented in the graph on the left of Figure 4.

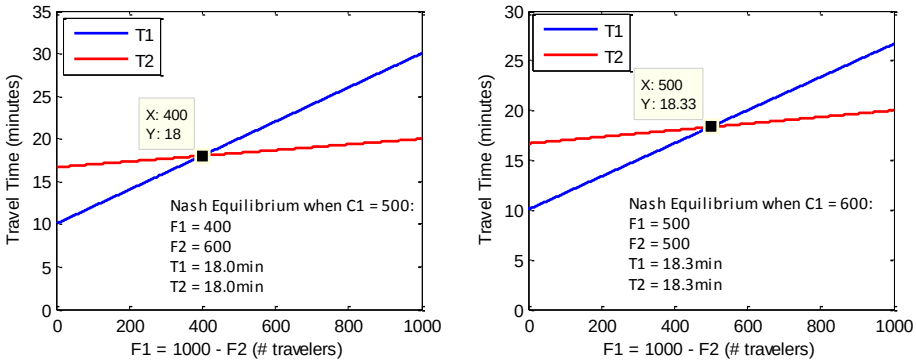

Figure 4
Nash Equilibrium when C_1 = 500 (left) and when C_1 = 600 (right)

If the road capacity C_1 is expanded to 600 vehicles/h, the equilibrium solution becomes:

$$F_1 = 500;\ F_2 = 500;\ T_1 = 10 + 10\ (500/600) = 18.33\text{min};\ T_2 = 20 - (500/300) = 18.33\text{min} \qquad (9)$$

In this situation, 500 travellers will take the road and 500 travellers will take the train. The travel time for everyone becomes 18.33 minutes. Here is the paradox: the increase of the road capacity (C_1) causes an increase on the travel time for all the users. The graphical solution for this situation is presented in the graph on the right of Figure 4.

The paradox does not occur for any increase of C_1. For instance, if C_1 is increased to 2,000 vehicles/h, all travellers will take the road, since the worst travel time using it is T_1 = 10 + 10 (1000/2000) = 15min, which is shorter than the best travel time by train T_2 = 20 – (1000/300) = 16.67min.

The relation between the equilibrium travel time and the road capacity C_1 is shown in Figure 5. Remark that if the road capacity C_1 is less than 1,500 vehicles/h, the equilibrium travel time can be reduced simply by closing it (i.e., making C_1 = 0). In other words, the Nash Equilibrium when $0 < C_1 < 1{,}500$ is not Pareto Optimal.

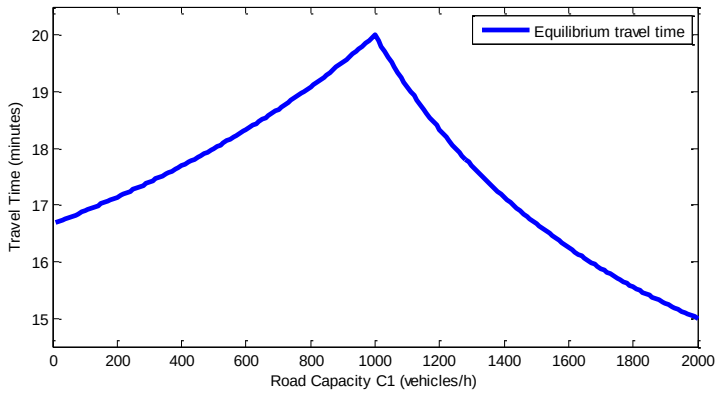

Figure 5
The equilibrium travel time in function of the road capacity C_1

4. The Braess Paradox

Suppose now that the 1,000 travellers from the previous examples use the network depicted in Figure 6.

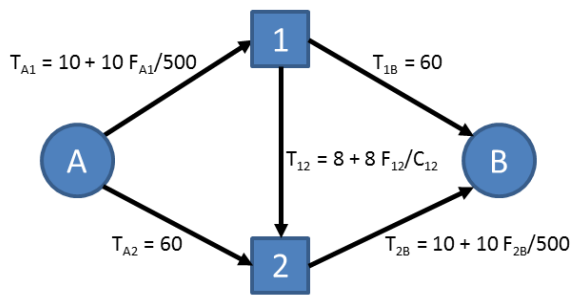

Figure 6
A network with three possible paths from A to B: A-1-B, A-2-B and A-1-2-B

The cities are separated by a river and the travellers have three options to go from city A to city B. In the first option, they can first take a narrow road from city A to point 1, in which the travel time increases with the number of vehicles, and then take a ferry from point 1 to city B. The frequency and capacity of the ferry can always suffice the demand, and the travel time is 60 minutes. The second option is similar to the first, but first they take a ferry from city A to point 2 and from there a narrow road to city B. A third option is to go from city A to point 1 using the same narrow road as in the first option, then take a narrow bridge from point 1 to point 2, and finally go from point 2 to city B using the same narrow road as in the second option. The travel time on the bridge (T_{12}) is 8 minutes without traffic, and it rises linearly with the ratio of traffic flow (F_{12}) to bridge capacity (C_{12}).

$$T_{A1} = 10 + 10\, F_{A1}/500 \tag{10}$$
$$T_{1B} = 60 \tag{11}$$
$$T_{A2} = 60 \tag{12}$$
$$T_{2B} = 10 + 10\, F_{2B}/500 \tag{13}$$
$$T_{12} = 8 + 8\, F_{12}/C_{12} \tag{14}$$

The travel times of the three possible paths, i.e. A-1-B, A-2-B and A-1-2-B, can be respectively written as:

$$T_{A1B} = T_{A1} + T_{1B} = 70 + 10\, F_{A1}/500 \tag{15}$$
$$T_{A2B} = T_{A2} + T_{2B} = 70 + 10\, F_{2B}/500 \tag{16}$$
$$T_{A12B} = T_{A1} + T_{12} + T_{2B} = 28 + 10\, F_{A1}/500 + 10\, F_{2B}/500 + 8\, F_{12}/C_{12} \tag{17}$$

Remark that a fourth option A-2-1-B does not have to be considered, since it would take at least 128 minutes ($T_{A2} + T_{12} + T_{1B}$), which is more than the maximum travel times for paths A-1-B and A-2-B. Since the path A-1-2-B share the arcs A-1 and 2-B respectively with the routes A-1-B and A-2-B, the flows F_{A1B}, F_{A2B} and F_{A12B} must satisfy the following equations:

$$F_{A1} = F_{A1B} + F_{A12B};\ F_{2B} = F_{A2B} + F_{A12B} \tag{18}$$

Path A-1-2-B is the only one which contains the arc 1-2. For this reason, the flow F_{A12B} must equal F_{12}.

$$F_{A12B} = F_{12} \tag{19}$$

The equilibrium solution can then be found by solving the following system of equations:

$$\begin{cases} T_{A1B} = T_{A2B} \Rightarrow\ 70 + 10\, F_{A1}/500 = \ 70 + 10\, F_{2B}/500 \\ T_{A1B} = T_{A12B} \Rightarrow\ 70 + 10\, F_{A1}/500 = \ 28 + 10\, F_{A1}/500 + 10\, F_{2B}/500\ + 8\, F_{12}/C_{12} \\ F_{A1B} + F_{A2B} + F_{A12B} = 1000 \\ F_{A1} = F_{A1B} + F_{A12B} \\ F_{2B} = F_{A2B} + F_{A12B} \\ F_{12} = F_{A12B} \end{cases} \tag{20}$$

Considering that the bridge capacity is initially C_{12} is 200 vehicles/h, the solution of the system is:

$$F_{A1B} = 180;\ F_{A2B} = 180;\ F_{A12B} = 640;\ T_{A1B} = T_{A2B} = T_{A12B} = 86.4 \text{ min} \tag{21}$$

In equilibrium, 180 travellers take the path A-1-B, 180 take the path A-2-B and 640 take the path A-1-2-B. The travel time for all of them is 86.4 minutes.

Since the paths A-1-B and A-2-B are equivalent, $F_{A1B} = F_{A2B}$ and $T_{A1B} = T_{A2B}$ in any equilibrium solution. This fact allows the problem to be solved using a two dimensional graph as demonstrated by Figure 7. The X-axis represents the sum of F_{A1B} and F_{A2B}, which is also the same as 1000 - F_{A12B}. On the left, the solution with C_{12} = 200 is presented.

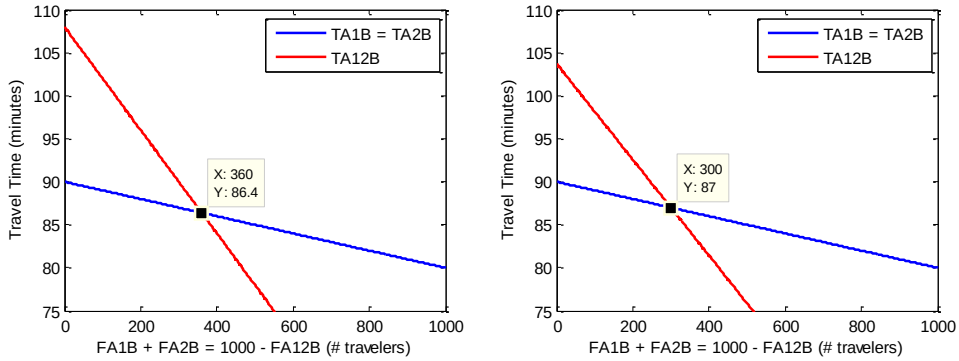

Figure 7
Nash Equilibrium when C_1 = 200 (left) and when C_1 = 224 (right)

If the bridge capacity C_{12} is increased to 224 vehicles/h (right portion of Figure 7), the solution of the system is:

$$F_{A1B} = 150;\ F_{A2B} = 150;\ F_{A12B} = 700;\ T_{A1B} = T_{A2B} = T_{A12B} = 87 \text{ min} \tag{22}$$

An increased bridge capacity motivates more travellers to use it, shifting users from paths A-1-B and A-2-B to the path A-1-2-B. The travel time, however, is also increased, and this is the paradox.

Similarly to the Downs-Thomson Paradox, the Braess Paradox does not occur for any increase of C_{12}. If C_{12} is increased to 500 vehicles/h, for instance, all travellers will take the bridge, and the travel time will be 84 minutes. This is less than the previous results found for C_{12} = 200 and C_{12} = 224. The relation between the equilibrium travel time and the bridge capacity C_{12} can be observed in Figure 8. If the bridge capacity is less than 667 vehicles/h (approximately), the equilibrium travel time can be shortened by making it unavailable (i.e., making C_{12} = 0). The Nash Equilibrium when 0 < C_{12} < 667 is thus not Pareto Optimal.

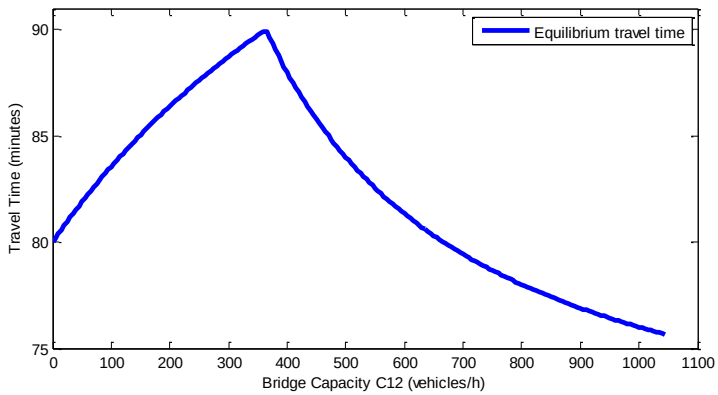

Figure 8
The equilibrium travel time in function of the bridge capacity C_{12}

5. The Downs-Thomson Paradox in Multimodal Networks

The paradox presented in this section represents a situation which can happen in multimodal transportation networks. To illustrate it, let us consider that the 1,000 travellers from the previous examples can go from city A to city B using cars or small buses. The frequency of buses is very high and users never have to wait long for them, although they are slower and the travel times without traffic are longer for them than for cars. There are dedicated lanes for them on some parts of the road; on the other parts, they share the road with cars. A network diagram of this situation is depicted in Figure 9.

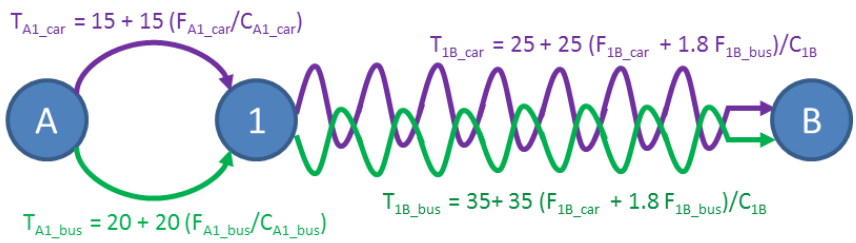

Figure 9
A multimodal network

The arcs A-1 represent the parts of the road in which there are dedicated bus lanes. On this arc, the travel time by car (T_{A1_car}) depends on the flow of cars (F_{A1_car}) and the capacity of the car lanes (C_{A1_car}), while the travel time by bus (T_{A1_bus}) depends on the flow of buses (F_{A1_bus}) and the capacity of the bus lanes (C_{A1_bus}).

$$T_{A1\_car} = 15 + 15\ F_{A1\_car}/C_{A1\_car};\ T_{A1\_bus} = 20 + 20\ F_{A1\_bus}/C_{A1\_bus} \tag{23}$$

The arcs 1-B represent the parts of the road shared by buses and cars. The travel time of each transportation mean depends on the flow of cars (F_{1B_car}) and buses (F_{1B_bus}) and the road capacity (C_{1B}). The congestion impact of buses is higher than the one of cars. This is taken into account in the calculation of the travel times by the inclusion of the Passenger Car Equivalent for buses ($PCE_{bus} = 1.8$) as a averaging factor for F_{1B_bus}. The travel times for cars and buses are respectively given by:

$$T_{1B\_car} = 25 + 25\ (F_{1B\_car} + 1.8\ F_{1B\_bus})/C_{1B};\ T_{1B\_bus} = 35 + 35\ (F_{1B\_car} + 1.8\ F_{1B\_bus})/C_{1B} \tag{24}$$

Remark that sections A-1 and 1-B on the network diagram do not necessarily represent real traffic roads. They can be seen as an illustration of the fragments of the road where the two distinct conditions apply, i.e., parts where there are dedicated bus lanes and parts where the lanes are shared by cars and buses share. In this sense, point 1 may actually not exist and travellers are not able to change from car to buses, nor vice-versa. Mathematically, this fact implies that:

$$F_{car} = F_{A1\_car} = F_{1B\_car};\ F_{bus} = F_{1B\_bus} \tag{25}$$

Made these considerations, the travel times by car and by bus can be rewritten as:

$$T_{car} = T_{A1\_car} + T_{1B\_car} = 40 + 15\ F_{car}/C_{A1\_car} + 25\ (F_{car} + 1.8\ F_{bus})/C_{1B} \tag{26}$$
$$T_{bus} = T_{A1\_bus} + T_{1B\_bus} = 55 + 20\ F_{bus}/C_{A1\_bus} + 35\ (F_{car} + 1.8\ F_{bus})/C_{1B} \tag{27}$$

Each bus can take 10 passengers and the total flow from city A to B is again 1,000 travellers/h. The flow equation is:

$$F_{car} + 10\ F_{bus} = 1000 \tag{28}$$

The equilibrium flow and travel times for this network can then be found by solving the following system of equations:

$$\begin{cases} T_{car} = T_{bus} \\ \quad \Rightarrow 40 + 15\ F_{car}/C_{A1car} + 25\ (F_{car} + 1.8F_{bus})/C_{1B} = 55 + 20\ F_{bus}/C_{A1bus} + 35\ (F_{car} + 1.8F_{bus})/C_{1B} \\ F_{car} + 10\ F_{bus} = 1000 \end{cases} \tag{29}$$

Considering that the capacities are $C_{A1\_car} = 140$; $C_{A1\_bus} = 100$; $C_{1B} = 200$, the system yields the following results:

$$F_{car} = 510;\ F_{bus} = 49;\ T_{car} = T_{bus} = 169.6\text{min} \tag{30}$$

If the capacity of arc A-1 is increased to $C_{A1\_car} = 190$, the solution of the system becomes:

$$F_{car} = 680;\ F_{bus} = 32;\ T_{car} = T_{bus} = 190.5\text{min} \tag{31}$$

Both solutions can be graphically observed in Figure 10. As in the paradoxes previously described, the increase of the capacity resulted in an increase of the equilibrium travel time. Figure 11 exhibits the relation between the equilibrium travel time and C_{A1_car}. Note that the minimum equilibrium travel time occurs for $C_{A1\_car} = 0$. This means that the shortest travel time would be achieved when all travellers take the buses. In other words, the Nash Equilibrium is never Pareto Optimum when $C_{A1\_car} > 0$.

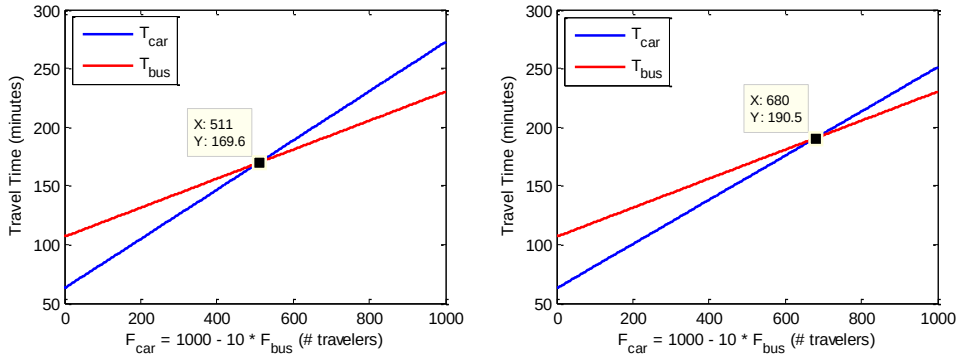

Figure 10
Nash Equilibrium when C_{A1_car} = 140 (left) and when C_{A1_car} = 190 (right)

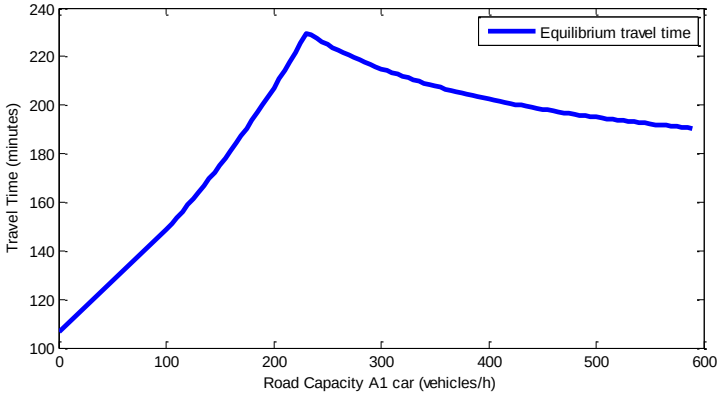

Figure 11
The equilibrium travel time in function of the road capacity C_{A1_car}

The paradox presented in this section can be seen as a version of the Down-Thomson Paradox, in which equilibrium travel times decrease as users shift from public to private transportation means. An important difference is that the worsening of the travel time of public transport in the original formulation is caused by a disinvestment due to a reduced demand for the service (e.g., less frequent trains), while in this version of the paradox it is directly caused by the increase of the number of cars competing with buses on the shared roads, which is a direct result of the network users' behaviour and not an external decision.

Consequently, in the original version of the paradox, if the public transport operator would not adapt its service level to the demand, either no paradox would occur (i.e., an increase of the private transport capacity would always decrease the equilibrium travel time) or the travel time would not change (we could refer here to the Pigou-Knight-Downs Paradox, which has not been discussed in this work). This means that the equilibrium travel time could be reduced if the public transport operator would receive subsides to improve its service level (e.g., more frequent trains). In the version presented in this section, however, no action can be taken by the public transport operator (e.g., more frequent buses) in order to avoid the worsening of the equilibrium travel time.

6. Concluding Remarks

Arnott, Richard and Kenneth Small (1994) have cleverly pointed out two reasons for the paradoxes. The first reason is latent demand, i.e., when the capacity of a route is expanded, it is likely that the number of users of that route will increase. The second reason is that congestion is mispriced, since drivers do not pay for the losses they impose on the others. The solution proposed by these authors was congestion pricing: users should pay to use (congested) roads. They have mathematically shown how it would avoid the paradoxes, highlighted the growing technology which could make it feasible and discussed about the economic and political issues which could rise by implementing such a system. From the game theoretical point of view, this approach creates a favourable setting for the competitive strategy that the network users tend to follow, and ensures that the Nash Equilibrium is always Pareto Optimum.

Among the transport network users, however, are the logistics operators, who act in an increasingly coordinated way, using the growing knowledge and technology to organize their activities (Benjelloun and Crainic, 2008). They appear as a special class of players, governing a substantial part of the total flow in the network and having more complex objectives than simply reducing the length of a single trip. The progressive organization, rationality and importance of these players allow new optimization possibilities through the cooperation between them and the traffic managers.

A small example of how cooperation can lead to a better utilization of the transportation network is this: large logistic operators avoid overloaded routes at busy periods, while the traffic managers, in return, give them priority at traffic lights or dedicated lanes, as frequently done for public transportation. It is important to highlight, however, that to perform cooperation in an optimal way, it is necessary to integrate the systems (Traffic Management and Logistic Optimization), and this requires a very good understanding of the interactions between the variables involved and the possible trade-offs.

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