A new Hilbert transform in Hermitean Clifford analysis

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Abstract

Orthogonal (or standard) Clifford analysis is a higher dimensional function theory offering a refinement of classical harmonic analysis. More recently, Hermitean Clifford analysis has emerged as a new and successful branch of Clifford analysis, offering yet a refinement of the orthogonal case. A new integral transform is introduced in this Hermitean context, obtained either as the composition of two orthogonal Clifford–Hilbert transforms or as the commutator of two new Hermitean Clifford–Hilbert transforms. The resulting operator is shown to possess the typical properties of a classical Hilbert transform as well. Its connections with standard Clifford–Hilbert transforms are explicitly investigated, and in particular new Hardy spaces associated to this operator are defined and characterized. Some results also allow for a nice geometric interpretation.

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1 Introduction

In engineering sciences, and in particular in signal analysis, the one–dimensional Hilbert transform of a real signal $u(t)$ of a one–dimensional time variable t has become a fundamental tool. For a suitable function or distribution $u(t)$ its Hilbert transform is given by the Cauchy Principal Value

$$\mathcal{H}[u](t) = -\frac{1}{\pi}\,\mathrm{Pv}\int_{-\infty}^{+\infty} \frac{u(\tau)}{\tau - t}\, d\tau$$

Though initiated by Hilbert, the concept of a "conjugated pair" $(u, \mathcal{H}[u])$, nowadays called a Hilbert pair, was developed mainly by Titchmarch and Hardy.

The multidimensional approach to the Hilbert transform usually is a tensorial one, considering the so–called Riesz transforms in each of the cartesian variables separately. As opposed to these tensorial approaches, Clifford analysis (see e.g. [5, 10, 11]) is particularly suited for a treatment of multidimensional phenomena where all dimensions are encompassed at the same time as an intrinsic feature. Clifford analysis essentially is a higher dimensional function theory offering both a generalization of the theory of holomorphic functions in the complex plane and a refinement of classical harmonic analysis. In the standard, also called orthogonal, case, Clifford analysis focusses on so–called monogenic functions, i.e. null solutions of the rotation invariant vector valued Dirac operator

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

where $(e_1, \ldots, e_m)$ forms an orthonormal basis for the quadratic space $\mathbb{R}^{0,m}$ underlying the construction of the real Clifford algebra $\mathbb{R}_{0,m}$. The theory of Hardy spaces and the multidimensional Hilbert transform in the orthogonal Clifford analysis framework is nowadays well established, see [11, 9, 4]. However we want to draw the reader's attention on the paper [12] of Horváth who, to our knowledge, was the first to define a Hilbert transform using Clifford algebra.

Recently another branch of Clifford analysis has emerged, offering yet a refinement of the orthogonal case; it is called Hermitean Clifford analysis and it focusses on the simultaneous null solutions of two complex Hermitean Dirac operators. Complex Dirac operators were already studied in [14, 13, 15]; however, a systematic development of the associated function theory including the invariance properties with respect to the underlying Lie groups and Lie algebras is still in full progress, see [8, 2, 3]. While studying Clifford–Hermite wavelets in the context of Hermitean Clifford analysis, see [6, 7], we came across a new kind of Hilbert operator. The aim of this paper is to further investigate the properties of this Hilbert–like transform and to show its similarities and connections with the standard as well as with newly introduced Hermitean Clifford–Hilbert transforms in Euclidean space. In particular, also new Hardy spaces associated to this operator are introduced and characterized.

2 Preliminaries

Let $\mathbb{R}^{0,m}$ be endowed with a non–degenerate quadratic form of signature $(0, m)$, let $(e_1, \ldots, e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$ and let $\mathbb{R}_{0,m}$ be the real Clifford algebra constructed over $\mathbb{R}^{0,m}$. The non–commutative multiplication in $\mathbb{R}_{0,m}$ is governed by

$$e_j e_k + e_k e_j = -2\delta_{jk} \quad , \quad j, k = 1, \ldots, m \tag{2.1}$$

A basis for $\mathbb{R}_{0,m}$ is obtained by considering for a set $A = \{j_1, \ldots, j_h\} \subset \{1, \ldots, m\}$ the element $e_A = e_{j_1} \ldots e_{j_h}$, with $1 \leq j_1 < j_2 < \ldots < j_h \leq m$. For the empty set $\emptyset$ one puts $e_\emptyset = 1$, the identity element. Any Clifford number a in $\mathbb{R}_{0,m}$ may thus be written as $a = \sum_A e_A a_A$, $a_A \in \mathbb{R}$, or still as $a = \sum_{k=0}^{m} [a]_k$, where $[a]_k = \sum_{|A|=k} e_A a_A$ is the so–called k–vector part of a $(k = 0, 1, \ldots, m)$. The Euclidean space $\mathbb{R}^{0,m}$ is embedded in $\mathbb{R}_{0,m}$ by identifying $(x_1, \ldots, x_m)$ with the Clifford vector $\underline{x}$ given by

$$\underline{x} = \sum_{j=1}^{m} e_j x_j$$

Note that the square of a vector $\underline{x}$ is scalar valued and equals the norm squared up to a minus sign: $\underline{x}^2 = - < \underline{x}, \underline{x} > = -|\underline{x}|^2$. The dual of the vector $\underline{x}$ is the vector valued first order differential operator

$$\partial_{\underline{x}} = \sum_{j=1}^{m} e_j \partial_{x_j}$$

called Dirac operator. It is precisely this Dirac operator which underlies the notion of monogenicity of a function, a notion which is the higher dimensional counterpart of holomorphy in the complex plane. A function f defined and differentiable in an open region Ω of $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$ is called left–monogenic in Ω if $\partial_{\underline{x}}[f] = 0$. Since the Dirac operator factorizes the Laplacian, $\Delta_m = -\partial_{\underline{x}}^2$, monogenicity can be regarded as a refinement of harmonicity.

We refer to this setting as the orthogonal case, since the fundamental group leaving the Dirac operator $\partial_{\underline{x}}$ invariant is the special orthogonal group $\mathrm{SO}(m)$, which is doubly covered by the $\mathrm{Spin}(m)$ group of the Clifford algebra $\mathbb{R}_{0,m}$. For this reason, the Dirac operator is called a rotation invariant operator.

When allowing for complex constants and moreover taking the dimension to be even, say $m = 2n$, the same set of generators as above, $(e_1, \ldots, e_{2n})$, still satisfying the defining relations (2.1), may in fact also produce the complex Clifford algebra $\mathbb{C}_{2n}$. As $\mathbb{C}_{2n}$ is the complexification of the real Clifford algebra $\mathbb{R}_{0,2n}$, i.e. $\mathbb{C}_{2n} = \mathbb{R}_{0,2n} \oplus i\mathbb{R}_{0,2n}$, any complex Clifford number $\lambda \in \mathbb{C}_{2n}$ may be written as $\lambda = a + ib$, $a, b \in \mathbb{R}_{0,2n}$, leading to the definition of the Hermitean conjugation $\lambda^\dagger = (a + ib)^\dagger = \overline{a} - i\overline{b}$, where the bar denotes the usual conjugation in $\mathbb{R}_{0,2n}$, i.e. the main anti–involution for which $\overline{e}_j = -e_j$, $j = 1, \ldots, 2n$. This Hermitean conjugation leads to a Hermitean inner product and its associated norm on $\mathbb{C}_{2n}$ given by $(\lambda, \mu) = [\lambda^\dagger \mu]_0$ and $|\lambda| = \sqrt{[\lambda^\dagger \lambda]_0} = (\sum_A |\lambda_A|^2)^{1/2}$. The above framework will be referred to as the Hermitean Clifford setting, as opposed to the traditional orthogonal Clifford one. Hermitean Clifford analysis then focusses on the null solutions of two Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}}^\dagger$, introduced below.

We first consider the so–called Witt basis for the complex Clifford algebra $\mathbb{C}_{2n}$, viz

$$\mathfrak{f}_j = \frac{1}{2}(e_j - ie_{n+j}) , \qquad \mathfrak{f}_j^\dagger = -\frac{1}{2}(e_j + ie_{n+j}) , \qquad j = 1, \ldots, n$$

These Witt basis elements satisfy the Grassmann identities

$$\mathfrak{f}_j\mathfrak{f}_k + \mathfrak{f}_k\mathfrak{f}_j = \mathfrak{f}_j^\dagger\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j^\dagger = 0 \quad , \quad j, k = 1, \ldots, n$$

and the duality identities

$$\mathfrak{f}_j\mathfrak{f}_k^\dagger + \mathfrak{f}_k^\dagger\mathfrak{f}_j = \delta_{jk} \quad , \quad j, k = 1, \ldots, n$$

The Grassmann subalgebras of $\mathbb{C}_{2n}$ generated by $(\mathfrak{f}_j)_{j=1}^n$ and $(\mathfrak{f}_j^\dagger)_{j=1}^n$ are denoted by $\mathbb{C}\Lambda_n$ and $\mathbb{C}\Lambda_n^\dagger$ respectively. Now take a vector $\underline{X} = (X_1, \ldots, X_{2n}) = (x_1, \ldots, x_n, y_1, \ldots, y_n)$ in $\mathbb{R}^{0,2n}$, identified with the Clifford vector $\underline{X} = \sum_{j=1}^n (e_j x_j + e_{n+j} y_j)$, and rewritten in terms of the Witt basis as

$$\underline{X} = \sum_{j=1}^n \mathfrak{f}_j z_j - \sum_{j=1}^n \mathfrak{f}_j^\dagger z_j^c$$

where n complex variables $z_j = x_j + iy_j$ have been introduced, with complex conjugates $z_j^c = x_j - iy_j$, $j = 1, \ldots, n$. In terms of the Hermitean vector variables, defined as

$$\underline{Z} = \sum_{j=1}^n \mathfrak{f}_j z_j \quad \text{and} \quad \underline{Z}^\dagger = (\underline{Z})^\dagger = \sum_{j=1}^n \mathfrak{f}_j^\dagger z_j^c \, ,$$

the Clifford vector $\underline{X}$ eventually takes the form

$$\underline{X} = \underline{Z} - \underline{Z}^\dagger$$

Similarly the traditional Dirac operator can be rewritten as

$$\partial_{\underline{X}} = \sum_{j=1}^n (e_j \partial_{x_j} + e_{n+j} \partial_{y_j}) = 2 \, (\sum_{j=1}^n \mathfrak{f}_j \partial_{z_j^c} - \sum_{j=1}^n \mathfrak{f}_j^\dagger \partial_{z_j}) = 2(\partial_{\underline{Z}}^\dagger - \partial_{\underline{Z}})$$

in terms of the Hermitean Dirac operators

$$\partial_{\underline{Z}} = \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \partial_{z_j} \quad \text{and} \quad \partial_{\underline{Z}}^\dagger = (\partial_{\underline{Z}})^\dagger = \sum_{j=1}^{n} \mathfrak{f}_j \partial_{z_j^c}$$

involving the classical Cauchy–Riemann operators $\partial_{z_j} = \frac{1}{2}(\partial_{x_j} - i\partial_{y_j})$ and their complex conjugates $\partial_{z_j^c} = \frac{1}{2}(\partial_{x_j} + i\partial_{y_j})$ in the complex z_j planes, $j = 1, \ldots, n$. In this framework also a second Clifford vector is considered, viz

$$\underline{X}| = \sum_{j=1}^{n}(e_j y_j - e_{n+j} x_j) = \frac{1}{i}\sum_{j=1}^{n} \mathfrak{f}_j z_j + \frac{1}{i}\sum_{j=1}^{n} \mathfrak{f}_j^\dagger z_j^c = \frac{1}{i}(\underline{Z} + \underline{Z}^\dagger)$$

with corresponding Dirac operator

$$\partial_{\underline{X}|} = \sum_{j=1}^{n}(e_j \partial_{y_j} - e_{n+j}\partial_{x_j}) = \frac{2}{i}\ (\sum_{j=1}^{n} \mathfrak{f}_j \partial_{z_j^c} + \sum_{j=1}^{n} \mathfrak{f}_j^\dagger \partial_{z_j}) = \frac{2}{i}(\partial_{\underline{Z}}^\dagger + \partial_{\underline{Z}})$$

Note that the vectors $\underline{X}$ and $\underline{X}|$ are orthogonal w.r.t. the standard Euclidean scalar product, which implies that the Clifford vectors $\underline{X}$ and $\underline{X}|$ anti–commute.

A continuously differentiable function g on $\mathbb{R}^{2n}$ with values in $\mathbb{C}_{2n}$ is called a Hermitean monogenic (or h–monogenic) function if and only if it satisfies the system

$$\partial_{\underline{X}} g = 0 = \partial_{\underline{X}|} g$$

or equivalently, the system

$$\partial_{\underline{Z}} g = 0 = \partial_{\underline{Z}}^\dagger g$$

The Hermitean Dirac operators $\partial_{\underline{Z}}$ and $\partial_{\underline{Z}}^\dagger$ are invariant under the action of a realisation, denoted $\widetilde{\mathrm{U}}(n)$, of the unitary group in terms of the Clifford algebra, see [8]. This group $\widetilde{\mathrm{U}}(n) \subset \mathrm{Spin}(2n)$ is given by

$$\widetilde{\mathrm{U}}(n) = \{s \in \mathrm{Spin}(2n) \mid \exists \theta \geq 0 : \overline{s} I = \exp(-i\theta) I\}$$

its definition involving the selfadjoint primitive idempotent $I = I_1 \ldots I_n$, with $I_j = \mathfrak{f}_j \mathfrak{f}_j^\dagger = \frac{1}{2}(1 - i e_j e_{n+j})$, $j = 1, \ldots, n$.

Finally observe that the Hermitean vector variables and Dirac operators are isotropic, i.e.

$$(\underline{Z})^2 = (\underline{Z}^\dagger)^2 = 0 \quad \text{and} \quad (\partial_{\underline{Z}})^2 = (\partial_{\underline{Z}}^\dagger)^2 = 0$$

whence the Laplacian $\Delta_{2n} = -\partial_{\underline{X}}^2 = -\partial_{\underline{X}|}^2$ allows for the decomposition

$$\Delta_{2n} = 4(\partial_{\underline{Z}} \partial_{\underline{Z}}^\dagger + \partial_{\underline{Z}}^\dagger \partial_{\underline{Z}})$$

and one also has that

$$\underline{Z}\underline{Z}^\dagger + \underline{Z}^\dagger \underline{Z} = |\underline{Z}|^2 = |\underline{Z}^\dagger|^2 = |\underline{X}|^2 = |\underline{X}||^2\,.$$

3 A pair of Hilbert transforms

Identifying $\mathbb{R}^{2n}$ with the hyperplane $\{(t, x_1, \ldots, x_n, y_1, \ldots, y_n) \ : \ t = 0\}$ in $\mathbb{R}^{2n+1}$ and introducing the supplementary unit vector e_0, squaring up to -1 and orthogonal to all of $(e_1, \ldots, e_n, e_{n+1}, \ldots, e_{2n})$, two associated Dirac operators in $\mathbb{R}^{2n+1}$ may also be defined, viz

$$\partial = \partial_t e_0 + \partial_{\underline{X}}$$

and

$$\partial| = \partial_t e_0 + \partial_{\underline{X}|}$$

In "upper halfspace" $\mathbb{R}^{2n+1}_+ = \{(t, x_1, \ldots, x_n, y_1, \ldots, y_n) \ : \ t > 0\}$ we then consider the Hardy spaces of ∂–monogenic, respectively $\partial|$–monogenic Clifford algebra valued functions

$$\begin{aligned} H^2(\mathbb{R}^{2n+1}_+) &= \{F(t, x_1, \ldots, x_n, y_1, \ldots, y_n) \ : \ \partial F = 0 \text{ and } \sup_{t>0} \int_{\mathbb{R}^{2n}} |F|^2 \, dV < +\infty\} \\ H|^2(\mathbb{R}^{2n+1}_+) &= \{F(t, x_1, \ldots, x_n, y_1, \ldots, y_n) \ : \ \partial| F = 0 \text{ and } \sup_{t>0} \int_{\mathbb{R}^{2n}} |F|^2 \, dV < +\infty\} \end{aligned}$$

It is well–known that $H^2(\mathbb{R}^{2n+1}_+)$ entails the Hardy space $H^2(\mathbb{R}^{2n})$ as the closure in $L_2(\mathbb{R}^{2n})$ of the space of all non–tangential boundary values for $t \to 0+$ of all functions in $H^2(\mathbb{R}^{2n+1}_+)$, and moreover, both spaces $H^2(\mathbb{R}^{2n+1}_+)$ and $H^2(\mathbb{R}^{2n})$ are isomorphic. In this way an orthogonal decomposition of $L_2(\mathbb{R}^{2n})$ w.r.t. the inner product

$$\langle f, g \rangle_{L_2} = \int_{\mathbb{R}^{2n}} f \overline{g} \, dV$$

is obtained, viz

$$\begin{array}{ccccc} L_2(\mathbb{R}^{2n}) & = & H^2(\mathbb{R}^{2n}) & \oplus & H^2(\mathbb{R}^{2n})^\perp \\ f & = & \mathbb{P}[f] & + & \mathbb{P}^\perp[f] \end{array}$$

where $\mathbb{P}$ and $\mathbb{P}^\perp$ are the so–called Szegö projections on $H^2(\mathbb{R}^{2n})$ and its orthogonal complement $H^2(\mathbb{R}^{2n})^\perp$ respectively. These projections may be further explicited in terms of the so–called Hilbert transform, given for a function $f \in L_2(\mathbb{R}^{2n})$ by its convolution with the so–called Hilbert kernel, viz

$$H_{\underline{X}}[f] = e_0 \, \frac{2}{a_{2n+1}} \, \mathrm{Pv} \frac{\underline{X}}{|\underline{X}|^{2n+1}} * f = e_0 \, \frac{2}{a_{2n+1}} \, \mathrm{Pv} \int_{\mathbb{R}^{2n}} \frac{\underline{X} - \underline{U}}{|\underline{X} - \underline{U}|^{2n+1}} \, f(\underline{U}) \, dV(\underline{U})$$

with a_{2n+1} denoting the area of the unit sphere S^{2n} in $\mathbb{R}^{2n+1}$. One then obtains that

$$\begin{aligned} \mathbb{P}[f] &= \frac{1}{2} \, (\mathbf{1} + H_{\underline{X}})[f] \\ \mathbb{P}^\perp[f] &= \frac{1}{2} \, (\mathbf{1} - H_{\underline{X}})[f] \end{aligned}$$

For the Dirac operator $\partial|$ and its corresponding Hardy space $H|^2(\mathbb{R}^{2n+1}_+)$ one arrives at a similar decomposition of $L_2(\mathbb{R}^{2n})$, with alternative Szegö projections $\mathbb{P}|$ and $\mathbb{P}|^\perp$and their associated Hilbert transform on $L_2(\mathbb{R}^{2n})$ given by

$$H_{\underline{X}|}[f] = e_0 \, \frac{2}{a_{2n+1}} \, \mathrm{Pv} \frac{\underline{X}|}{|\underline{X}|^{2n+1}} * f$$

where we have taken into account that $|\underline{X}| = |\underline{X}||$. This pair of Hilbert transforms on $L_2(\mathbb{R}^{2n})$, already introduced in [7, 1], enjoys the following properties.

Proposition 3.1 *One has*

(i) $H_{\underline{X}}$ *and* $H_{\underline{X}|}$ *are bounded linear operators on* $L_2(\mathbb{R}^{2n})$*;*

(ii) $H^2_{\underline{X}} = H^2_{\underline{X}|} = \mathbf{1}$*;*

(iii) $H^*_{\underline{X}} = H_{\underline{X}}$*,* $H^*_{\underline{X}|} = H_{\underline{X}|}$*;*

(iv) $\langle H_{\underline{X}}[f], H_{\underline{X}}[g]\rangle_{L_2} = \langle f, g\rangle_{L_2} = \langle H_{\underline{X}|}[f], H_{\underline{X}|}[g]\rangle_{L_2}$*.*

It follows that f and $H_{\underline{X}}[f]$ have the same Szegö projection on $H^2(\mathbb{R}^{2n})$:

$$\mathbb{P}\left[H_{\underline{X}}[f]\right] = \frac{1}{2}(\mathbf{1} + H_{\underline{X}})\left[H_{\underline{X}}[f]\right] = \frac{1}{2}(H_{\underline{X}} + \mathbf{1})[f] = \mathbb{P}[f]$$

which can be rephrased in geometrical terms as: f and $H_{\underline{X}}[f]$ lie symmetrically w.r.t. $H^2(\mathbb{R}^{2n})$. This implies the following characterization of $H^2(\mathbb{R}^{2n})$.

Proposition 3.2 *A function* $f \in L_2(\mathbb{R}^{2n})$ *belongs to the Hardy space* $H^2(\mathbb{R}^{2n})$ *(respectively to its orthogonal complement* $H^2(\mathbb{R}^{2n})^\perp$*) if and only if* $H_{\underline{X}}[f] = f$ *(respectively* $H_{\underline{X}}[f] = -f$*).*

Similar observations hold for f and $H_{\underline{X}|}[f]$ with respect to $H|^2(\mathbb{R}^{2n})$.

4 Hermitean Hilbert transforms

One of the ways for introducing Hermitean Clifford analysis is by considering the complex Clifford algebra $\mathbb{C}_{2n}$ and a so–called complex structure on it, i.e. an $\mathrm{SO}(2n;\mathbb{R})$–element J for which $J^2 = -\mathbf{1}$, see [2]. Let us recall that the main objects of the Hermitean setting are then generated conceptually by considering the projection operators $\frac{1}{2}(\mathbf{1} \pm iJ)$ and letting them act on the corresponding protagonists of the orthogonal framework, the obtained definitions being in agreement with those given in Section 2 when J is chosen to be represented by the matrix

$$\begin{bmatrix} 0 & \mathbb{E}_n \\ -\mathbb{E}_n & 0 \end{bmatrix}$$

where $\mathbb{E}_n$ denotes the identity matrix of order n, or equivalently, $J[e_j] = -e_{n+j}$ and $J[e_{n+j}] = e_j$, $j = 1, \ldots, n$. Indeed, the Witt basis elements are obtained through the action of $\frac{1}{2}(\mathbf{1} \pm iJ)$ on the orthogonal basis elements e_j:

$$\begin{aligned} \mathfrak{f}_j &= \frac{1}{2}(\mathbf{1} + iJ)[e_j] = \frac{1}{2}(e_j - i\, e_{n+j}), \quad j = 1, \ldots, n \\ \mathfrak{f}_j^\dagger &= -\frac{1}{2}(\mathbf{1} - iJ)[e_j] = -\frac{1}{2}(e_j + i\, e_{n+j}), \quad j = 1, \ldots, n \end{aligned}$$

while the Hermitean Clifford variables arise through the action on the standard Clifford vector $\underline{X}$:

$$\begin{aligned} \underline{Z} &= \frac{1}{2}(\mathbf{1} + iJ)[\underline{X}] = \frac{1}{2}(\underline{X} + i\,\underline{X}|) \\ \underline{Z}^\dagger &= -\frac{1}{2}(\mathbf{1} - iJ)[\underline{X}] = -\frac{1}{2}(\underline{X} - i\,\underline{X}|) \end{aligned}$$

and the Hermitean Dirac operators through the action on the orthogonal Dirac operator $\partial_{\underline{X}}$:

$$\begin{aligned}\partial_{\underline{Z}}^{\dagger} &= \frac{1}{4}(\mathbf{1}+iJ)[\partial_{\underline{X}}] &= \frac{1}{4}(\partial_{\underline{X}}+i\,\partial_{\underline{X}|})\\ \partial_{\underline{Z}} &= -\frac{1}{4}(\mathbf{1}-iJ)[\partial_{\underline{X}}] &= -\frac{1}{4}(\partial_{\underline{X}}-i\,\partial_{\underline{X}|})\end{aligned}$$

In the same order of ideas we now introduce two Hermitean Hilbert transforms on $L_2(\mathbb{R}^{2n})$ by letting $\frac{1}{2}(\mathbf{1}\pm iJ)$ act on the Hilbert transform $H_{\underline{X}}$, resulting into

$$\begin{aligned}H_{\underline{Z}} &= \frac{1}{2}(\mathbf{1}+iJ)[H_{\underline{X}}] &= \frac{1}{2}(H_{\underline{X}}+iH_{\underline{X}|})\\ H_{\underline{Z}^{\dagger}} &= -\frac{1}{2}(\mathbf{1}-iJ)[H_{\underline{X}}] &= -\frac{1}{2}(H_{\underline{X}}-iH_{\underline{X}|})\end{aligned}$$

or more explicitly

$$\begin{aligned}H_{\underline{Z}}[f] &= e_0\,\frac{2}{a_{2n+1}}\,\mathrm{Pv}\frac{\underline{Z}}{r^{2n+1}}*f\\ H_{\underline{Z}^{\dagger}}[f] &= e_0\,\frac{2}{a_{2n+1}}\,\mathrm{Pv}\frac{\underline{Z}^{\dagger}}{r^{2n+1}}*f\end{aligned}$$

with $r=|\underline{Z}|=|\underline{Z}^{\dagger}|=|\underline{X}|=|\underline{X}|\,|$. We now list a number of properties of these Hermitean Hilbert transforms, the proofs of which are straightforward.

Proposition 4.1 *One has*

(i) $H_{\underline{Z}}$ *and* $H_{\underline{Z}^{\dagger}}$ *are bounded linear operators on* $L_2(\mathbb{R}^{2n})$*;*

(ii) $H_{\underline{Z}}^2=0,\ H_{\underline{Z}^{\dagger}}^2=0$*;*

(iii) $H_{\underline{Z}}^*=H_{\underline{Z}},\ H_{\underline{Z}^{\dagger}}^*=H_{\underline{Z}^{\dagger}}$*;*

(iv) $H_{\underline{Z}}\,H_{\underline{Z}^{\dagger}}+H_{\underline{Z}^{\dagger}}\,H_{\underline{Z}}=-\mathbf{1}$*;*

(v) $H_{\underline{Z}}\,H_{\underline{Z}^{\dagger}}-H_{\underline{Z}^{\dagger}}\,H_{\underline{Z}}=\frac{1}{2}(iH_{\underline{X}}\,H_{\underline{X}|}-iH_{\underline{X}|}\,H_{\underline{X}})$*.*

Observe in particular property (ii) expressing the isotropy of the Hermitean Hilbert transforms, and property (v) which will lead to an new Hilbert type transform in the Hermitean Clifford analysis setting, as introduced below.

5 A new Hilbert transform

As a first step towards the definition of a new Hilbert transform the following lemma is crucial; for a proof we refer to [7].

Lemma 5.1 *The Hilbert transforms* $H_{\underline{X}}$ *and* $H_{\underline{X}|}$ *anti–commute.*

As in [7], we then put

$$\mathcal{K} = iH_{\underline{X}} H_{\underline{X}|} = -iH_{\underline{X}|} H_{\underline{X}} \tag{5.1}$$

As the operator $\mathcal{K}$ results from the composition of two convolution operators, it is itself a convolution operator, i.e.

$$\mathcal{K}[f] = K * f$$

A precise calculation reveals that its kernel $K(\underline{X})$ is given by

$$K(\underline{X}) = i\,\frac{(n-1)!}{\pi^n}\,\mathrm{Pv}\left(n\,\frac{\underline{X}\,\underline{X}|}{|\underline{X}|^{2n+2}} + \frac{\sum_{j=1}^{n} e_j\,e_{n+j}}{|\underline{X}|^{2n}}\right)$$

We now list a number of properties of this new $\mathcal{K}$ transform, which are proven in a rather straightforward way.

Proposition 5.1 *One has*

(i) $\mathcal{K}$ *is a bounded linear operator on* $L_2(\mathbb{R}^{2n})$*;*

(ii) $\mathcal{K}$ *squares to unity, i.e.* $\mathcal{K}^2 = \mathbf{1}$*;*

(iii) $\mathcal{K}$ *is selfadjoint, i.e.* $\mathcal{K}^* = \mathcal{K}$*;*

(iv) $\mathcal{K}$ *preserves the* L_2 *inner product, i.e.* $\langle \mathcal{K}[f], \mathcal{K}[g]\rangle_{L_2} = \langle f, g\rangle_{L_2}$*;*

(v) in frequency space, $\mathcal{K}$ *takes the form*

$$\mathcal{F}\left[\mathcal{K}[f]\right](\Xi) = i\,\frac{\Xi|\;\Xi}{|\Xi|^2}\,\mathcal{F}[f](\Xi)$$

where $\mathcal{F}$ *denotes the standard Fourier transform in* $\mathbb{R}^{2n}$*, given by*

$$\mathcal{F}[f](\underline{U}) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^{2n}} \exp\left(-i < \underline{X}, \underline{U} >\right)\, f(\underline{X})\; dV(\underline{X})\,,$$

(vi) $\mathcal{K}$ *anti–commutes with both* $H_{\underline{X}}$ *and* $H_{\underline{X}|}$*, i.e.*

$$H_{\underline{X}}\,\mathcal{K} + \mathcal{K}\,H_{\underline{X}} = 0\,, \qquad H_{\underline{X}|}\,\mathcal{K} + \mathcal{K}\,H_{\underline{X}|} = 0$$

Corollary 5.1 *For the Hardy spaces* $H^2(\mathbb{R}^{2n})$ *and* $H|^2(\mathbb{R}^{2n})$*, defined in terms of the Dirac operators* $\partial_{\underline{X}}$ *and* $\partial_{\underline{X}|}$ *respectively, one has*

$$H^2(\mathbb{R}^{2n}) \cap H|^2(\mathbb{R}^{2n}) = \{0\}$$

Proof.
Let g belong to both Hardy spaces. It then should hold simultaneously that $H_{\underline{X}}[g] = g$ and $H_{\underline{X}|}[g] = g$, from which we infer that

$$\mathcal{K}[g] = i\,H_{\underline{X}}\,H_{\underline{X}|}[g] = i\,H_{\underline{X}}[g] = ig$$

Hence

$$\mathcal{K}^2[g] = \mathcal{K}[ig] = i\,\mathcal{K}[g] = -g$$

However, as $\mathcal{K}^2 = \mathbf{1}$, we also have that $\mathcal{K}^2[g] = g$, whence $g = 0$. ∎

Corollary 5.2 *The operators $\frac{1}{2}(\mathbf{1} \pm \mathcal{K})$ are projection operators on $L_2(\mathbb{R}^{2n})$.*

Proof.
One can immediately check that

$$\left(\frac{1}{2}(\mathbf{1} \pm \mathcal{K})\right)^2 = \frac{1}{2}(\mathbf{1} \pm \mathcal{K})$$

and that

$$\frac{1}{2}(\mathbf{1} + \mathcal{K})\,\frac{1}{2}(\mathbf{1} - \mathcal{K}) = \frac{1}{2}(\mathbf{1} - \mathcal{K})\,\frac{1}{2}(\mathbf{1} + \mathcal{K}) = 0$$

■

In this way a new orthogonal decomposition of $L_2(\mathbb{R}^{2n})$ is obtained. Indeed, putting

$$\begin{aligned} K^2(\mathbb{R}^{2n}) &= \frac{1}{2}\,(\mathbf{1} + \mathcal{K})[L_2(\mathbb{R}^{2n})] \\ K^2(\mathbb{R}^{2n})^\perp &= \frac{1}{2}\,(\mathbf{1} - \mathcal{K})[L_2(\mathbb{R}^{2n})] \end{aligned}$$

we obtain

$$\begin{array}{ccccc} L_2(\mathbb{R}^{2n}) & = & K^2(\mathbb{R}^{2n}) & \oplus & K^2(\mathbb{R}^{2n})^\perp \\ f & = & \frac{1}{2}\,(\mathbf{1} + \mathcal{K})[f] & + & \frac{1}{2}\,(\mathbf{1} - \mathcal{K})[f] \end{array}$$

with $\langle \frac{1}{2}\,(\mathbf{1} + \mathcal{K})[f], \frac{1}{2}\,(\mathbf{1} - \mathcal{K})[f] \rangle_{L_2} = 0$. Moreover the closed subspaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ may be characterized in a similar way as the traditional Hardy spaces.

Proposition 5.2 *One has*

(i) $f \in K^2(\mathbb{R}^{2n})$ *if and only if* $\mathcal{K}[f] = f$;

(ii) $f \in K^2(\mathbb{R}^{2n})^\perp$ *if and only if* $\mathcal{K}[f] = -f$;

(iii) $f \in H^2(\mathbb{R}^{2n})$ *if and only if* $\mathcal{K}[f] = -i\, H_{\underline{X}|}[f]$;

(iv) $f \in H^2(\mathbb{R}^{2n})^\perp$ *if and only if* $\mathcal{K}[f] = i\, H_{\underline{X}|}[f]$

(v) $f \in H|^2(\mathbb{R}^{2n})$ *if and only if* $\mathcal{K}[f] = i\, H_{\underline{X}}[f]$;

(vi) $f \in H|^2(\mathbb{R}^{2n})^\perp$ *if and only if* $\mathcal{K}[f] = -i\, H_{\underline{X}}[f]$.

Quite remarkably, although directly following from Proposition 4.1 and the definition (5.1) of $\mathcal{K}$, the projection operators defining the subspaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ may be expressed in terms of the Hermitean Hilbert transforms introduced in Section 4.

Proposition 5.3 *One has*

(i) $\frac{1}{2}\,(\mathbf{1} + \mathcal{K}) = -H_{\underline{Z}^\dagger}\, H_{\underline{Z}}$;

(ii) $\frac{1}{2}\,(\mathbf{1} - \mathcal{K}) = -H_{\underline{Z}}\, H^\dagger_{\underline{Z}}$;

(iii) $H_{\underline{Z}}\, H_{\underline{Z}^\dagger} - H_{\underline{Z}^\dagger}\, H_{\underline{Z}} = \mathcal{K}$.

Even more remarkable, however, is the following observation. The Hilbert transforms $H_{\underline{X}}$ and $H_{\underline{X}|}$, as well as the operator $\mathcal{K}$ are bijective on $L_2(\mathbb{R}^{2n})$, since they are bounded linear operators on $L_2(\mathbb{R}^{2n})$ with $H_{\underline{X}}^{-1} = H_{\underline{X}}$, $H_{\underline{X}|}^{-1} = H_{\underline{X}|}$ and $\mathcal{K}^{-1} = \mathcal{K}$. The Hermitean Hilbert transforms on the contrary can not be injective since they are isotropic, so their kernels should be nontrivial; they are determined in the following proposition.

Proposition 5.4 *One has*

$$\ker H_{\underline{Z}^\dagger} = K^2(\mathbb{R}^{2n})\,, \qquad \ker H_{\underline{Z}} = K^2(\mathbb{R}^{2n})^\perp$$

Proof.
If $f \in \ker H_{\underline{Z}^\dagger}$, then it holds that

$$\frac{1}{2}\,(\mathbf{1} - \mathcal{K})[f] = -H_{\underline{Z}}\, H_{\underline{Z}^\dagger}[f] = 0$$

or $\mathcal{K}[f] = f$, whence $f \in K^2(\mathbb{R}^{2n})$. Conversely, if $f \in K^2(\mathbb{R}^{2n})$, then

$$f = \mathcal{K}[f] = i\, H_{\underline{X}}\, H_{\underline{X}|}[f]$$

from which it follows that

$$H_{\underline{X}}[f] = i\, H_{\underline{X}}^2\, H_{\underline{X}|}[f] = i\, H_{\underline{X}|}[f]$$

whence $H_{\underline{Z}^\dagger}[f] = -\frac{1}{2}(H_{\underline{X}}[f] - i\, H_{\underline{X}|}[f]) = 0$. A similar argument may be applied to $\ker H_{\underline{Z}}$. ■

Summarizing, we obtain the following characterizations of the Hardy–like spaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$.

Theorem 5.1

(a) A function f belongs to $K^2(\mathbb{R}^{2n})$ if and only if one of the following conditions is satisfied:

- *(i)* $\mathcal{K}[f] = f$;
- *(ii)* $H_{\underline{Z}^\dagger}[f] = 0$;
- *(iii)* $H_{\underline{Z}}[f] = H_{\underline{X}}[f] = i\, H_{\underline{X}|}[f]$.

(b) A function f belongs to $K^2(\mathbb{R}^{2n})^\perp$ if and only if one of the following conditions is satisfied:

- *(i)* $\mathcal{K}[f] = -f$;
- *(ii)* $H_{\underline{Z}}[f] = 0$;
- *(iii)* $H_{\underline{Z}^\dagger}[f] = -H_{\underline{X}}[f] = i\, H_{\underline{X}|}[f]$.

Corollary 5.3 *One has*

(i) $k \in K^2(\mathbb{R}^{2n})$ *if and only if* $H_{\underline{X}}[k] \in K^2(\mathbb{R}^{2n})^\perp$;

(ii) $k \in K^2(\mathbb{R}^{2n})$ *if and only if* $H_{\underline{X}|}[k] \in K^2(\mathbb{R}^{2n})^\perp$;

(iii) $\ell \in K^2(\mathbb{R}^{2n})^\perp$ *if and only if* $H_{\underline{X}}[\ell] \in K^2(\mathbb{R}^{2n})$;

(iv) $\ell \in K^2(\mathbb{R}^{2n})^\perp$ *if and only if* $H_{\underline{X}|}[\ell] \in K^2(\mathbb{R}^{2n})$.

Proof.
We only prove property (i), the proofs of (ii), (iii) and (iv) proceeding along similar lines. If $k \in K^2(\mathbb{R}^{2n})$ then $i\, H_{\underline{X}|}[k] = H_{\underline{X}}[k]$ and hence

$$\mathcal{K}\,[H_{\underline{X}}[k]] = -i\, H_{\underline{X}|}\, H_{\underline{X}}\, H_{\underline{X}}[k] = -i\, H_{\underline{X}|}[k] = -H_{\underline{X}}[k]$$

from which it follows that $H_{\underline{X}}[k] \in K^2(\mathbb{R}^{2n})^\perp$. Conversely, if $H_{\underline{X}}[k] \in K^2(\mathbb{R}^{2n})^\perp$ then

$$i\, H_{\underline{X}|}\, [H_{\underline{X}}[k]] = -H_{\underline{X}}\, [H_{\underline{X}}[k]] = -k$$

or $-\mathcal{K}[k] = -k$ meaning that $k \in K^2(\mathbb{R}^{2n})$. ∎

There is a nice geometric interpretation of the above corollary. Indeed, let us recall that a function $f \in L_2(\mathbb{R}^{2n})$ and its Hilbert transform $H_{\underline{X}}[f]$ lie symmetrically w.r.t. $H^2(\mathbb{R}^{2n})$. If in particular f is chosen to belong to $K^2(\mathbb{R}^{2n})$ then $H_{\underline{X}}[f] \in K^2(\mathbb{R}^{2n})^\perp$, i.e. a function $f \in K^2(\mathbb{R}^{2n})$ and its Hilbert transform $H_{\underline{X}}[f]$ are orthogonal. The spaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ may thus be considered as being the bisector spaces of $H^2(\mathbb{R}^{2n})$ and $H^2(\mathbb{R}^{2n})^\perp$. At the same time, $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^\perp$ are also the bisector spaces of the associated Hardy spaces $H|^2(\mathbb{R}^{2n})$ and $H|^2(\mathbb{R}^{2n})^\perp$.

Finally, the $L_2(\mathbb{R}^{2n})$ decompositions w.r.t. the Hilbert transform $H_{\underline{X}}$ and w.r.t. the new integral transform $\mathcal{K}$ can be matched together, resulting into the following schemes. Take $f \in L_2(\mathbb{R}^{2n})$, then on the one hand

$$f = h + g\,, \qquad h \in H^2(\mathbb{R}^{2n}),\ g \in H^2(\mathbb{R}^{2n})^\perp$$

and on the other

$$f = k + \ell\,, \qquad k \in K^2(\mathbb{R}^{2n}),\ \ell \in K^2(\mathbb{R}^{2n})^\perp$$

where moreover the components h, g, k and ℓ may be decomposed themselves as well, viz

$$\begin{aligned} h &= h_k + h_\ell\,, \qquad h_k \in K^2(\mathbb{R}^{2n}),\ h_\ell \in K^2(\mathbb{R}^{2n})^\perp \\ g &= g_k + g_\ell\,, \qquad g_k \in K^2(\mathbb{R}^{2n}),\ g_\ell \in K^2(\mathbb{R}^{2n})^\perp \end{aligned}$$

and

$$\begin{aligned} k &= k_h + k_g\,, \qquad k_h \in H^2(\mathbb{R}^{2n}),\ k_g \in H^2(\mathbb{R}^{2n})^\perp \\ \ell &= \ell_h + \ell_g\,, \qquad \ell_h \in H^2(\mathbb{R}^{2n}),\ \ell_g \in H^2(\mathbb{R}^{2n})^\perp \end{aligned}$$

where obviously the following relations hold:

$$k = h_k + g_k,\ \ell = h_\ell + g_\ell \quad \text{and} \quad h = k_h + \ell_h,\ g = k_g + \ell_g$$

We thus obtain

$$f = h_k + g_k + h_\ell + g_\ell \tag{5.2}$$
$$H_{\underline{X}}[f] = h_k - g_k + h_\ell - g_\ell \tag{5.3}$$
$$\mathcal{K}[f] = h_k + g_k - h_\ell - g_\ell \tag{5.4}$$

and

$$H_{\underline{X}}\mathcal{K}[f] = i\, H_{\underline{X}|}[f] = -h_k + g_k + h_\ell - g_\ell \tag{5.5}$$

since

$$H_{\underline{X}}[h_k] = h_\ell\,,\ H_{\underline{X}}[g_k] = -g_\ell\,,\ H_{\underline{X}}[h_\ell] = h_k\,,\ H_{\underline{X}}[g_\ell] = -g_k$$

and

$$\mathcal{K}[h_k] = h_k\,,\ \mathcal{K}[g_k] = g_k\,,\ \mathcal{K}[h_\ell] = -h_\ell\,,\ \mathcal{K}[g_\ell] = -g_\ell$$

Furthermore, the above results (5.2)–(5.5) show that

$$\begin{aligned} H_{\underline{Z}}[f] &= \frac{1}{2}(H_{\underline{X}} + i\,H_{\underline{X}|})[f] = h_\ell - g_\ell \\ H_{\underline{Z}^\dagger}[f] &= -\frac{1}{2}(H_{\underline{X}} - i\,H_{\underline{X}|})[f] = g_k - h_k \end{aligned}$$

as it should, since

$$H_{\underline{Z}}[f] = H_{\underline{Z}}[h_k + g_k] = H_{\underline{X}}[h_k + g_k] = h_\ell - g_\ell$$

and

$$H_{\underline{Z}^\dagger}[f] = H_{\underline{Z}^\dagger}[h_\ell + g_\ell] = -H_{\underline{X}}[h_\ell + g_\ell] = -h_k + g_k$$

on account of Theorem 5.1. Invoking (5.2)–(5.5), we may also express the components (5.2) of f in terms of the respective projection operators $\frac{1}{2}\,(\mathbf{1} \pm \mathcal{K})$ and $\frac{1}{2}\,(\mathbf{1} \pm H_{\underline{X}})$, leading to

$$\begin{aligned} h_k &= \frac{1}{4}\,(\mathbf{1} + H_{\underline{X}} + \mathcal{K} - H_{\underline{X}}\,\mathcal{K})\,[f] &= \frac{(\mathbf{1}+\mathcal{K})}{2}\,\frac{(\mathbf{1}+H_{\underline{X}})}{2}[f] \\ g_k &= \frac{1}{4}\,(\mathbf{1} - H_{\underline{X}} + \mathcal{K} + H_{\underline{X}}\,\mathcal{K})\,[f] &= \frac{(\mathbf{1}+\mathcal{K})}{2}\,\frac{(\mathbf{1}-H_{\underline{X}})}{2}[f] \\ h_\ell &= \frac{1}{4}\,(\mathbf{1} + H_{\underline{X}} - \mathcal{K} + H_{\underline{X}}\,\mathcal{K})\,[f] &= \frac{(\mathbf{1}-\mathcal{K})}{2}\,\frac{(\mathbf{1}+H_{\underline{X}})}{2}[f] \\ g_\ell &= \frac{1}{4}\,(\mathbf{1} - H_{\underline{X}} - \mathcal{K} - H_{\underline{X}}\,\mathcal{K})\,[f] &= \frac{(\mathbf{1}-\mathcal{K})}{2}\,\frac{(\mathbf{1}-H_{\underline{X}})}{2}[f] \end{aligned}$$

which is in accordance with the definitions of these projections. Similarly, we also have

$$\begin{aligned} f &= k_h + k_g + \ell_h + \ell_g && (5.6) \\ H_{\underline{X}}[f] &= k_h - k_g + \ell_h - \ell_g && (5.7) \\ \mathcal{K}[f] &= k_h + k_g - \ell_h - \ell_g && (5.8) \end{aligned}$$

and

$$H_{\underline{X}}\mathcal{K}[f] \;=\; i\,H_{\underline{X}|}[f] \;=\; k_h - k_g - \ell_h + \ell_g \qquad (5.9)$$

since

$$H_{\underline{X}}[k_h] = k_h\,,\ H_{\underline{X}}[k_g] = -k_g\,,\ H_{\underline{X}}[\ell_h] = \ell_h\,,\ H_{\underline{X}}[\ell_g] = -\ell_g$$

and

$$\mathcal{K}[k_h] = k_g\,,\ \mathcal{K}[k_g] = k_h\,,\ \mathcal{K}[\ell_h] = -\ell_g\ \ \mathcal{K}[\ell_g] = -\ell_h$$

This yields, as an alternative decomposition of f by subsequent projections,

$$\begin{aligned} k_h &= \frac{1}{4}\,(\mathbf{1} + H_{\underline{X}} + \mathcal{K} + H_{\underline{X}}\,\mathcal{K})\,[f] &= \frac{(\mathbf{1}+H_{\underline{X}})}{2}\,\frac{(\mathbf{1}+\mathcal{K})}{2}[f] \\ k_g &= \frac{1}{4}\,(\mathbf{1} - H_{\underline{X}} + \mathcal{K} - H_{\underline{X}}\,\mathcal{K})\,[f] &= \frac{(\mathbf{1}-H_{\underline{X}})}{2}\,\frac{(\mathbf{1}+\mathcal{K})}{2}[f] \\ \ell_h &= \frac{1}{4}\,(\mathbf{1} + H_{\underline{X}} - \mathcal{K} - H_{\underline{X}}\,\mathcal{K})\,[f] &= \frac{(\mathbf{1}+H_{\underline{X}})}{2}\,\frac{(\mathbf{1}-\mathcal{K})}{2}[f] \\ \ell_g &= \frac{1}{4}\,(\mathbf{1} - H_{\underline{X}} - \mathcal{K} + H_{\underline{X}}\,\mathcal{K})\,[f] &= \frac{(\mathbf{1}-H_{\underline{X}})}{2}\,\frac{(\mathbf{1}-\mathcal{K})}{2}[f] \end{aligned}$$

which also was to be expected. Furthermore we then have

$$\begin{aligned} H_{\underline{Z}}[f] &= k_h - k_g \\ H_{\underline{Z}^\dagger}[f] &= \ell_g - \ell_h \end{aligned}$$

again in agreement with Theorem 5.1.

6 Analytic signals

In one–dimensional signal analysis the concept of "analytic signal" is fundamental. If $u(t)$ is a real signal of the time variable $t \in \mathbb{R}$ and $\mathcal{H}[u]$ is its one–dimensional Hilbert transform, see Section 1, then the complex signal

$$A(t) = u(t) + i\mathcal{H}[u](t)$$

is called the analytic signal associated to $u(t)$. As $\mathcal{H}^2 = -\mathbf{1}$, this analytic signal satisfies the condition

$$i\mathcal{H}[A] = A$$

which means that if the signal $u(t)$ has "finite energy", i.e. $u(t) \in L_2(\mathbb{R})$, then its associated analytic signal $A(t)$ belongs to the Hardy space $H^2(\mathbb{R})$. At the same time the complex conjugated signal

$$A^c(t) = u(t) - i\mathcal{H}[u](t)$$

will then belong to the orthogonal complement $H^2(\mathbb{R})^\perp$ since

$$i\mathcal{H}[A^c] = -(u - i\mathcal{H}[u]) = -A^c$$

So one could say that in the one–dimensional case complex conjugation maps the Hardy spaces $H^2(\mathbb{R})$ and $H^2(\mathbb{R})^\perp$ onto each other.

In the multidimensional case it is directly seen from the definitions of the Hilbert transforms $H_{\underline{X}}$ and $H_{\underline{X}|}$ themselves, see Section 3, that for functions $f : \mathbb{R}^{2n} \to \mathbb{C}_{2n}$

$$H_{\underline{X}}[e_0\, f] = -e_0\, H_{\underline{X}}[f]$$

and

$$H_{\underline{X}|}[e_0\, f] = -e_0\, H_{\underline{X}|}[f]$$

It follows that left multiplication by e_0, i.e. $T_{e_0} : f \to e_0\, f$, will map the Hardy spaces $H^2(\mathbb{R}^{2n})$ and $H|^2(\mathbb{R}^{2n})$ onto their orthogonal complements and vice versa. Indeed, for $f : \mathbb{R}^{2n} \to \mathbb{C}_{2n}$ its associated analytic signal is given by

$$A(\underline{X}) = f(\underline{X}) + H_{\underline{X}}[f](\underline{X})$$

which clearly belongs to $H^2(\mathbb{R}^{2n})$ since

$$H_{\underline{X}}[A] = H_{\underline{X}}[f] + H_{\underline{X}}^2[f] = f + H_{\underline{X}}[f] = A$$

Note that A takes values in the Clifford algebra $\mathbb{C}_{2n} \oplus e_0\, \mathbb{C}_{2n} = \mathbb{C}_{2n+1}$. The action of the T_{e_0} map then results into

$$T_{e_0}[A] = e_0\, f + e_0\, H_{\underline{X}}[f] = -H_{\underline{X}}[e_0\, f] + e_0\, f$$

where now $e_0\, f$ takes values in $e_0\, \mathbb{C}_{2n}$, while $-H_{\underline{X}}[e_0\, f] = e_0\, H_{\underline{X}}[f]$ takes values in $\mathbb{C}_{2n}$. Moreover

$$H_{\underline{X}}\left[T_{e_0}[A]\right] = H_{\underline{X}}[e_0\, f] - e_0\, f = -T_{e_0}[A]$$

which means that $T_{e_0}[A]$ belongs to $H^2(\mathbb{R}^{2n})^{\perp}$.

However, for the new spaces $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^{\perp}$ of Hardy type, it holds that

$$\begin{aligned} \mathcal{K}[e_0\, f] &= i\, H_{\underline{X}}\, H_{\underline{X}|}[e_0\, f] \;=\; i\, H_{\underline{X}}\left[-e_0\, H_{\underline{X}|}[f]\right] \\ &= e_0\, i\, H_{\underline{X}}\, H_{\underline{X}|}[f] \;=\; e_0\, \mathcal{K}[f] \end{aligned}$$

implying that T_{e_0} will map both $K^2(\mathbb{R}^{2n})$ and $K^2(\mathbb{R}^{2n})^{\perp}$ onto themselves.

7 A Cauchy integral

As was already mentioned in Section 3, the Hardy spaces $H^2(\mathbb{R}^{2n+1}_+)$ and $H^2(\mathbb{R}^{2n})$ are isomorphic, the isomorphism being obtained explicitly by means of the Cauchy integral, which, for a given $h \in H^2(\mathbb{R}^{2n})$, reads:

$$\mathcal{C}[h](t, x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{1}{a_{2n+1}} \int_{\mathbb{R}^{2n}} \frac{t + e_0\,(\underline{X} - \underline{U})}{|te_0 + \underline{X} - \underline{U}|^{2n+1}}\, h(\underline{U})\, dV(\underline{U})$$

with $\mathcal{C}[h] \in H^2(\mathbb{R}^{2n+1}_+)$ and

$$\lim_{t\to 0+} \mathcal{C}[h] = h \tag{7.1}$$

in the L_2 sense of non–tangential boundary values, such that $\mathcal{C}[h]$ may be seen as a monogenic extension of h and both functions may be identified with each other.

More generally, for $f \in L_2(\mathbb{R}^{2n})$, its Cauchy integral $\mathcal{C}[f]$ still exists and belongs to $H^2(\mathbb{R}^{2n+1}_+)$ (and in fact also to $H^2(\mathbb{R}^{2n+1}_-)$, defined similarly), but one obtains

$$\begin{aligned} \lim_{t\to 0+} \mathcal{C}[f] &= \mathcal{C}^+[f] = \frac{1}{2} f + \frac{1}{2} H_{\underline{X}}[f] \\ \lim_{t\to 0-} \mathcal{C}[f] &= \mathcal{C}^-[f] = -\frac{1}{2} f + \frac{1}{2} H_{\underline{X}}[f] \end{aligned} \tag{7.2}$$

for its non–tangential boundary values, also called Hardy projections. Note that these results are in agreement with Proposition 3.2, since (7.2) indeed reduces to (7.1) when $f \in H^2(\mathbb{R}^{2n})$. Quite similarly, using the $\underline{X}|$ variable, one can define the associated Cauchy integral

$$\mathcal{C}|[f](t, x_1, \ldots, x_n, y_1, \ldots, y_n) = \frac{1}{a_{2n+1}} \int_{\mathbb{R}^{2n}} \frac{t + e_0\,(\underline{X}| - \underline{U}|)}{|te_0 + \underline{X} - \underline{U}|^{2n+1}}\, f(\underline{U})\, dV(\underline{U})$$

establishing the isomorphism between the Hardy spaces $H|^2(\mathbb{R}^{2n+1}_+)$ and $H|^2(\mathbb{R}^{2n})$. For $f \in L_2(\mathbb{R}^{2n})$, $\mathcal{C}|[f]$ will be $\partial|$–monogenic in $\mathbb{R}^{2n+1} \setminus \mathbb{R}^{2n}$ and show the non–tangential boundary values or associated Hardy projections

$$\begin{aligned} \lim_{t\to 0+} \mathcal{C}|[f] &= \mathcal{C}|^+[f] = \frac{1}{2} f + \frac{1}{2} H_{\underline{X}|}[f] \\ \lim_{t\to 0-} \mathcal{C}|[f] &= \mathcal{C}|^-[f] = -\frac{1}{2} f + \frac{1}{2} H_{\underline{X}|}[f] \end{aligned}$$

Observe that, in the present case of halfspace, the Hardy projections coincide with the Szegö projections, which is a well–known fact. Explicitly, one has

$$\begin{array}{rclrcl} \mathcal{C}^+[f] & = & \mathbb{P}[f] & \mathcal{C}^-[f] & = & -\mathbb{P}^\perp[f] \\ \mathcal{C}|^+[f] & = & \mathbb{P}|[f] & \mathcal{C}|^-[f] & = & -\mathbb{P}|^\perp[f] \end{array}$$

This leads to an expression of the newly introduced $\mathcal{K}$–transform as a commutator of the Hardy projections considered.

Proposition 7.1

$$\frac{1}{2}\mathcal{K} = [\mathcal{C}^+, i\mathcal{C}|^+] = [\mathcal{C}^-, i\mathcal{C}|^-]$$

Proof.
For a function $f \in L_2(\mathbb{R}^{2n})$, the result is directly obtained from the following calculations:

$$\begin{array}{rcl} \mathcal{C}^+\left(i\mathcal{C}|^+\right)[f] & = & \frac{1}{4}\left(i\mathbf{1} + iH_{\underline{X}|} + iH_{\underline{X}} + iH_{\underline{X}}H_{\underline{X}|}\right)[f] \\ \left(i\mathcal{C}|^+\right)\mathcal{C}^+[f] & = & \frac{1}{4}\left(i\mathbf{1} + iH_{\underline{X}} + iH_{\underline{X}|} + iH_{\underline{X}|}H_{\underline{X}}\right)[f] \\ \mathcal{C}^-\left(i\mathcal{C}|^-\right)[f] & = & \frac{1}{4}\left(i\mathbf{1} - iH_{\underline{X}|} - iH_{\underline{X}} + iH_{\underline{X}}H_{\underline{X}|}\right)[f] \\ \left(i\mathcal{C}|^-\right)\mathcal{C}^-[f] & = & \frac{1}{4}\left(i\mathbf{1} - iH_{\underline{X}} - iH_{\underline{X}|} + iH_{\underline{X}|}H_{\underline{X}}\right)[f] \end{array}$$

■

Finally it is also possible to make the Hermitean Hilbert transforms $H_{\underline{Z}}$ and $H_{\underline{Z}^\dagger}$ apparent as part of a boundary value of a suitable combination of a ∂–monogenic and a $\partial|$–monogenic function in halfspace.

Proposition 7.2 *One has*

(i) $(\mathcal{C}^+ + i\mathcal{C}|^+)[f] = \frac{1+i}{2}f + H_{\underline{Z}}[f]$;

(ii) $(\mathcal{C}^+ - i\mathcal{C}|^+)[f] = \frac{1-i}{2}f - H_{\underline{Z}^\dagger}[f]$;

(iii) $(\mathcal{C}^- + i\mathcal{C}|^-)[f] = -\frac{1+i}{2}f + H_{\underline{Z}}[f]$;

(iv) $(\mathcal{C}^- - i\mathcal{C}|^-)[f] = -\frac{1-i}{2}f - H_{\underline{Z}^\dagger}[f]$.

References

[1] R. Abreu Blaya, J. Bory Reyes, D. Peña Peña and F. Sommen, A boundary value problem for Hermitian monogenic functions, submitted.

[2] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen and V. Souček, Fundaments of Hermitean Clifford Analysis. Part I: Complex structure, accepted for publication in *Complex Analysis and Operator Theory*.

[3] F. Brackx, J. Bureš, H. De Schepper, D. Eelbode, F. Sommen and V. Souček, Fundaments of Hermitean Clifford Analysis. Part II: Splitting of *h*-monogenic equations, submitted.

[4] F. Brackx, B. De Knock, H. De Schepper and D. Eelbode, On the interplay between the Hilbert transform and conjugate harmonic functions, *Mathematical Methods in the Applied Sciences* **29**(12) (2006), 1435–1450.

[5] F. Brackx, R. Delanghe and F. Sommen, *Clifford Analysis*, Pitman Publishers, Boston-London-Melbourne, 1982.

[6] F. Brackx, H. De Schepper, N. De Schepper and F. Sommen, The Hermitian Clifford–Hermite wavelets. In: 17th International Conference on the Application of Computer Science and Mathematics in Architecture and Civil Engineering; K. Gürlebeck and C. Könke (eds.); Weimar, Germany, 12–14 July 2006.

[7] F. Brackx, H. De Schepper, N. De Schepper and F. Sommen, Hermitean Clifford–Hermite wavelets: an alternative approach, submitted.

[8] F. Brackx, H. De Schepper and F. Sommen, The Hermitean Clifford analysis toolbox, accepted for publication in *Proceedings of ICCA7*.

[9] R. Delanghe, On Some Properties of the Hilbert Transform in Euclidean Space, *Bull. Belg. Math. Soc. - Simon Stevin* **11** (2004), 163–180.

[10] R. Delanghe, F. Sommen and V. Souček, *Clifford Algebra and Spinor–Valued Functions*, Kluwer Academic Publishers, Dordrecht, 1992.

[11] J. Gilbert and M. Murray, *Clifford Algebras and Dirac Operators in Harmonic Analysis*, Cambridge University Press, Cambridge, 1991.

[12] J. Horváth, Sur les fonctions conjuguées à plusieurs variables, *Nederl. Akad. Wetensch. Proc. Ser. A.* **56** = *Indagationes Math.* **15** (1953), 17–29.

[13] R. Rocha–Chavez, M. Shapiro and F. Sommen, *Integral theorems for functions and differential forms in* $\mathbb{C}_m$, Research Notes in Math. **428**, Chapman & Hall/ CRC, New York, 2002.

[14] J. Ryan, Complexified Clifford analysis, *Complex Variables: Theory & Application* **1** (1982), 119–149.

[15] I. Sabadini and F. Sommen, Hermitian Clifford analysis and resolutions, *Math. Meth. Appl. Sci.* **25**(16–18) (2002), 1395–1414.