ULRICH BUNDLES ON SOME THREEFOLD SCROLLS OVER $\mathbb{F}_e$

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ABSTRACT. We investigate the existence of Ulrich vector bundles on suitable 3-fold scrolls X_e over Hirzebruch surfaces $\mathbb{F}_e$, for any $e \geqslant 0$, which arise as tautological embedding of projectivization of very-ample vector bundles on $\mathbb{F}_e$ which are *uniform* in the sense of Brosius and Aprodu–Brinzanescu, cf. [8] and [3] respectively. We explicitely describe components of moduli spaces of rank $r \geqslant 1$ Ulrich vector bundles whose general point is a slope-stable, indecomposable vector bundle.

We moreover determine the dimension of such components as well as we prove that they are generically smooth. As a direct consequence of these facts, we also compute the *Ulrich complexity* of any such X_e and give an effective proof of the fact that such X_e's turn out to be *geometrically Ulrich wild*.

INTRODUCTION

Let X be a smooth irreducible projective variety of dimension $n \geqslant 1$ and let H be a very ample divisor on X. A vector bundle $\mathcal{U}$ on X is said to be an *Ulrich vector bundle with respect to* H if it satisfies suitable cohomological conditions involving some multiples of the polarization induced by H (cf. Definition 1.1 below for precise statement and, e.g. [5, Thm. 2.3], for equivalent conditions).

Ulrich vector bundles first appeared in Commutative Algebra in the paper [23] by B. Ulrich from 1984, since these bundles enjoy suitable extremal cohomological properties. After that, the attention on Ulrich bundles entered in the realm of Algebraic Geometry with the paper [16] where, among other things, the authors compute the Chow form of a projective variety X using Ulrich vector bundles on X, under the assumption that X supports Ulrich bundles.

In recent years there has been a huge amount of work on Ulrich bundles (for nice surveys the reader is referred to e.g. [13, 14]), mainly investigating the following problems:

- Given any polarization H on a variety X, does there exist a vector bundle $\mathcal{U}$ which is Ulrich with respect to H?
- Or even more generally, given a variety X does there exist a very ample divisor H, inducing a polarization on X, and a vector bundle $\mathcal{U}$ on X which is Ulrich with respect to H?
- What is the smallest possible rank for an Ulrich bundle on a given polarized variety (X, H) (the so called *Ulrich complexity of* X *w.r.t.* H, denoted by $uc_H(X)$, cf. Remark 1.2-(i) below)?
- If Ulrich bundles on (X, H) do exist, are they *stable* bundles? If not empty, are their moduli spaces $\mathcal{M}$ either smooth or at least reduced?
- What is $\dim(\mathcal{M})$?

Although something is known about these problems for some specific classes of varieties (e.g. curves, Segre, Veronese, Grassmann varieties, rational normal scrolls, hypersurfaces, some classes of surfaces and threefolds, cf. e.g. [5, 9, 13, 14] for overviews) the above questions are still open in their full generality even for surfaces.

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In the present paper we investigate the case when X is a 3-fold scroll over a Hirzebruch surface $\mathbb{F}_e$, with $e \geqslant 0$. More precisely we focus on 3-fold scrolls X_e arising as embedding, via very-ample tautological line bundles $\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)$, of projective bundles $\mathbb{P}(\mathcal{E}_e)$, where $\mathcal{E}_e$ are very-ample rank-2 vector bundles on $\mathbb{F}_e$ with Chern classes $c_1(\mathcal{E}_e)$ numerically equivalent to $3C_e + b_e f$ and $c_2(\mathcal{E}_e) = k_e$, where C_e and f are the generators of $\mathrm{Num}(\mathbb{F}_e)$ and where b_e and k_e are integers satisfying some natural numerical conditions (cf. Assumptions 1.7 and Remark 1.8 below).

In this set-up one gets 3-fold scrolls $X_e \subset \mathbb{P}^{n_e}$, with $n_e = 4b_e - k_e - 6e + 4$, which are of degree $\deg(X_e) = 6b_e - 9e - k_e$ (cf. (2.2) below), whose hyperplane section divisor we denote by ξ. The aim of this paper is to study the behaviour of 3-fold scrolls (X_e, ξ) as above in terms of Ulrich bundles they can support.

In [18] the existence of Ulrich bundles of rank one and two on low degree smooth three-dimensional scrolls over a surface, was investigated. Among such three-dimensional scrolls over a surface, that are scrolls over $\mathbb{F}_e$ with $e = 0, 1$. Here we extend these results to 3-fold scrolls (X_e, ξ), $e \geq 0$ by proving the following:

Main Theorem *For any integer $e \geqslant 0$, consider the Hirzebruch surface $\mathbb{F}_e$ and let $\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$ denote the line bundle $\alpha C_e + \beta f$ on $\mathbb{F}_e$, where C_e and f the generators of* $\mathrm{Num}(\mathbb{F}_e)$.

Let (X_e, ξ) be a 3-fold scroll over $\mathbb{F}_e$ as above, where $\varphi : X_e \to \mathbb{F}_e$ denote the scroll map. Then:

(a) X_e does not support any Ulrich line bundle w.r.t. ξ unless $e = 0$. In this latter case, the unique Ulrich line bundles on X_0 are the following:

(i) *$L_1 := \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, -1)$ and $L_2 := \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, b_0 - 1)$;*

(ii) *for any integer $t \geq 1$, $M_1 := 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, -t-1)$ and $M_2 := \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t-1)$, which only occur for $b_0 = 2t, k_0 = 3t$.*

(b) Set $e = 0$ and let $r \geqslant 2$ be any integer. Then the moduli space of rank-r vector bundles $\mathcal{U}_r$ on X_0 which are Ulrich w.r.t. ξ and with first Chern class

$$c_1(\mathcal{U}_r) = \begin{cases} r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(\frac{r+3}{2}, \frac{(r-1)}{2}b_0 - r), & \text{if } r \text{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(\frac{r}{2}, \frac{r}{2}b_0 - r), & \text{if } r \text{ is even.} \end{cases}$$

is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \frac{(r^2-1)}{4}(6b_0 - 4), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_0 - 4) + 1, & \text{if } r \text{ is even.} \end{cases}$$

The general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 8b_0 - k_0 - 3$. If moreover $r = 2$, then $\mathcal{U}_2$ is also special *(cf. Def. 1.3 below).*

(c) When $e > 0$, let $h \geqslant 1$ be any integer. Then the moduli space of rank-$2h$ vector bundles $\mathcal{U}_{2h}$ on X_e which are Ulrich w.r.t. ξ and with first Chern class

$$c_1(\mathcal{U}_{2h}) = 2h\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(h, h(b_e - e - 2))$$

is not empty and it contains a generically smooth component $\mathcal{M}(2h)$ of dimension

$$\dim(\mathcal{M}(2h)) = h^2(6b_e - 9e - 4) + 1.$$

The general point $[\mathcal{U}_{2h}] \in \mathcal{M}(2h)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 8b_e - k_e - 12e - 3$. If moreover $h = 1$, then $\mathcal{U}_2$ is also special.

The proof of the **Main Theorem** will be the collection of those of Theorems 2.1, 3.1, 3.2, 4.8 and 4.13.

Recall that, as suggested by an analogous definition in [15], there is a notion of *Ulrich wildness* for a given polarized variety X. To be more precise for a projective variety $X \subset \mathbb{P}^n$ the notion of being *Ulrich wild* can be defined both:

• *algebraically*, i.e. in terms of functorial behavior of suitable modules over the homogeneous coordinate ring of the variety X, we refer the reader to [17, Section 2.2] for more precise details,
• *geometrically*, namely if it possesses families of dimension r of pairwise non–isomorphic, indecomposable, Ulrich vector bundles for arbitrarily large r, cf. e.g. [15, Introduction].

Moreover, if X is Ulrich wild in the *algebraic sense*, then it is also Ulrich wild in the *geometric sense* (cf. [17, Rem. 2.6–(iii)]).

We must point out that the 3-fold scrolls (X_e, ξ) studied in this paper are *algebraically Ulrich wild* (and thus, from above, also *geometrically Ulrich wild*) and this follows from the results in [17]. In fact, when $e = 0$, the Ulrich line bundles L_1 and L_2 as in **Main Theorem**, (i), satisfy the conditions of [17, Theorem A, Corollary 3.1] as well as, when $e > 0$, then two general Ulrich rank 2 vector bundles as in Theorem 3.2, which are not isomorphic, satisfy the same conditions in [17, Theorem A, Corollary 3.1]. These facts imply that X_e is *(strictly) algebraically Ulrich wild* for any $e \geqslant 0$, see [17, Def. 2.5] for precise definition.

In this perspective, **Main Theorem** not only computes the Ulrich complexity of the 3-fold scrolls (X_e, ξ) that we are considering but it also gives a constructive proof of the fact that (X_e, ξ) is *geometrically Ulrich wild*, for any integer $e \geqslant 0$, explicitely describing families of pairwise non–isomorphic, indecomposable, Ulrich vector bundles of arbitrarily large dimension and rank, with further details. Hence one has:

Main Corollary *For any $e \geqslant 0$, the moduli spaces $\mathcal{M}(r)$ constructed in Main Theorem, (a)-(b)-(c), give rise to explicit families of arbitrarily large dimension and ranks of slope-stable, pairwise non–isomorphic, indecomposable, Ulrich vector bundles on (X_e, ξ), an effective proof of the geometric Ulrich wildness of such varieties. Moreover,*

(a) when $e = 0$, the Ulrich complexity of X_0 w.r.t. ξ is $uc_\xi(X_0) = 1$ and X_0 supports Ulrich vector bundles w.r.t. ξ of any rank $r \geqslant 1$, with no gaps on r;

(b) for $e > 0$, the Ulrich complexity of X_e w.r.t. ξ is $uc_\xi(X_e) = 2$ and X_e certainly supports Ulrich vector bundles w.r.t. ξ of any even rank $2h$, for any integer $h \geqslant 1$.

It is an open question whether there are no odd-rank gaps for $e > 0$ as soon as the rank is higher than one.

To conclude we observe that the 3-fold scrolls (X_e, ξ) are varieties not of *minimal degree* in $\mathbb{P}^{n_e}$, being $d_e \neq n_e - 2$ (see 2.2), which are *(strictly) Ulrich wild*, as in [17, Def. 2.5], therefore (X_e, ξ), for any $e \geqslant 0$, is a class of varieties which satisfy [17, Conjecture 1.].

The paper consists of four sections. In Section 1 we recall some generalities on Ulrich vector bundles on projective varieties, which will be used in the sequel, as well as preliminaries from [1, 7, 8] to properly define 3-fold scrolls (X_e, ξ) which are the core of the paper. Sect. 2 deals with Ulrich line bundles on scrolls (X_e, ξ), cf. Theorem 2.1, whereas Sect. 3 focuses on the rank-2 case, using extensions suitably defined (cf. Theorems 3.1, 3.2). Finally Sect. 4 deals with the general case of any rank $r \geqslant 1$, via inductive processes, extensions, deformation and modular theory (cf. Theorems 4.8 and 4.13).

Notation and terminology. We work throughout over the field $\mathbb{C}$ of complex numbers. All schemes will be endowed with the Zariski topology. By *variety* we mean an integral algebraic scheme. We say that a property holds for a *general* point of a variety V if it holds for any point in a Zariski open non–empty subset of V. We will interchangeably use the terms *rank-r vector bundle* on a variety V and *rank-r locally free sheaf* on V; in particular for the case $r = 1$ of line bundles (equiv. invertible sheaves), to ease the notation and if no confusion arises, we sometimes identify line bundles with Cartier divisors interchangeably using additive notation instead of multiplicative notation and tensor products. Thus, if L and M are line bundles on V, the *dual* of L will be denoted by either $L^\vee$, or L^{-1} or even $-L$, so that $M \otimes L^\vee$ will be also denoted by either $M \otimes L^{-1}$ or just $M - L$. If $\mathcal{P}$ is either a *parameter space* of a flat family of geometric objects $\mathcal{E}$ defined on V (e.g. vector bundles, extensions, etc.) or a *moduli space* parametrizing geometric objects modulo a given equivalence relation, we will denote by $[\mathcal{E}]$ the parameter point (resp., the moduli point) corresponding to the geometric object $\mathcal{E}$ (resp., associated to the equivalence class of $\mathcal{E}$). For further non-reminded terminology, we refer the reader to [20].

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1. Preliminaries

We first remind some general definitions concerning Ulrich bundles on projective varieties.

Definition 1.1. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension n and let H be a hyperplane section of X. A vector bundle $\mathcal{U}$ on X is said to be* Ulrich *with respect to H if*

$$H^i(X, \mathcal{U}(-jH)) = 0 \quad \textit{for} \quad i = 0, \cdots, n \quad \textit{and} \quad 1 \leq j \leq \dim X.$$

Remark 1.2. (i) If X supports Ulrich bundles w.r.t. H, then one sets $uc_H(X)$, called the *Ulrich complexity of X w.r.t. H*, to be the minimum rank among possible Ulrich vector bundles on X.
(ii) If $\mathcal{U}_1$ is a vector bundle on X, which is Ulrich w.r.t. H then $\mathcal{U}_2 := \mathcal{U}_1^\vee(K_X + (n+1)H)$ is also Ulrich w.r.t. H. The vector bundle $\mathcal{U}_2$ is called the *Ulrich dual* of $\mathcal{U}_1$. From this we see that, if Ulrich bundles of some rank r on X do exist, then they come in pairs.

Definition 1.3. *Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension n polarized by H, where H is a hyperplane section of X, and let $\mathcal{U}$ be a rank-2 Ulrich vector bundle on X. Then $\mathcal{U}$ is said to be* special *if $c_1(\mathcal{U}) = K_X + (n+1)H$.*

Notice that, because $\mathcal{U}$ in Definition 1.3 is of rank-2, then $\mathcal{U}^\vee \cong \mathcal{U}(-c_1(\mathcal{U}))$ therefore being special is equivalent to $\mathcal{U}$ to be isomorphic to its Ulrich dual bundle.

We now remind facts concerning (semi)stability and slope-(semi)stability properties of these bundles (cf. [9, Def. 2.7]). Let $\mathcal{E}$ be a vector bundle on X; recall that $\mathcal{E}$ is said to be *semistable* if for every non-zero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$, with $0 < \mathrm{rk}(\mathcal{F}) :=$ rank of $\mathcal{F} < \mathrm{rk}(\mathcal{E})$, the inequality $\frac{P_{\mathcal{F}}}{\mathrm{rk}(\mathcal{F})} \leqslant \frac{P_{\mathcal{E}}}{\mathrm{rk}(\mathcal{E})}$ holds true, where $P_{\mathcal{F}}$ and $P_{\mathcal{E}}$ are the Hilbert polynomials of the sheaves. Furthermore, $\mathcal{E}$ is *stable* if the strict inequality above holds.

Similarly, recall that the *slope* of a vector bundle $\mathcal{E}$ (w.r.t. $\mathcal{O}_X(H)$) is defined to be $\mu(\mathcal{E}) := \frac{c_1(\mathcal{E}) \cdot H^{n-1}}{\mathrm{rk}(\mathcal{E})}$; the bundle $\mathcal{E}$ is said to be *μ-semistable*, or even *slope-semistable*, if for every non-zero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ with $0 < \mathrm{rk}(\mathcal{F}) < \mathrm{rk}(\mathcal{E})$, one has $\mu(\mathcal{F}) \leqslant \mu(\mathcal{E})$. The bundle $\mathcal{E}$ is *μ-stable*, or *slope-stable*, if the strict inequality holds.

The two definitions of (semi)stability are related as follows (cf. e.g. [9, § 2]):

$$\text{slope-stability} \Rightarrow \text{stability} \Rightarrow \text{semistability} \Rightarrow \text{slope-semistability}.$$

When the bundle in question is in particular Ulrich, the following more precise situation holds:

Theorem 1.4. *(cf.* [9, Thm. 2.9]*) Let $X \subset \mathbb{P}^N$ be a smooth variety of dimension n and let H be a hyperplane section of X. Let $\mathcal{U}$ be a rank-r vector bundle on X which is Ulrich w.r.t. H. Then:*
(a) $\mathcal{U}$ is semistable, so also slope-semistable;
(b) If $0 \to \mathcal{F} \to \mathcal{U} \to \mathcal{G} \to 0$ is an exact sequence of coherent sheaves with $\mathcal{G}$ torsion-free, and $\mu(\mathcal{F}) = \mu(\mathcal{U})$, then $\mathcal{F}$ and $\mathcal{G}$ are both Ulrich vector bundles.
(c) If $\mathcal{U}$ is stable then it is also slope-stable. In particular, the notions of stability and slope-stability coincide for Ulrich bundles.

In the sequel, we will focus on $n = \dim(X) = 3$; in such a case, the following notation will be used throughout this work.

X is a smooth, irreducible, projective variety of dimension 3 (or simply a 3-*fold*);

$\chi(\mathcal{F}) = \sum_{i=0}^{3}(-1)^i h^i(\mathcal{F})$, the Euler characteristic of $\mathcal{F}$, where $\mathcal{F}$ is any vector bundle of rank $r \geqslant 1$ on X;
K_X the canonical bundle of X. When the context is clear, X may be dropped, so $K_X = K$;
$c_i = c_i(X)$, the i^{th} Chern class of X;
$d = \deg X = L^3$, the degree of X in the embedding given by a very-ample line bundle L;
$g = g(X)$, the sectional genus of (X, L) defined by $2g - 2 = (K + 2L)L^2$;
if S is a smooth surface, $\equiv$ will denote the numerical equivalence of divisors on S.

For non-reminded terminology and notation, we basically follow [20].

Definition 1.5. *A pair (X, L), where X is a 3-fold and L is an ample line bundle on X, is a* scroll *over a normal variety Y if there exist an ample line bundle M on Y and a surjective morphism $\varphi : X \to Y$ with connected fibers such that $K_X + (4 - \dim Y)L = \varphi^*(M)$.*

In particular, if Y is a smooth surface and (X, L) is a scroll over Y, then (see [6, Prop. 14.1.3]) $X \cong \mathbb{P}(\mathcal{E})$, where $\mathcal{E} = \varphi_*(L)$ is a vector bundle on Y and L is the tautological line bundle on $\mathbb{P}(\mathcal{E})$. Moreover, if $S \in |L|$ is a smooth divisor, then (see e.g. [6, Thm. 11.1.2]) S is the blow up of Y at $c_2(\mathcal{E})$ points; therefore $\chi(\mathcal{O}_Y) = \chi(\mathcal{O}_S)$ and

$$d := L^3 = c_1^2(\mathcal{E}) - c_2(\mathcal{E}). \tag{1.1}$$

For the reader convenience we recall the following Theorem [18, Theorem 2.4] that will be used in the paper.

Theorem 1.6. *Let (Y, H) be a polarized surface with H very ample and let $\mathcal{E}$ be a rank two vector bundle on Y such that $\mathcal{E}$ is (very) ample and spanned. Let $\mathcal{F}$ be a rank r vector bundle satisfying:*

$$H^i(Y, \mathcal{F}) = 0 \quad \text{and} \quad H^i(Y, \mathcal{F}(-c_1(\mathcal{E}))) = 0, \tag{1.2}$$

for $i = 0, 1, 2$. Then on the 3-fold scroll $X \cong \mathbb{P}(\mathcal{E}) \xrightarrow{\pi} Y$, the vector bundle $\mathcal{U} := \pi^*\mathcal{F} \otimes \xi$ is Ulrich with respect to ξ, where ξ denotes the tautological line bundle on X, with $(X, \xi) \cong (\mathbb{P}(\mathcal{E}), \mathcal{O}_{\mathbb{P}(\mathcal{E})}(1))$.

Throughout this work, the base Y of the scroll X in Definition 1.5 will be the Hirzebruch surface $\mathbb{F}_e := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))$, with $e \geq 0$ an integer.

Let $\pi_e : \mathbb{F}_e \to \mathbb{P}^1$ be the natural projection onto the base. Then $\mathrm{Num}(\mathbb{F}_e) = \mathbb{Z}[C_e] \oplus \mathbb{Z}[f]$, where:
• C_e denotes the unique section corresponding to the morphism $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e) \twoheadrightarrow \mathcal{O}_{\mathbb{P}^1}(-e)$ on $\mathbb{P}^1$, and
• $f = \pi^*(p)$, for any $p \in \mathbb{P}^1$.
In particular

$$C_e^2 = -e, \; f^2 = 0, \; C_e f = 1.$$

Let $\mathcal{E}_e$ be a rank-two vector bundle over $\mathbb{F}_e$ and let $c_i(\mathcal{E}_e)$ be its i^{th}-Chern class. Then $c_1(\mathcal{E}_e) \equiv aC_e + bf$, for some $a, b \in \mathbb{Z}$, and $c_2(\mathcal{E}_e) \in \mathbb{Z}$. For the line bundle $\mathcal{L} \equiv \alpha C_e + \beta f$ we will also use the notation $\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$.

From now on, we will consider the following:

Assumptions 1.7. *Let $e \geq 0$, b_e, k_e be integers. Let $\mathcal{E}_e$ be a rank-two vector bundle over $\mathbb{F}_e$, with*

$$c_1(\mathcal{E}_e) \equiv 3C_e + b_e f \;\text{ and }\; c_2(\mathcal{E}_e) = k_e,$$

such that

(i) $h^0(\mathcal{E}_e) \geq 7$
(ii) $b_e \geq 3e + 1$
(iii) $b_e - e < k_e < 2b_e - 4e$

Moreover, there exists an exact sequence

$$0 \to A_e \to \mathcal{E}_e \to B_e \to 0, \tag{1.3}$$

where A_e and B_e are line bundles on $\mathbb{F}_e$ such that

$$A_e \equiv 2C_e + (2b_e - k_e - 2e)f \text{ and } B_e \equiv C_e + (k_e - b_e + 2e)f \tag{1.4}$$

From (1.3), in particular, one has $c_1(\mathcal{E}_e) = A_e + B_e$ and $c_2(\mathcal{E}_e) = A_e B_e$.

Remark 1.8. We give the explanation of the above assumptions. Conditions (i), (ii), and the inequality $b_e - e < k_e$ in (iii) give necessary conditions for the very ampleness of $\mathcal{E}_e$ (cf. [1, Prop.7.2]). In particular, $b_e - e < k_e$ in (iii) ensures that B_e is a very ample line bundle (cf. [20, § V, Prop.2.20]). For the existence of the exact sequence (1.3), with A_e, B_e as in (1.4), cf. [1, Prop.7.2] and [8]. Finally the last hypothesis in (iii), i.e. $k_e < 2b_2 - 4e$, ensures that also A_e is a very ample line bundle (cf. [20, § V, Prop.2.20]) which, together with $b_e - e < k_e$ in (iii), obviously implies that the vector bundle $\mathcal{E}_e$ is very ample (cf. [19, Remark 4.2 -(1)], where computations therein hold also for $e = 0, 1$).

2. Ulrich line bundles on 3-fold scrolls over $\mathbb{F}_e$

In this section, we consider 3-dimensional scrolls over $\mathbb{F}_e$, with $e \geq 0$, in projective spaces satisfying the conditions as in Assumptions 1.7.

Let therefore $\mathcal{E}_e$ be a very ample, rank-two vector bundle over $\mathbb{F}_e$ such that

$$c_1(\mathcal{E}_e) \equiv 3C_e + b_e f, \;\; c_2(\mathcal{E}_e) = k_e,$$

with b_e and k_e as in Assumptions 1.7-(ii) and (iii). Let $(\mathbb{P}(\mathcal{E}_e), \mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1))$ be the 3-fold scroll over $\mathbb{F}_e$, and let $\pi_e : \mathbb{F}_e \to \mathbb{P}^1$ and $\varphi : \mathbb{P}(\mathcal{E}_e) \to \mathbb{F}_e$ be the usual projections. Then $\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)$ defines an embedding

$$\Phi_e := \Phi_{|\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1)|} : \mathbb{P}(\mathcal{E}_e) \hookrightarrow X_e \subset \mathbb{P}^{n_e}, \tag{2.1}$$

where $X_e = \Phi_e(\mathbb{P}(\mathcal{E}_e))$ is smooth, non-degenerate, of degree d_e, with

$$n_e = 4b_e - k_e - 6e + 4 \geqslant 6 \;\; \text{and} \;\; d_e = 6b_e - 9e - k_e. \tag{2.2}$$

We set $(X_e, \xi) \cong (\mathbb{P}(\mathcal{E}_e), \mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1))$. Our aim in this section is to determine line bundles on such scrolls which are Ulrich w.r.t. ξ.

Theorem 2.1. *Let $e \geqslant 0$ be an integer and let (X_e, ξ) be a 3-fold scroll as above. Then X_e does not support any Ulrich line bundle w.r.t. ξ unless $e = 0$, in which case the following are the unique Ulrich line bundles on X_0:*

- (i) *$L_1 := \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, -1)$ and its Ulrich dual $L_2 := \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, b_0 - 1)$;*
- (ii) *for any integer $t \geq 1$, $M_1 := 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, -t - 1)$ and its Ulrich dual $M_2 := \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t - 1)$, which only occur for $b_0 = 2t, k_0 = 3t$.*

Proof. Let $\mathcal{L} = a\xi + \varphi^*\mathcal{O}_{F_e}(\alpha, \beta)$ be an Ulrich line bundle on X_e. From [18, Corollary 2.2] we know that $a = 0, 1, 2$.

Case I: If $a = 1$ then, by [18, Corollary 2.2], $\mathcal{L} = \xi + \varphi^*\mathcal{O}_{F_e}(\alpha, \beta)$ is Ulrich with respect to ξ if and only if

$$H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)) = H^i(\mathbb{F}_e, \mathcal{O}_{F_e}(\alpha, \beta) - c_1(\mathcal{E}_e)) = 0 \quad \text{for } i = 0, 1, 2.$$

Thus $\chi(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)) = \chi(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta) - c_1(\mathcal{E}_e)) = 0$. By Riemann-Roch we get, respectively,

$$(\alpha + 1)(e\alpha - 2\beta - 2) = 0 \tag{2.3}$$

and

$$(\alpha - 2)(e\alpha - 2\beta + 2b_e - 3e - 2) = 0 \tag{2.4}$$

Thus either $\alpha = -1$ which, along with (2.4), gives $\beta = b_e - 2e - 1$ or $\alpha = 2$ which, along with (2.3), gives $\beta = e - 1$. We need to check if $H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)) = H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta) - c_1(\mathcal{E}_e)) = 0$ for $i \geq 0$, with $(\alpha, \beta) = (-1, b_e - 2e - 1)$ or $(\alpha, \beta) = (2, e - 1)$.

If $e = 0$ then the vanishings follow by the Künneth formula, hence we get $\mathcal{L} = L_2$ in the first case whereas $\mathcal{L} = L_1$ in the latter case, where L_1 and L_2 are as in the statement. If otherwise $e > 0$, the cohomology groups are not all zero therefore there are no Ulrich line bundles with $a = 1$ in these cases.

Case II: If $a = 2$ then, by [18, Corollary 2.2], $\mathcal{L} = 2\xi + \varphi_e^*\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)$ is Ulrich with respect to ξ if and only if

$$H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)) = H^i(\mathbb{F}_e, \mathcal{E}_e(\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta))) = 0 \quad \text{for } i = 0, 1, 2.$$

Thus $\chi(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)) = \chi(\mathbb{F}_e, \mathcal{E}_e(\mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)))) = 0$. By Riemann-Roch we get (2.3) and

$$-e\alpha^2 + 2\alpha\beta + \alpha b_e - 4e\alpha + 2\alpha + 5\beta + 4b_e - 6e - k_e + 5 = 0, \tag{2.5}$$

respectively. From (2.3) either $\alpha = -1$ or $\beta = \frac{\alpha e}{2} - 1$.

Case II-a: $\alpha = -1$.

Plugging such value in (2.5) we get $\beta = -b_e + e + \frac{k_e}{3} - 1$, which forces $k_e = 3t$ for some $t \in \mathbb{Z}$, hence $\beta = -b_e + e + t - 1$. We compute $H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(\alpha, \beta)) = H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1))$ and $H^i(\mathbb{F}_e, \mathcal{E}_e(\mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1)))$ for $i = 0, 1, 2$.

Now $R^i\pi_{e*}(\mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1)) = 0$, for $i \geq 0$, hence, from Leray's isomorphism we have $H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1)) \cong H^i(\mathbb{P}^1, 0) = 0$, for $i = 0, 1, 2$.

To compute $H^i(\mathbb{F}_e, \mathcal{E}_e(\mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1)))$ we recall that the vector bundle $\mathcal{E}_e$ sits in the exact sequence (1.3), where $A_e \in |\mathcal{O}_{\mathbb{F}_e}(2, 2b_e - k_e - 2e)|$ and $B_e \in |\mathcal{O}_{\mathbb{F}_e}(1, k_e - b_e + 2e)|$ and after twisting (1.3) with $\mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1)$ we have

$$0 \to \mathcal{O}_{\mathbb{F}_e}(1, b_e - 2t - e - 1) \to \mathcal{E}_e(\mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1)) \to \mathcal{O}_{\mathbb{F}_e}(0, 4t - 2b_e + 3e - 1) \to 0. \tag{2.6}$$

Now

$$R^i\pi_{e*}\mathcal{O}_{\mathbb{F}_e}(1, b_e - 2t - e - 1) = 0, \quad \text{for } i > 0, \text{ and}$$
$$\pi_{e*}\mathcal{O}_{\mathbb{F}_e}(1, b_e - 2t - e - 1) \cong (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e)) \otimes \mathcal{O}_{\mathbb{P}^1}(b_e - 2t - e - 1),$$

hence, from Leray's isomorphism we have

$$\begin{aligned} H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(1, b_e - 2t - e - 1)) &\cong H^i(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))(b_e - 2t - e - 1)) \\ &= H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_e - 2t - e - 1)) \oplus H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_e - 2t - 2e - 1)) \end{aligned} \tag{2.7}$$

and similarly

$$H^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(0, 4t - 2b_e + 3e - 1)) \quad \cong \quad H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_e + 3e - 1)) \tag{2.8}$$

We consider first the case $e = 0$ and then the case $e \geq 1$.

If $e = 0$, then (2.7) and (2.8) become

$$H^i(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(1, b_0 - 2t - 1)) \quad = \quad H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_0 - 2t - 1)^{\oplus 2})$$

$$H^i(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(0, 4t - 2b_0 - 1)) \quad = \quad H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_0 - 1))$$

If $b_0 - 2t - 1 \geq 0$ then

$$\begin{aligned} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_0 - 2t - 1)^{\oplus 2}) &= 2(b_0 - 2t) \\ h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_0 - 2t - 1)^{\oplus 2}) &= 0, \quad \text{by Serre's duality on } \mathbb{P}^1. \end{aligned}$$

Note that if $b_0 - 2t - 1 \geq 0$ then $4t - 2b_0 - 1 \leq -3$ hence $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_0 - 1)) = 0$ and, by Serre duality, $h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_0 - 1)) \cong h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2b_0 - 4t - 1)) = 2b_0 - 4t$.

These computations, along with the cohomology sequence associated to (2.6), give

$$\begin{aligned} h^i(\mathcal{E}_0(\mathcal{O}_{\mathbb{F}_0}(-1, -b_0 + t - 1))) &= 2b_0 - 4t \geq 2 \text{ (by assumption), for } i = 0, 1, \\ h^2(\mathcal{E}_0(\mathcal{O}_{\mathbb{F}_0}(-1, -b_0 + t - 1))) &= 0, \text{ trivially.} \end{aligned}$$

If $b_0 - 2t - 1 < 0$ then

$$\begin{aligned} h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_0 - 2t - 1)^{\oplus 2}) &= 0 \\ h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_0 - 2t - 1)^{\oplus 2}) &\cong 2h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2t - b_0 - 1)), \quad \text{by Serre's duality on } \mathbb{P}^1. \end{aligned}$$

Note that

$$h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2t - b_0 - 1)) = \begin{cases} 0 & \text{if } b_0 = 2t \\ 2t - b_0 & \text{if } 2t - b_0 - 1 \geq 0. \end{cases}$$

Note also that $b_0 - 2t - 1 < 0$ implies that $4t - 2b_0 - 1 \geq -2$, hence
if $4t - 2b_0 - 1 \geq 0$, $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_0 - 1)) = 4t - 2b_0$ and $h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_0 - 1)) = 0$;
if $4t - 2b_0 - 1 = -1$, $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_0 - 1)) = h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_0 - 1)) = 0$;
if $4t - 2b_0 - 1 = -2$, $h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_0 - 1)) = 0$ and $h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_0 - 1)) = 1$.
These facts, along with (2.6), give that

$$\begin{aligned} &h^i(\mathcal{E}_0(\mathcal{O}_{\mathbb{F}_0}(-1, -b_0 + t - 1))) = 0, \text{ for } i = 0, 1, 2, \quad \text{if } b_0 = 2t, \\ &h^0(\mathcal{E}_0(\mathcal{O}_{\mathbb{F}_0}(-1, -b_0 + t - 1))) \neq 0, \text{ in the remaining two cases,} \end{aligned}$$

the latter case holds because otherwise from the cohomology sequence associated to (2.6) it would follow that $h^1(\mathcal{E}_0(\mathcal{O}_{\mathbb{F}_0}(-1, -b_0 + t - 1))) < 0$, which is impossible. Thus we are left with the cases $b_0 = 2t, k_0 = 3t$, where $t \geq 1$, as it follows from Assumptions 1.7 - (ii). Hence in this case we get $\mathcal{L} = M_1$ is Ulrich and its Ulrich dual is M_2.

If $e \geq 1$ in order to compute the cohomology groups in (2.7) and (2.8) we consider first the case in which $b_e - 2t - 2e - 1 \geq 0$. Note that in this case also $b_e - 2t - e - 1 \geq 0$ and thus in (2.7) we have

$$\begin{aligned} h^0(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(1, b_e - 2t - e - 1)) = h^0(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))(b_e - 2t - e - 1)) &= 2b_e - 4t - 3e, \\ h^1(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(1, b_e - 2t - e - 1)) = h^1(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))(b_e - 2t - e - 1)) &= 0. \end{aligned}$$

As for (2.8) note that $4t - 2b_e + 3e - 1 \leq -3 - e$, by assumption, and thus

$$\begin{aligned} H^0(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(0, 4t - 2b_e + 3e - 1)) &\cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_e + 3e - 1)) = 0 \quad \text{and} \\ h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(4t - 2b_e + 3e - 1))) &= h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2b_e - 4t - 3e - 1))) = 2b_e - 4t - 3e. \end{aligned}$$

From the cohomology sequence associated to (2.6) it follows that

$$h^0(\mathcal{E}_e(\mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1))) = h^1(\mathcal{E}_e(\mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1))) = 2b_e - 4t - 3e \geq 2 + e,$$

because $4t - 2b_e + 3e - 1 \leq -3 - e$.

If $b_e - 2t - e - 1 \geq 0$ and $b_e - 2t - 2e - 1 < 0$ (the case $b_e - 2t - 2e - 1 \geq 0$ was just treated), then

$$h^0(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(1, b_e - 2t - e - 1)) = h^0(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-e))(b_e - 2t - e - 1)) = b_e - 2t - e \geq 1.$$

Thus $h^0(\mathcal{E}_e(\mathcal{O}_{\mathbb{F}_e}(-1, -b_e + e + t - 1)) \geq h^0(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(1, b_e - 2t - e - 1)) = b_e - 2t - e \geq 1$.

Case II-b: $\beta = \frac{\alpha e}{2} - 1$.

Plugging such value in (2.5) we get

$$\alpha = \frac{-8b_e + 12e + 2k_e}{2b_e - 3e} \tag{2.9}$$

which implies that

$$(\alpha + 4)(2b_e - 3e) = 2k_e.$$

By Assumptions 1.7-(ii) we get $2b_e - 3e \geq 3e + 2 \geq 2$. Moreover by Assumptions 1.7-(iii) we get $k_e > b_e - e \geq 2e + 1$. Thus $\alpha + 4 > 0$, that is $\alpha \geq -3$. Notice that if $\alpha = -3$ then (2.9) gives $b_e = \frac{3}{2}e + k_e = \frac{1}{2}e + e + k_e > \frac{1}{2}e + b_e$ by Assumption 1.7-(iii), which is a contradiction. Hence $\alpha \geq -2$ and therefore from (2.9) it follows that $k_e \geq 2b_e - 3e$ which contradicts the condition $k_e < 2b_e - 4e$ in Assumptions 1.7-(iii).

The proof is complete since the case $a = 0$ is the Ulrich dual of the case $a = 2$. □

3. Rank-2 Ulrich vector bundles on 3-fold scrolls over $\mathbb{F}_e$

As in the previous section, we consider here 3-dimensional scrolls over $\mathbb{F}_e$, with $e \geq 0$, in projective spaces, satisfying the conditions as in Assumptions 1.7. Our aim is to prove the existence of some moduli spaces of rank-2 Ulrich bundles on such 3-fold scrolls and to study their basic properties. As a matter of notation, in the sequel F will always denote the fiber of the natural scroll map $\varphi : X_e \cong \mathbb{P}(\mathcal{E}_e) \to \mathbb{F}_e$.

3.1. **Rank-2 Ulrich vector bundles on 3-fold scrolls over $\mathbb{F}_0$.** From Theorem 2.1 we know that on X_0 there exist Ulrich line bundles. Using these line bundles, we will construct rank two Ulrich vector bundles arising as non-trivial extensions of them.

Case L: Let L_1 and L_2 be line bundles on X_0 as in Theorem 2.1-(i). Notice that

$$\begin{aligned}\operatorname{Ext}^1(L_2, L_1) &\cong H^1(X_0, L_1 - L_2) = H^1(X_0, \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, -b_0)) \cong H^1(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(3, -b_0))\\ &\cong H^1(\mathbb{P}^1, S^3(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})(-b_0)) \cong H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus 4}(-b_0)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus 4}(b_0 - 2)).\end{aligned}$$

Hence $\dim \operatorname{Ext}^1(L_2, L_1) = 4b_0 - 4 > 0$, being $b_0 \geq 2$ by Assumptions 1.7. Thus there are non-trivial extensions $\mathcal{F}_1$

$$0 \to L_1 \to \mathcal{F}_1 \to L_2 \to 0 \tag{3.1}$$

of L_2 by L_1. Similarly

$$\begin{aligned}\operatorname{Ext}^1(L_1, L_2) &\cong H^1(X_0, L_2 - L_1) = H^1(X_0, \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3, b_0)) \cong H^1(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(-3, b_0))\\ &\cong H^1(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(1, -2 - b_0)) \cong H^1(\mathbb{P}^1, (\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1})(-2 - b_0)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(b_0)).\end{aligned}$$

Hence $\dim \operatorname{Ext}^1(L_1, L_2) = 2b_0 + 2 > 0$ and thus there are non-trivial extensions $\mathcal{F}_1'$

$$0 \to L_2 \to \mathcal{F}_1' \to L_1 \to 0 \tag{3.2}$$

of L_1 by L_2. Notice that the vector bundles $\mathcal{F}_1$ and $\mathcal{F}_1'$ are both Ulrich rank two vector bundles with

$$c_1(\mathcal{F}_1) = c_1(\mathcal{F}_1') = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, b_0 - 2) \text{ and } c_2(\mathcal{F}_1) = c_2(\mathcal{F}_1') = \xi^2 + \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, b_0 - 2) + (2b_0 - 1)F,$$

also $c_2(\mathcal{F}_1) = c_2(\mathcal{F}_1') = \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 2b_0 - 2) + (2b_0 - k_0 - 1)F$. Moreover, since L_1 and L_2 are non-isomorphic line bundles with the same slope

$$\mu(L_1) = \mu(L_2) = 8b_0 - k_0 - 3$$

with respect to ξ then, by [9, Lemma 4.2], $\mathcal{F}_1$ and $\mathcal{F}_1'$ are simple vector bundles.

The family of non-trivial extensions of (3.1) is of dimension $4b_0 - 4$ while the one of (3.2) is $2b_0 + 2$, which are different positive integers unless $b_0 = 3$.

Case M: Let M_1 and M_2 be line bundles on X_0 as in Theorem 2.1-(ii). As above one computes

$$\operatorname{Ext}^1(M_2, M_1) \cong H^1(\mathbb{P}(\mathcal{E}_0), M_1 - M_2) = H^1(2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3, -4t)) \cong H^1(\mathbb{F}_0, S^2(\mathcal{E}_0)(-3, -4t)),$$

hence we need to compute $H^1(\mathbb{F}_0, S^2(\mathcal{E}_0)(-3, -4t))$, where $S^2(\mathcal{E}_0)$ denotes the second symmetric power of $\mathcal{E}_0$. The vector bundle $\mathcal{E}_0$ fits in the exact sequence (1.3), with A_0 and B_0 as in (1.4) and with $b_0 = 2t$ and $k_0 = 3t$, $t \geq 1$. By [20, 5.16.(c), p. 127], there is a finite filtration of $S^2(\mathcal{E}_0)$,

$$S^2(\mathcal{E}_0) = F^0 \supseteq F^1 \supseteq F^2 \supseteq F^3 = 0$$

with quotients

$$F^p/F^{p+1} \cong S^p(A_0) \otimes S^{2-p}(B_0),$$

for each $0 \leq p \leq 2$. Hence

$$\begin{aligned}F^0/F^1 &\cong S^0(A_e) \otimes S^2(B_0) = 2B_0\\ F^1/F^2 &\cong S^1(A_0) \otimes S^1(B_0) = A_0 + B_0\\ F^2/F^3 &\cong S^2(A_0) \otimes S^0(B_0) = 2A_0, \text{ that is } F^2 = 2A_0,\end{aligned}$$

since $F^3 = 0$. Thus, we get the following exact sequences

$$0 \to F^1 \to S^2(\mathcal{E}_0) \to 2B_0 \to 0 \tag{3.3}$$

$$0 \to F^2 \to F^1 \to A_0 + B_0 \to 0 \tag{3.4}$$

$$F^2 = 2A_0 \tag{3.5}$$

Twisting (3.3), (3.4) with $\mathcal{O}_{\mathbb{F}_0}(-3,-4t)$ and using (3.5) we get

$$0 \to F^1(-3,-4t) \to S^2(\mathcal{E}_0) \otimes \mathcal{O}_{\mathbb{F}_0}(-3,-4t) \to \mathcal{O}_{\mathbb{F}_0}(-1,-2t) \to 0 \tag{3.6}$$

$$0 \to \mathcal{O}_{F_0}(1,-2t) \to F^1 \otimes \mathcal{O}_{\mathbb{F}_0}(-3,-4t) \to \mathcal{O}_{F_0}(0,-2t) \to 0 \tag{3.7}$$

First we focus on (3.7).

$$h^i(\mathcal{O}_{F_0}(1,-2t)) = h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2t)^{\oplus 2})$$

so, for dimension reasons, $h^i(\mathcal{O}_{F_0}(1,-2t)) = 0$, for any $i \geq 2$. Since $t \geq 1$, $h^0(\mathcal{O}_{F_0}(1,-2t)) = 0$ and $h^1(\mathcal{O}_{F_0}(1,-2t)) = 4t-2$. Similarly

$$h^i(\mathcal{O}_{F_0}(0,-2t)) = h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2t))$$

so, $h^i(\mathcal{O}_{F_0}(0,-2t)) = 0$, for any $i \geq 2$, $h^0(\mathcal{O}_{F_0}(0,-2t)) = 0$ and $h^1(\mathcal{O}_{F_0}(0,-2t)) = 2t-1$ then (3.7) gives

$$h^1(F^1(-3,-4t)) = 6t-3, \quad h^i(F^1(-3,-4t)) = 0, \text{ for } i = 0, 2.$$

Passing to (3.6) observe that, $h^i(\mathcal{O}_{F_0}(-1,-2t)) = 0$, for any $i \geq 0$. This, along with (3.8) and (3.6) gives

$$\begin{aligned} h^1(2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3,-4t)) &= h^1(\mathbb{F}_0, S^2(\mathcal{E}_0(-3,-4t)) = 6t-3 = 3b_0 - 3, \\ h^i(2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3,-4t)) &= h^i(\mathbb{F}_0, S^2(\mathcal{E}_0)(-3,-4t)) = 0, \quad \text{for } i = 0,2,3. \end{aligned} \tag{3.8}$$

Hence $\dim(\mathrm{Ext}^1(M_2, M_1)) = 3b_0 - 3 > 0$ because $b_0 \geq 2$ by Assumptions 1.7. Thus there are non-trivial extensions $\mathcal{F}_2$

$$0 \to M_1 \to \mathcal{F}_2 \to M_2 \to 0 \tag{3.9}$$

of M_2 by M_1. Similarly,

$$\begin{aligned} \mathrm{Ext}^1(M_1, M_2) &\cong H^1(\mathbb{P}(\mathcal{E}_0), M_2 - M_1) = H^1(-2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3,4t)) \cong H^2(\varphi^*\mathcal{O}_{\mathbb{F}_0}(-2,-b_0-2)) \\ &\cong H^2(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(-2,-b_0-2)) \cong H^0(\mathbb{F}_0, \mathcal{O}_{\mathbb{F}_0}(0,b_0)) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_0)). \end{aligned}$$

Hence $\dim(\mathrm{Ext}^1(M_1, M_2)) = b_0 + 1 > 0$ and thus there are non-trivial extensions $\mathcal{F}'_2$

$$0 \to M_2 \to \mathcal{F}'_2 \to M_1 \to 0 \tag{3.10}$$

of M_1 by M_2. Notice that the vector bundles $\mathcal{F}_2$ and $\mathcal{F}'_2$ are both Ulrich rank two vector bundles with

$$c_1(\mathcal{F}_2) = c_1(\mathcal{F}'_2) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 2t-2) \ \text{ and } \ c_2(\mathcal{F}_2) = c_2(\mathcal{F}'_2) = \xi\varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 6t-2) - (5t+1)F.$$

Moreover, since M_1 and M_2 are non-isomorphic line bundles with the same slope

$$\mu(M_1) = \mu(M_2) = 13t - 3$$

with respect to ξ then, by [9, Lemma 4.2], $\mathcal{F}_2$ and $\mathcal{F}'_2$ are simple vector bundles. The family of non-trivial extensions of (3.1) is of dimension $3b_0 - 3$ while the one of (3.2) is $b_0 + 1$ which are different positive integers unless $b_0 = 2$.

Case L-M: If we consider extensions using both line bundles of type L_i and M_j, with $i, j = 1, 2$, one can easily see that for some of them we get only trivial extensions, precisely:

$$\mathrm{Ext}^1(M_1, L_1) \cong H^1(X_0, L_1 - M_1) = H^1(X_0, -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3,t)) \cong H^1(\mathbb{F}_0, 0) = 0,$$

$$\mathrm{Ext}^1(L_1, M_2) \cong H^1(X_0, M_2 - L_1) = H^1(X_0, -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0,3t)) \cong H^1(\mathbb{F}_0, 0) = 0,$$

$\mathrm{Ext}^1(M_1, L_2) \cong H^1(X_0, L_2 - M_1) = H^1(X_0, -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0, 3t)) \cong H \cong H^1(\mathbb{F}_0, 0) = 0,$

$\mathrm{Ext}^1(L_2, M_2) \cong H^1(X_0, M_2 - L_2) = H^1(X_0, -\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, 3t - b_0) \cong H^1(\mathbb{F}_0, 0) = 0.$

On the contrary, in the remaining possibilities we get non-trivial extensions and precisely:

$$\mathrm{Ext}^1(L_1, M_1) \cong H^1(X_0, M_1 - L_1) = H^1(X_0, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3, -t)) \cong H^1(\mathbb{F}_0, \mathcal{E}_0(-3, -t));$$

an easy computation gives that $\dim(\mathrm{Ext}^1(L_1, M_1)) = 1$ and thus there are non-trivial extensions $\mathcal{F}_3$ such that $c_1(\mathcal{F}_3) = 3\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, -t - 2)$ and $c_2(\mathcal{F}_3) = \xi\varphi^*\mathcal{O}_{\mathbb{F}_0}9, 3t - 3) - (8t + 1)F$.

$$\mathrm{Ext}^1(M_2, L_1) \cong H^1(X_0, L_1 - M_2) = H^1(X_0, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0, -3t)) \cong H^1(\mathbb{F}_0, \mathcal{E}_0(0, -t));$$

in this case $\dim(\mathrm{Ext}^1(M_2, L_1)) = 5b_0 - 5$ and so there are non-trivial extensions $\mathcal{F}_4$ (because $b_0 > 1$) such that $c_1(\mathcal{F}_4) = \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(4, 3t - 2)$ and $c_2(\mathcal{F}_4) = \xi\varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t - 1) + (6t - 4)F$.

$$\mathrm{Ext}^1(L_2, M_1) \cong H^1(X_0, M_1 - L_2) = H^1(X_0, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(0, -3t)) \cong H^1(\mathbb{F}_0, \mathcal{E}_0(0, -t)),$$

thus $\dim(\mathrm{Ext}^1(L_2, M_1)) = 5b_0 - 5$ and thus there are non-trivial extensions $\mathcal{F}_5$ with $c_1(\mathcal{F}_5) = 3\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-2, t - 2)$ and $c_2(\mathcal{F}_5) = \xi\varphi^*\mathcal{O}_{\mathbb{F}_0}(3, 7t - 3) + (2 - 7t)F$.

$$\mathrm{Ext}^1(M_2, L_2) \cong H^1(X_0, L_2 - M_2) = H^1(X_0, \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3, -t)) \cong H^1(\mathbb{F}_0, \mathcal{E}_0(-3, -t)) \cong \mathbb{C};$$

thus there are non-trivial extensions $\mathcal{F}_6$ with $c_1(\mathcal{F}_6) = \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, 5t - 2)$ and $c_2(\mathcal{F}_6) = \xi\varphi^*\mathcal{O}_{\mathbb{F}_0}(2, 3t - 1) + (t - 1)F$.

Previous computations show that there are Ulrich rank-2 vector bundles belonging to different moduli spaces, since their Chern classes are different. For simplicity, in the sequel we will focus only on extensions of type (3.1) and, in particular, we prove the following theorem.

Theorem 3.1. *Let $(X_0, \xi) \cong (\mathbb{P}(\mathcal{E}_0), \mathcal{O}_{\mathbb{P}(\mathcal{E}_0)}(1))$ be a 3-fold scroll over $\mathbb{F}_0$, with $\mathcal{E}_0$ as in Assumptions 1.7. Let $\varphi : X_0 \to \mathbb{F}_0$ be the scroll map and F be the φ-fibre. Then the moduli space of rank-2 vector bundles $\mathcal{U}$ on X_0 which are Ulrich w.r.t. ξ and with Chern classes*

$$c_1(\mathcal{U}) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, b_0 - 2) \text{ and } c_2(\mathcal{U}) = \xi^2 + \xi \cdot \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, b_0 - 2) + (2b_0 - 1)F, \tag{3.11}$$

is not empty and it contains a generically smooth component $\mathcal{M}$ of dimension $6b_0 - 3$, whose general point $[\mathcal{U}]$ corresponds to a special and slope-stable vector bundle, of slope w.r.t. ξ

$$\mu(\mathcal{U}) = 8b_0 - k_0 - 3. \tag{3.12}$$

Proof. We consider non–trivial extensions (3.1) in Case L. Recall that $\dim \mathrm{Ext}^1(L_2, L_1) = 4b_0 - 4 > 0$, being $b_0 > 1$ by Assumptions 1.7, and moreover that an indecomposable vector bundle $\mathcal{F}_1$ as in (3.1) is Ulrich and simple, that is $h^0(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = 1$. Since $\mu(L_1) = \mu(L_2) = 8b_0 - k_0 - 3$, as computed in § 3.1-Case L, the same holds true for $\mu(\mathcal{F}_1)$.

We now want to show that $h^2(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = 0 = h^3(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee)$ and that $\chi(\mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = -6b_0 + 3$. Tensoring (3.1) with $\mathcal{F}_1^\vee$ we get

$$0 \to L_1 \otimes \mathcal{F}_1^\vee \to \mathcal{F}_1 \otimes \mathcal{F}_1^\vee \to L_2 \otimes \mathcal{F}_1^\vee \to 0. \tag{3.13}$$

Dualizing (3.1) gives the following exact sequence

$$0 \to L_2^\vee \to \mathcal{F}_1^\vee \to L_1^\vee \to 0 \tag{3.14}$$

Tensoring (3.14) with L_1 and L_2, respectively gives

$$0 \to L_2^\vee \otimes L_1 (= \varphi^*\mathcal{O}_{\mathbb{F}_0}(3, -b_0)) \to L_1 \otimes \mathcal{F}_1^\vee \to \mathcal{O}_{X_0} \to 0 \tag{3.15}$$

$$0 \to \mathcal{O}_{X_0} \to L_2 \otimes \mathcal{F}_1^\vee \to L_2 \otimes L_1^\vee (= \varphi^*\mathcal{O}_{\mathbb{F}_0}(-3, b_0)) \to 0 \tag{3.16}$$

Because $\mathcal{F}_1$ is simple, then $h^0(X, \mathcal{F}_1 \otimes \mathcal{F}_1^\vee) = 1$. The remaining cohomology $H^i(X, \mathcal{F}_1 \otimes \mathcal{F}_1^\vee)$ can be easily computed from the cohomology sequence associated to (3.15) and (3.16). Clearly

$h^i(\mathcal{O}_{X_0}) = 0$ if $i \geq 1$ and $h^0(\mathcal{O}_{X_0}) = 1$. It remains to compute $H^i(\varphi^*\mathcal{O}_{\mathbb{F}_0}(3,-b_0))$ and $H^i(\varphi^*\mathcal{O}_{\mathbb{F}_0}(-3,b_0))$.

$$(3.17)\quad H^i(X_0,\varphi^*\mathcal{O}_{\mathbb{F}_0}(3,-b_0)) \cong H^i(\mathbb{F}_0,\mathcal{O}_{\mathbb{F}_0}(3,-b_0)) \cong H^i(\mathbb{P}^1, S^3(\mathcal{O}_{\mathbb{P}^1}\oplus\mathcal{O}_{\mathbb{P}^1})(-b_0))$$
$$H^i(\mathbb{P}^1,\mathcal{O}^{\oplus 4}(-b_0)) = \begin{cases} 0 & \text{if } i=0,2,3 \\ 4b_0-4 & \text{if } i=1 \end{cases}$$

Similarly

$$(3.18)\quad H^i(X,\varphi^*\mathcal{O}_{\mathbb{F}_0}(-3,b_0)) \cong H^i(\mathbb{F}_0,\mathcal{O}_{\mathbb{F}_0}(-3,b_0)) \cong H^{2-i}((\mathbb{F}_0,\mathcal{O}_{\mathbb{F}_0}(1,-2-b_0))$$
$$H^{2-i}(\mathbb{P}^1,\mathcal{O}_{\mathbb{P}^1}^{\oplus 2}(-2-b_0)) = \begin{cases} 0 & \text{if } i=0,2,3 \\ 2b_0+2 & \text{if } i=1 \end{cases}$$

It thus follows that

$$h^2(\mathcal{F}_1\otimes\mathcal{F}_1^\vee) = 0 = h^3(\mathcal{F}_1\otimes\mathcal{F}_1^\vee).$$

From (3.13) we have that

$$\chi(\mathcal{F}_1\otimes\mathcal{F}_1^\vee) = \chi(L_1\otimes\mathcal{F}_1^\vee) + \chi(L_2\otimes\mathcal{F}_1^\vee) = -6b_0+4.$$

By [9, Proposition 2.10], since $h^2(\mathcal{F}_1\otimes\mathcal{F}_1^\vee) = 0$ and from the fact that Ulrichness is an open condition by semi-continuity, it follows that $[\mathcal{F}_1]$ corresponds to a smooth point of a unique component $\mathcal{M}$ of the moduli space of rank two Ulrich vector bundles with Chern classes

$$c_1(\mathcal{F}_1) = 2\xi+\varphi^*\mathcal{O}_{\mathbb{F}_0}(1,b_0-2) \text{ and } c_2(\mathcal{F}_1) = \xi^2+\xi\cdot\varphi^*\mathcal{O}_{\mathbb{F}_0}(1,b_0-2)+(2b_0-1)F,$$

as computed in § 3.1-Case L.

Moreover, as $h^3(\mathcal{F}_1\otimes\mathcal{F}_1^\vee) = 0$ and $\mathcal{F}_1$ is simple, we have that

$$\dim(\mathcal{M}) = h^1(\mathcal{F}_1\otimes\mathcal{F}_1^\vee) = -\chi(\mathcal{F}_1\otimes\mathcal{F}_1^\vee)+1 = 6b_0-3.$$

By semicontinuity and the invariants of the Chern classes, same assertions hold for the general point $[\mathcal{U}]\in\mathcal{M}$ so $\mathcal{M}$ is generically smooth. Note further that

$$K_{X_0}+4\xi = -2\xi+\varphi^*\mathcal{O}_{\mathbb{F}_0}(-2,-2)+\varphi^*\mathcal{O}_{\mathbb{F}_0}(3,b_0)+4\xi = 2\xi+\varphi^*\mathcal{O}_{\mathbb{F}_0}(1,b_0-2) = c_1(\mathcal{F})$$

thus $\mathcal{U}$ is special, as stated.

It only remains to prove that $\mathcal{U}$ is slope-stable. By [5, Sect 3, (3.2)], if $\mathcal{U}$ were not stable, it would be presented as an extension of Ulrich line bundles on X_0. In such a case, by the classification of Ulrich line bundles in Theorem 2.1 and all the possible non-trivial extensions computed in § 3.1-Case M or Case L-M, we see that the only possibilities should be either (3.1) or (3.2), by Chern classes reasons. In both cases the dimension of (the projectivization) of the families of extensions are either $4b_0-5$ or $2b_0+1$, which are both strictly smaller than $6b_0-2 = \dim\mathcal{M}$. This shows that $[\mathcal{U}]\in\mathcal{M}$ general corresponds to a stable and also slope-stable bundle (cf. Theorem 1.4-(c) above).

Finally, the slope $\mu(\mathcal{U}) = \frac{c_1(\mathcal{U})\cdot\xi^2}{2}$ of the vector bundle $\mathcal{U}$ with respect to ξ is $\mu(\mathcal{U}) = 8b_0-k_0-3$, as $c_1(\mathcal{U}) = c_1(\mathcal{F}_1)$. □

3.2. **Rank-2 Ulrich vector bundles on 3-fold scrolls over $\mathbb{F}_e$, $e>0$.** In this section, we will focus on the case $e>0$.

Theorem 3.2. *Let $(X_e,\xi)\cong(\mathbb{P}(\mathcal{E}_e),\mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1))$ be a 3-fold scroll over $\mathbb{F}_e$, with $e>0$ and $\mathcal{E}_e$ as in Assumptions 1.7. Let $\varphi: X_e\to\mathbb{F}_e$ be the scroll map and F be the φ- fibre. Then the moduli space of rank-2 vector bundles $\mathcal{U}$ on X_e, which are Ulrich w.r.t. ξ, with Chern classes*

$$(3.19)\quad c_1(\mathcal{U}) = 2\xi+\varphi^*\mathcal{O}_{\mathbb{F}_e}(1,b_e-e-2) \text{ and } c_2(\mathcal{U}) = \xi^2+\xi\varphi^*\mathcal{O}_{F_e}(1,b_e-e-2)+(2b_e-3e-1)F,$$

is not empty and it contains a generically smooth component $\mathcal{M}$ of dimension $6b_e-9e-3$, whose general point $[\mathcal{U}]$ corresponds to a special and slope-stable vector bundle of slope w.r.t. ξ

$$(3.20)\qquad \mu(\mathcal{U}) = 8b_e-k_e-12e-3.$$

Proof. By [4, Theorem 3.4], we know that there exist rank two vector bundles $\mathcal{H}_1$ on $\mathbb{F}_e$, which are Ulrich with respect to $c_1(\mathcal{E}_e) = \mathcal{O}_{\mathbb{F}_e}(3, b_e)$, given by extensions

$$0 \to \mathcal{O}_{\mathbb{F}_e}(3, b_e - 1) \to \mathcal{H}_1 \to I_Z \otimes \mathcal{O}_{\mathbb{F}_e}(4, 2b_e - 1 - e) \to 0, \tag{3.21}$$

where Z is a general zero-dimensional subscheme of $\mathbb{F}_e$ of length $\ell(Z) = 2b_e - 3e$. Such a bundle $\mathcal{H}_1$ is stable, [4, Remark 3.7], hence simple, that is $h^0(\mathcal{H}_1 \otimes \mathcal{H}_1^\vee) = 1$,

Let $\mathcal{H} := \mathcal{H}_1(-c_1(\mathcal{E}_e))$. Note that $\mathcal{H}$ is stable, being a twist of a stable vector bundle, so it is also simple, i.e. $h^0(\mathcal{H} \otimes \mathcal{H}^\vee) = 1$. Because the vector bundle $\mathcal{H}$ satisfies (1.2), then by Theorem 1.6 we know that the vector bundle $\mathcal{V} := \varphi^*(\mathcal{H}) \otimes \xi$ is a rank two vector bundle on $\mathbb{P}(\mathcal{E}_e)$ which is Ulrich with respect to ξ.

From (3.21) we see that $c_1(\mathcal{H}) = \mathcal{O}_{\mathbb{F}_e}(1, b_e - 2 - e)$ and $c_2(\mathcal{H}) = 2b_2 - 3e - 1$. Easy Chern classes computations give that

$$c_1(\mathcal{V}) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(1, b_e - e - 2) \text{ and } c_2(\mathcal{V}) = \xi^2 + \xi\varphi^*\mathcal{O}_{F_e}(1, b_e - e - 2) + (2b_e - 3e - 1)F.$$

Moreover by [5, Sect 3, (3.2)] such a bundle $\mathcal{V}$ is stable, so slope-stable by Theorem 1.4-(c), since there are no Ulrich line bundles on (X_e, ξ) as it follows from Theorem 2.1.

Our next step is to compute the cohomology groups $H^i(X_e, \mathcal{V} \otimes \mathcal{V}^\vee)$ for $i = 0, 1, 2, 3$. Because $H^i(X_e, \mathcal{V}\otimes\mathcal{V}^\vee) = H^i(X_e, \varphi^*(\mathcal{H}\otimes\mathcal{H}^\vee)) \cong H^i(\mathbb{F}_e, \mathcal{H}\otimes\mathcal{H}^\vee)$ we will focus on computations of $H^i(\mathbb{F}_e, \mathcal{H} \otimes \mathcal{H}^\vee)$, $i = 0, 1, 2, 3$.

First of all $h^3(\mathbb{F}_e, \mathcal{H} \otimes \mathcal{H}^\vee) = 0$, as $\mathbb{F}_e$ is a surface, and $h^0(\mathbb{F}_e, \mathcal{H} \otimes \mathcal{H}^\vee) = 1$, as $\mathcal{H}$ is simple. For the other cohomology groups, we then tensor (3.21) with $-c_1(\mathcal{E}_e) = \mathcal{O}_{\mathbb{F}_e}(-3, -b_e)$ and we get

$$0 \to \mathcal{O}_{\mathbb{F}_e}(0, -1) \to \mathcal{H} \to I_Z \otimes \mathcal{O}_{\mathbb{F}_e}(1, b_e - 1 - e) \to 0 \tag{3.22}$$

Because $\mathcal{H}$ is of rank 2 and $c_1(\mathcal{H}) = \mathcal{O}_{\mathbb{F}_e}(1, b_e - 2 - e)$, we have that $\mathcal{H}^\vee \cong \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(-1, -b_e + 2 + e)$ and thus after tensoring (3.22) with $\mathcal{H}^\vee$ we get

$$0 \to \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(-1, 1 + e - b_e) \to \mathcal{H} \otimes \mathcal{H}^\vee \to I_Z \otimes \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0, 1) \to 0. \tag{3.23}$$

In order to compute the cohomology groups of $\mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(-1, 1 + e - b_e)$ we will use the short exact sequence (3.22) twisted with $\mathcal{O}_{\mathbb{F}_e}(-1, 1 + e - b_e)$ which gives

$$0 \to \mathcal{O}_{\mathbb{F}_e}(-1, e - b_e) \to \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(-1, 1 + e - b_e) \to I_Z \to 0. \tag{3.24}$$

From this we can easily see that $H^0(\mathcal{F}\otimes\mathcal{O}_{\mathbb{F}_e}(-1, 1+e-b_e)) = 0 = H^2(\mathcal{F}\otimes\mathcal{O}_{\mathbb{F}_e}(-1, 1+e-b_e))$ and $h^1(\mathcal{F} \otimes \mathcal{O}_{\mathbb{F}_e}(-1, 1 + e - b_e)) = 2b_e - 3e - 1$.

Our next task is to compute the cohomology groups of $I_Z \otimes \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0, 1)$. We tensor the sequence

$$0 \to I_Z \to \mathcal{O}_{\mathbb{F}_e} \to \mathcal{O}_Z \to 0$$

with $\mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0, 1)$ and $\mathcal{O}_{\mathbb{F}_e}(1, b_e - e)$, respectively, and we get

$$0 \to I_Z \otimes \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0, 1) \to \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0, 1) \to (\mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0, 1))_{|Z} \to 0 \tag{3.25}$$

$$0 \to I_Z \otimes \mathcal{O}_{\mathbb{F}_e}(1, b_e - e) \to \mathcal{O}_{\mathbb{F}_e}(1, b_e - e) \to \mathcal{O}_Z \to 0. \tag{3.26}$$

We tensor (3.22) with $\mathcal{O}_{\mathbb{F}_e}(0, 1)$ and we get

$$0 \to \mathcal{O}_{\mathbb{F}_e} \to \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0, 1) \to I_Z \otimes \mathcal{O}_{\mathbb{F}_e}(1, b_e - e) \to 0. \tag{3.27}$$

Now use the cohomology sequence associated to the short exact sequences (3.25), (3.26) and (3.27). Note that

$$h^0(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(1, b_e - e)) \cong h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_e - e) \oplus \mathcal{O}_{\mathbb{P}^1}(b_e - 2e)) = 2b_e - 3e + 2,$$

and

$$h^i(\mathbb{F}_e, \mathcal{O}_{\mathbb{F}_e}(1, b_e - e)) \cong h^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b_e - e) \oplus \mathcal{O}_{\mathbb{P}^1}(b_e - 2e)) = 0 \quad \text{for } i = 1, 2$$

because by assumption $b_e > 3e$. Notice that $\dim(|\mathcal{O}_{\mathbb{F}_e}(1, b_e - e)|) = 2b_e - 3e + 1$, so $h^0(I_Z \otimes \mathcal{O}_{\mathbb{F}_e}(1, b_e - e)) = 2$, being Z general of length $\ell(Z) = 2b_e - 3e$. Therefore from (3.26) it follows that $h^i(I_Z \otimes \mathcal{O}_{\mathbb{F}_e}(1, b_e - e)) = 0$ for $i = 1, 2$. Now using (3.27) it follows that $h^0(\mathbb{F}_e, \mathcal{H} \otimes$

$\mathcal{O}_{\mathbb{F}_e}(0,1)) = 3$ and $h^i(\mathbb{F}_e, \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0,1)) = 0$ for $i = 1, 2$. Thus $h^2(I_Z \otimes \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0,1)) = 0$ and this, combined with the cohomology sequence associated to (3.24), gives that $h^2(\mathcal{H} \otimes \mathcal{H}^\vee) = 0$.

Thus from (3.25), since $h^0(\mathbb{F}_e, (\mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0,1))_{|Z}) = 2(2b_2 - 3e)$ and $h^i(\mathbb{F}_e, \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0,1)_{|Z}) = 0$ for $i = 1, 2$, it follows that $\chi(I_Z \otimes \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0,1)) = \chi(\mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0,1)) - \chi(\mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0,1)_{|Z}) = 3 - 4b_e + 6e$. From the cohomology sequence associated to (3.24) it follows that $\chi(\mathcal{H} \otimes \mathcal{H}^\vee) = \chi(I_Z \otimes \mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(0,1)) + \chi(\mathcal{H} \otimes \mathcal{O}_{\mathbb{F}_e}(-1, 1 + e - b_e)) = 3 - 4b_e + 6e - 2b_2 + 3e + 1 = 4 - 6b_e + 9e$ and thus $h^1(\mathcal{H} \otimes \mathcal{H}^\vee) = 1 - \chi(\mathcal{H} \otimes \mathcal{H}^\vee) = 6b_e - 9e - 3$.

As already observed, $H^i(X_e, \mathcal{V} \otimes \mathcal{V}^\vee) \cong H^i(\mathbb{F}_e, \mathcal{H} \otimes \mathcal{H}^\vee)$, $i = 0, 1, 2, 3$, hence the above computations give us the dimensions of all the cohomology groups $H^i(X_e, \mathcal{V} \otimes \mathcal{V}^\vee)$, $i = 0, 1, 2, 3$.

By [9, Proposition 2.10], since $h^2(\mathcal{V} \otimes \mathcal{V}^\vee) = 0$, it follows that $[\mathcal{V}]$ corresponds to a smooth point of a unique component $\mathcal{M}$ of the moduli space of rank two Ulrich vector bundles with Chern classes $c_1(\mathcal{V}) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(1, b_e - 2 - e)$ and $c_2(\mathcal{V}) = \xi^2 + \xi\varphi^*\mathcal{O}_{\mathbb{F}_e}(1, b_e - 2 - e) + (2b_e - 3e - 1)F = \xi\varphi^*\mathcal{O}_{\mathbb{F}_e}(4, b_e - 2 - e) + (2b_e - 3e - k_e - 1)F$. Moreover we have that

$$\dim(\mathcal{M}) = h^1(\mathcal{V} \otimes \mathcal{V}^\vee) = 6b_e - 9e - 3.$$

Since Ulrichness, slope-stability, simplicity are open conditions as well as Chern classes are constant for vector bundles varying in $\mathcal{M}$, it follows that all the properties satisfied by $\mathcal{V}$ hold true for the general point $[\mathcal{U}] \in \mathcal{M}$, in particular $\mathcal{M}$ is also generically smooth.

Note further that $K_{X_e} + 4\xi = -2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(-2, -2-e) + \varphi^*\mathcal{O}_{\mathbb{F}_e}(3, b_e) + 4\xi = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, b_e - e - 2) = c_1(\mathcal{V})$, thus $\mathcal{U}$ is a special Ulrich bundle. Finally, the slope of $\mathcal{U}$ with respect to ξ is

$$\mu(\mathcal{U}) = \frac{c_1(\mathcal{U}) \cdot \xi^2}{2} = \frac{(2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(1, b_e - e - 2)) \cdot \xi^2}{2} = 8b_e - k_e - 12e - 3.$$

□

Remark 3.3. In [18, Theorems 5.8, 5.9] it was shown the existence of stable rank two Ulrich vector bundle w.r.t. ξ on X_0 and X_1 of low degree. In [21, Corollary 5.17] it was shown the existence of rank two Ulrich vector bundle on $\mathbb{P}(\mathcal{E}) \to \mathbb{F}_e$ w.r.t. a different very ample polarization $D = \pi^*(A) + \xi$ with A such that $rk(\mathcal{E})A + c_1(\mathcal{E})$ is also very ample, $rk(\mathcal{E}) \geq 2$ and $\pi : \mathbb{P}(\mathcal{E}) \to \mathbb{F}_e$ the natural projection. But nothing was said about their moduli space.

4. Higher rank Ulrich vector bundles on 3-fold scrolls over $\mathbb{F}_e$

In this section we will construct higher rank slope-stable Ulrich vector bundles on X_e, where $e \geqslant 0$. We will moreover compute the dimensions of the moduli spaces of the constructed bundles, completely proving the **Main Theorem** and the **Main Corollary**, stated in the Introduction.

To do so, we will use Theorems 2.1, 3.1, 3.2, as well as inductive procedures and deformation arguments.

4.1. Higher rank Ulrich vector bundles on 3-fold scrolls over $\mathbb{F}_0$.

We will first concentrate on the case $e = 0$. From Theorem 2.1 we know that, under Assumptions 1.7, the case $e = 0$ is the only case where Ulrich line bundles on (X_0, ξ) actually exist. We will focus on the line bundles

$$L_1 = \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(2, -1) \text{ and its Ulrich dual } L_2 = \xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(-1, b_0 - 1), \tag{4.1}$$

as in Theorem 2.1-(i), which are Ulrich w.r.t. ξ.

Recalling computations in § 3.1-**Case L** and the fact that $b_0 > 1$ by Assumptions 1.7, we have that:

$$\begin{aligned} \dim(\mathrm{Ext}^1(L_2, L_1)) &= h^1(L_1 - L_2) = 4(b_0 - 1) \geqslant 4, \text{ whereas} \\ \dim(\mathrm{Ext}^1(L_1, L_2)) &= h^1(L_2 - L_1) = 2(b_0 + 1) \geqslant 6. \end{aligned} \tag{4.2}$$

In Theorem 3.1 we used such extensions to construct rank-2 Ulrich vector bundles. To construct higher rank Ulrich bundles on X_0 we proceed with an iterative strategy as follows.

Set $\mathcal{G}_1 := L_1$; from (4.2) the general $[\mathcal{G}_2] \in \mathrm{Ext}^1(L_2, \mathcal{G}_1) = \mathrm{Ext}^1(L_2, L_1)$ is associated to a non-splitting extension

$$0 \to \mathcal{G}_1 = L_1 \to \mathcal{G}_2 \to L_2 \to 0, \tag{4.3}$$

where $\mathcal{G}_2$ is a rank-2 Ulrich and simple vector bundle on X_0 with

$$c_1(\mathcal{G}_2) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1, b_0 - 2)$$

(cf. (3.1), where $\mathcal{G}_2 := \mathcal{F}_1$ therein, and the proof of Theorem 3.1). If, in the next step, we considered further extensions $\mathrm{Ext}^1(L_2, \mathcal{G}_2)$, it is easy to see that the dimension of such an extension space drops by one with respect to that of $\mathrm{Ext}^1(L_2, \mathcal{G}_1)$. Therefore, proceeding in this way, after finitely many steps we would have only splitting bundles in $\mathrm{Ext}^1(L_2, \mathcal{G}_k)$ for any $k \geqslant k_0$, for some positive integer k_0.

To avoid this, similarly as in [10, § 4], we proceed by taking extensions

$$0 \to \mathcal{G}_2 \to \mathcal{G}_3 \to L_1 \to 0, \;\; 0 \to \mathcal{G}_3 \to \mathcal{G}_4 \to L_2 \to 0, \; \ldots,$$

and so on, that is, defining

$$\epsilon_r := \begin{cases} 1, & \text{if } r \text{ is odd,} \\ 2, & \text{if } r \text{ is even,} \end{cases} \tag{4.4}$$

we take successive extensions $[\mathcal{G}_r] \in \mathrm{Ext}^1(L_{\epsilon_r}, \mathcal{G}_{r-1})$ for all $r \geqslant 2$:

$$0 \to \mathcal{G}_{r-1} \to \mathcal{G}_r \to L_{\epsilon_r} \to 0. \tag{4.5}$$

The fact that we can always take *non–trivial* such extensions will be proved in a moment in Corollary 4.2 below. In any case, all vector bundles $\mathcal{G}_r$ recursively defined as in (4.5) are of rank r and Ulrich w.r.t. ξ, since extensions of Ulrich bundles are again Ulrich. The first Chern class is given by

$$c_1(\mathcal{G}_r) := \begin{cases} r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(\frac{r+3}{2}, \frac{(r-1)}{2}b_0 - r), & \text{if } r \text{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(\frac{r}{2}, \frac{r}{2}b_0 - r), & \text{if } r \text{ is even.} \end{cases} \tag{4.6}$$

Thus, for any $r \geqslant 1$, the slope is

$$\mu(\mathcal{G}_r) = 8b_0 - k_0 - 3, \tag{4.7}$$

as in § 3.1-**Case L** and in (3.12). Moreover, from Theorem 1.4-(a), any such $\mathcal{G}_r$ is strictly semistable and slope-semistable, being extensions of Ulrich bundles of the same slope $\mu(\mathcal{G}_{r-1}) = \mu(L_{\epsilon_r}) = 8b_0 - k_0 - 3$.

Lemma 4.1. *Let L denote any of the two line bundles L_1 and L_2 as in* (4.1)*. Then, for all integers $r \geqslant 1$, we have*

(i) $h^2(\mathcal{G}_r \otimes L^\vee) = h^3(\mathcal{G}_r \otimes L^\vee) = 0$,
(ii) $h^2(\mathcal{G}_r^\vee \otimes L) = h^3(\mathcal{G}_r^\vee \otimes L) = 0$,
(iii) $h^1(\mathcal{G}_r \otimes L^\vee_{\epsilon_{r+1}}) \geqslant \min\{4b_0 - 4,\ 2b_0 + 2\} \geqslant 4$.

Proof. For $r = 1$ we have $\mathcal{G}_1 = L_1$; therefore $\mathcal{G}_1 \otimes L^\vee$ and $\mathcal{G}_1^\vee \otimes L$ are either equal to $\mathcal{O}_{X_0}$, if $L = L_1$, or equal to $L_1 - L_2$ and $L_2 - L_1$, respectively, if $L = L_2$. Therefore (i) and (ii) hold true by computations as in § 3.1-**Case L**. As for (iii), by (4.4) we have that $L_{\epsilon_2} = L_2$ thus $h^1(\mathcal{G}_1 \otimes L_2^\vee) = h^1(L_1 - L_2) = 4b_0 - 4$, as is § 3.1-**Case L**, the latter being always greater than or equal to $\min\{4b_0 - 4,\ 2b_0 + 2\} \geqslant 4$ as $b_0 \geqslant 2$.

Therefore, we will assume $r \geqslant 2$ and proceed by induction. Regarding (i), since it holds for $r = 1$, assuming it holds for $r - 1$ then by tensoring (4.5) with $L^\vee$ we get that

$$h^j(\mathcal{G}_r \otimes L^\vee) = 0, \;\; j = 2, 3,$$

because $h^j(\mathcal{G}_{r-1} \otimes L^\vee) = 0$, for $j = 2, 3$, by induction hypothesis whereas $h^j(L_{\epsilon_r} \otimes L^\vee) = 0$, for $j = 2, 3$, since $L_{\epsilon_r} \otimes L^\vee$ is either $\mathcal{O}_{X_0}$, or $L_2 - L_1$, or $L_1 - L_2$.

A similar reasoning, tensoring the dual of (4.5) by L, proves (ii).

To prove (iii), tensor (4.5) by $L^\vee_{\epsilon_{r+1}}$ and use that $h^2(\mathcal{G}_{r-1} \otimes L^\vee_{\epsilon_{r+1}}) = 0$ by (i). Thus we have the surjection

$$H^1(\mathcal{G}_r \otimes L^\vee_{\epsilon_{r+1}}) \twoheadrightarrow H^1(L_{\epsilon_r} \otimes L^\vee_{\epsilon_{r+1}}),$$

which implies that $h^1(\mathcal{G}_r \otimes L^\vee_{\epsilon_{r+1}}) \geqslant h^1(L_{\epsilon_r} \otimes L^\vee_{\epsilon_{r+1}})$. According to the parity of r, we have that $L_{\epsilon_r} \otimes L^\vee_{\epsilon_{r+1}}$ equals either $L_1 - L_2$ or $L_2 - L_1$. From computations as in § 3.1-**Case L**, $h^1(L_1 - L_2) = 4b_0 - 4$ whereas $h^1(L_2 - L_1) = 2b_0 + 2$. Notice that

$$\min\{4b_0 - 4,\ 2b_0 + 2\} := \begin{cases} 4b_0 - 4 = 4, & \text{if } b_0 = 2,\\ 4b_0 - 4 = 2b_0 + 2 = 8, & \text{if } b_0 = 3,\\ 2b_0 + 2 \geqslant 10, & \text{if } b_0 \geqslant 4.\end{cases}$$

Therefore one concludes. □

Corollary 4.2. *For any integers $r \geqslant 1$ there exist on X_0 rank-r vector bundles $\mathcal{G}_r$, which are Ulrich w.r.t. ξ, with first Chern class $c_1(\mathcal{G}_r)$ as in (4.6), which are morover simple, i.e. $h^0(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = 1$, indecomposable and of slope $\mu(\mathcal{G}_r) = 8b_0 - k_0 - 3$.*

Proof. For $r = 1$, we have $\mathcal{G}_1 = L_1$ and the statement holds true from Theorem 2.1 and computations in § 3.1-**Case L**.

For any $r \geqslant 2$, notice that

$$\mathrm{Ext}^1(L_{\epsilon_r}, \mathcal{G}_{r-1}) \cong H^1(\mathcal{G}_{r-1} \otimes L^\vee_{\epsilon_r}).$$

Therefore, from Lemma 4.1-(iii) there exist non–splitting extensions as in (4.5), which are therefore Ulrich with respect to ξ and whose Chern class $c_1(\mathcal{G}_r)$ is exactly as in (4.6).

By induction $\mu(\mathcal{G}_{r-1}) = \mu(L_{\epsilon_r}) = 8b_0 - k_0 - 3$; then $\mathcal{G}_r$ has the same slope w.r.t. ξ. Moreover $\mathcal{G}_{r-1}$ and L_{ϵ_r} are not isomorphic, in fact if $r > 2$ they have different ranks, while if $r = 2$ then $L_{\epsilon_2} = L_2$ is not isomorphic to $\mathcal{G}_1 = L_1$ as it follows from their expressions in Theorem 2.1. Hence by [9, Lemma 4.2] we get that $\mathcal{G}_r$ is a simple bundle. In particular, it must be indecomposable. □

From Corollary 4.2, at any step we can always pick *non–splitting* extensions of the form (4.5). We will henceforth do so.

Lemma 4.3. *Let $r \geqslant 1$ be an integer. Then we have*

(i) $h^1(\mathcal{G}_{r+1} \otimes L^\vee_{\epsilon_{r+1}}) = h^1(\mathcal{G}_r \otimes L^\vee_{\epsilon_{r+1}}) - 1$,

(ii) $h^1(\mathcal{G}_r \otimes L^\vee_{\epsilon_{r+1}}) = \begin{cases} \frac{(r+1)}{2} h^1(L_1 - L_2) - \frac{(r-1)}{2} = 2(r+1)(b_0 - 1) - \frac{(r-1)}{2}, & \textit{if } r \textit{ is odd},\\ \frac{r}{2} h^1(L_2 - L_1) - \frac{(r-2)}{2} = r(b_0 + 1) - \frac{(r-2)}{2}, & \textit{if } r \textit{ is even}.\end{cases}$

(iii) $h^2(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = h^3(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = 0$,

(iv) $\chi(\mathcal{G}_r \otimes L^\vee_{\epsilon_{r+1}}) = \begin{cases} \frac{(r+1)}{2}(1 - h^1(L_1 - L_2)) - 1 = \frac{(r+1)}{2}(5 - 4b_0) - 1, & \textit{if } r \textit{ is odd},\\ \frac{r}{2}(1 - h^1(L_2 - L_1)) = \frac{r}{2}(-1 - 2b_0), & \textit{if } r \textit{ is even}.\end{cases}$

(v) $\chi(L_{\epsilon_r} \otimes \mathcal{G}_r^\vee) = \begin{cases} \frac{(r-1)}{2}(1 - h^1(L_1 - L_2)) + 1 = \frac{(r-1)}{2}(5 - 4b_0) + 1, & \textit{if } r \textit{ is odd},\\ \frac{r}{2}(1 - h^1(L_2 - L_1)) = \frac{r}{2}(-1 - 2b_0), & \textit{if } r \textit{ is even}.\end{cases}$

(vi) $\chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = \begin{cases} \frac{(r^2-1)}{4}(2 - h^1(L_1 - L_2) - h^1(L_2 - L_1)) + 1 = \frac{(r^2-1)}{4}(4 - 6b_0) + 1, & \textit{if } r \textit{ is odd},\\ \frac{r^2}{4}(2 - h^1(L_1 - L_2) - h^1(L_2 - L_1)) = \frac{r^2}{4}(4 - 6b_0), & \textit{if } r \textit{ is even}.\end{cases}$

Proof. (i) Consider the exact sequence (4.5), where r is replaced by $r + 1$. From $\mathrm{Ext}^1(L_{\epsilon_{r+1}}, \mathcal{G}_r) \cong H^1(\mathcal{G}_r \otimes L^\vee_{\epsilon_{r+1}})$ and the fact that the exact sequence defining $\mathcal{G}_{r+1}$ is constructed by taking a non–zero vector $[\mathcal{G}_{r+1}]$ in $\mathrm{Ext}^1(L_{\epsilon_{r+1}}, \mathcal{G}_r)$, it follows that the coboundary map

$$H^0(\mathcal{O}_{X_0}) \xrightarrow{\partial} H^1(\mathcal{G}_r \otimes L^\vee_{\epsilon_{r+1}})$$

of the exact sequence

$$0 \to \mathcal{G}_r \otimes L^\vee_{\epsilon_{r+1}} \to \mathcal{G}_{r+1} \otimes L^\vee_{\epsilon_{r+1}} \to \mathcal{O}_{X_0} \to 0, \tag{4.8}$$

is non–zero so it is injective. Thus, (i) follows from the cohomology of (4.8).

(ii) We use induction on r. For $r=1$, the right hand side of the formula yields $4(b_0-1)$ which is exactly $h^1(\mathcal{G}_1\otimes L_2^\vee)=h^1(L_1-L_2)$ as in § 3.1-**Case L**.

When $r=2$, the right hand side of the formula is $2(b_0+1)$ which is $h^1(\mathcal{G}_2\otimes L_1^\vee)=h^1(L_2-L_1)=2b_0+2$, as it follows from computations in § 3.1-**Case L**, from the exact sequence

$$0\to \mathcal{O}_{X_0}\to \mathcal{G}_2\otimes L_1^\vee \to L_2-L_1\to 0,$$

obtained by (4.5) with $r=2$ and tensored with $L_1^\vee$, and the fact that $h^j(\mathcal{O}_{X_0})=0$, for $j=1,2$.

Assume now that the formula holds true up to some integer $r\geqslant 2$; we have to show that it holds also for $r+1$. Consider the exact sequence (4.5), with r replaced by $r+1$, and tensor it by $L_{\epsilon_{r+2}}^\vee$. We thus obtain

$$0\to \mathcal{G}_r\otimes L_{\epsilon_{r+2}}^\vee\to \mathcal{G}_{r+1}\otimes L_{\epsilon_{r+2}}^\vee\to L_{\epsilon_{r+1}}\otimes L_{\epsilon_{r+2}}^\vee\to 0 \tag{4.9}$$

If r is even, then $L_{\epsilon_{r+2}}=L_2$ whereas $L_{\epsilon_{r+1}}=L_1$. Thus $h^0(L_{\epsilon_{r+1}}\otimes L_{\epsilon_{r+2}}^\vee)=h^0(L_1-L_2)=0$ and $h^1(L_{\epsilon_{r+1}}\otimes L_{\epsilon_{r+2}}^\vee)=h^1(L_1-L_2)=4b_0-4$. On the other hand, by Lemma 4.1-(i), $h^2(\mathcal{G}_r\otimes L_{\epsilon_{r+2}}^\vee)=0$. Thus, from (4.9), we get:

$$h^1(\mathcal{G}_{r+1}\otimes L_{\epsilon_{r+2}}^\vee)=(4b_0-4)+h^1(\mathcal{G}_r\otimes L_{\epsilon_{r+2}}^\vee)=(4b_0-4)+h^1(\mathcal{G}_r\otimes L_{\epsilon_r}^\vee),$$

as r and $r+2$ have the same parity. Using (i), we have $h^1(\mathcal{G}_r\otimes L_{\epsilon_r}^\vee)=h^1(\mathcal{G}_{r-1}\otimes L_{\epsilon_r}^\vee)-1$ therefore, by inductive hypothesis with $r-1$ odd, we have $h^1(\mathcal{G}_{r-1}\otimes L_{\epsilon_r}^\vee)=\frac{r}{2}(4b_0-4)-\frac{(r-2)}{2}$. Summing up, we have

$$h^1(\mathcal{G}_{r+1}\otimes L_{\epsilon_{r+2}}^\vee)=(4b_0-4)+\frac{r}{2}(4b_0-4)-\frac{(r-2)}{2}-1,$$

which is easily seen to be equal to the right hand side expression in (ii), when r is replaced by $r+1$.

If r is odd, the same holds for $r+2$ whereas $r+1$ is even. In this case $L_{\epsilon_{r+2}}=L_1$, $L_{\epsilon_{r+1}}=L_2$ so $h^1(L_{\epsilon_{r+1}}\otimes L_{\epsilon_{r+2}}^\vee)=h^1(L_2-L_1)=2b_0+2$ and one applies the same procedure as in the previous case.

(iii) We again use induction on r. For $r=1$, formula (iii) states that $h^j(L_1-L_1)=h^j(\mathcal{O}_{X_0})=0$, for $j=2,3$, which is certainly true.

Assume now that (iii) holds up to some integer $r\geqslant 1$; we have to prove that it holds also for $r+1$. Consider the exact sequence (4.5), where r is replaced by $r+1$, and tensor it by $\mathcal{G}_{r+1}^\vee$. From this we get that, for $j=2,3$,

$$h^j(\mathcal{G}_{r+1}\otimes \mathcal{G}_{r+1}^\vee)\leqslant h^j(\mathcal{G}_r\otimes \mathcal{G}_{r+1}^\vee)+h^j(L_{\epsilon_{r+1}}\otimes \mathcal{G}_{r+1}^\vee)=h^j(\mathcal{G}_r\otimes \mathcal{G}_{r+1}^\vee), \tag{4.10}$$

the latter equality follows from $h^j(L_{\epsilon_{r+1}}\otimes \mathcal{G}_{r+1}^\vee)=0$, $j=2,3$, as in Lemma 4.1-(ii).

Consider the dual exact sequence of (4.5), where r is replaced by $r+1$, and tensor it by $\mathcal{G}_r$. Thus, Lemma 4.1-(i) yields that, for $j=2,3$, one has

$$h^j(\mathcal{G}_r\otimes \mathcal{G}_{r+1}^\vee)\leqslant h^j(\mathcal{G}_r\otimes L_{\epsilon_{r+1}}^\vee)+h^j(\mathcal{G}_r\otimes \mathcal{G}_r^\vee)=h^j(\mathcal{G}_r\otimes \mathcal{G}_r^\vee). \tag{4.11}$$

Now (4.10)–(4.11) and the inductive hypothesis yield $h^j(\mathcal{G}_{r+1}\otimes \mathcal{G}_{r+1}^\vee)=0$, for $j=2,3$, as desired.

(iv) For $r=1$, (iv) reads $\chi(L_1-L_2)=-h^1(L_1-L_2)=4-4b_0$, which is true since $h^j(L_1-L_2)=0$ for $j=0,2,3$.

For $r=2$, (iv) reads $\chi(\mathcal{G}_2\otimes L_1^\vee)=1-h^1(L_2-L_1)=-1-2b_0$ and this holds true because if we take the exact sequence (4.5), with $r=2$, tensored by $L_1^\vee$ then

$$\chi(\mathcal{G}_2\otimes L_1^\vee)=\chi(\mathcal{O}_{X_0})+\chi(L_2-L_1)=1-h^1(L_2-L_1)=1-(2b_0+2),$$

as $h^j(L_2-L_1)=0$ for $j=0,2,3$.

Assume now that the formula holds up to a certain integer $r \geq 2$, we have to prove that it also holds for $r+1$. From (4.9) we get

$$\chi(\mathcal{G}_{r+1} \otimes L^{\vee}_{\epsilon_{r+2}}) = \chi(\mathcal{G}_r \otimes L^{\vee}_{\epsilon_{r+2}}) + \chi(L_{\epsilon_{r+1}} \otimes L^{\vee}_{\epsilon_{r+2}}).$$

If r is even, the same is true for $r+2$ whereas $r+1$ is odd. Therefore,

$$\chi(\mathcal{G}_{r+1} \otimes L^{\vee}_{\epsilon_{r+2}}) = \chi(\mathcal{G}_r \otimes L^{\vee}_2) + \chi(L_1 - L_2) = \chi(\mathcal{G}_r \otimes L^{\vee}_2) - h^1(L_1 - L_2). \tag{4.12}$$

Then (4.8), with r replaced by $r-1$, yields

$$\chi(\mathcal{G}_r \otimes L^{\vee}_2) = \chi(\mathcal{G}_{r-1} \otimes L^{\vee}_2) + \chi(\mathcal{O}_{X_0}) = \chi(\mathcal{G}_{r-1} \otimes L^{\vee}_2) + 1. \tag{4.13}$$

Substituting (4.13) into (4.12) and using the inductive hypothesis with $r-1$ odd, we get

$$\begin{aligned}\chi(\mathcal{G}_{r+1} \otimes L^{\vee}_2) &= \chi(\mathcal{G}_{r-1} \otimes L^{\vee}_2) + 1 - h^1(L_1 - L_2)\\ &= \frac{(r)}{2}(1 - h^1(L_1 - L_2)) - h^1(L_1 - L_2)\\ &= \frac{(r+2)}{2}(1 - h^1(L_2 - L_1)) - 1,\end{aligned}$$

proving that the formula holds also for $r+1$ odd.

Similar procedure can be used to treat the case when r is odd. In this case, $L_{\epsilon_{r+1}} = L_2$ whereas $L_{\epsilon_{r+2}} = L_1$. Thus, from the above computations,

$$\chi(\mathcal{G}_{r+1} \otimes L^{\vee}_1) = \chi(\mathcal{G}_r \otimes L^{\vee}_1) + \chi(L_2 - L_1) = \chi(\mathcal{G}_r \otimes L^{\vee}_1) - h^1(L_2 - L_1).$$

As in the previous case, $\chi(\mathcal{G}_r \otimes L^{\vee}_1) = 1 + \chi(\mathcal{G}_{r-1} \otimes L^{\vee}_1)$ so, applying inductive hypothesis with $r-1$ even, we get $\chi(\mathcal{G}_r \otimes L^{\vee}_1) = 1 + \frac{(r-1)}{2}(1 - h^1(L_2 - L_1))$. Adding up all these quantities, we get

$$\chi(\mathcal{G}_{r+1} \otimes L^{\vee}_{\epsilon_{r+2}}) = \chi(\mathcal{G}_{r+1} \otimes L^{\vee}_1) = \frac{r+1}{2}(1 - h^1(L_2 - L_1)),$$

so formula (iv) holds true also for $r+1$ even.

(v) For $r = 1$, (v) reads $\chi(L_1 - L_1) = \chi(\mathcal{O}_{X_0}) = 1$, which is correct. For $r = 2$, (v) reads $\chi(L_2 \otimes \mathcal{G}^{\vee}_2) = 1 - h^1(L_2 - L_1)$, which is once again correct as it follows from the dual of sequence (4.5) tensored by L_2.

Assume now that the formula holds up to a certain integer $r \geqslant 2$ and we need to proving it for $r+1$. Dualizing (4.5), replacing r by $r+1$ and tensoring it by $L_{\epsilon_{r+1}}$ we find that

$$\begin{aligned}\chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_{r+1}) &= \chi(L_{\epsilon_{r+1}} \otimes L^{\vee}_{\epsilon_{r+1}}) + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_r)\\ &= \chi(\mathcal{O}_{X_n}) + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_r) = 1 + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_r).\end{aligned} \tag{4.14}$$

The dual of sequence (4.5), with r replaced by $r-1$, tensored by $L_{\epsilon_{r+1}}$ yields

$$\chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_r) = \chi(L_{\epsilon_{r+1}} \otimes L^{\vee}_{\epsilon_r}) + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_{r-1}). \tag{4.15}$$

Substituting (4.15) into (4.14) and using the fact that $r+1$ and $r-1$ have the same parity, we get

$$\chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_{r+1}) = 1 + \chi(L_{\epsilon_{r+1}} \otimes L^{\vee}_{\epsilon_r}) + \chi(L_{\epsilon_{r-1}} \otimes \mathcal{G}^{\vee}_{r-1}).$$

If r is even, then $\chi(L_{\epsilon_{r+1}} \otimes L^{\vee}_{\epsilon_r}) = \chi(L_1 - L_2) = -h^1(L_1 - L_2)$ whereas, from the inductive hypothesis with $r-1$ odd, $\chi(L_{\epsilon_{r-1}} \otimes \mathcal{G}^{\vee}_{r-1}) = 1 + \frac{(r-2)}{2}(1 - h^1(L_1 - L_2))$. Thus

$$\chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}^{\vee}_{r+1}) = 1 - h^1(L_1 - L_2) + 1 + \frac{(r-2)}{2}(1 - h^1(L_1 - L_2)),$$

the latter equals $1 + \frac{r}{2}(1 - h^1(L_1 - L_2))$, proving that the formula holds also for $r+1$ odd.

If r is odd, the strategy is similar; in this case one has $\chi(L_{\epsilon_{r+1}} \otimes L^{\vee}_{\epsilon_r}) = \chi(L_2 - L_1) = -h^1(L_2 - L_1)$ and, by the inductive hypothesis with $r-1$ even, $\chi(L_{\epsilon_{r-1}} \otimes \mathcal{G}^{\vee}_{r-1}) = \frac{(r-1)}{2}(1 - h^1(L_2 - L_1))$ so one can conclude.

(vi) We first check the given formula for $r = 1, 2$. We have $\chi(\mathcal{G}_1 \otimes \mathcal{G}_1^\vee) = \chi(L_1 - L_1) = \chi(\mathcal{O}_{X_0}) = 1$, which fits with the given formula for $r = 1$. From (4.5), with $r = 2$, tensored by $\mathcal{G}_2^\vee$ we get

$$\chi(\mathcal{G}_2 \otimes \mathcal{G}_2^\vee) = \chi(L_1 \otimes \mathcal{G}_2^\vee) + \chi(L_2 \otimes \mathcal{G}_2^\vee) \overset{(v)}{=} \chi(L_1 \otimes \mathcal{G}_2^\vee) + 1 - h^1(L_2 - L_1). \tag{4.16}$$

From the dual of (4.5), with $r = 2$, tensored by L_1 we get

$$\chi(L_1 \otimes \mathcal{G}_2^\vee) = \chi(L_1 - L_1) + \chi(L_1 - L_2) = \chi(\mathcal{O}_{X_0}) - h^1(L_1 - L_2) = 1 - h^1(L_1 - L_2). \tag{4.17}$$

Combining (4.16) and (4.17), we get

$$\chi(\mathcal{G}_2 \otimes \mathcal{G}_2^\vee) = 2 - h^1(L_1 - L_2) - h^1(L_2 - L_1),$$

which again fits with the given formula for $r = 2$.

Assume now that the given formula is valid up to a certain integer $r \geqslant 2$; we need to prove it holds for $r+1$. From (4.5), in which r is replaced by $r+1$, tensored by $\mathcal{G}_{r+1}^\vee$ and successively the dual of (4.5), with r replaced by $r+1$, tensored by $\mathcal{G}_r$ we get

$$\chi(\mathcal{G}_{r+1} \otimes \mathcal{G}_{r+1}^\vee) \;\; = \;\; \chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) + \chi(\mathcal{G}_r \otimes L_{\epsilon_{r+1}}^\vee) + \chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}_{r+1}^\vee).$$

If r is even, then $r+1$ is odd and $L_{\epsilon_{r+1}} = L_1$. From (v) with $(r+1)$ odd, we get $\chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}_{r+1}^\vee) = 1 + \frac{r}{2}(1 - h^1(L_1 - L_2))$, whereas from (iv) with r even $\chi(\mathcal{G}_r \otimes L_{\epsilon_{r+1}}^\vee) = \frac{r}{2}(1 - h^1(L_2 - L_1))$. Finally, by the inductive hypothesis with r even, $\chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = \frac{r^2}{4}(2 - h^1(L_1 - L_2) - h^1(L_2 - L_1))$. Summing–up the three quantities, one gets

$$\chi(\mathcal{G}_{r+1} \otimes \mathcal{G}_{r+1}^\vee) = 1 + \frac{(r+1)^2 - 1}{4}(2 - h^1(L_1 - L_2) - h^1(L_2 - L_1)),$$

proving that the formula holds for $r+1$ odd.

If r is odd, then $\chi(L_{\epsilon_{r+1}} \otimes \mathcal{G}_{r+1}^\vee) = \frac{r+1}{2}(1 - h^1(L_2 - L_1))$, as it follows from (v) with $(r+1)$ even, whereas $\chi(\mathcal{G}_r \otimes L_{\epsilon_{r+1}}^\vee) = \frac{(r+1)}{2}(1 - h^1(L_1 - L_2)) - 1$, as predicted by (iv) with r odd. Finally, form the inductive hypothesis with r odd, we have $\chi(\mathcal{G}_r \otimes \mathcal{G}_r^\vee) = 1 + \frac{(r^2-1)}{4}(2 - h^1(L_1 - L_2) - h^1(L_2 - L_1))$. If we add up the three quantities, we get

$$\chi(\mathcal{G}_{r+1} \otimes \mathcal{G}_{r+1}^\vee) = \frac{(r+1)^2}{4}(2 - h^1(L_1 - L_2) - h^1(L_2 - L_1)),$$

finishing the proof. □

We now define, for any integer $r \geqslant 1$, the irreducible scheme $\mathcal{M}(r)$ to be the modular family of vector bundles $\mathcal{G}_r$ recursively defined by (4.5). For $r \geqslant 2$ the scheme $\mathcal{M}(r)$ contains a subscheme, denoted by $\mathcal{M}(r)^{\text{ext}}$, which parametrizes bundles $\mathcal{F}_r$ that are non–trivial extensions of the form

$$0 \to \mathcal{U}_{r-1} \to \mathcal{F}_r \to L_{\epsilon_r} \to 0, \tag{4.18}$$

with $[\mathcal{U}_{r-1}] \in \mathcal{M}(r-1)$.

Lemma 4.4. *Let $r \geqslant 2$ be an integer and let $[\mathcal{U}_r] \in \mathcal{M}(r)$ be a general point. Then $\mathcal{U}_r$ is a vector bundle of rank r, which is Ulrich with respect to ξ with slope w.r.t. ξ given by $\mu := 8b_0 - k_0 - 3$. Moreover,*

(i) $\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = \begin{cases} \frac{(r^2-1)}{4}(2-h^1(L_1-L_2)-h^1(L_2-L_1))+1=\frac{(r^2-1)}{4}(4-6b_0)+1, & \textit{if } r \textit{ is odd,} \\ \frac{r^2}{4}(2-h^1(L_1-L_2)-h^1(L_2-L_1))=\frac{r^2}{4}(4-6b_0), & \textit{if } r \textit{ is even.} \end{cases}$

(ii) $h^j(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$, *for $j = 2, 3$.*

Proof. Ulrichness is an open property in irreducible families, as well as the rank and the slope are constant, since c_1 is. Thus the general member $[\mathcal{U}_r] \in \mathcal{M}(r)$ is Ulrich of rank r and slope μ as each $\mathcal{G}_r$ defined by (4.5) (cf. Corollary 4.2).

Property (ii) follows by specializing $\mathcal{F}_r$ to a vector bundle $\mathcal{G}_r$ constructed above, and using semicontinuity and Lemma 4.3-(iii) and (ii), respectively. Property (i) follows by Lemma 4.3-(vi), since the given χ depends only on the Chern classes of the two factors and on X_0, which are constant in the irreducible family $\mathcal{M}(r)$. $\square$

We want to prove that the general member of $\mathcal{M}(r)$ is also slope–stable. To this aim we will first need the following auxiliary results.

Lemma 4.5. *Let $r \geqslant 2$ be an integer and assume that $[\mathcal{F}_r] \in \mathcal{M}(r)^{\rm ext}$ sits in a non–splitting sequence like* (4.18) *with $[\mathcal{U}_{r-1}] \in \mathcal{M}(r-1)$ being slope–stable w.r.t. ξ. Then*

(i) *$\mathcal{U}_r$ is simple;*
(ii) *if $\mathcal{D}$ is a destabilizing subsheaf of $\mathcal{U}_r$, then $\mathcal{D}^\vee \cong \mathcal{U}_{r-1}^\vee$ and $(\mathcal{F}_r/\mathcal{D})^\vee \cong L_{\epsilon_r}^\vee$; if furthermore $\mathcal{F}_r/\mathcal{D}$ is torsion–free, then $\mathcal{D} \cong \mathcal{F}_{r-1}$ and $\mathcal{F}_r/\mathcal{D} \cong L_{\epsilon_r}$.*

Proof. The reasoning is similar to [10, Lemma 4.5], we will describe it for the reader's convenience. We first prove (ii); assume that $\mathcal{D}$ is a destabilizing subsheaf of $\mathcal{U}_r$, that is $0 < \mathrm{rk}(\mathcal{D}) < \mathrm{rk}(\mathcal{U}_r) = r$ and $\mu(\mathcal{D}) \geqslant \mu = \mu(\mathcal{U}_r)$. Define the sheaves

$$\mathcal{Q} := \mathrm{Im}\{\mathcal{D} \subset \mathcal{U}_r \to L_{\epsilon_r}\} \ \text{ and } \ \mathcal{K} := \mathrm{Ker}\{\mathcal{D} \to \mathcal{Q}\}$$

so that (4.18) may be put into the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathcal{K} & \to & \mathcal{D} & \to & \mathcal{Q} & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathcal{U}_{r-1} & \to & \mathcal{U}_r & \to & L_{\epsilon_r} & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 \to & & \mathcal{K}' & \to & \mathcal{U}_r/\mathcal{D} & \to & \mathcal{Q}' & \to 0 & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

defining the sheaves $\mathcal{K}'$ and $\mathcal{Q}'$. We have $\mathrm{rk}(\mathcal{Q}) \leqslant 1$.

Assume that $\mathrm{rk}(\mathcal{Q}) = 0$. Then $\mathcal{Q} = 0$, whence $\mathcal{K} \cong \mathcal{D}$ and $\mathcal{Q}' \cong L_{\epsilon_r}$. Since $\mu(\mathcal{K}) = \mu(\mathcal{D}) \geqslant \mu = \mu(\mathcal{U}_{r-1})$ and $\mathcal{U}_{r-1}$ is slope–stable, we must have $\mathrm{rk}(\mathcal{K}) = \mathrm{rk}(\mathcal{U}_{r-1}) = r-1$. It follows that $\mathrm{rk}(\mathcal{K}') = 0$. As

$$c_1(\mathcal{K}) = c_1(\mathcal{U}_{r-1}) - c_1(\mathcal{K}') = c_1(\mathcal{U}_{r-1}) - D',$$

where D' is an effective divisor supported on the codimension one locus of the support of $\mathcal{K}'$, we have

$$\mu \leqslant \mu(\mathcal{K}) = \frac{(c_1(\mathcal{U}_{r-1}) - D') \cdot \xi^2}{r-1} = \frac{c_1(\mathcal{U}_{r-1}) \cdot \xi^2}{r-1} - \frac{D' \cdot \xi^2}{r-1} = \mu - \frac{D' \cdot \xi^2}{r-1}.$$

Hence $D' = 0$, which means that $\mathcal{K}'$ is supported in codimension at least two. Thus, the sheaves $\mathfrak{ext}^i(\mathcal{K}', \mathcal{O}_{X_0})$ are zero, for $i \leqslant 1$, and it follows that

$$\mathcal{D}^\vee \cong \mathcal{K}^\vee \cong \mathcal{U}_{r-1}^\vee \ \text{ and } \ (\mathcal{U}_r/\mathcal{D})^\vee \cong \mathcal{Q}'^\vee \cong L_{\epsilon_r}^\vee,$$

as desired. If furthermore $\mathcal{U}_r/\mathcal{D}$ is torsion–free, then we must have $\mathcal{K}' = 0$, whence $\mathcal{D} \cong \mathcal{U}_{r-1}$ and $\mathcal{U}_r/\mathcal{D} \cong L_{\epsilon_r}$.

Next we prove that $\mathrm{rk}(\mathcal{Q}) = 1$ cannot happen. Indeed, if $\mathrm{rk}(\mathcal{Q}) = 1$, then $\mathrm{rk}(\mathcal{K}) = \mathrm{rk}(\mathcal{D}) - 1 \leqslant r-2 < r-1 = \mathrm{rk}(\mathcal{U}_{r-1})$ and $\mathrm{rk}(\mathcal{Q}') = 0$; in particular $\mathcal{Q}'$ is a torsion sheaf. Since

$$c_1(\mathcal{K}) = c_1(\mathcal{D}) - c_1(\mathcal{Q}) = c_1(\mathcal{D}) - c_1(L_{\epsilon_r}) + c_1(\mathcal{Q}') = c_1(\mathcal{D}) - c_1(L_{\epsilon_r}) + D,$$

where D is an effective divisor supported on the codimension-one locus of the support of $\mathcal{Q}'$, we have

$$\begin{aligned} \mu(\mathcal{K}) &= \frac{\Big(c_1(\mathcal{D}) - c_1(L_{\epsilon_r}) + D\Big) \cdot \xi^2}{\mathrm{rk}(\mathcal{K})} \geqslant \frac{\Big(c_1(\mathcal{D}) - c_1(L_{\epsilon_r})\Big) \cdot \xi^2}{\mathrm{rk}(\mathcal{K})} \\ &= \frac{\mu(\mathcal{D})\mathrm{rk}(\mathcal{D}) - c_1(L_{\epsilon_r}) \cdot \xi^2}{\mathrm{rk}(\mathcal{K})} = \frac{\mu(\mathcal{D})\mathrm{rk}(\mathcal{D}) - \mu}{\mathrm{rk}(\mathcal{D}) - 1} \geq \frac{\mu\mathrm{rk}(\mathcal{D}) - \mu}{\mathrm{rk}(\mathcal{D}) - 1} = \mu \end{aligned}$$

This contradicts the slope–stability of $\mathcal{U}_{r-1}$.

To prove (i), assume that $\mathcal{U}_r$ is not simple, that is, it admits a non-trivial endomorphism. This implies there exists a non-zero endomorphism $\varphi : \mathcal{U}_r \to \mathcal{U}_r$ dropping rank everywhere; indeed, take any endomorphism α that is not a constant times the identity, pick an eigenvalue λ of $\alpha(x)$, for some $x \in X_0$, and set $\varphi := \alpha - \lambda \mathrm{Id}$; then $\det(\varphi) \in H^0(\det(\mathcal{U}_r^\vee) \otimes \det(\mathcal{U}_r)) = H^0(\mathcal{O}_{X_0}) \cong \mathbb{C}$ vanishes at the point x, whence it is identically zero. Both $\mathrm{Ker}(\varphi)$ and $\mathrm{Im}(\varphi)$, being subsheaves of $\mathcal{U}_r$, are torsion–free, and one easily checks that at least one of them is destabilizing. By part (ii), it follows that either $\mathrm{Ker}(\varphi) \cong \mathcal{U}_{r-1}$ or $\mathrm{Im}(\varphi)^\vee \cong \mathcal{U}_{r-1}^\vee$. In the first case, φ factors through L_{ϵ_r}, whence the map $\mathcal{U}_r \to L_{\epsilon_r}$ in (4.18) splits, a contradiction. In the second case, the natural injection $\mathrm{Im}(\varphi) \subset \mathrm{Im}(\varphi)^{\vee\vee} \cong \mathcal{U}_{r-1}^{\vee\vee} \cong \mathcal{U}_{r-1}$ shows that φ factors through $\mathcal{U}_{r-1}$, whence the map $\mathcal{U}_{r-1} \to \mathcal{U}_r$ in (4.18)splits, again a contradiction. □

Lemma 4.6. *Let $r \geqslant 2$ be an integer. Assume that the general member of $\mathcal{M}(r-1)$ is slope–stable. Then $\mathcal{M}(r)$ is generically smooth of dimension*

$$\dim(\mathcal{M}(r)) = \begin{cases} \frac{(r^2-1)}{4}(6b_0 - 4), & \text{if } r \text{ is odd}, \\ \frac{r^2}{4}(6b_0 - 4) + 1, & \text{if } r \text{ is even}. \end{cases}$$

Furthermore $\mathcal{M}(r)$ properly contains the locally closed subscheme $\mathcal{M}(r)^{\mathrm{ext}}$, namely $\dim(\mathcal{M}(r)^{\mathrm{ext}}) < \dim(\mathcal{M}(r))$.

Proof. Consider the general member $[\mathcal{U}_r] \in \mathcal{M}(r)$. It satisfies $h^0(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 1$, by Lemma 4.5-(i), and $h^j(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$ for $j = 2, 3$, by Lemma 4.4-(ii).

From the fact that $h^2(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$, it follows that $\mathcal{M}(r)$ is generically smooth of dimension $\dim(\mathcal{M}(r)) = h^1(\mathcal{U}_r \otimes \mathcal{U}_r^\vee)$ (cf. e.g. [9, Prop. 2.10]). On the other hand, since $h^3(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 0$ and $h^0(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = 1$, we have $h^1(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) = -\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) + 1$. Therefore, the formula concerning $\dim(\mathcal{M}(r))$ directly follows from Lemma 4.4-(i).

Similarly, being slope-stable by assumptions, also the general member $\mathcal{U}_{r-1}$ of $\mathcal{M}(r-1)$ satisfies $h^0(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) = 1$. Thus, using Lemma 4.4-(ii), the same reasoning as above shows that

$$\dim(\mathcal{M}(r-1)) = h^1(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) = -\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) + 1, \tag{4.19}$$

where $\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee)$ as in Lemma 4.4-(i) (with r replaced by $r-1$). Morover, by specialization of $\mathcal{U}_{r-1}$ to $\mathcal{G}_{r-1}$ and semi-continuity, we have

$$\dim(\mathrm{Ext}^1(L_{\epsilon_r}, \mathcal{U}_{r-1})) = h^1(\mathcal{U}_{r-1} \otimes L_{\epsilon_r}^\vee) \leqslant h^1(\mathcal{G}_{r-1} \otimes L_{\epsilon_r}^\vee), \tag{4.20}$$

where the latter is as in Lemma 4.3-(ii) (with r replaced by $r-1$). Therefore, by the very definition of $\mathcal{M}(r)^{\mathrm{ext}}$ and by (4.19)-(4.20), we have

$$\begin{aligned} \dim(\mathcal{M}(r)^{\mathrm{ext}}) &\leqslant \dim(\mathcal{M}(r-1)) + \dim(\mathbb{P}(\mathrm{Ext}^1(L_{\epsilon_r}, \mathcal{U}_{r-1})) \\ &= -\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) + 1 + h^1(\mathcal{U}_{r-1} \otimes L_{\epsilon_r}^\vee) - 1 \\ &\leqslant -\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) + h^1(\mathcal{G}_{r-1} \otimes L_{\epsilon_r}^\vee). \end{aligned}$$

On the other hand, from the above discussion,

$$\dim(\mathcal{M}(r)) = -\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) + 1.$$

Therefore to prove that $\dim(\mathcal{M}(r)^{\rm ext}) < \dim(\mathcal{M}(r))$ it is enough to show that for any integer $r \geqslant 2$ the following inequality

$$-\chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) + h^1(\mathcal{G}_{r-1} \otimes L_{\epsilon_r}^\vee) < -\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) + 1$$

holds true. Notice that the previous inequality reads also

$$-\chi(\mathcal{U}_r \otimes \mathcal{U}_r^\vee) + 1 + \chi(\mathcal{U}_{r-1} \otimes \mathcal{U}_{r-1}^\vee) - h^1(\mathcal{G}_{r-1} \otimes L_{\epsilon_r}^\vee) > 0, \tag{4.21}$$

which is satisfied for any $r \geqslant 2$, as we can easily see.

Indeed use Lemmas 4.4-(i) and 4.3-(ii): if r is even, the left hand side of (4.21) reads $rb_0 + 2 + \frac{(r-2)}{2}$ which obviously is positive since $r, b_0 \geqslant 2$; if r is odd, then $r \geqslant 3$ and the left hand side of (4.21) reads $(r-1)(2b_0 - 3) + \frac{(r-3)}{2}$ which obviously is positive under the assumptions $r \geqslant 3$, $b_0 \geqslant 2$. □

We can now prove slope–stability of the general member of $\mathcal{M}(r)$.

Proposition 4.7. *Let $r \geqslant 1$ be an integer. The general member of $\mathcal{M}(r)$ is slope–stable.*

Proof. We use induction on r, the result being obviously true for $r = 1$.

Assume therefore $r \geqslant 2$ and that the general member of $\mathcal{M}(r)$ is not slope–stable, whereas the general member of $\mathcal{M}(r-1)$ is. Then, similarly as in [10, Prop. 4.7], we may find a one-parameter family of bundles $\{\mathcal{U}_r^{(t)}\}$ over the unit disc Δ such that $\mathcal{U}_r^{(t)}$ is a general member of $\mathcal{M}(r)$ for $t \neq 0$ and $\mathcal{U}_r^{(0)}$ lies in $\mathcal{M}(r)^{\rm ext}$, and such that we have a destabilizing sequence

$$0 \to \mathcal{D}^{(t)} \to \mathcal{U}_r^{(t)} \to \mathcal{Q}^{(t)} \to 0 \tag{4.22}$$

for $t \neq 0$, which we can take to be saturated, that is, such that $\mathcal{Q}^{(t)}$ is torsion free, whence so that $\mathcal{D}^{(t)}$ and $\mathcal{Q}^{(t)}$ are (Ulrich) vector bundles (see [9, Thm. 2.9] or [5, (3.2)]).

The limit of $\mathbb{P}(\mathcal{Q}^{(t)}) \subset \mathbb{P}(\mathcal{U}_r^{(t)})$ defines a subvariety of $\mathbb{P}(\mathcal{U}_r^{(0)})$ of the same dimension as $\mathbb{P}(\mathcal{Q}^{(t)})$, whence a coherent sheaf $\mathcal{Q}^{(0)}$ of rank ${\rm rk}(\mathcal{Q}^{(t)})$ with a surjection $\mathcal{U}_r^{(0)} \to \mathcal{Q}^{(0)}$. Denoting by $\mathcal{D}^{(0)}$ its kernel, we have ${\rm rk}(\mathcal{D}^{(0)}) = {\rm rk}(\mathcal{D}^{(t)})$ and $c_1(\mathcal{D}^{(0)}) = c_1(\mathcal{D}^{(t)})$. Hence, (4.29) specializes to a destabilizing sequence for $t = 0$. Lemma 4.5 yields that $\mathcal{D}^{(0)^\vee}$ (respectively, $\mathcal{Q}^{(0)^\vee}$) is the dual of a member of $\mathfrak{U}(r-1)$ (resp., the dual of L_{ϵ_r}). It follows that $\mathcal{D}^{(t)^\vee}$ (resp., $\mathcal{Q}^{(t)^\vee}$) is a deformation of the dual of a member of $\mathfrak{U}(r-1)$ (resp., a deformation of $L_{\epsilon_r}^\vee$), whence that $\mathcal{D}^{(t)}$ is a deformation of a member of $\mathfrak{U}(r-1)$, as both are locally free, and $\mathcal{Q}^{(t)} \cong L_{\epsilon_r}$, for the same reason.

In other words, the general member of $\mathcal{M}(r)$ is an extension of L_{ϵ_r} by a member of $\mathcal{M}(r-1)$. Hence $\mathcal{M}(r) = \mathcal{M}(r)^{\rm ext}$, contradicting Lemma 4.6. □

The collection of the previous results gives the following

Theorem 4.8. *Let $(X_0, \xi) \cong (\mathbb{P}(\mathcal{E}_0), \mathcal{O}_{\mathbb{P}(\mathcal{E}_0)}(1))$ be a 3-fold scroll over $\mathbb{F}_0$, with $\mathcal{E}_0$ as in Assumptions 1.7. Let $\varphi : X_0 \to \mathbb{F}_0$ be the scroll map and F be the φ-fibre. Let $r \geqslant 2$ be any integer. Then the moduli space of rank-r vector bundles $\mathcal{U}_r$ on X_0 which are Ulrich w.r.t. ξ and with first Chern class*

$$c_1(\mathcal{U}_r) = \begin{cases} r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(\frac{r+3}{2}, \frac{(r-1)}{2}b_0 - r), & \text{if } r \text{ is odd,} \\ r\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(\frac{r}{2}, \frac{r}{2}b_0 - r), & \text{if } r \text{ is even.} \end{cases}$$

is not empty and it contains a generically smooth component $\mathcal{M}(r)$ of dimension

$$\dim(\mathcal{M}(r)) = \begin{cases} \frac{(r^2-1)}{4}(6b_0 - 4), & \text{if } r \text{ is odd,} \\ \frac{r^2}{4}(6b_0 - 4) + 1, & \text{if } r \text{ is even.} \end{cases}$$

The general point $[\mathcal{U}_r] \in \mathcal{M}(r)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 8b_0 - k_0 - 3$.

Proof. It directly follows from Theorem 3.1, (4.6), (4.7) and from Lemmas 4.4, 4.6 and Proposition 4.7. □

If in particular we set $b_0 = 3$ and $r = 2$, then from Theorem 4.8 one gets

$$c_1(\mathcal{U}_2) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_0}(1,1),$$

$$\dim(\mathcal{M}(2)) = \frac{7}{2}(4) + 1 = 15$$

and

$$\mu(\mathcal{U}_2) = 21 - k_0,$$

which is what was obtained in [18, Proposition 5.7] with $k_0 = 7, 8, 9, 10$.

4.2. **Higher rank Ulrich vector bundles on 3-fold scrolls over $\mathbb{F}_e$, $e > 0$.** Here we focus on the case $e > 0$. The strategy we will use is similar to that in § 4.1, thus we will be brief. We will inductively define irreducible families of vector bundles on (X_e, ξ) whose general members will be slope–stable Ulrich bundles, obtained by induction as deformations of extensions of lower ranks Ulrich bundles.

The main difference with respect to the case $e = 0$ is that there are no Ulrich line bundles w.r.t. ξ on (X_e, ξ) when $e > 0$, as it follows from Theorem 2.1. Therefore our starting point for the inductive process will be rank-2 Ulrich vector bundles as in Theorem 3.2 which, using extensions, recursive procedures, deformations and moduli theory, will allow us to construct slope-stable Ulrich vector bundles on (X_e, ξ) of even ranks $2h$, for any $h \geqslant 1$, and to study their moduli spaces.

We start by defining the irreducible scheme $\mathcal{M}(2)$ to be the component $\mathcal{M}$ as in Theorem 3.2. Recall that $\mathcal{M}(2) = \mathcal{M}$ is generically smooth, of dimension $\dim(\mathcal{M}(2)) = 6b_e - 9e - 3$ and that the general member $[\mathcal{U}_2] \in \mathcal{M}(2)$ is a rank-2 Ulrich vector on X_e, which is slope-stable of slope $\mu(\mathcal{U}_2) = 8b_e - k_e - 12e - 3$ w.r.t. ξ, and whose first Chern class is $c_1(\mathcal{U}_2) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(1, b_e - e - 2)$.

Assume by induction, to have defined an irreducible scheme $\mathcal{M}(2h-2)$, for some $h \geqslant 2$, similarly as in [12] we define $\mathcal{M}(2h)$ to be the (possibly empty a priori) component of the moduli space of Ulrich bundles on (X_e, ξ) containing bundles $\mathcal{F}_{2h}$ that are *non-splitting* extensions of the form

$$0 \to \mathcal{U}'_2 \to \mathcal{F}_{2h} \to \mathcal{U}_{2h-2} \to 0, \tag{4.23}$$

with $[\mathcal{U}'_2] \in \mathcal{M}(2)$ and $[\mathcal{U}_{2h-2}] \in \mathcal{M}(2h-2)$, and such that $\mathcal{U}'_2 \not\cong \mathcal{U}_{2h-2}$ when $h = 2$. As in § 4.1, we let $\mathcal{M}(2h)^{\text{ext}}$ denote the locus in $\mathcal{M}(2h)$ of bundles that are non-splitting extensions of the form (4.23).

In the next results we will prove that non-trivial extensions as in (4.23) always exist and that $\mathcal{M}(2h)^{\text{ext}} \neq \emptyset$, so in particular $\mathcal{M}(2h) \neq \emptyset$, for any $h \geqslant 2$. In statements and proofs below we will use the following notation: $\mathcal{U}'_2$ will correspond to a general member of $\mathcal{M}(2)$ and $\mathcal{U}_{2h-2}$ to a general member of $\mathcal{M}(2h-2)$, with $\mathcal{U}'_2 \not\cong \mathcal{U}_{2h-2}$ when $h = 2$. We will denote by $\mathcal{F}_{2h}$ a general member of $\mathcal{M}(2h)^{\text{ext}}$ and, in bounding cohomologies, we will use the fact that $\mathcal{U}_{2h}$ *specializes* to $\mathcal{F}_{2h}$ in an irreducible flat family.

All vector bundles $\mathcal{F}_{2h}$, $h \geqslant 2$, recursively defined as in (4.23) are of rank $2h$ and Ulrich w.r.t. ξ, since extensions of Ulrich bundles are again Ulrich. Their first Chern class is given by

$$c_1(\mathcal{F}_{2h}) := 2h\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(h, h(b_e - e - 2)) = h\, c_1(\mathcal{U}_2) \tag{4.24}$$

and its slope w.r.t. ξ is

$$\mu(\mathcal{F}_{2h}) = 8b_e - k_e - 12e - 3. \tag{4.25}$$

Moreover, from Theorem 1.4-(a), any such $\mathcal{F}_{2h}$ is strictly semistable and slope-semistable, being extensions of Ulrich bundles of the same slope.

Lemma 4.9. *Let $h \geqslant 1$ be an integer and assume $\mathcal{M}(2k) \neq \emptyset$ for all $1 \leqslant k \leqslant h$. Then*

(i) $h^j(\mathcal{U}_{2h} \otimes \mathcal{U}'^{\vee}_2) = h^j(\mathcal{U}'_2 \otimes \mathcal{U}^{\vee}_{2h}) = 0$ *for* $j = 2, 3$,

(ii) $\chi(\mathcal{U}_{2h} \otimes \mathcal{U}'^{\vee}_2) = \chi(\mathcal{U}'_2 \otimes \mathcal{U}^{\vee}_{2h}) = h(4 + 9e - 6b_e)$,

(iii) $h^j(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) = 0$ *for* $j = 2, 3$,
(iv) $\chi(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) = h^2(4 + 9e - 6b_e)$.

Proof. For $h = 1$, (iii) and (iv) follow from the proof of Theorem 3.2. As for (i), the vanishings hold when $\mathcal{U}_2' = \mathcal{U}_2$ once again by the proof of Theorem 3.2, and thus, by semicontinuity, they also hold for a general pair $([\mathcal{U}_2'], [\mathcal{U}_2]) \in \mathcal{M}(2) \times \mathcal{M}(2)$. Similarly, (ii) follows from the proof of Theorem 3.2, since the given χ is constant as $\mathcal{U}_2$ and $\mathcal{U}_2'$ vary in $\mathcal{M}(2)$.

We now prove the statements for any integer $h \geqslant 2$ by induction. Assume therefore that they are satisfied for all positive integers k less than h.

(i) Let $j \in \{2, 3\}$. By specialization and (4.23) we have

$$h^j(\mathcal{U}_{2h} \otimes \mathcal{U}_2'^\vee) \leqslant h^j(\mathcal{F}_{2h} \otimes \mathcal{U}_2'^\vee) \leqslant h^j(\mathcal{U}_2' \otimes \mathcal{U}_2'^\vee) + h^j(\mathcal{U}_{2h-2} \otimes \mathcal{U}_2'^\vee),$$

and the latter are 0 by induction. Similarly, by specialization and the dual of (4.23) we have

$$h^j(\mathcal{U}_2' \otimes \mathcal{U}_{2h}^\vee) \leqslant h^j(\mathcal{U}_2' \otimes \mathcal{F}_{2h}^\vee) \leqslant h^j(\mathcal{U}_2' \otimes \mathcal{U}_2'^\vee) + h^j(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee),$$

which are again 0 by induction.

(ii) By specialization, (4.23) and induction we have

$$\begin{aligned} \chi(\mathcal{U}_{2h} \otimes \mathcal{U}_2'^\vee) &= \chi(\mathcal{F}_{2h} \otimes \mathcal{U}_2'^\vee) = \chi(\mathcal{U}_2' \otimes \mathcal{U}_2'^\vee) + \chi(\mathcal{U}_{2h-2} \otimes \mathcal{U}_2'^\vee) \\ &= (4 + 9e - 6b_e) + (h-1)(4 + 9e - 6b_e) = h(4 + 9e - 6b_e). \end{aligned}$$

Likewise, by specialization, the dual of (4.23) and induction, the same holds for $\chi(\mathcal{U}_2' \otimes \mathcal{U}_{2h}^\vee)$.

(iii) Let $j = 2, 3$; by specialization, (4.23) and its dual we have

$$\begin{aligned} h^j(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) &\leqslant h^j(\mathcal{F}_{2h} \otimes \mathcal{F}_{2h}^\vee) \leq h^j(\mathcal{U}_2' \otimes \mathcal{F}_{2h}^\vee) + h^j(\mathcal{U}_{2h-2} \otimes \mathcal{F}_{2h}^\vee) \\ &\leqslant h^j(\mathcal{U}_2' \otimes \mathcal{U}_2'^\vee) + h^j(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) + h^j(\mathcal{U}_{2h-2} \otimes \mathcal{U}_2'^\vee) + h^j(\mathcal{U}_{2h-2} \otimes \mathcal{U}_{2h-2}^\vee), \end{aligned}$$

which are all 0 by induction.

(iv) By specialization, (4.23) and its dual we have

$$\begin{aligned} \chi(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) &= \chi(\mathcal{F}_{2h} \otimes \mathcal{F}_{2h}^\vee) = \chi(\mathcal{U}_2' \otimes \mathcal{F}_{2h}^\vee) + \chi(\mathcal{U}_{2h-2} \otimes \mathcal{F}_{2h}^\vee) \\ &= \chi(\mathcal{U}_2' \otimes \mathcal{U}_2'^\vee) + \chi(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) + \chi(\mathcal{U}_{2h-2} \otimes \mathcal{U}_2'^\vee) + \chi(\mathcal{U}_{2h-2} \otimes \mathcal{U}_{2h-2}^\vee). \end{aligned}$$

By induction, this equals $(4 + 9e - 6b_e) + 2(h-1)(4 + 9e - 6b_e) + (h-1)^2(4 + 9e - 6b_e) = h^2(4 + 9e - 6b_e)$. □

Proposition 4.10. *For all integers* $h \geqslant 1$ *the scheme* $\mathcal{M}(2h)$ *is nonempty and its general member* $\mathcal{U}_{2h}$ *is an Ulrich vector bundle w.r.t.* ξ *which satisfies* $\mathrm{rk}(\mathcal{U}_{2h}) = 2h$, $c_1(\mathcal{U}_{2h}) = 2h\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(h, h(b_e - e - 2))$ *and* $h^j(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) = 0$ *for* $j = 2, 3$.

Proof. We prove this by induction on h, the case $h = 1$ being satisfied by the choice of $\mathcal{M}(2)$. Therefore, let $h \geqslant 2$; for general $[\mathcal{U}_{2h-2}] \in \mathcal{M}(2h-2)$ and $[\mathcal{U}_2'] \in \mathcal{M}(2)$, one has

$$\dim(\mathrm{Ext}^1(\mathcal{U}_{2h-2}, \mathcal{U}_2')) = h^1(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee).$$

By Lemma 4.9(i) we have that $h^j(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) = 0$, for $j = 2, 3$. Therefore

$$\chi(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) = h^0(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) - h^1(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee)$$

so, by specialization and invariance of χ in irreducible families, we have

$$\begin{aligned} \dim(\mathrm{Ext}^1(\mathcal{U}_{2h-2}, \mathcal{U}_2')) &= h^1(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) && (4.26) \\ &= -\chi(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) + h^0(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) \\ &\geqslant -\chi(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) \\ &= -\chi(\mathcal{U}_2' \otimes \mathcal{F}_{2h-2}^\vee) && (4.27) \\ &= (h-1)(6b_e - 9e - 4) > 0 \end{aligned}$$

the latter equality following from Lemma 4.9-(ii) (with h replaced by $h-1$) whereas the last strict inequality following from $h \geqslant 2$. Hence, by its very definition, one has that $\mathcal{M}(2h)^{\mathrm{ext}}$, and so also $\mathcal{M}(2h)$, is not empty.

The members of $\mathcal{M}(2h)$ have rank $2h$ and first Chern class $2h\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(h, h(b_e - e - 2))$ as in (4.24), since $c_1(\mathcal{U}_{2h}) = c_1(\mathcal{F}_{2h})$ being constant in $\mathcal{M}(2h)$. It is immediate that extensions of Ulrich bundles are still Ulrich, so the general member $\mathcal{U}_{2h}$ of $\mathcal{M}(2h)$ is an Ulrich bundle. It also satisfies $h^j(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) = 0$ for $j = 2, 3$ by Lemma 4.9-(iii). □

We need to prove that the general member of $\mathcal{M}(2h)$ corresponds to a slope–stable vector bundle, that $\mathcal{M}(2h)$ is generically smooth and we need to compute the dimension at its general point $[\mathcal{U}_{2h}]$. We will again prove all these facts by induction on h.

Similarly as in the case $e = 0$, we need the following auxiliary result.

Lemma 4.11. *Let $\mathcal{F}_{2h}$ correspond to a general member of $\mathcal{M}(2h)^{\text{ext}}$, sitting in an extension like* (4.23). *Assume furthermore that $\mathcal{U}_2'$ and $\mathcal{U}_{2h-2}$ are slope–stable. Let $\mathcal{D}$ be a destabilizing subsheaf of $\mathcal{F}_{2h}$. Then $\mathcal{D}^* \cong {\mathcal{U}_2'}^\vee$ and $(\mathcal{F}_{2h}/\mathcal{D})^\vee \cong \mathcal{U}_{2h-2}^\vee$.*

Proof. The proof is almost identical to that of Lemma 4.5, so it is left to the reader. □

Proposition 4.12. *For all integers $h \geqslant 1$ the scheme $\mathcal{M}(2h)$ is not empty, generically smooth of dimension*

$$\dim(\mathcal{M}(2h)) = 1 + h^2(6b_e - 12e - 3).$$

Its general member corresponds to a slope-stable bundle $\mathcal{U}_{2h}$ whose slope w.r.t. ξ is $\mu(\mathcal{U}_{2h}) = 8b_e - k_e - 12e - 3$. Furthermore, $\mathcal{M}(2h)$ properly contains the locally closed subscheme $\mathcal{M}(2h)^{\text{ext}}$, namely $\dim(\mathcal{M}(2h)^{\text{ext}}) < \dim(\mathcal{M}(2h))$.

Proof. We prove this by induction on h, the case $h = 1$ being satisfied by $\mathcal{M}(2)$ as in Theorem 3.2.

Let therefore $h \geqslant 2$ and assume that we have proved the lemma for all positive integers $k \leqslant h - 1$; we will prove it for h.

The slope of the members of $\mathcal{M}(2)$ and $\mathcal{M}(2h-2)$ are both equal to $8b_e - k_e - 12e - 3$ as in (4.25). Thus, by [9, Lemma 4.2], the general member $[\mathcal{F}_{2h}] \in \mathcal{M}(2h)^{\text{ext}}$ corresponds to a simple bundle. Hence, by semi-continuity, also the general member $\mathcal{U}_{2h}$ of $\mathcal{M}(2h)$ is simple and it also satisfies $h^j(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) = 0$ for $j = 2, 3$ by Lemma 4.9-(iii).

Therefore $\mathcal{M}(2h)$ is smooth at $[\mathcal{U}_{2h}]$ (see, e.g., [9, Prop. 2.10]) with

$$\begin{aligned} \dim(\mathcal{M}(2h)) &= h^1(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) = -\chi(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) + h^0(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) \\ &= h^2(6b_e - 9e - 4) + 1, \end{aligned} \tag{4.28}$$

using the facts that $h^0(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) = 1$ as $\mathcal{U}_{2h}$ is simple, and that $\chi(\mathcal{U}_{2h} \otimes \mathcal{U}_{2h}^\vee) = h^2(4 + 9e - 6b_e)$ by Lemma 4.9-(iv). This proves that $\mathcal{M}(2h)$ is generically smooth of the stated dimension.

Finally, we prove that $\mathcal{U}_{2h}$ general is slope–stable and that $\dim(\mathcal{M}(2h)^{\text{ext}}) < \dim(\mathcal{M}(2h))$. If $\mathcal{U}_{2h}$ general is not slope-stable, then as in the proof of Proposition 4.7, we may find a one-parameter family of bundles $\{\mathcal{U}_{2h}^{(t)}\}$ over the disc Δ such that $\mathcal{U}_{2h}^{(t)}$ is a general member of $\mathcal{M}(2h)$ for $t \neq 0$ and $\mathcal{U}_{2h}^{(0)}$ lies in $\mathcal{M}(2h)^{\text{ext}}$, and such that we have a destabilizing sequence

$$0 \to \mathcal{D}^{(t)} \to \mathcal{U}_{2h}^{(t)} \to \mathcal{G}^{(t)} \to 0 \tag{4.29}$$

for $t \neq 0$, which we can take to be saturated, that is, such that $\mathcal{G}^{(t)}$ is torsion free, whence so that $\mathcal{D}^{(t)}$ and $\mathcal{G}^{(t)}$ are (Ulrich) vector bundles (see [9, Thm. 2.9] or [5, (3.2)]). The limit of $\mathbb{P}(\mathcal{G}^{(t)}) \subset \mathbb{P}(\mathcal{U}_{2h}^{(t)})$ defines a subvariety of $\mathbb{P}(\mathcal{U}_{2h}^{(0)})$ of the same dimension as $\mathbb{P}(\mathcal{G}^{(t)})$, whence a coherent sheaf $\mathcal{G}^{(0)}$ of rank $\text{rk}(\mathcal{G}^{(t)})$ with a surjection $\mathcal{U}_{2h}^{(0)} \twoheadrightarrow \mathcal{G}^{(0)}$. Denoting by $\mathcal{D}^{(0)}$ its kernel, we have $\text{rk}(\mathcal{D}^{(0)}) = \text{rk}(\mathcal{D}^{(t)})$ and $c_1(\mathcal{D}^{(0)}) = c_1(\mathcal{D}^{(t)})$. Hence, (4.29) specializes to a destabilizing sequence for $t = 0$.

Lemma 4.11 yields that ${\mathcal{D}^{(0)}}^\vee$ (resp., ${\mathcal{G}^{(0)}}^\vee$) is the dual of a member of $\mathcal{M}(2)$ (resp., of $\mathcal{M}(2h)$). It follows that ${\mathcal{D}^{(t)}}^\vee$ (resp., ${\mathcal{G}^{(t)}}^\vee$) is a deformation of the dual of a member of $\mathcal{M}(2)$)

(resp., of $\mathcal{M}(2h)$), whence that $\mathcal{D}^{(t)}$ (resp., $\mathcal{G}^{(t)}$) is a deformation of a member of $\mathcal{M}(2)$ (resp., $\mathcal{M}(2h)$), as both are locally free. It follows that $[\mathcal{U}_{2h}^{(t)}] \in \mathcal{M}(2h)^{\rm ext}$ for $t \neq 0$. Thus,

$$\mathcal{M}(2h)^{\rm ext} = \mathcal{M}(2h). \tag{4.30}$$

On the other hand we have

$$\dim(\mathcal{M}(2h)^{\rm ext}) \leqslant \dim(\mathbb{P}(\mathrm{Ext}^1(\mathcal{U}_{2h-2}, \mathcal{U}_2'))) + \dim(\mathcal{M}(2h-2)) + \dim(\mathcal{M}(2)), \tag{4.31}$$

for general $[\mathcal{U}_{2h-2}] \in \mathcal{M}(2h-2)$ and $[\mathcal{U}_2'] \in \mathcal{M}(2)$. As $\mathcal{U}_{2h-2}$ and $\mathcal{U}_2'$ are slope–stable by induction, of the same slope, we have $h^0(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) = 0$. Lemma 4.9-(i), (ii) and (iii) thus yield

$$h^1(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) = -\chi(\mathcal{U}_2' \otimes \mathcal{U}_{2h-2}^\vee) = (h-1)(6b_e - 9e - 4).$$

Hence, by (4.31) and (4.28) we have

$$\begin{aligned} \dim(\mathcal{M}(2h)^{\rm ext}) &\leqslant (h-1)(6b_e - 9e - 4) - 1 + \left[(h-1)^2(6b_e - 9e - 4) + 1\right] + (6b_e - 9e - 3) \\ &= (h^2 - h + 1)(6b_e - 9e - 4) + 1 < h^2(6b_e - 9e - 4) + 1 = \dim(\mathcal{M}(2h)), \end{aligned}$$

as it easily follows from the fact that $h \geqslant 2$. The previous inequality shows that $\dim(\mathcal{M}(2h)^{\rm ext}) < \dim(\mathcal{M}(2h))$, as stated; in particular (4.30) is a contradiction, which forces also $\mathcal{U}_{2h}$ general to be slope-stable. □

The collection of the previous results gives the following

Theorem 4.13. *Let $(X_e, \xi) \cong (\mathbb{P}(\mathcal{E}_e), \mathcal{O}_{\mathbb{P}(\mathcal{E}_e)}(1))$ be a 3-fold scroll over $\mathbb{F}_e$, with $e > 0$ and $\mathcal{E}_e$ as in Assumptions 1.7. Let $\varphi : X_e \to \mathbb{F}_e$ be the scroll map and F be the φ-fibre. Let $h \geqslant 1$ be any integer. Then the moduli space of rank-$2h$ vector bundles $\mathcal{U}_{2h}$ on X_e which are Ulrich w.r.t. ξ and with first Chern class*

$$c_1(\mathcal{U}_{2h}) = 2h\xi + \varphi^*\mathcal{O}_{\mathbb{F}_e}(h, h(b_e - e - 2))$$

is not empty and it contains a generically smooth component $\mathcal{M}(2h)$ of dimension

$$\dim(\mathcal{M}(2h)) = h^2(6b_e - 9e - 4) + 1.$$

The general point $[\mathcal{U}_{2h}] \in \mathcal{M}(2h)$ corresponds to a slope-stable vector bundle, of slope w.r.t. ξ given by $\mu(\mathcal{U}_r) = 8b_e - k_e - 12e - 3$.

Proof. It directly follows from Theorem 3.2, (4.24), (4.25) and from Propositions 4.10, 4.12. □

If in particular we set $e = 1$, $r = 2$, $b_e = 5$ and $k_1 = 10, 11$ then from Theorem 4.13 one gets

$$c_1(\mathcal{U}_2) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_1}(1, 5 - 1 - 2)) = 2\xi + \varphi^*\mathcal{O}_{\mathbb{F}_1}(1, 2))$$

$$\dim(\mathcal{M}(2)) = h^2(6b_e - 9e - 4) + 1 = 1^2(30 - 9 - 4) + 1 = 18$$

and

$$\mu(\mathcal{U}_2) = 40 - k_1 - 12 - 3 = 25 - k_1.$$

The 3-fold X_1 has degree, $\deg(X_1)$, either 11 or 10 because $\deg(X_1) = 21 - k_1$, and such threefolds have been considered in [18, Theorem 5.9] where it was only shown the existence of rank two Ulrich bundles on them, but nothing was said about their moduli space.

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