# Robustness in Queueing Systems

## WPMSIIP 2014

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Ghent University, SYSTeMS

12th September 2014

# Robustness

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What do we mean?

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Which performance measures?

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Why robustness? ⇒ Model uncertainty

Our Purpose? ⇒ Lower & Upper expected values of various performance measures

Which performance measures?

- Expected queue length
- Probability of a certain length
- Turning on the server (probability having 1 given 0)
- Averages

# Our Queueing System

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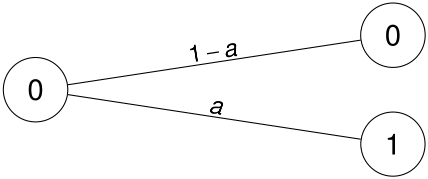
Discrete Time, Single-server (1) queue with finite capacity ($L$)

Other assumptions

- A departure occurs prior to an arrival
- Service obeys the *FCFS* principle
- Item stays till served!

# Our Queueing System



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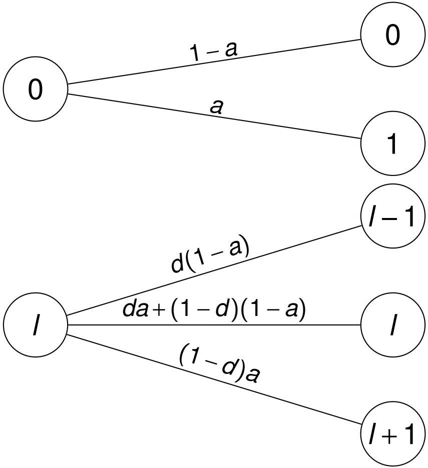

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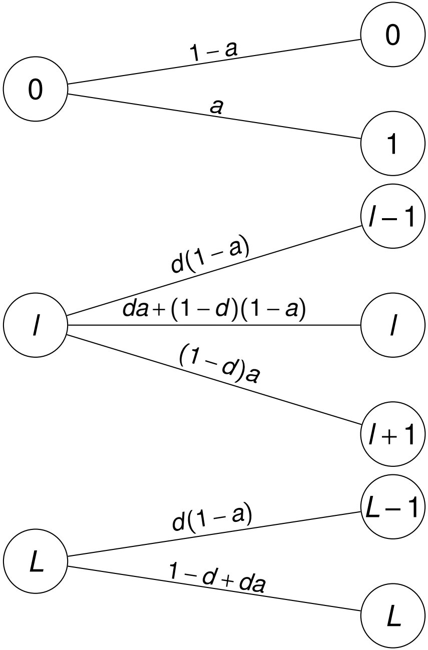

# Notation

State space ⇒ $\mathscr{X} = \{0, \ldots, L\}$

local/conditional probability ⇒ $p(\cdot|x_n, a, d)$ with $x_n \in \mathscr{X}$ at any time point $n$

probability mass functions

$$p(x_1) \prod_{i=1}^{n-1} p(x_{i+1}|x_{1:i}) = p(x_1) \prod_{i=1}^{n-1} p(x_{i+1}|x_i, a, d)$$

denoted by $p_{1:n,a,d}$

# Expectations

Let $f$ be a function on $\mathscr{X}^n := \underbrace{\mathscr{X} \times \cdots \times \mathscr{X}}_{n}$ then,

$$E(f) = \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_{1:n}) = \sum_{x_{1:n} \in \mathscr{X}^n} f(x_{1:n}) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_{1:i})$$

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$$= \sum_{x_1 \in \mathscr{X}} p(x_1) \sum_{x_2 \in \mathscr{X}} p(x_2 | x_1, a, d) \cdots \sum_{x_n \in \mathscr{X}} f(x_1, \ldots, x_n) p(x_n | x_{n-1}, a, d)$$

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\end{aligned}
$$

which is the Law of Iterated Expectation (LIE)

$$
E_{1:n}(f) = E(f) = E(E(\ldots E(f | X_{1:n-1}) \ldots | X_1) | \Box)
$$

with $\Box$ being the initial state

# Expectations

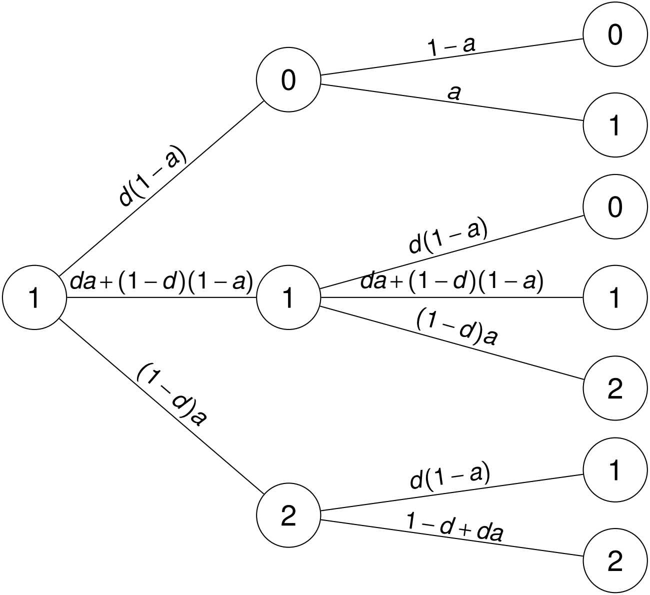
Functions on $\mathscr{X}$ will be denoted by $h$

$$E_n(h) = \sum_{x_n \in \mathscr{X}} h(x_n) p(x_1) \prod_{i=1}^{n-1} p(x_{i+1}|x_i) =$$

$$E(f) = E(E(\ldots E(f|X_{n-1})\ldots|X_1)|\square)$$

For probabilities we use *indicator functions*
i.e. $\mathbb{I}_A$ assigns 1 when $A$ happens else 0

# Calculating with Backwards Recursion



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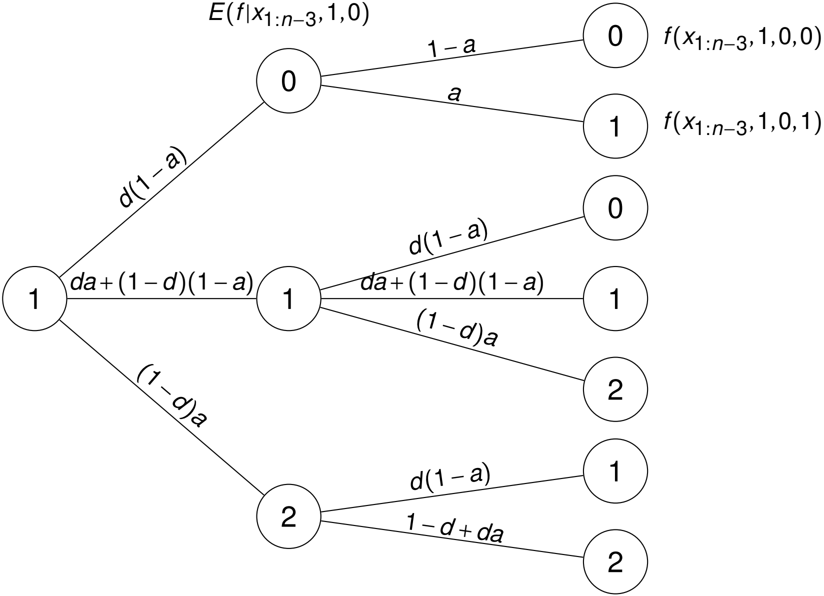

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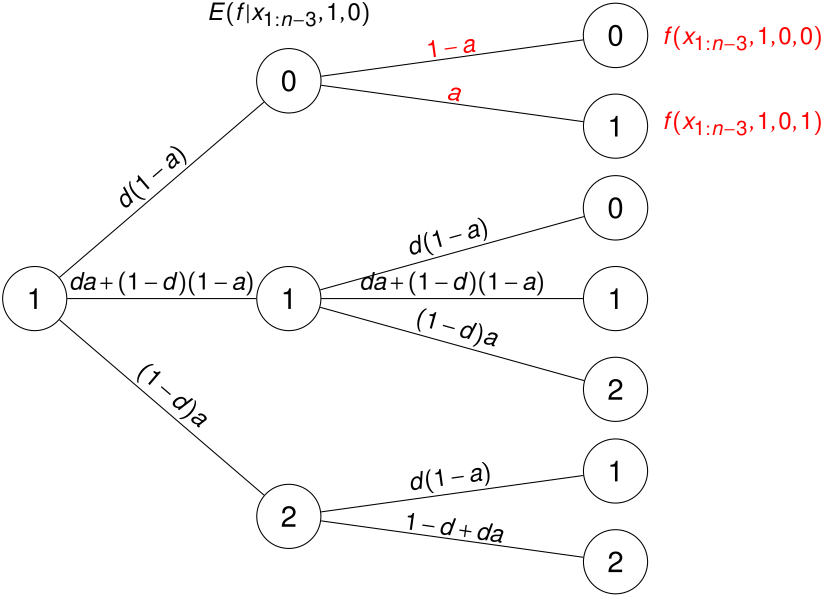

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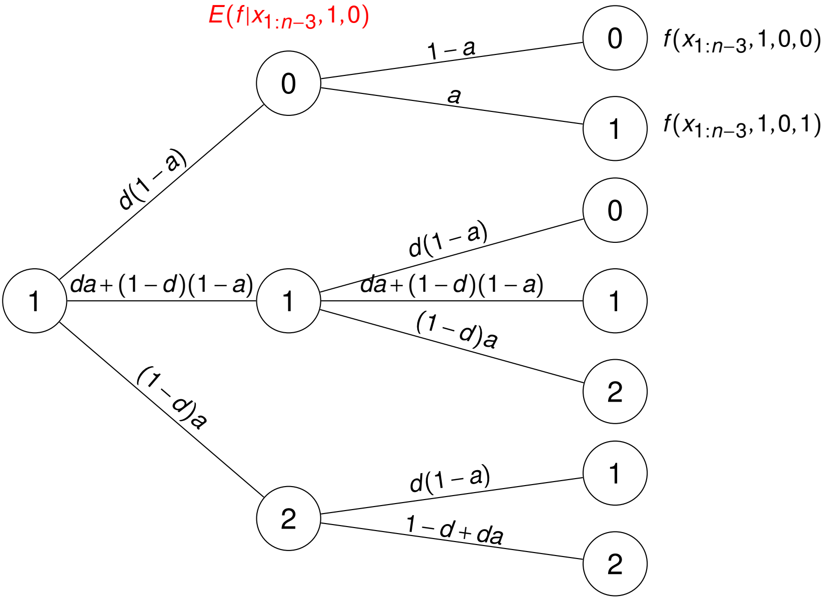

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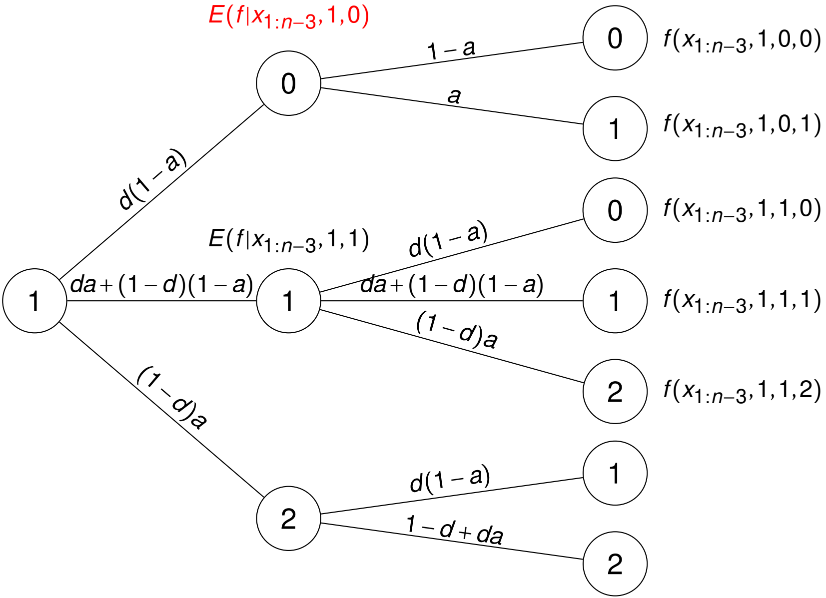

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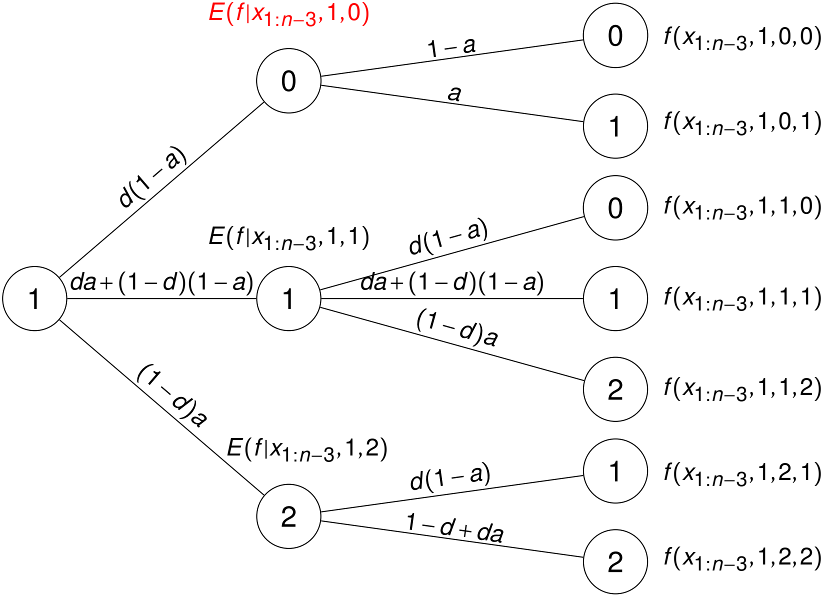

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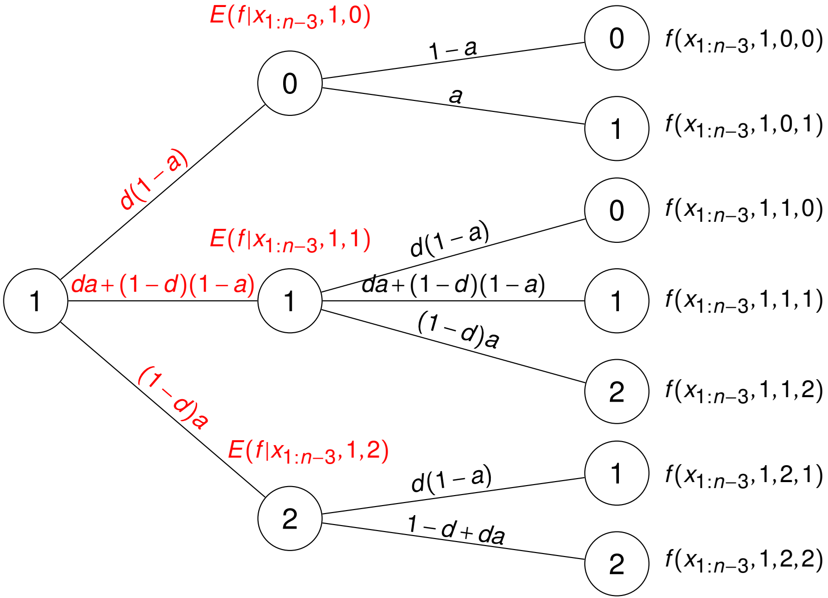

# Uncertainty

Uncertainty in the parameters of the model

Calculate bounds (Lower & Upper Expectations)

$$\underline{E}^{\mathscr{P}}(g) := \min\left\{E^{P}(g) : P \in \mathscr{P}\right\} \quad \text{and} \quad \overline{E}^{\mathscr{P}}(g) := \max\left\{E^{P}(g) : P \in \mathscr{P}\right\}$$

Combining with our notation ⇒ $\underline{E}_n$, $\overline{E}_n$ & $\underline{E}_{1:n}$, $\overline{E}_{1:n}$

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Combining with our notation ⇒ $\underline{E}_n$, $\overline{E}_n$ & $\underline{E}_{1:n}$, $\overline{E}_{1:n}$

*Geo/Geo/1/L* ⇒ *interval probabilities* $a \rightarrow [\underline{a}, \overline{a}]$ & $d \rightarrow [\underline{d}, \overline{d}]$

where each $P$ has form

$$p(x_1) \prod_{i=1}^{n-1} p(x_{i+1} | x_{1:i}, a_{x_{1:i}}, d_{x_{1:i}}) \text{ with } a_{x_{1:i}} \in [\underline{a}, \overline{a}],\ d_{x_{1:i}} \in [\underline{d}, \overline{d}]\ (p_{1:n,A,D})$$

Two approaches to deal with uncertainty

# 1st approach - Strong independence

Tree corresponding to lower (or upper) expectation consists of time-homogeneous/stationary probabilities of arrival and departure

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$$\underline{E}^s_{1:n}(f) = \min\left\{E^{p_{1:n,a,d}}(f) : a \in [\underline{a}, \overline{a}], d \in [\underline{d}, \overline{d}]\right\}$$

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We are mainly interested in $n \to \infty$

# Calculations under strong independence

Given a function $h$ on $\mathscr{X}$, w.r.t to lower expectation in the limit

$$\lim_{n\to\infty} \underline{E}_n^s(h) = \min\left\{\sum_{x\in\mathscr{X}} h(x)P[X_n = x] : a \in [\underline{a},\overline{a}], d \in [\underline{d},\overline{d}]\right\} \quad (1)$$

$$P[X=0] = \frac{d-a}{d - \frac{(1-d)^L a^{L+1}}{(d(1-a))^L}}. \quad (2)$$

$$P[X=l] = \frac{(1-d)^{l-1} a^l}{(d(1-a))^l} P[X=0] \quad (3)$$

We solve (1), where the parameters of (2) and (3) vary in $[\underline{a},\overline{a}]$ and $[\underline{d},\overline{d}]$

# Calculations under strong independence

For functions on $\mathscr{X}^n$ which represent averages of a function $h$ on $\mathscr{X}$ ⇒ the lower expectation in the limit approaches the value of (1)

For general $f$ on $\mathscr{X}^n$ it is difficult to formulate and solve a similar to (1) optimization problem

We approximate lower and upper expectations by

- choosing a number of probabilities from $[\underline{a}, \overline{a}]$ and $[\underline{d}, \overline{d}]$
- and calculating for all combinations using LIE in backwards recursion in combination with formulas (2) and (3)

# 2nd Approach - Epistemic Irrelevance

We drop stationarity

The tree corresponding to lower (or upper) expectation can have any probability of arrival and departure, from the respective sets, at any time point and given any sequence of queue lengths

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$$\underline{E}^{ei}_{1:n}(f) = \min\left\{E^{p_{1:n,A,D}}(f) : (\forall i \leq n)(\forall x_{1:i} \in \mathscr{X}^i) a_{x_{1:i}} \in [\underline{a}, \overline{a}], d_{x_{1:i}} \in [\underline{d}, \overline{d}]\right\}$$

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# Calculations under epistemic irrelevance

What is important...

- for any $n$ (approaching or not infinity)
- for any function (on $\mathscr{X}$ or $\mathscr{X}^n$)

we can always use LIE for calculating efficiently lower and upper expectations

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## Proposition

For any real-valued function $f$ on $\mathscr{X}^n$, with $n \in \mathbb{N}_0$

$$\underline{E}^{ei}_{1:n}(f) = \underline{E}_1(\underline{E}_2(\ldots\underline{E}_n(f|X_{1:n-1})\ldots|X_1)|\square)$$

- Linear complexity in the number of steps $n$
- In each iteration we can calculate conditional expectations by using only the extreme points $(\underline{a}, \overline{a}, \underline{d}, \overline{d})$

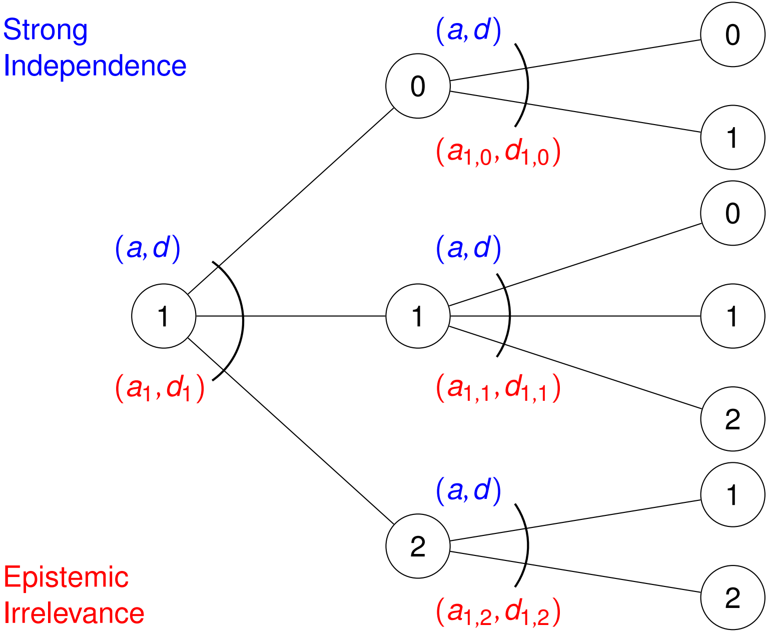
## Lemma

For any real-valued map $f$ on $\mathscr{X}^n$, with $n \in \mathbb{N}_0$, and any $x_{1:i} \in \mathscr{X}^i$ with $i \in \{1, \ldots, n\}$, it holds that

$$\underline{E}^{ei}_{i:n}(f|x_{1:i}) \leq \underline{E}^{s}_{i:n}(f|x_{1:i}) \text{ and } \overline{E}^{ei}_{i:n}(f|x_{1:i}) \geq \overline{E}^{s}_{i:n}(f|x_{1:i}).$$

Epistemic irrelevance is associated with all the possible probability trees, whereas the first one only with the stationary ones

# Probability trees under both approaches



# Experiments and Discussion

# Some useful properties

Interested in $n \to \infty$

Under strong independence

For functions on $\mathscr{X}$ we have convergence independent of the initial model

Functions on $\mathscr{X}^n$ convergence to a value affected by the initial model

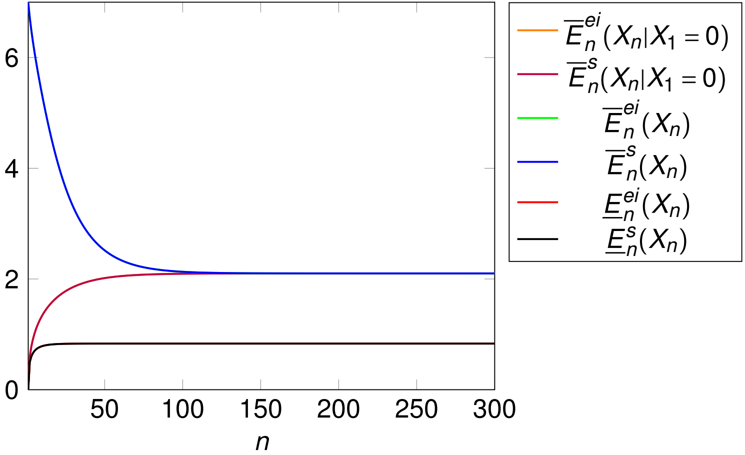
Under epistemic irrelevance

The same convergence properties hold, but for the bounds

# Our setting

- queue length = 7
- arrival $\in [0.5, 0.6]$
- departure $\in [0.7, 0.8]$
- for the initial model, we use the *vacuous* one

# Expected (Average) Queue Length




Lower and upper expected queue length

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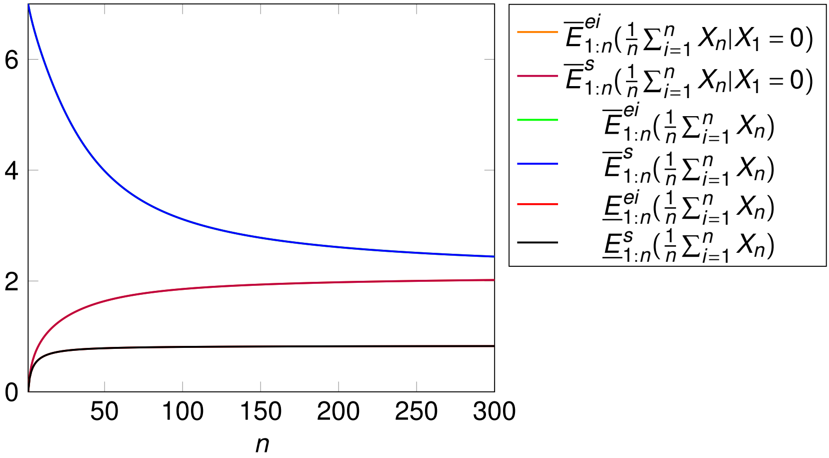


Lower and upper expected average queue length

# Expected (Average) Queue Length

Both approaches lead to the same corresponding tree

- For lower expected (average) queue length largest departure rate, lowest arrival rate
- For upper expected (average) queue length lowest departure rate, largest arrival rate

Due to the monotonicity of the function

# (Average) Probability of queue length

| $k$ | 0 | 1 | 2 | 7 |
|---|---|---|---|---|
| $\underline{E}_n^{ei}(\mathbb{I}_k(X_n))$ | 0.148638 | 0.290906 | 0.108098 | 0.000114 |
| $\underline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$ | 0.148301 | 0.308642 | 0.109882 | 0.000114 |
| $\overline{E}_n^{ei}(\mathbb{I}_k(X_n))$ | 0.375014 | 0.534395 | 0.268357 | 0.022481 |
| $\overline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$ | 0.375084 | 0.517315 | 0.257773 | 0.022987 |
| $\underline{E}_n^{s}(\mathbb{I}_k(X_n))$ | 0.148638<br>(0.6,0.7) | 0.31815<br>(0.6,0.7) | 0.117192<br>(0.5,0.8) | 0.000114<br>(0.5,0.8) |
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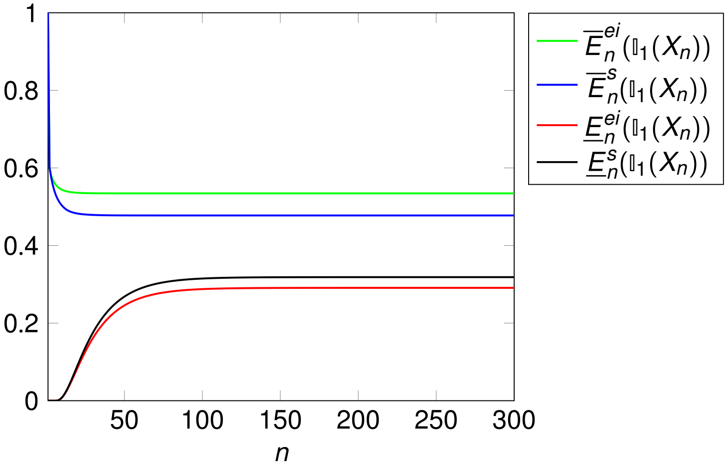
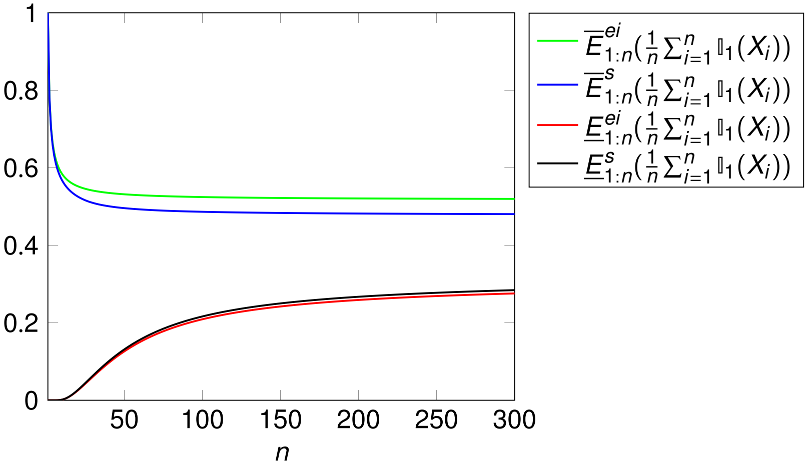
|---|---|---|---|---|
| $\underline{E}_n^{ei}(\mathbb{I}_k(X_n))$ | 0.148638 | 0.290906 | 0.108098 | 0.000114 |
| $\underline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$ | 0.148301 | 0.308642 | 0.109882 | 0.000114 |
| $\overline{E}_n^{ei}(\mathbb{I}_k(X_n))$ | 0.375014 | 0.534395 | 0.268357 | 0.022481 |
| $\overline{E}_{1:n}^{ei}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$ | 0.375084 | 0.517315 | 0.257773 | 0.022987 |
| $\underline{E}_n^{s}(\mathbb{I}_k(X_n))$ | 0.148638<br>(0.6,0.7) | 0.31815<br>(0.6,0.7) | 0.117192<br>(0.5,0.8) | 0.000114<br>(0.5,0.8) |
| $\underline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$ | 0.148301<br>(0.6,0.7) | 0.317824<br>(0.6,0.7) | 0.117162<br>(0.5,0.8) | 0.000114<br>(0.5,0.8) |
| $\overline{E}_n^{s}(\mathbb{I}_k(X_n))$ | 0.375014<br>(0.5,0.8) | 0.477512<br>(0.55,0.8) | 0.206501<br>(0.6,0.72) | 0.022481<br>(0.6,0.7) |
| $\overline{E}_{1:n}^{s}(\frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_k(X_i))$ | 0.375084<br>(0.5,0.8) | 0.477569<br>(0.55,0.8) | 0.206624<br>(0.6,0.72) | 0.022987<br>(0.6,0.7) |

# Probability of queue length 1



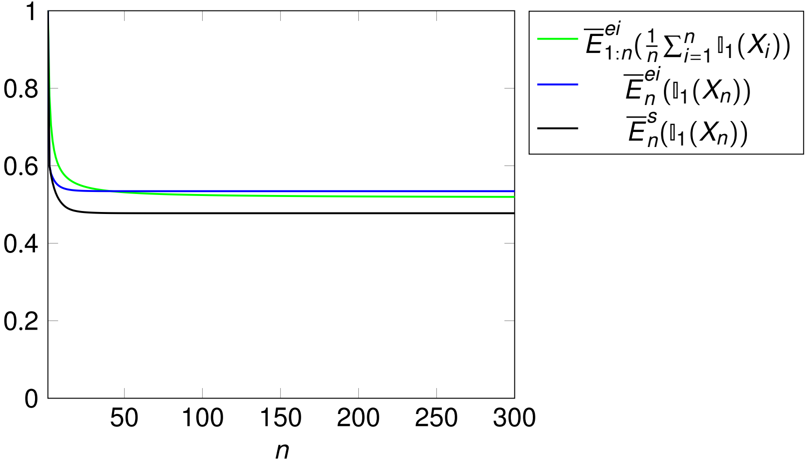

Lower and upper probability of queue length 1

# Average Probability of queue length 1




Lower and upper average probability of queue length 1

# (Average) Probability of queue length 1




Upper (average) probability of queue length 1

# A useful theorem

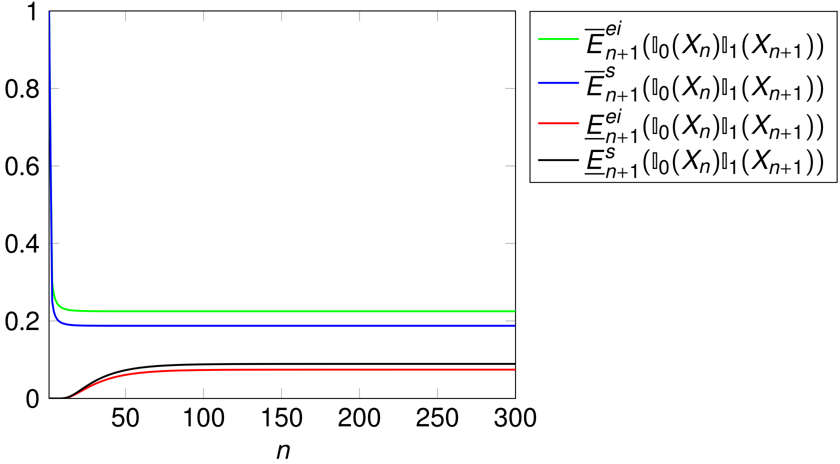
### Theorem

*Let* $L \in \mathbb{N}_0$. *Then, for all* $k \in \{1, \ldots, L-1\}$ *it holds that*

$$\lim_{n\to\infty} \underline{E}_n^{ei}(\mathbb{I}_k(X_n)) \leq \lim_{n\to\infty} \underline{E}_{1:n}^{ei}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_k(X_i)\right) \text{ and}$$

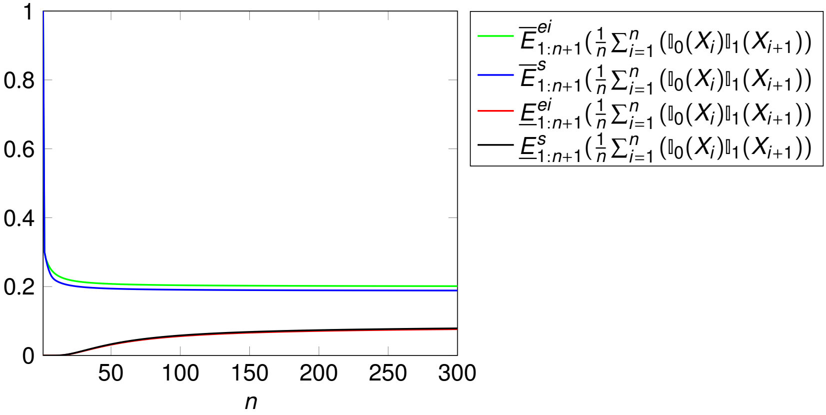
$$\lim_{n\to\infty} \overline{E}_n^{ei}(\mathbb{I}_k(X_n)) \geq \lim_{n\to\infty} \overline{E}_{1:n}^{ei}\left(\frac{1}{n}\sum_{i=1}^{n} \mathbb{I}_k(X_i)\right)$$

# Turning on the server



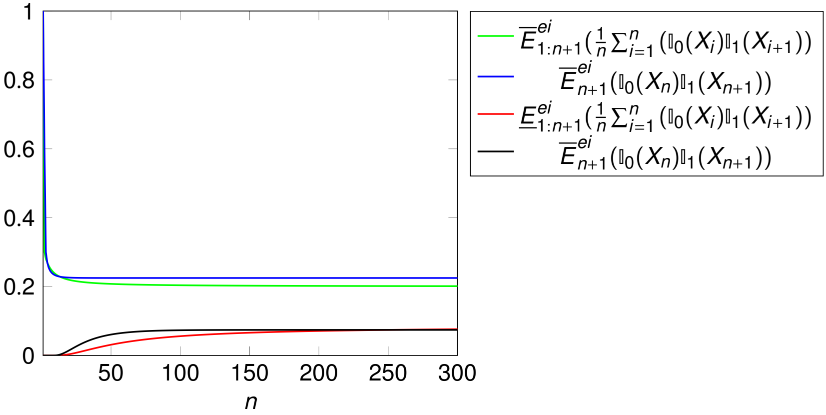

Lower and upper probability of turning on the server

# Turning on the server




Lower and upper average probability of turning on the server

# Turning on the server




Lower and upper (average) probability of turning on the server

# Conclusions & Future (Ongoing) work

When we are uncertain about the model, an average might not represent the actual situation

Compare the approaches with the state dependent model