

VIBRATION ANALYSIS OF CRACKED PLATE USING HIGHER-ORDER SHEAR DEFORMATION THEORY

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Abstract: A novel and effective formulation that combines the eXtended IsoGeometric Approach (XIGA) and Higher-order Shear Deformation Theory (HSDT) is proposed to study the free vibration of cracked plates. XIGA utilizes the Non-Uniform Rational B-Spline (NURBS) functions with their inherent arbitrary high order smoothness, which permit the C^1 requirement of the HSDT model. Two numerical examples are provided to show excellent performance of the proposed method compared with other published solutions in the literature.

Keywords: cracked plate; non-uniform rational b-spline; isogeometric analysis; higher-order shear deformation theory

1 INTRODUCTION

Plate structures play an increasing important role in engineering applications. They thus were researched since long time from static to dynamic and buckling analysis [1-5]. From the literature, these works are carried out for designing the plate structures without the presences of cracks or flaws. It is known that in service, the cracks are generated and grown when subjected to large cyclic loading. They lead to a reduction of the load carrying capacity of the structures. To clearly understand the dynamic behaviour of plate with initial cracks is hence too necessary. This problem has been interesting by many scientists with various methods: Stahl and Keer [6] and Liew et al. [7] with analytical solutions, Finite Element Method (FEM) [9], eXtended Finite Element Method (XFEM) [8,10], etc. Among them, XFEM is known as a robust numerical method, which uses enrichment functions to model discontinuities independent of the finite element mesh. Recently, Natarajan et al. [11] extended this method for dynamic analysis of FGM plate based on the FSDT model. This model is simple to implement and is applicable to both thick and thin FGM plates. However, the accuracy of solutions will be strongly dependent on the Shear Correction Factors (SCF) of which their values are quite dispersed through each problem.

In this paper, we develop the Higher-order Shear Deformation Theory (HSDT) model that includes higher-order terms in the approximation of the displacement field for modelling the plates. It is worth mentioning that this model requires C^1 -continuity of the generalized displacements leading to the second-order derivative of the stiffness formulation which causes some obstacles in standard C^0 finite formulations. Fortunately, it is shown that such a C^1 -HSDT formulation can be easily achieved using a NURBS-based isogeometric approach [12]. In addition, to capture the discontinuous phenomenon in the cracked plates, the enrichment functions is incorporated with NURBS basic functions to create a novel method as so-called eXtended IsoGeometric Analysis (XIGA) [13]. Herein, our study focuses on investigating the vibration of the cracked plates with an initial crack. Two numerical examples are given to show the performance of the proposed method and results obtained are compared to other published methods in the literature.

2 GOVERNING EQUATIONS FOR PLATES

To consider the effect of shear deformation directly, the five-parameter displacement field based on higher-order shear deformation theory is defined as

$$\begin{aligned} u(x, y, z) &= u_0 - zw_{,x} + \left(z - \frac{4}{3h^2} z^3\right) \beta_x \\ v(x, y, z) &= v_0 - zw_{,y} + \left(z - \frac{4}{3h^2} z^3\right) \beta_y, \quad \left(-\frac{h}{2} \leq z \leq \frac{h}{2}\right) \\ w(x, y) &= w_0 \end{aligned} \quad (1)$$

where u_0, v_0, w are the axial displacements and β_x, β_y are the rotations in the x and y axes, respectively.

The strains of the mid-surface deformation are given by

$$\begin{Bmatrix} \epsilon \\ \gamma \end{Bmatrix} = \begin{Bmatrix} \epsilon_0 + z\kappa_1 + \left(z - \frac{4}{3h^2} z^3\right) \kappa_2 \\ \left(1 - \frac{4}{h^2} z^2\right) \beta \end{Bmatrix} \quad (2)$$

Using the Hamilton principle, the weak form for free vibration analysis of a plate can be expressed as:

$$\int_{\Omega} \delta \mathbf{\epsilon}^T \mathbf{D}^b \mathbf{\epsilon} d\Omega + \int_{\Omega} \delta \mathbf{\gamma}^T \mathbf{D}^s \mathbf{\gamma} d\Omega = \int_{\Omega} \delta \mathbf{\tilde{u}}^T \mathbf{m} \ddot{\mathbf{u}} d\Omega \quad (3)$$

where \mathbf{D}^b and \mathbf{D}^s are the material matrices

Herein, the mass matrix \mathbf{m} is calculated according to consistent form as follow

$$\mathbf{m} = \begin{bmatrix} I_1 & I_2 & I_4 \\ I_2 & I_3 & I_5 \\ I_4 & I_5 & I_6 \end{bmatrix} \quad \text{with} \quad I_i = \int_{-h/2}^{h/2} \rho_e \left(1, z, z^2, \left(z - \frac{4}{3h^2} z^3\right), z \left(z - \frac{4}{3h^2} z^3\right), \left(z - \frac{4}{3h^2} z^3\right)^2\right) dz \quad (4)$$

and

$$\mathbf{\tilde{u}} = \begin{Bmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{Bmatrix}, \quad \mathbf{u}_1 = \begin{Bmatrix} u_0 \\ v_0 \\ w \end{Bmatrix}; \mathbf{u}_2 = -\begin{Bmatrix} w_{,x} \\ w_{,y} \\ 0 \end{Bmatrix}; \mathbf{u}_3 = \begin{Bmatrix} \beta_x \\ \beta_y \\ 0 \end{Bmatrix} \quad (5)$$

3 AN EXTENDED ISOGEOMETRIC CRACKED PLATE FORMULATION

3.1 A brief of B-spline/NURBS functions

A knot vector $\Xi = \{\xi_1, \xi_2, \dots, \xi_{n+p+1}\}$ is defined as a sequence of knot value $\xi_i \in R$, $i = 1, \dots, n+p$. If the first and the last knots are repeated $p+1$ times, the knot vector is called an open knot. Through open knot, the B-spline basis functions $N_{i,p}(\xi)$ are defined by the following recursion formula

$$N_{i,p}(\xi) = \frac{\xi - \xi_i}{\xi_{i+p} - \xi_i} N_{i,p-1}(\xi) + \frac{\xi_{i+p+1} - \xi}{\xi_{i+p+1} - \xi_{i+1}} N_{i+1,p-1}(\xi) \quad (6)$$

$$\text{as } p = 0, \quad N_{i,0}(\xi) = \begin{cases} 1 & \text{if } \xi_i < \xi < \xi_{i+1} \\ 0 & \text{otherwise} \end{cases}$$

To present exactly some curved geometries (e.g. circles, cylinders, spheres, etc.) the Non-Uniform Rational B-Splines (NURBS) functions are used. Be different from B-spline, each control point of NURBS has additional value called an individual weight w_i [12]. Then the NURBS functions can be expressed as

$$R_i(\xi, \eta) = \frac{N_i w_i}{\sum N_i(\xi, \eta) w_i} \quad (7)$$

3.2 Cracked plate formulation eXtended IsoGeometric Analysis (XIGA)

To capture the local discontinuous and singular fields, the new enriched functions are added according to idea of XFEM as follow:

$$\mathbf{u}^h(\mathbf{x}) = \sum_{I \in S} R_I(\xi) \mathbf{q}_I^{std} + \sum_{J \in S^{enr}} R_J^{enr}(\xi) \mathbf{q}_J^{enr} \quad (8)$$

Here, the NURBS basis functions are utilized instead of the Lagrange polynomials to create a new numerical procedure – so-called eXtended IsoGeometric Analysis (XIGA) [13,14]. R_J^{enr} are the enrichment functions associated with node J located in enriched domain S^{enr} which is splitted up two parts including: a set S^c for Heaviside enriched control points and a set S^f for crack tip enriched control points as shown in Fig. 1.

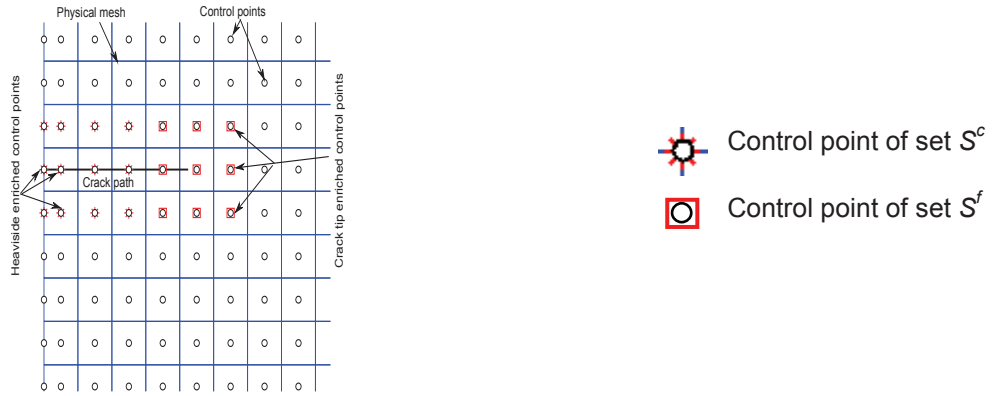


Fig. 1 Illustration of the nodal sets S^c , S^f for a quadratic NURBS mesh

To describe the discontinuous displacement field, the enrichment function is given by

$$R_J^{enr}(\xi) = R_J(\xi)(H(\mathbf{x}) - H(\mathbf{x}_J)), \quad J \in S^c \quad (9)$$

where $H(\mathbf{x})$ is the Heaviside function and the singularity field near crack tip is modified by the branching functions (see more details in Ref [10]) as follow

$$R_J^{enr}(\xi) = R_J(\xi) \left(\sum_{L=1}^4 (G_L(r, \theta) - G_L(r_J, \theta_J)) \right), \quad J \in S^f \quad (10)$$

where

$$G_L(r, \theta) = \begin{cases} r^{3/2} \begin{bmatrix} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & \sin \frac{3\theta}{2} & \cos \frac{3\theta}{2} \end{bmatrix} & \text{for } u_0, v_0, w \text{ variables} \\ r^{1/2} \begin{bmatrix} \sin \frac{\theta}{2} & \cos \frac{\theta}{2} & \sin \frac{\theta}{2} \sin \theta & \cos \frac{\theta}{2} \cos \theta \end{bmatrix} & \text{for } \beta_x, \beta_y \text{ variables} \end{cases} \quad (11)$$

in which r and θ are polar coordinates in the local crack tip coordinate system.

3.3 Cracked plate formulation based on HSDT

Now, using the displacement field approximated in Eq. (8), the strain matrices including in-plane and shear strains can be rewritten as:

$$\begin{bmatrix} \epsilon_0^T & \kappa_1^T & \kappa_2^T & \epsilon_s^T \end{bmatrix}^T = \sum_{I=1}^{m \times n} \begin{bmatrix} (\mathbf{B}_I^m)^T & (\mathbf{B}_I^{b1})^T & (\mathbf{B}_I^{b2})^T & (\mathbf{B}_I^s)^T \end{bmatrix}^T \mathbf{q}_I \quad (12)$$

in which the unknown vector \mathbf{q} contains both displacements and enriched DOFs, and

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}^{std} & \mathbf{B}^{enr} \end{bmatrix} \quad (13)$$

where \mathbf{B}^{std} and \mathbf{B}^{enr} are the standard and enriched strain matrices of \mathbf{B} .

Substituting Eq. (2) with relation in Eq. (12) into Eq. (3), the formulations of free vibration problem can be rewritten as follow:

$$(\mathbf{K} - \omega^2 \mathbf{M}) \mathbf{d} = \mathbf{0} \quad (14)$$

where $\omega \in \mathbb{R}^+$ are the natural frequency and the global stiffness matrix \mathbf{K} is given by:

$$\mathbf{K} = \int_{\Omega} \left\{ \mathbf{B}^m \quad \mathbf{B}^{b1} \quad \mathbf{B}^{b2} \right\} \mathbf{D}^b \left\{ \mathbf{B}^m \quad \mathbf{B}^{b1} \quad \mathbf{B}^{b2} \right\}^T + \mathbf{B}^{sT} \mathbf{D}^s \mathbf{B}^s d\Omega \quad (15)$$

4 NUMERICAL RESULTS

First, let study the natural frequencies of a thin isotropic cracked plate ($\nu=0.3$) with dimension $L \times W \times h$ has an initial crack at center as shown in Fig. 2a.

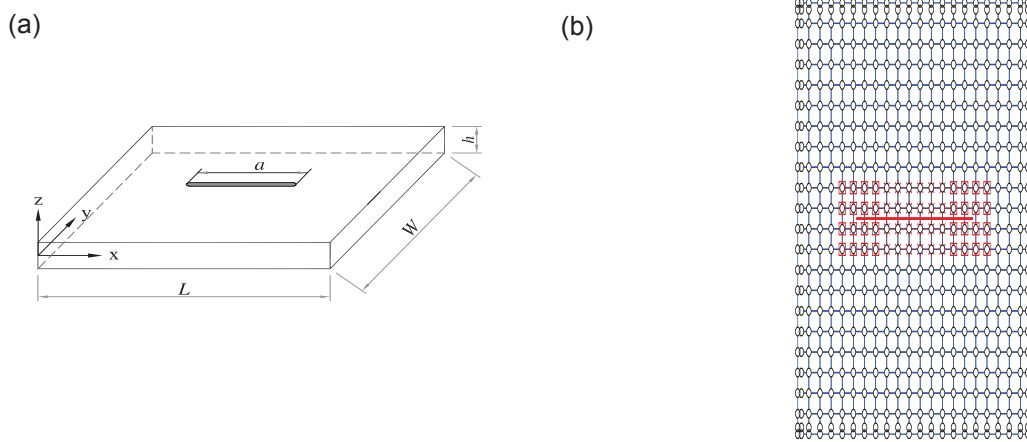
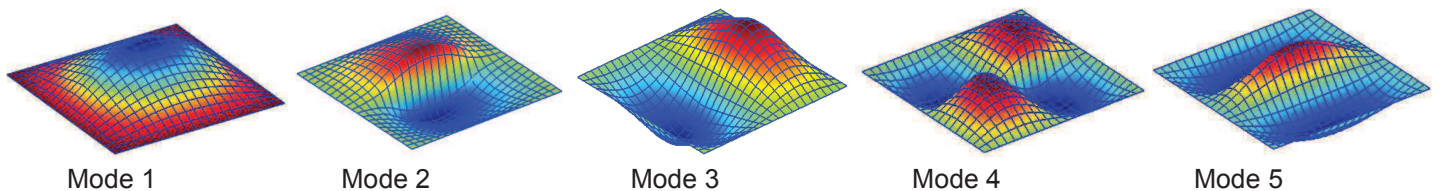


Fig. 2 The plate with a center crack: (a) model; (b) mesh of 21x21 cubic NURBS elements

The relation between non-dimension frequencies $\hat{\omega} = \omega L^2 \sqrt{\rho h / D}$ and crack length ratio according to mesh 21x21 as shown in Fig2b is reported in Table1. The obtained results from XIGA are in good agreement compared to both analytical solution [6,7] and XFEM [8] using 20x20 nine-node Lagrange elements. For clearer vision, the comparison of first five frequencies between the present result and that of Stahl [6] and Liew [7] is depicted in Fig.3. It is revealed that the frequencies decrease via increase in crack length ratio. For example, the values of frequency according to change of mode from 1 to 5 drop up to 18.3%, 67.3%, 5.3%, 40.8% and 23.7% of its initial values corresponding to an intact plate, respectively. It is concluded that the magnitude of the frequency according to anti-symmetric modes through the y-axis, which is perpendicular with cracked path (e.g. mode 2, mode 4, shown in Fig. 4), is much more affected by the crack length. Here, the discontinuous displacement is shown clearly along crack path.

Table 1 Non dimensional natural frequency of an isotropic square plate with central crack $L/h=1000$

Mode Number	Source	Crack length ratio a/L						
		0	0.2	0.4	0.5	0.6	0.8	1
1	Stahl [6]	19.7390	19.3050	18.2790	17.7060	17.1930	16.4030	16.1270
	Liew [7]	19.7400	19.3800	18.4400	17.8500	17.3300	16.4700	16.1300
	XFEM [8]	19.7390	19.3050	18.2780	17.7070	17.1800	16.4060	16.1330
	XIGA	19.7392	19.3846	18.4617	17.8989	17.3576	16.5157	16.1345
2	Stahl [6]	49.3480	49.1700	46.6240	43.0310	37.9780	27.7730	16.1270
	Liew [7]	49.3500	49.1600	46.4400	42.8200	37.7500	27.4300	16.1300
	XFEM [8]	49.3480	49.1810	46.6350	43.0420	37.9870	27.7530	17.8260
	XIGA	49.3501	49.1906	47.1197	44.7124	39.3469	29.1186	16.1345
3	Stahl [6]	49.3480	49.3280	49.0320	48.6970	48.2230	47.2560	46.7420
	Liew [7]	49.3500	49.3100	49.0400	48.7200	48.2600	47.2700	46.7400
	XFEM [8]	49.3480	49.3240	49.0320	48.6850	48.2140	47.2010	46.7340
	XIGA	49.3501	49.3292	49.0903	48.6328	48.3547	47.3448	46.7376
4	Stahl [6]	78.9570	78.9570	78.6020	77.7330	75.5810	65.7320	46.7420
	Liew [7]	78.9600	78.8100	78.3900	77.4400	75.2300	65.1900	46.7400
	XFEM [8]	78.9550	78.9450	78.6000	77.7100	75.5790	65.7150	49.0990
	XIGA	78.9589	78.9452	78.6507	77.6642	76.0779	67.2308	46.7376
5	Stahl [6]	98.6960	93.9590	85.5100	82.1550	79.5880	76.3710	75.2850
	Liew [7]	98.7000	94.6900	86.7100	83.0100	80.3200	76.6000	75.2800
	XFEM [8]	98.6980	93.8930	85.4500	82.1080	79.5560	76.3510	75.2750
	XIGA	98.7292	94.5834	86.6987	82.7347	80.3835	76.7866	75.2823

**Fig. 3** Variation of first five mode frequencies via length crack ratio**Fig. 4** The first five mode shapes of simply supported plate having center crack with $a/L = 0.8$

Next, an annular plate with uniform thickness h , outer radius R and inner one r as shown in Fig. is studied. The plate is made of $\text{Al}/\text{Al}_2\text{O}_3$ FGM with properties : $E_c=70$ GPa , $E_m=380$ GPa , $\nu_c=\nu_m=0.3$, $\rho_c=2707$ kg/m³, $\rho_m=3800$ kg/m³. The FGM is homogenized according to the Mori-Tanaka scheme with the effective values of Young's modulus E and Poisson's ratio ν are calculated from the effective bulk and shear modulus [4] as below :

$$E_e = \frac{9K_e\mu_e}{3K_e + \mu_e}, \quad \nu_e = \frac{3K_e - 2\mu_e}{2(3K_e + \mu_e)} \quad (16)$$

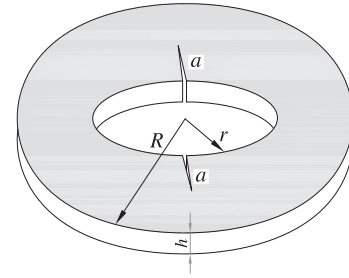
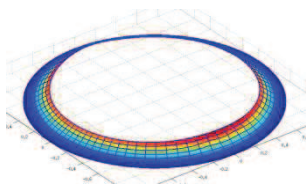


Fig. 5 The model of annular plate.

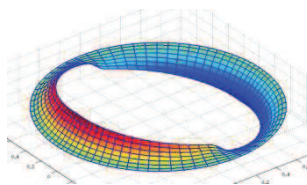
Table 2 tabulates the frequency parameter of the annular $\text{Al}/\text{Al}_2\text{O}_3$ plates via outer radius to inner radius ratio R/r and radius to thickness ratio R/h according to $n=1$. It is concluded that the frequency parameters decrease sequentially following to increase in inner radius to outer radius ratio r/R . To enclose this section the first four mode shapes of annular FGM plate are depicted in Fig. 6.

Table 2 The frequency parameter $\tilde{\omega} = \omega(R-r)^2 / h\sqrt{\rho_c / E_c}$ of an annular pate via inner radius to outer radius ratio r/R and radius to thickness ratio R/h according to $n=1$.

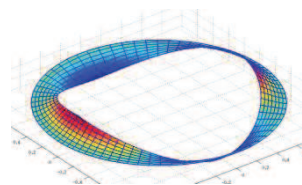
R/h	r/R	Mode number				
		1	2	3	4	5
2	0	1.2786	1.7682	2.3336	2.7352	2.8058
	0.2	0.8438	1.0109	1.7316	1.8588	1.9021
	0.5	0.5516	0.5896	0.7308	0.8458	0.9817
	0.8	0.2760	0.2771	0.2800	0.2896	0.2905
10	0	1.8480	3.5185	4.0473	5.9916	6.3512
	0.2	1.1563	1.5352	3.1932	4.1404	4.8849
	0.5	0.8621	0.9560	1.3533	1.6388	2.3101
	0.8	0.6470	0.6540	0.6730	0.6975	0.7545
20	0	1.8379	3.1941	4.1533	6.1526	6.4956
	0.2	1.1793	1.598	3.3309	4.4139	5.2045
	0.5	0.8877	0.9954	1.4507	1.7937	2.5902
	0.8	0.7279	0.7371	0.7655	0.7999	0.8775
100	0	1.8649	3.3264	4.2398	6.3270	6.7435
	0.2	1.1922	1.6442	3.4202	4.6187	5.3415
	0.5	0.8973	1.0154	1.5007	1.8656	2.6187
	0.8	0.7646	0.7753	0.8124	0.8567	0.9362



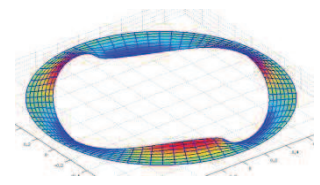
Mode 1



Mode 2



Mode 3



Mode 4

Fig. 6 The first four mode shapes of the annular plate with $r/R=0.8$, $R/h=10$.

5 CONCLUSIONS

In this paper, a novel and effective formulation based on combining XIGA and HSDT has been applied to dynamic analysis of the cracked plates. The present method, utilizing NURBS basis functions, allows us to achieve easily the smoothness with arbitrary continuity order compared with the traditional FEM. Consequently, it naturally fulfills the C^1 -continuity of HSDT model. Furthermore, the special enrichment functions are applied for describing the singularity behaviors along the crack. The obtained results are in excellent agreement with others in the literature.

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