Phase transitions related to the pigeonhole principle

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Abstract. Since Jeff Paris introduced them in the late seventies [Par78], densities turned out to be useful for studying independence results. Motivated by their simplicity and surprising strength we investigate the combinatorial complexity of two such densities which are strongly related to the pigeonhole principle. The aim is to miniaturise Ramsey's Theorem for 1-tuples. The first principle uses an unlimited amount of colours, whereas the second has a fixed number of two colours. We show that these principles give rise to Ackermannian growth. After parameterising these statements with respect to a function $f\colon \mathbb{N} \to \mathbb{N}$, we investigate for which functions f Ackermannian growth is still preserved.

Keywords Ackermann function, pigeonhole principle, Ramsey theory, phase transitions.

1   Introduction

The pigeonhole principle is one of the most well-know combinatorial principles, due to both its simplicity and usefulness. The principle is also known as the chest-of-drawers principle or Schubfachprinzip and is attributed to Dirichlet in 1834. The pigeonhole principle can also be considered as a finite instance of Ramsey's theorem for 1-tuples. So, if RT_k^n stands for Ramsey's Theorem for n dimensions and k colours, i.e.

$$\mathrm{RT}_k^n \leftrightarrow \text{For every } G\colon [\mathbb{N}]^n \to k \text{ there exists an infinite set } H$$
$$\text{such that } G \restriction [H]^n \text{ is constant,}$$

then the pigeonhole principle is a finite instance of $\mathrm{RT}^1_{<\infty} = \forall k\, \mathrm{RT}^1_k$. In this paper we will investigate miniaturisations of the statements $\mathrm{RT}^1_{<\infty}$ and RT^1_2. Let us recall some results from reverse mathematics: for any fixed natural number k, $\mathrm{RCA}_0 \vdash \mathrm{RT}^1_k$, whereas $\mathrm{WKL}_0 \nvdash \mathrm{RT}^1_{<\infty}$. Both results are due to Hirst (see [Hir87], Theorem 6.3 and Theorem 6.5). In addition, there it is also proved that $\mathrm{RT}^1_{<\infty}$ does not imply ACA_0 over RCA_0. As it does not fit nicely into the programme of reverse mathematics, one might be tempted to think that $\mathrm{RT}^1_{<\infty}$ is of little importance. However, it pops up every now and then in the literature. It is, for instance, equivalent to Rado's Lemma over RCA_0 (see [Hir87], Theorem 6.6).

For miniaturising $\mathrm{RT}^1_{<\infty}$ and RT^1_2 we define two notions of density, n-density and $(\alpha, 2)$-density, which are parametrised by a function $f\colon \mathbb{N} \to \mathbb{N}$. Using these notions we define two first order assertions and study their provability with respect to $\mathrm{I}\Sigma_1$, the first-order part of RCA_0.

We show which f give rise to Ackermannian growth and determine the exact phase transition. In case of n-density Ackermannian growth is obtained for $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$, whereas for $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$ it is not. Here A_d denotes the d-th branch of the Ackermann function A_ω. Our proof will show that in these results A_ω (A_d) could be replaced by any non decreasing unbounded non primitive recursive function (resp. by any non decreasing unbounded primitive recursive function). In the case of $(\alpha, 2)$-density we restrict ourselves to only two colours and strength disappears, as expected. Surprisingly, iterating up to ω^2 suffices to gain proof theoretic strength again. It turns out that $f(i) = \frac{1}{A_d^{-1}(i)} \log(i)$ gives rise to no more than primitive recursive growth, but $f(i) = \frac{1}{A_\omega^{-1}(i)} \log(i)$ does. Our proof will show that also in these results A_ω (A_d) could be replaced by any non decreasing unbounded non primitive recursive function (resp. by any non decreasing unbounded primitive recursive function).

We would like to mention that the n-density threshold functions are exactly the same as those for the parameterised Kanamori-McAloon principle, whereas the $(\omega^2, 2)$-density functions equal those for the parameterised Paris-Harrington principle [KLOW08,WVH2012]. It is our hope that by investigating miniaturisations of $\mathrm{RT}^1_{<\infty}$ and RT^2_2 one could obtain insights into the seemingly difficult question whether RT^2_2 does or does not prove the totality of the Ackermann function (the so called Ramsey for pairs problem).

For related work we also we also refer to [DSW08] and the unpublished PhD thesis of the first author [DS11].

2 n-Density

Henceforth, let $f\colon \mathbb{N} \to \mathbb{N}$ be any (elementary recursive function), such that $1 \leq f(x) \leq x$, for x large enough. We define the functions $F_{f,k}$ and F_f, depending

on f, by

$$F_{f,0}(n) := n + 1$$

$$F_{f,k+1}(n) := \underbrace{F_{f,k}(\ldots(F_{f,k}(n))\ldots)}_{f(n) \text{ times}} := F_{f,k}^{f(n)}(n)$$

$$F_f(n) := F_{f,n}(n),$$

for every $k, n \in \mathbb{N}$.

If it is clear which f we are working with, we leave out the subscript f and simply write F_k and F, instead of $F_{f,k}$ and F_f, respectively. We also consider functions f with non integer values. It is then understood that we round a value $f(i)$ down to $\lfloor f(i) \rfloor$, the biggest natural number below $f(i)$. Moreover we assume that f has always values at least as big as 1. It is easy to verify that the functions $F_{f,k}$ and F_f are strictly monotonic increasing if the parameter function f is non decreasing.

In case of $f(i) = i$ we write A_ω for F_f and A_d for $F_{f,d}$. A_ω is a standard choice for the well known Ackermann function, which is a recursive but not a primitive recursive function. The function A_d is called the d-th branch of the Ackermann function. Every function A_d is primitive recursive. In [OW09] a classification is given of those functions f for which F_f is primitive or non primitive recursive.

Let us define n-density, the first density notion related to the pigeonhole principle. In this case the number of colours depends on the minimum of X and the function f.

Definition 1 X is called 0-dense(f) if $|X| \geq \max\{f(\min X), 3\}$. X is called $(n+1)$-dense(f) if for all $G : X \to f(\min X)$, there exists $Y \subseteq X$, such that Y is homogeneous for G and Y is n-dense(f).

Lemma 1 *Assume that $k \leq l$ and that $X \subseteq [k, l]$ and that $f, g : [k, l] \to \mathbb{N}$ are two functions such that $f(i) \leq g(i)$ for all $i \in [k, l]$. If X is n-dense(g) then X is n-dense(f).*

Proof. One verifies the claim easily by induction on n.

2.1 Upper bound

Lemma 2 *Let f be non decreasing. Let $n \in \mathbb{N}$ and $X \subseteq \mathbb{N}$ be a finite set. If X is n-dense(f), then $\max X \geq F_{f,n}(\min X)$.*

Proof. Being of no importance for the proof itself, we leave out the subscript f. Henceforth, let $x_0 = \min X$ and $c = f(x_0)$. The proof goes by induction on n.

If X is 0-dense(f), then $|X| \geq \max\{f(x_0), 3\}$. Thus, $\max X \geq x_0 + 2 \geq F_0(x_0)$.

Secondly, assume the statement is proven for n and X is $(n+1)$-dense(f). Consider the following partition of $X = \cup_{0 \leq i < c} Y_i$, where Y_i is defined by

$$Y_i = \{x \in X | F_n^i(x_0) \leq x < F_n^{i+1}(x_0)\}$$

for $0 \leq i < c - 1$ and $Y_{c-1} = \{x \in X | F_n^{c-1}(x_0) \leq x\}$. Now, define $G : X \to c$, as follows

$$G(x) := i,$$

for $x \in Y_i$. Since X is $(n+1)$-dense(f), there exists a subset Y of X, such that Y is n-dense(f) and homogeneous for G. By contradiction assume $Y \subseteq Y_{i_0}$ for some i_0 with $0 \leq i_0 < c - 1$. The n-density of Y and the monotonicity of F_n yield

$$F_n^{i_0+1}(x_0) - 1 \geq \max Y_{i_0} \geq \max Y \geq F_n(\min Y) \geq$$
$$F_n(\min Y_{i_0}) \geq F_n(F_n^{i_0}(x_0)) = F_n^{i_0+1}(x_0),$$

a contradiction. So $Y \subseteq Y_{c-1}$, which implies

$$\max X = \max Y_{c-1} \geq \max Y \geq F_n(\min Y) \geq$$
$$F_n(\min Y_{c-1}) \geq F_n(F_n^{c-1}(x_0)) = F_n^c(x_0) = F_n^{f(x_0)}(x_0) = F_{n+1}(x_0),$$

by the n-density of Y. This concludes the induction argument.

Definition 2 *Define* $\mathrm{PHP}_f : \mathbb{N} \to \mathbb{N}$ *by*

$$\mathrm{PHP}_f(n) := \min\{n' \in \mathbb{N} | [n, n'] \text{ is } n\text{-dense}(f)\}.$$

Let $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$, where A_ω denotes the Ackermann fuction. Then F_f is Ackermannian, due to Theorem 1 in [OW09]. If f would be non decreasing then Lemma 2 would yield

$$\mathrm{PHP}_f(n) \geq F_{f,n}(n) = F_f(n),$$

for all $n \in \mathbb{N}$, hence also PHP_f would Ackermannian. Since the provably total functions of $\mathrm{I}\Sigma_1$ are exactly the primitive recursive functions, we would immediately obtain that PHP_f would not be provably recursive in $\mathrm{I}\Sigma_1$. We now show how to overcome this problem.

Theorem 1 *If* $f(i) = i^{\frac{1}{A_\omega^{-1}(i)}}$, *then*

$$\mathrm{I}\Sigma_1 \nvdash (\forall n)(\forall a)(\exists b)([a, b] \text{ is } n\text{-dense}(f)).$$

Proof. Let $p(n) := 4 + 3^{n+1} + (n+1)^{n+1}$ and let $f_k(i) := i^{\frac{1}{k}}$. It suffices to show that $\mathrm{PHP}_f(p(n)) > A_\omega(n)$ Assume that $\mathrm{PHP}_f(p(n)) \leq A_\omega(n)$. Then for $i \leq A_\omega(n)$ one has that $A_\omega^{-1}(i) \leq n$ which yields $f(i) \geq f_n(i)$ for all $i \leq \mathrm{PHP}_f(p(n))$. The proof of Claim 2.12 from [KLOW08] yields $F_{f_{n+1}, n+n^2+4n+5}(p(n) > A_\omega(n)$. Together with Lemma 1 this yields

$$\mathrm{PHP}_f(p(n)) \geq \mathrm{PHP}_{f_n}(p(n)) \geq F_{f_{n+1}, n+n^2+4n+5}(p(n)) > A_\omega(n)$$

which is a contradiction.

2.2 Lower bound

Let $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$, where A_d denotes the d-th branch of the Ackermann function A_ω. This function is not weakly increasing on its domain. For $i \in [A_d(k), A_d(k+1)-1]$ one has that $A_d^{-1}(i) = k$ and on such intervals f will be non decreasing. In intervals of the form $i \in [A_d(k)-1, A_d(k)]$ the function A_d^{-1} jumps from $k-1$ to k. But since the intervals of the form $[A_d(k), A_d(k+1)-1]$ are rather long it is very easy to find enough points b such that $f(b) \geq f(i)$ for all $i \leq b$. One simply has to choose b so large that f majorizes $f(c)$ where c is the initial point of a last jump interval which comes before b. With this caveat we can consider f basically as non decreasing function although it in fact is not.

Theorem 2 *If* $f(i) = i^{\frac{1}{A_d^{-1}(i)}}$, *then*

$$I\Sigma_1 \vdash (\forall n)(\forall a)(\exists b)([a,b] \text{ is } n\text{-dense}(f)).$$

Proof. Assume that n and a are given. Put $b := 2^{A_d(a2^{n+1})2^{n+1}}$. Then $f(i) \leq f(b)$ for all $i \leq b$. We claim that any $Y \subseteq [a,b]$ with $|Y| > 2^{A_d(a2^{n+1})2^k}$ is k-dense(f). To prove the claim we proceed by induction on k.

Assume the claim holds for $k-1$ and consider $Y \subseteq [a,b]$ with $|Y| > 2^{A_d(a2^{n+1})2^k}$. Since $2^{A_d(a2^{n+1})2^{n+1}} > A_d(2^{n+1})$, we have

$$f(\min Y) < f(b) = \left(2^{A_d(a2^{n+1})2^{n+1}}\right)^{\frac{1}{A_d^{-1}(2^{A_d(a2^{n+1})2^{n+1}})}}$$

$$\leq \left(2^{A_d(a2^{n+1})2^{n+1}}\right)^{\frac{1}{A_d^{-1}(A_d(2^{n+1}))}}$$

$$= \left(2^{A_d(a2^{n+1})2^{n+1}}\right)^{\frac{1}{2^{n+1}}} = 2^{A_d(a2^{n+1})} \leq 2^{A_d(a2^{n+1})2^{k-1}}.$$

Let $c = f(\min Y)$ and $G : Y \to c$ be any function. Consider the partition of Y induced by G, i.e.

$$Y = \cup_{0 \leq i < c} Y_i,$$

with $Y_i = \{y \in Y | G(y) = i\}$. By contradiction, assume that $|Y_i| \leq 2^{A_d(a2^{n+1})2^{k-1}}$ for every $0 \leq i < c$. Then

$$2^{A_d(a2^{n+1})2^k} < |Y| \leq c \cdot 2^{A_d(a2^{n+1})2^{k-1}} = f(\min Y) \cdot 2^{A_d(a2^{n+1})2^{k-1}}$$

$$< 2^{A_d(a2^{n+1})2^{k-1}} \cdot 2^{A_d(a2^{n+1})2^{k-1}} = 2^{A_d(a2^{n+1})2^{k-1}+A_d(a2^{n+1})2^{k-1}} = 2^{A_d(a2^{n+1})2^k},$$

a contradiction. Thus, there exists an index $i_0 \in \{0, \ldots, c-1\}$, such that $|Y_{i_0}| > 2^{A_d(a2^{n+1})2^{k-1}}$. The induction hypothesis yields that Y_{i_0} is $(k-1)$-dense(f) and by definition Y_{i_0} is homogeneous for G, so Y is k-dense(f).

If $k = 0$ then $|Y| > 2^{A_d(a2^{n+1})} > f(\min Y)$, which completes the induction argument and proves the claim.

Now return to $[a,b]$. $[a,b]$ is n-dense(f) since $|[a,b]| \geq 2^{A_d(a2^{n+1})2^{n+1}} - a \geq 2^{A_d(a2^{n+1})2^n}$. Remarking that the function $E : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, defined by $E(a,n) = 2^{A_d(a2^{n+1})2^{n+1}}$ is primitive recursive, completes the proof.

Let PHP$_f$ stand for "$(\forall n)(\forall a)(\exists b)([a,b]$ is n-dense$(f))$". Then we obtain the following picture.
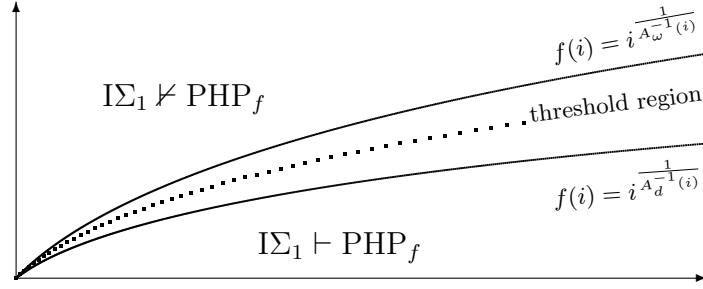
Fig. 1. *Phase transition for* PHP_f.

3 $(\alpha, 2)$-Density

In this section we work with a fixed number of colours, namely two. For a limit ordinal not exceeding ω^2 we define the fundamental sequence as follows. We put $\omega \cdot (k+1)[n] = \omega \cdot k + n$ and we set $\omega^2[n] = \omega \cdot n$.

Definition 3 *X is called* $(0,2)$-*dense(f) if* $|X| \geq \max\{f(\min X), 3\}$. *X is called* $(\alpha + 1, 2)$-*dense(f) if for all* $G : X \to 2$ *there exists* $Y \subseteq X$, *such that* Y *is* $(\alpha, 2)$-*dense(f) and* Y *is homogeneous for* G. *If* λ *is a limit ordinal, then* X *is called* $(\lambda, 2)$-*dense(f) if for all* $G : X \to 2$ *there exists* $Y \subseteq X$, *such that* Y *is* $(\lambda[\min X], 2)$-*dense(f) and* Y *is homogeneous for* G.

Lemma 3 *Let* $k \leq l$ *and* $f, g : [k, l] \to \mathbb{N}$ *be non decreasing such that* $f(i) \leq g(i)$ *for all* $i \in [k, l]$. *If* $X \subseteq [k, l]$ *is* $(n, 2)$-*dense(g) then* X *is is* $(n, 2)$-*dense(g)*.

3.1 Upper bound

In this section we will use another hierarchy which we call $B_{f,\alpha}$ and which turns out to be related to $F_{f,k}$. We define B_α, depending on f, by

$$B_{f,0}(n) := n + 1$$
$$B_{f,\alpha+1}(n) := B_{f,\alpha}(B_{f,\alpha}(n)) := B_{f,\alpha}^2(n)$$
$$B_{f,\lambda}(n) := B_{f,\lambda[f(n)]}(n),$$

for all $n \in \mathbb{N}$ and ordinals α and λ, with the latter a limit ordinal. As for F_f, we leave out the subscript f and write B_α if it is clear which f we are working with. We first show a simple lemma concerning the relation between the two hierarchies defined in this paper. This lemma might be considered as folklore.

Lemma 4 *Let* k, l *and* m *be natural numbers. Then* $B_{f,\omega \cdot k + l}(m) = F_{2^f, k}^{2^l}(m)$.

Proof. We proceed by main induction on k and subsidiary induction on l.

If k equals l equals zero, we have $B_{f,0}(m) = m + 1 = F_{2^f, 0}(m)$.

Assume the statement is proven for $k-1$, we will prove it for k by subsidiary induction on l.

If $l = 0$, then the main induction hypothesis yields

$$B_{f,\omega\cdot(k-1)}(m) = F_{2^f,k-1}(m).$$

Assume the claim is proven for $l-1$. We have

$$B_{f,\omega\cdot(k-1)+l}(m) = B_{f,\omega\cdot(k-1)+l-1}(B_{f,\omega\cdot(k-1)+l-1}(m))$$
$$= F_{2^f,k-1}^{2^{l-1}}(F_{2^f,k-1}^{2^{l-1}}(m)) = F_{2^f,k-1}^{2^l}(m),$$

which proves the statement for $k-1$ and every l. Using this fact, we obtain

$$B_{f,\omega\cdot k}(m) = B_{f,\omega\cdot(k-1)+\omega[f(m)]}(m)) = B_{f,\omega\cdot(k-1)+f(m)}(m))$$
$$= F_{2^f,k-1}^{2^{f(m)}}(m) = F_{2^f,k}(m),$$

which concludes the main induction and proves the statement.

Lemma 5 *Let f be non decreasing. Let n be a natural number and α be any ordinal. If $X \subseteq \mathbb{N}$ is $(\alpha, 2)$-dense(f), then $\max X \geq B_{f,\alpha}(\min X)$.*

Proof. Being of no importance for the proof itself, we leave out the subscript f. Henceforth, let $x_0 = \min X$. The proof goes by transfinite induction on α.

If X is $(0,2)$-dense(f), then $|X| \geq \max\{f(x_0), 3\}$. Thus, $\max X \geq x_0 + 2 > x_0 + 1 = B_0(x_0)$.

Assume the statement is proven for α and X is $(\alpha+1, 2)$-dense(f). Define $G : X \to 2$ as follows

$$G(x) := \begin{cases} 0 & \text{if } x_0 \leq x < B_\alpha(x_0) \\ 1 & \text{if } B_\alpha(x_0) \leq x \end{cases},$$

for all $x \in X$. Since X is $(\alpha+1, 2)$-dense(f), there exists a subset Y of X, such that Y is $(\alpha, 2)$-dense(f) and Y is homogeneous with respect to G. By contradiction, assume G takes colour 0 on Y. Then, by the induction hypothesis,

$$B_\alpha(x_0) - 1 \geq \max Y \geq B_\alpha(\min Y) = B_\alpha(x_0),$$

a contradiction. So, the colour needs to be 1, which implies

$$Y \subseteq \{x \in X | B_\alpha(x_0) \leq x\}.$$

The induction hypothesis yields

$$\max X \geq \max Y \geq B_\alpha(\min Y) = B_\alpha(B_\alpha(x_0)) = B_{\alpha+1}(x_0).$$

Finally, assume the statement is proven for all $\alpha < \lambda$, with λ a limit ordinal, and X is $(\lambda, 2)$-dense(f). There exists a subset Y which is $(\lambda[f(x_0)], 2)$-dense(f). We obtain by the induction hypothesis

$$\max X \geq \max Y \geq B_{\lambda[f(x_0)]}(\min Y) \geq B_{\lambda[f(x_0)]}(x_0) = B_\lambda(x_0).$$

This completes the proof.

Definition 4 *Define* $\mathrm{PHP2}_f : \mathbb{N} \to \mathbb{N}$ *by*

$$\mathrm{PHP2}_f(n) := \min\{n' \in \mathbb{N}|[n,n'] \ is \ (\omega^2,2)\text{-}dense(f)\}.$$

Fix $f(i) = \frac{1}{A_\omega^{-1}(i)} \log(i)$ for the rest of this subsection.

Lemma 6 *Let* $f_n(i) := \frac{1}{n} \cdot \log_2(i)$. *Then* $\mathrm{PHP2}_{f_n}(2^{n^2}) \geq F_{2^{f_n}}(n)$ *holds for every* $n \in \mathbb{N}$.

Proof. Let $X = [2^{n^2}, \mathrm{PHP2}_f(2^{n^2})]$. Define $G : X \to 2$ by $G(x) = 0$ for every $x \in X$. Since X is $(\omega^2,2)$-dense(f_n) there exists $Y \subseteq X$, such that Y is $(\omega^2[f_n(\min X)],2)$-PHP2-dense$(f_n)$, i.e. $(\omega \cdot f_n(2^{n^2}),2)$-dense$(f_n)$. Lemma 4 and Lemma 5 yield

$$\mathrm{PHP2}(2^{n^2}) \geq \max Y \geq B_{f_n,\omega \cdot f(\min Y)}(\min Y) \geq B_{f_n,\omega \cdot f(2^{n^2})}(2^{n^2})$$

$$= F_{2^{f_n},f_n(2^{n^2})}(2^{n^2}) \geq F_{2^{f_n},n}(n) = F_{2^{f_n}}(n),$$

since $f_n(2^{n^2}) = n$.

Corollary 1 *If* $f(i) = \frac{1}{A_\omega^{-1}(i)} \log(i)$, *then*

$$I\Sigma_1 \nvdash (\forall a)(\exists b)([a,b] \ is \ (\omega^2,2)\text{-}dense(f)).$$

Proof. Let $p(n) = 4 + 3^{n+1} + (n+1)^{n+1}$ and $f_k(i) := \frac{1}{k} \log_2(i)$. It suffices to show that $\mathrm{PHP2}_f(2^{p(n)^2}) > A_\omega(n)$. Assume for a contradiction that $\mathrm{PHP2}_f(2^{p(n)^2}) \leq A_\omega(n)$. For $i \leq A_\omega(n)$ one has $A_\omega^{-1}(i) \leq n$ hence $f(i) \geq f_n(i)$ for all $i \leq \mathrm{PHP2}_f(2^{p(n)^2})$. This yields $\mathrm{PHP2}_f(2^{p(n)^2}) \geq \mathrm{PHP2}_{f_n}(2^{p(n)^2}) \geq F_{2^{f_n}}(p(n) > A_\omega(n)$. Contradiction!

3.2 Lower bound

As in Section 2.2 let $f(i) = \frac{1}{A_d^{-1}(i)} \log(i)$, where A_d denotes the dth branch of the Ackermann function A_ω. Recall from Section 2.2 that f is almost non decreasing and that it is easy to identify the jumps for f.

Theorem 3 *If* $f(i) = \frac{1}{A_d^{-1}(i)} \log(i)$, *then*

$$I\Sigma_1 \vdash (\forall a)(\exists b)([a,b] \ is \ (\omega^2,2)\text{-}dense(f)).$$

Proof. Assume that a is given. Put $b := 2^{A_d(2^{a+2})2^{a+1}}$. Then $f(i) \leq f(b)$ for all $i \leq b$. We claim that any $Y \subseteq [a,b]$, with $|Y| > 2^{A_d(2^{a+2})2^k}$ is $(\omega \cdot k,2)$-dense(f). The proof goes by induction on k.

Let $k = 0$. Since $2^{A_d(2^{a+2})2^{a+1}} > A_d(2^{a+2})$, we have

$$f(\min Y) < f(b) = \frac{1}{A_d^{-1}(2^{A_d(2^{a+2})2^{a+1}})} \log(2^{A_d(2^{a+2})2^{a+1}})$$

$$< \frac{1}{2^{a+2}} A_d(2^{a+2})2^{a+1} < A_d(2^{a+2}),$$

so, $|Y| > 2^{A_d(2^{a+2})} > \max\{f(\min Y), 3\}$, i.e. Y is $(0,2)$-dense(f).

Assume the assertion holds for $k-1$ and consider $Y \subseteq [a,b]$ with $|Y| > 2^{A_d(2^{a+2})2^k}$. We claim that if $Z \subseteq Y$ and $|Z| > 2^{A_d(2^{a+2})2^{k-1}+l}$, then Z is $(\omega \cdot (k-1)+l, 2)$-dense$(f)$. The proof goes by subsidiary induction on l.

If $l = 0$, then the claim follows by the main induction hypothesis. Assume the claim holds for $l-1$ and $|Z| > 2^{A_d(2^{a+2})2^{k-1}+l}$. Let $G : Z \to 2$ be any function. Consider the partition of Z induced by G, i.e.

$$Z = Z_0 \cup Z_1,$$

with $Z_i = \{z \in Z | G(z) = i\}$. By contradiction, assume that

$$|Z_i| \leq 2^{A_d(2^{a+2})2^{k-1}+l-1},$$

for $i = 0, 1$. Then

$$2^{A_d(2^{a+2})2^{k-1}+l} < |Z| \leq 2 \cdot 2^{A_d(2^{a+2})2^{k-1}+l-1} = 2^{A_d(2^{a+2})2^{k-1}+l},$$

a contradiction. Thus, there exists an index $i_0 \in \{0,1\}$, such that $|Z_{i_0}| > 2^{A_d(2^{a+2})2^{k-1}+l-1}$. The induction hypothesis yields Z_{i_0} is $(\omega \cdot (k-1)+l-1, 2)$-dense$(f)$, and so Z is $(\omega \cdot (k-1)+l, 2)$-dense$(f)$, since Z_{i_0} is homogeneous for G. This proves the latter claim.

Now return to Y. Let $G : Y \to 2$ be any function. Consider the partition of Y induced by G, i.e.

$$Y = Y_0 \cup Y_1,$$

with $Y_i = \{y \in Y | G(y) = i\}$. In the same way as above, one can prove there exists an index $i_0 \in \{0,1\}$, such that

$$|Y_{i_0}| > 2^{A_d(2^{a+2})2^k - 1} = 2^{A_d(2^{a+2})2^{k-1}+A_d(2^{a+2})2^{k-1}-1}.$$

Since

$$A_d(2^{a+2})2^{k-1} \geq A_d(2^{a+2}) \geq f(\min Y) + 1,$$

we have $|Y_{i_0}| > 2^{A_d(2^{a+2})2^{k-1}+f(\min Y)}$. The latter claim yields Y_{i_0} is $(\omega \cdot (k-1)+f(\min Y), 2)$-dense$(f)$, i.e. $(\omega \cdot k[f(\min Y)], 2)$-dense$(f)$. Thus Y is $(\omega \cdot k, 2)$-dense(f), since Y_{i_0} is homogeneous for G.

We finally prove that $[a,b]$ is $(\omega^2, 2)$-dense(f). Let $G : [a,b] \to 2$ be any function and consider the partition of $[a,b]$ induced by G, i.e.

$$[a,b] = Y_0 \cup Y_1,$$

with $Y_i = \{y \in [a,b] | G(y) = i\}$. Remark that $|[a,b]| > 2^{A_d(2^{a+2})2^{a+1}} - a \geq 2^{A_d(2^{a+2})2^a+1}$. Similarly as before, there exists an index $i_0 \in \{0,1\}$, such that

$$|Y_{i_0}| > 2^{A_d(2^{a+2})2^a} \geq 2^{A_d(2^{a+2})2^{f(a)}}.$$

The main claim yields Y is $(\omega \cdot f(a), 2)$-dense(f), i.e. $(\omega^2[f(a)], 2)$-dense(f). In combination with Y_{i_0} being homogeneous for G, this implies $[a,b]$ is $(\omega^2, 2)$-dense(f).
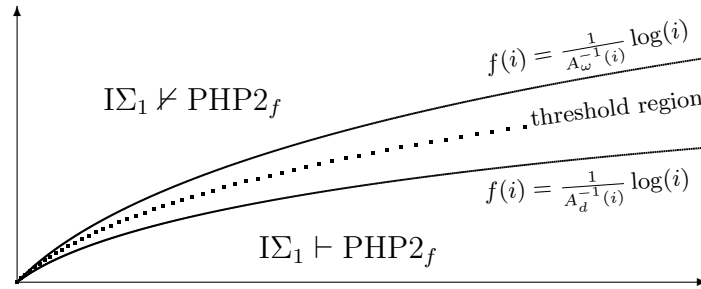
Fig. 2. *Phase transition for* $\mathrm{PHP}2_f$.

Let $\mathrm{PHP}2_f$ stand for "$(\forall n)(\forall a)(\exists b)([a,b]$ is $(\omega^2,2)$-dense$(f))$". Once again, we obtain the following picture.

In accordance with the referees (for which we are grateful for valuable comments) we expect that it will be not too hard to show that for natural choices of f the principles $\mathrm{PHP}_{\log \circ f}$ and $\mathrm{PHP}2_f$ are equivalent over $I\Sigma_1$.

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