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Higher order Borel-Pompeiu representations

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Abstract

We establish a higher order Borel-Pompeiu formula for monogenic functions associated to an arbitrary orthogonal basis (called structural set) of Euclidean space, or combinations of such structural sets.

Key words: Borel-Pompeiu representation, Cauchy-type integral, structural set
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1 Introduction and preliminaries

Clifford analysis in its original form (see [6]) deals with the study of so-called monogenic functions (null solutions of the Cauchy Riemann operator), which are a higher dimensional analogue of holomorphic functions in the complex plane. One of the key results in this function theory is the Borel-Pompeiu representation formula, leading to various higher-dimensional analogues of well-known classical theorems. Integral representation formulas

δ_{ij} is Kronecker's symbol. Then for $A = \{h_1, \ldots, h_k\}$ $(1 \leq h_1 < \cdots < h_k \leq n)$, $e_A = e_{h_1} \cdots e_{h_k}$ and $e_\emptyset = 1$ define a basis of $\mathbb{R}_{0,n}$, whence any $a \in \mathbb{R}_{0,n}$ may be written as $a = \sum_{A \subseteq \mathbb{N}_n} a_A e_A$ where $a_A \in \mathbb{R}$ or still as $a = \sum_{k=0}^{n} [a]_k$, where $[a]_k = \sum_{|A|=k} a_A e_A$ is a so-called k-vector. Denoting the space of k-vectors by $\mathbb{R}_{0,n}^{(k)}$, it holds that $\mathbb{R}_{0,n} = \bigoplus_{k=0}^{n} \mathbb{R}_{0,n}^{(k)}$. In this way, the spaces $\mathbb{R}$ and $\mathbb{R}^n$ are identified with $\mathbb{R}_{0,n}^{(0)}$ and $\mathbb{R}_{0,n}^{(1)}$. Moreover, each element $x = (x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ can be written as $x = x_0 + \sum_{i=1}^{n} x_i e_i \in \mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$ and is called a *paravector*. It holds that $x\bar{x} = \bar{x}x = x_0^2 + x_1^2 + \cdots + x_n^2 = |x|^2$. We will use the *conjugation*, defined by $\bar{a} = \sum_{A \subseteq \mathbb{N}_n} a_A \bar{e}_A$, where $\bar{e}_A := (-1)^k e_{h_k} \cdots e_{h_1} = (-1)^{\frac{k(k+1)}{2}} e_A$, if $e_A = e_{h_1} \cdots e_{h_k}$. We consider functions $f : \Omega \to \mathbb{R}_{0,n}$, where Ω is a bounded domain in $\mathbb{R}^{n+1}$, with sufficiently smooth boundary Γ.

Let $\psi := \{\psi^0, \psi^1, \ldots, \psi^n\} \subset \mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$, and let $\overline{\psi} := \{\overline{\psi^0}, \overline{\psi^1}, \ldots, \overline{\psi^n}\}$. On $C^1(\Omega, \mathbb{R}_{0,n})$ we define the left and the right ψ-Cauchy-Riemann operator by

$$^{\psi}D[f] := \sum_{i=0}^{n} \psi^i \frac{\partial f}{\partial x_i}, \qquad D^{\psi}[f] := \sum_{i=0}^{n} \frac{\partial f}{\partial x_i} \psi^i$$

Let Δ_{n+1} be the $(n+1)$-dimensional Laplace operator, then the factorization

$$^{\psi}D \cdot {}^{\overline{\psi}}D = {}^{\overline{\psi}}D \cdot {}^{\psi}D = D^{\psi} \cdot D^{\overline{\psi}} = D^{\overline{\psi}} \cdot D^{\psi} = \Delta_{n+1},$$

will hold if and only if $\psi^i \cdot \overline{\psi^j} + \psi^j \cdot \overline{\psi^i} = 2\delta_{ij}$, $i, j \in \mathbb{N}_n^0$, implying that ψ should be an orthonormal basis, called *structural set*, of $\mathbb{R}^{n+1} \cong \mathbb{R}_{0,n}^{(0)} \oplus \mathbb{R}_{0,n}^{(1)}$. Note that ψ and $\overline{\psi}$ are structural sets simultaneously. Choosing as a structural set the standard basis of $\mathbb{R}^{n+1}$, we obtain the standard Cauchy-Riemann operator D. Basic properties of the structural sets can be found in [8] and the references therein.

Null solutions of the left , respectively the right ψ-Cauchy-Riemann operator are called left, respectively right ψ-hyperholomorphic functions. Let $\Theta_{n+1}^{(k)}$ be the fundamental solution of the iterated Laplace operator Δ_{n+1}^k, $k \in \mathbb{N}$:

$$\Theta_{n+1}^{(k)}(x) = \frac{|x|^{2k-(n+1)}}{|\mathbb{S}^n| 2^{k-1} (k-1)! \prod_{v=1}^{k} (2v - n - 1)}.$$

Theorem 1 (Cauchy integral theorem). *Let* $f \in {}^{\psi}\mathfrak{M}(\Omega, \mathbb{R}_{0,n}) \cap C^0(\overline{\Omega}, \mathbb{R}_{0,n})$ *and* $g \in \mathfrak{M}^{\psi}(\Omega, \mathbb{R}_{0,n}) \cap C^0(\overline{\Omega}, \mathbb{R}_{0,n})$*. Then*

$$\int_{\Gamma} g(\xi)\, n_{\psi}(\xi)\, f(\xi)\, dS_{\xi} = 0$$

Theorem 2 (Borel-Pompeiu formula). *Let* $f \in C^1(\Omega, \mathbb{R}_{0,n}) \cap C^0(\overline{\Omega}, \mathbb{R}_{0,n})$*. Then*

$$\int_{\Gamma} K_{\psi}(\xi - x)\, n_{\psi}(\xi)\, f(\xi)\, d\Gamma_{\xi} - \int_{\Omega} K_{\psi}(\xi - x)^{\psi} D[f](\xi) d\xi = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^{n+1} \setminus \overline{\Omega} \end{cases}$$

where the first term at the left hand side is a Cauchy type integral.

Theorem 3 *Let* $f \in L_p(\Omega, \mathbb{R}_{0,n})$, $p \in (1, \infty)$*. Then*

$${}^{\psi}D^{\psi}T_{\Omega}[f](x) = \begin{cases} f(x) & \text{if } x \in \Omega, \\ 0 & \text{if } x \in \mathbb{R}^{n+1} \setminus \overline{\Omega} \end{cases}$$

where the ψ*-Téodorescu transform is given by (see [1]):*

$${}^{\psi}T_{\Omega}[f](x) = -\int_{\Omega} K_{\psi}(\xi - x)\, f(\xi)\, d\xi, \qquad x \in \mathbb{R}^{n+1}$$

2 Higher order Borel-Pompeiu representations

Let $\Psi = \{\psi_1, \psi_2, \ldots, \psi_k\}$ be a collection of k structural sets, denote $\overline{\Psi} = \{\overline{\psi}_k, \overline{\psi}_{k-1}, \ldots, \overline{\psi}_1\}$, and consider the left and right Ψ-iterated Cauchy-Riemann operators given by

$${}^{\Psi}D = {}^{\psi_k}D^{\psi_{k-1}}D \cdots {}^{\psi_1}D, \qquad D^{\Psi} = D^{\psi_k} D^{\psi_{k-1}} \cdots D^{\psi_1}$$

Then it holds that ${}^{\Psi}D \cdot {}^{\overline{\Psi}}D = {}^{\overline{\Psi}}D \cdot {}^{\Psi}D = D^{\Psi} \cdot D^{\overline{\Psi}} = D^{\overline{\Psi}} \cdot D^{\Psi} = \Delta^k_{n+1}$, and the fundamental solution of ${}^{\Psi}D$ is given by

$$\begin{aligned} K_{\Psi}(x) &= {}^{\overline{\Psi}}D\left[\Theta^{(k)}_{n+1}\right] = {}^{\overline{\psi}_1}D^{\overline{\psi}_2}D \cdots {}^{\overline{\psi}_k}D\left[\Theta^{(k)}_{n+1}\right] = D^{\overline{\psi}_k} \cdots D^{\overline{\psi}_2} \cdot D^{\overline{\psi}_1}\left[\Theta^{(k)}_{n+1}\right] = \\ &= \frac{P_{\Psi}(x)}{|\mathbb{S}^n| 2^{k-1}(k-1)! |x|^{n+1}} \end{aligned}$$

where $P_{\Psi}(x)$ is an homogeneous polynomial of degree k. The following integral operators

Theorem 4 *Let $f \in L_1(\Omega, \mathbb{R}_{0,n})$. Then the integral ${}^{\Psi}T_{\Omega}[f](x)$ exists for every $x \in \mathbb{R}^{n+1}$.*

Theorem 5 *Let $f \in C(\overline{\Omega}, \mathbb{R}_{0,n})$. Then*

(i) ${}^{\psi_1}D^{\Psi}T_{\Omega}[f] = {}^{\tilde{\Psi}}T_{\Omega}[f]$ *where* $\tilde{\Psi} = \{\psi_2, \ldots, \psi_k\}$, $k \geq 2$.

(ii) ${}^{\Psi}D^{\Psi}T_{\Omega}[f](x) = f(x)$ *for every* $x \in \Omega$.

Theorem 6 [Higher order Borel-Pompeiu formula] *Let $f \in C^k(\overline{\Omega}, \mathbb{R}_{0,n})$. Then for $x \in \Omega$*

$$f(x) = \sum_{i=1}^{k} (-1)^{i-1} {}^{\Psi^i}K_{\Gamma} {}^{\Psi^{i-1}}D[f](x) + {}^{\Psi}T_{\Omega} {}^{\Psi}D[f](x)$$

where ${}^{\Psi^0}D[f] = f$.

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