# A Robustness Analysis of the Numerical Computation of Green's Dyadics in Bianisotropic Media

**I. Bogaert**

Ghent University, Belgium
Ignace.Bogaert@intec.UGent.be

**Abstract**— The robustness of a previously developed numerical algorithm for the computation of the four Green's dyadics in homogeneous bianisotropic materials is analyzed. The lossy bianisotropic materials are identified as a class of materials for which the numerical computation is provably robust. This is of practical importance because many bianisotropic materials are lossy.

The most general linear constitutive equations in classical frequency-domain electromagnetics are those of bianisotropic materials:

$$\boldsymbol{d}(\boldsymbol{r}) = \bar{\bar{\varepsilon}} \cdot \boldsymbol{e}(\boldsymbol{r}) + \bar{\bar{\xi}} \cdot \boldsymbol{h}(\boldsymbol{r}), \tag{1a}$$

$$\boldsymbol{b}(\boldsymbol{r}) = \bar{\bar{\zeta}} \cdot \boldsymbol{e}(\boldsymbol{r}) + \bar{\bar{\mu}} \cdot \boldsymbol{h}(\boldsymbol{r}). \tag{1b}$$

Therefore, piecewise homogeneous bianisotropic scatterers are the most general scatterers that can be handled with boundary integral equation methods. However, the absence of analytical expressions for the Green's dyadics poses a serious practical problem and has severely hindered the development of boundary integral equations for bianisotropic materials. It is worthwhile to point out that analytical expressions for Green's dyadics exist for many materials [1–3], but not for all of them.

To circumvent the need for analytical expressions, an all-numerical scheme to compute the Bianisotropic Scalar Green's Function (BSGF) was introduced in [4, 5]. The BSGF is defined as

$$G(\boldsymbol{r}) = \frac{1}{8\pi^3} \int_{\mathbb{R}^3} \frac{e^{j\boldsymbol{s}\cdot\boldsymbol{r}}}{D(\boldsymbol{s})} \mathrm{d}\boldsymbol{s}, \tag{2}$$

with $D(\boldsymbol{s})$ the so-called Helmholtz determinant [6]

$$D(\boldsymbol{s}) = \mathrm{Det}\left[\mathsf{P}(\boldsymbol{s})\right], \tag{3}$$

which is defined by means of the $6 \times 6$ matrix

$$\mathsf{P}(\boldsymbol{s}) = \begin{bmatrix} \bar{\bar{\varepsilon}} & \bar{\bar{\xi}} - \boldsymbol{s} \times \mathbb{1} \\ \bar{\bar{\zeta}} + \boldsymbol{s} \times \mathbb{1} & \bar{\bar{\mu}} \end{bmatrix}. \tag{4}$$

The symbol $\mathbb{1}$ denotes the $3 \times 3$ unit matrix, such that

$$\boldsymbol{s} \times \mathbb{1} = \begin{bmatrix} 0 & -s_z & s_y \\ s_z & 0 & -s_x \\ -s_y & s_x & 0 \end{bmatrix}. \tag{5}$$

It can be shown that the components of the Green's dyadics, which relate electric and magnetic currents with the electric and magnetic fields they generate, are linear combinations of at most fourth-order partial derivatives of this BSGF [4]. Hence, the numerical computation of the Green's dyadics can be accomplished if the BSGF and its derivatives (up to fourth order) can be numerically computed in a stable and efficient way.

It will now be shown that the integral in (2) can be computed in a robust manner if the bianisotropic material is lossy. A bianisotropic material is called lossy if

$$\Im\left\{\boldsymbol{V}^H \cdot \begin{bmatrix} \bar{\bar{\varepsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} & \bar{\bar{\mu}} \end{bmatrix} \cdot \boldsymbol{V}\right\} < 0, \forall \boldsymbol{V} \in \mathbb{C}^6 \setminus \{\boldsymbol{0}\}. \tag{6}$$

The proof hinges on two properties:

- a sufficient speed of convergence to zero of $\frac{1}{D(\boldsymbol{s})}$ for $||\boldsymbol{s}|| \to \infty$ in every direction of $\mathbb{R}^3$.
- the nonexistence of real zeros for the Helmholtz determinant $D(\boldsymbol{s})$,

It is clear that, if both properties are satisfied, an adaptive integration routine can be applied to (2), which is very robust. The first property will be proved in the final version of this paper. The second property is proved by assuming that there exists a real zero $\boldsymbol{s}_0$ of the Helmholtz determinant and showing that this results in a contradiction. If

$$D(\boldsymbol{s}_0) = 0, \tag{7}$$

then, by the definition of the Helmholtz determinant (3), there exists a complex vector $\boldsymbol{V}_0 \in \mathbb{C}^6$ such that

$$\begin{bmatrix} \bar{\bar{\varepsilon}} & \bar{\bar{\xi}} - \boldsymbol{s}_0 \times \mathbb{1} \\ \bar{\bar{\zeta}} + \boldsymbol{s}_0 \times \mathbb{1} & \bar{\bar{\mu}} \end{bmatrix} \cdot \boldsymbol{V}_0 = \boldsymbol{0}. \tag{8}$$

Taking the imaginary part of the inner product with $\boldsymbol{V}_0^H$ yields

$$\Im\left\{ \boldsymbol{V}_0^H \cdot \begin{bmatrix} \bar{\bar{\varepsilon}} & \bar{\bar{\xi}} - \boldsymbol{s}_0 \times \mathbb{1} \\ \bar{\bar{\zeta}} + \boldsymbol{s}_0 \times \mathbb{1} & \bar{\bar{\mu}} \end{bmatrix} \cdot \boldsymbol{V}_0 \right\} = 0. \tag{9}$$

However, if $\boldsymbol{s}_0$ is real, then

$$\Im\left\{ \boldsymbol{V}_0^H \cdot \begin{bmatrix} 0 & -\boldsymbol{s}_0 \times \mathbb{1} \\ \boldsymbol{s}_0 \times \mathbb{1} & 0 \end{bmatrix} \cdot \boldsymbol{V}_0 \right\} = 0, \tag{10}$$

regardless of what $\boldsymbol{V}_0$ may be, because the matrix in between the two vectors is hermitian. Therefore, (9) simplifies to

$$\Im\left\{ \boldsymbol{V}_0^H \cdot \begin{bmatrix} \bar{\bar{\varepsilon}} & \bar{\bar{\xi}} \\ \bar{\bar{\zeta}} & \bar{\bar{\mu}} \end{bmatrix} \cdot \boldsymbol{V}_0 \right\} = 0, \tag{11}$$

which contradicts the criterium for lossyness (6). This robustness result for (2) immediately extends to the approach in [4, 5].

To conclude, the numerical computation of the BSGF and its derivatives is robust for all lossy bianisotropic materials. Since all passive bianisotropic materials exhibit at least some loss at nonzero frequencies, this shows that the numerical computation strategy introduced in [4, 5] has wide applicability, e.g. for modeling circulators or other microwave components.

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## REFERENCES

1. F. Olyslager and I. V. Lindell, "A pedigree of bianisotropic media." in *Proceedings of the 8th International Conference on Electromagnetics of Complex Media (Bianisotropics 2000), 27-29 September 2000, Lisbon, Portugal*, 2000, pp. 153–158.
2. F. Olyslager and I. Lindell, "Electromagnetics and exotic media: a quest for the holy grail," *Antennas and Propagation Magazine, IEEE*, vol. 44, no. 2, pp. 48 –58, apr 2002.
3. I. Lindell and F. Olyslager, "Generalized decomposition of electromagnetic fields in bi-anisotropic media," *Antennas and Propagation, IEEE Transactions on*, vol. 46, no. 10, pp. 1584 –1585, oct 1998.
4. I. Bogaert, "Accurate Computation and Tabulation of the Scalar Green Function for'Bi-anisotropic Media and its Derivatives," in *Proceedings of the conference of the Applied Computational Electromagnetics Society*, Columbus, Ohio, USA, 10-14 April 2012.
5. ——, "Computing Green's Functions for Bianisotropic Materials," in *Proceedings of the International Conference on Electromagnetics in Advanced Applications*, Cape Town, South Africa, 2–7 September 2012.
6. B. Jakoby and F. Olyslager, "Asymptotic expansions for greens dyadics in bianisotropic media," *Progress on Electromagnetics Research*, vol. 12, pp. 277–302, 1996.