Translation and convolution in Clifford analysis

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Abstract In this paper we study an integral transform in Clifford analysis with a general kernel expressed as an infinite series in terms of Bessel functions and Gegenbauer polynomials. After giving some examples, we construct the inverse of this general transform on the Schwartz space. Moreover, we define a generalized translation operator and four types of generalized convolution associated to the general kernel.

1 Introduction

Recently, several generalizations of the classical Fourier transform (see e.g. [12]) and its fractional version (see e.g. [10]) to the setting of Clifford analysis (see e.g. [1, 8]) have been introduced by the authors. They are all examples of the general transform studied in this paper. For this general transform, the kernel of which is expressed as an infinite series in terms of Bessel functions and Gegenbauer polynomials, we first determine the inverse on a basis of the Schwartz space in Clifford analysis. Next we define a generalized translation operator in terms of the general transform and two types of generalized convolution in terms of the translation operator. However, taking into account the interaction of the convolution with the Fourier transform in the classical case, two other types of convolution are introduced. The examination of the properties of the translation operator and the four types of convolution is postponed to a subsequent publication.

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The paper is organized as follows. In Section 2 we repeat some basic knowledge of Clifford analysis. In Section 3 we introduce a general kernel, which takes the form of an infinite series in terms of Gegenbauer polynomials and Bessel functions. We briefly discuss some examples and calculate the action of the associated transform on a basis of the Clifford Schwartz space. In Section 4, we construct the inverse of the general transform on the Schwartz basis. Next, in Section 5, we define a generalized translation operator related to the general integral transform. Finally, in subsection 6.1, we introduce two types of convolution based on the translation operator. However, also another type of convolution structure, based on the interaction of the convolution with the Fourier transform in the classical case, seems natural and is hence introduced in subsection 6.2.

2 Preliminaries

The Clifford algebra $\mathscr{C}l_{0,m}$ over $\mathbb{R}^m$ is the algebra generated by e_i, $i = 1,\dots,m$, under the relations

$$e_i e_j + e_j e_i = 0, \quad i \neq j, \qquad e_i^2 = -1.$$

This algebra has dimension 2^m as a vector space over $\mathbb{R}$. It can be decomposed as $\mathscr{C}l_{0,m} = \oplus_{k=0}^m \mathscr{C}l_{0,m}^k$ with $\mathscr{C}l_{0,m}^k$ the space of k-vectors defined by

$$\mathscr{C}l_{0,m}^k := \operatorname{span}\{e_{i_1} \dots e_{i_k}, i_1 < \dots < i_k\}.$$

In the sequel, we will always consider functions f taking values in $\mathscr{C}l_{0,m}$, unless explicitly mentioned. Such functions can be decomposed as

$$f = f_0 + \sum_{i=1}^m e_i f_i + \sum_{i<j} e_i e_j f_{ij} + \dots + e_1 \dots e_m f_{1\dots m}$$

with $f_0, f_i, f_{ij}, \dots, f_{1\dots m}$ all real-valued functions on $\mathbb{R}^m$.

The Dirac operator is given by $\partial_{\underline{x}} = \sum_{j=1}^m \partial_{x_j} e_j$ and the vector variable by $\underline{x} = \sum_{j=1}^m x_j e_j$. The square of the Dirac operator equals, up to a minus sign, the Laplace operator in $\mathbb{R}^m$: $\partial_{\underline{x}}^2 = -\Delta$.

We further introduce the so-called Gamma operator (see e.g. [8])

$$\Gamma_{\underline{x}} := -\sum_{j<k} e_j e_k (x_j \partial_{x_k} - x_k \partial_{x_j}).$$

Note that $\Gamma_{\underline{x}}$ commutes with radial functions, i.e. $[\Gamma_{\underline{x}}, f(|\underline{x}|)] = 0$.

Denote by $\mathscr{P}$ the space of polynomials taking values in $\mathscr{C}l_{0,m}$, i.e.

$$\mathscr{P} := \mathbb{R}[x_1, \dots, x_m] \otimes \mathscr{C}l_{0,m}.$$

The space of homogeneous polynomials of degree k is then denoted by $\mathscr{P}_k$. The space $\mathscr{M}_k := \ker \partial_{\underline{x}} \cap \mathscr{P}_k$ is called the space of spherical monogenics of degree k. Similarly, $\mathscr{H}_k := \ker \Delta \cap \mathscr{P}_k$ is the space of spherical harmonics of degree k.

Next we define the inner product and the wedge product of two vectors $\underline{x}$ and $\underline{y}$

$$\langle \underline{x}, \underline{y} \rangle := \sum_{j=1}^{m} x_j y_j \qquad \underline{x} \wedge \underline{y} := \sum_{j<k} e_j e_k (x_j y_k - x_k y_j).$$

We introduce a basis $\{\psi_{j,k,\ell}\}$ for the space $\mathscr{S}(\mathbb{R}^m) \otimes \mathscr{C}l_{0,m}$, where $\mathscr{S}(\mathbb{R}^m)$ denotes the Schwartz space. Define the functions $\psi_{j,k,\ell}(\underline{x})$ by

$$\begin{aligned} \psi_{2j,k,\ell}(\underline{x}) &:= L_j^{\frac{m}{2}+k-1}(|\underline{x}|^2) M_k^{(\ell)} e^{-|\underline{x}|^2/2}, \\ \psi_{2j+1,k,\ell}(\underline{x}) &:= L_j^{\frac{m}{2}+k}(|\underline{x}|^2) \underline{x} M_k^{(\ell)} e^{-|\underline{x}|^2/2}, \end{aligned} \tag{1}$$

where $j,k \in \mathbb{N}$, $\{M_k^{(\ell)} \in \mathscr{M}_k : \ell = 1,\dots,\dim \mathscr{M}_k\}$ is a basis for $\mathscr{M}_k$, and L_j^α are the Laguerre polynomials. The set $\{\psi_{j,k,\ell}\}$ forms a basis of $\mathscr{S}(\mathbb{R}^m) \otimes \mathscr{C}l_{0,m}$, see [11].

3 Generalized Fourier transforms: examples and eigenvalues

In this section we consider a general kernel of the following form

$$K(\underline{x}, \underline{y}) = \big(A(w, \widetilde{z}) + (\underline{x} \wedge \underline{y})\, B(w, \widetilde{z})\big)\ e^{\frac{i}{2}(\cot \alpha)(|\underline{x}|^2 + |\underline{y}|^2)} \tag{2}$$

with

$$\begin{aligned} A(w, \widetilde{z}) &= \sum_{k=0}^{+\infty} \alpha_k\ (\widetilde{z})^{-\lambda} J_{k+\lambda}(\widetilde{z}) C_k^\lambda(w) \\ B(w, \widetilde{z}) &= \sum_{k=1}^{+\infty} \beta_k\ (\widetilde{z})^{-\lambda-1} J_{k+\lambda}(\widetilde{z}) C_{k-1}^{\lambda+1}(w) \end{aligned}$$

and $\alpha_k, \beta_k \in \mathbb{C}$, $\widetilde{z} = (|\underline{x}||\underline{y}|)/\sin\alpha$, $w = \langle \underline{\xi}, \underline{\eta} \rangle$ ($\underline{x} = |\underline{x}|\underline{\xi}$, $\underline{y} = |\underline{y}|\underline{\eta}$, $\underline{\xi}, \underline{\eta} \in S^{m-1}$), $\lambda = (m-2)/2$, $\alpha \in [-\pi, \pi]$. Here, J_ν is the Bessel function and C_k^λ the Gegenbauer polynomial. We exclude the case where $\alpha = 0$ or $\alpha = \pm\pi$.

The integral transform associated with this kernel is defined by

$$\mathscr{F}[f](\underline{y}) = \frac{1}{(\pi(1 - e^{-2i\alpha}))^{m/2}} \int_{\mathbb{R}^m} K(\underline{x}, \underline{y})\ f(\underline{x})\ dx \tag{3}$$

with dx the standard Lebesgue measure on $\mathbb{R}^m$.

3.1 Examples

3.1.1 The case $\alpha = \frac{\pi}{2}$: the class of Clifford-Fourier transforms

In the special case where $\alpha = \frac{\pi}{2}$, the kernel takes the form

$$K(\underline{x},\underline{y}) = A(w,z) + (\underline{x} \wedge \underline{y})\, B(w,z) \tag{4}$$

with

$$A(w,z) = \sum_{k=0}^{+\infty} \alpha_k \, z^{-\lambda} J_{k+\lambda}(z) C_k^{\lambda}(w)$$
$$B(w,z) = \sum_{k=1}^{+\infty} \beta_k \, z^{-\lambda-1} J_{k+\lambda}(z) C_{k-1}^{\lambda+1}(w)$$

and $z = |\underline{x}||\underline{y}|$, $w = \langle \underline{\xi}, \underline{\eta} \rangle$ ($\underline{x} = |\underline{x}|\underline{\xi}$, $\underline{y} = |\underline{y}|\underline{\eta}$, $\underline{\xi}, \underline{\eta} \in S^{m-1}$), $\lambda = (m-2)/2$. The corresponding integral transform is given by

$$\mathscr{F}[f](\underline{y}) = \frac{1}{(2\pi)^{m/2}} \int_{\mathbb{R}^m} K(\underline{x},\underline{y})\; f(\underline{x})\; dx.$$

Note that the classical Fourier transform, which can for example be expressed as the operator exponential $\mathscr{F}_{cl} = e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta - |\underline{x}|^2)}$, takes the form (4) with

$$\alpha_k = 2^{\lambda} \Gamma(\lambda)(k+\lambda)(-i)^k \qquad \text{and} \qquad \beta_k = 0.$$

Also the Clifford-Fourier transform (see [2, 3, 4, 7]), a generalization of the classical Fourier transform in the framework of Clifford analysis, takes this form. It is defined by the following exponential operator $\mathscr{F}_{\pm} := e^{\frac{i\pi m}{4}} e^{\frac{i\pi}{4}(\Delta - |\underline{x}|^2 \mp 2\Gamma)}$. In case of the Clifford-Fourier transform $\mathscr{F}_{-}$, the coefficients α_k and β_k take the form:

$$\alpha_k = 2^{\lambda-1}\Gamma(\lambda+1)(i^{2\lambda+2} + (-1)^k) - 2^{\lambda-1}\Gamma(\lambda)\;(k+\lambda)(i^{2\lambda+2} - (-1)^k)$$
$$\beta_k = -2^{\lambda}\Gamma(\lambda+1)(i^{2\lambda+2} + (-1)^k).$$

In case of the transform $\mathscr{F}_{+}$, similar expressions hold.

Moreover, in [5] we have determined a whole class of kernels of the form (4) yielding new integral transforms that have the same calculus properties as the original Clifford-Fourier transform, but with different spectrum.

3.1.2 The fractional Clifford-Fourier transform

The fractional Fourier transform is a generalization of the classical Fourier transform. It is usually defined using the operator expression

$$\mathscr{F}_\alpha = e^{\frac{i\alpha m}{2}} e^{\frac{i\alpha}{2}(\Delta - |\underline{x}|^2)}, \qquad \alpha \in [-\pi, \pi].$$

Recently, we have introduced a fractional version of the Clifford-Fourier transform (see [6]). It is defined by the following exponential operator

$$\mathscr{F}_{\alpha,\beta} = e^{\frac{i\alpha m}{2}} e^{i\beta\Gamma} e^{\frac{i\alpha}{2}(\Delta - |\underline{x}|^2)}, \qquad \alpha, \beta \in [-\pi, \pi].$$

The integral kernel of this transform takes the form (2) with

$$\alpha_k = -2^{\lambda-1}\Gamma(\lambda+1) i^{-k}(e^{i\beta(k+2\lambda)} - e^{-i\beta k}) + 2^{\lambda-1}\Gamma(\lambda)(k+\lambda) i^{-k}(e^{i\beta(k+2\lambda)} + e^{-i\beta k})$$
$$\beta_k = \frac{2^\lambda \Gamma(\lambda+1)}{\sin\alpha} i^{-k}(e^{i\beta(k+2\lambda)} - e^{-i\beta k}).$$

3.2 Eigenvalues

Now we calculate the action of the transform (3) on the basis (1) of $\mathscr{S}(\mathbb{R}^m)\otimes\mathscr{C}l_{0,m}$. In a similar manner as in Theorem 6.4 of [7] we can calculate the radial behavior of the integral transform, which in its turn leads to the following result.

Theorem 1. *One has, putting $\beta_0 = 0$,*

$$\mathscr{F}[\psi_{2j,k,\ell}](\underline{y}) = \frac{2^{-\lambda}}{\Gamma(\lambda+1)} \left(\frac{\lambda}{\lambda+k}\alpha_k - \sin\alpha\, \frac{k}{2(\lambda+k)}\beta_k\right) i^k e^{-i\alpha(k+2j)}\ \psi_{2j,k,\ell}(\underline{y})$$
$$\mathscr{F}[\psi_{2j+1,k,\ell}](\underline{y}) = \frac{2^{-\lambda}}{\Gamma(\lambda+1)} \left(\frac{\lambda}{\lambda+k+1}\alpha_{k+1} + \sin\alpha\, \frac{k+1+2\lambda}{2(\lambda+k+1)}\beta_{k+1}\right) i^{k+1}$$
$$\times\, e^{-i\alpha(k+2j+1)}\ \psi_{2j+1,k,\ell}(\underline{y}).$$

4 Inverse transform

In order to construct the inverse of the general transform $\mathscr{F}$ on the basis $\{\psi_{j,k,\ell}\}$ we consider the following integral transform:

$$\mathscr{F}^*[f](\underline{y}) = \frac{1}{(\pi(1-e^{2i\alpha}))^{m/2}} \int_{\mathbb{R}^m} K^*(\underline{x},\underline{y})\ f(\underline{x})\ dx,$$

where the kernel is given by

$$K^*(\underline{x},\underline{y}) = \left(A^*(w,\widetilde{z}) + (\underline{x}\wedge\underline{y})\ B^*(w,\widetilde{z})\right)\ e^{-\frac{i}{2}(\cot\alpha)(|\underline{x}|^2+|\underline{y}|^2)}$$

with

$$A^*(w,\widetilde{z}) = \sum_{k=0}^{+\infty} (-1)^k \gamma_k \, (\widetilde{z})^{-\lambda} J_{k+\lambda}(\widetilde{z}) C_k^{\lambda}(w)$$
$$B^*(w,\widetilde{z}) = \sum_{k=1}^{+\infty} (-1)^{k+1} \, \delta_k \, (\widetilde{z})^{-\lambda-1} J_{k+\lambda}(\widetilde{z}) C_{k-1}^{\lambda+1}(w)$$

and $\gamma_k, \delta_k \in \mathbb{C}$, $\widetilde{z} = \frac{|\underline{x}| \ |\underline{y}|}{\sin\alpha}$, $w = \langle \underline{\xi}, \underline{\eta} \rangle$ ($\underline{x} = |\underline{x}|\underline{\xi}$, $\underline{y} = |\underline{y}|\underline{\eta}$, $\underline{\xi}, \underline{\eta} \in S^{m-1}$), $\lambda = (m-2)/2$.

Similarly as for the transform $\mathscr{F}$, we can consecutively calculate the radial behavior of the transform $\mathscr{F}^*$ and determine its action on the basis $\{\psi_{j,k,\ell}\}$.

Theorem 2. *One has, putting $\delta_0 = 0$,*

$$\mathscr{F}^*[\psi_{2j,k,\ell}](\underline{y}) = \frac{2^{-\lambda}}{\Gamma(\lambda+1)} \left(\frac{\lambda}{\lambda+k} \gamma_k + \sin\alpha \, \frac{k}{2(\lambda+k)} \delta_k \right) i^k e^{i\alpha(k+2j)} \ \psi_{2j,k,\ell}(\underline{y})$$
$$\mathscr{F}^*[\psi_{2j+1,k,\ell}](\underline{y}) = \frac{2^{-\lambda}}{\Gamma(\lambda+1)} \left(\frac{\lambda}{\lambda+k+1} \gamma_{k+1} - \sin\alpha \, \frac{k+1+2\lambda}{2(\lambda+k+1)} \delta_{k+1} \right) i^{k+1} \times e^{i\alpha(k+2j+1)} \ \psi_{2j+1,k,\ell}(\underline{y}).$$

Combining Theorem 1 and 2, we are now able to construct the inverse of the general transform $\mathscr{F}$ on the basis $\{\psi_{j,k,\ell}\}$.

Theorem 3. *The inverse of $\mathscr{F}$ on the basis $\{\psi_{j,k,\ell}\}$ is given by*

$$\mathscr{F}^{-1}[f](\underline{y}) = \frac{1}{(\pi(1-e^{2i\alpha}))^{m/2}} \int_{\mathbb{R}^m} \widetilde{K(\underline{x},\underline{y})} \ f(\underline{x}) \ dx,$$

with

$$\widetilde{K(\underline{x},\underline{y})} = \left(\widetilde{A(w,\widetilde{z})} + (\underline{x} \wedge \underline{y}) \ \widetilde{B(w,\widetilde{z})} \right) \ e^{-\frac{i}{2}(\cot\alpha)(|\underline{x}|^2+|\underline{y}|^2)}$$

given by

$$\widetilde{A(w,\widetilde{z})} = \sum_{k=0}^{+\infty} \frac{1}{N} (\alpha_k + \beta_k \ \sin\alpha) \ (\widetilde{z})^{-\lambda} J_{k+\lambda}(\widetilde{z}) C_k^{\lambda}(w)$$
$$\widetilde{B(w,\widetilde{z})} = -\sum_{k=1}^{+\infty} \frac{1}{N} \beta_k \ (\widetilde{z})^{-\lambda-1} J_{k+\lambda}(\widetilde{z}) C_{k-1}^{\lambda+1}(w),$$

where

$$N = \frac{1}{2^{2\lambda}(\Gamma(\lambda+1))^2} \left(\frac{\lambda}{\lambda+k} \alpha_k - \sin\alpha \ \frac{k}{2(\lambda+k)} \beta_k \right) \left(\frac{\lambda}{\lambda+k} \alpha_k + \sin\alpha \frac{k+2\lambda}{2(\lambda+k)} \beta_k \right).$$

Proof. Put $\widetilde{K(\underline{x},\underline{y})} = \left(\widetilde{A(w,\widetilde{z})} + (\underline{x} \wedge \underline{y}) \ \widetilde{B(w,\widetilde{z})} \right) e^{-\frac{i}{2}(\cot\alpha)(|\underline{x}|^2+|\underline{y}|^2)}$ where

$$\widetilde{A(w,\widetilde{z})} = \sum_{k=0}^{+\infty} (-1)^k \gamma_k \, (\widetilde{z})^{-\lambda} J_{k+\lambda}(\widetilde{z}) C_k^{\lambda}(w)$$
$$\widetilde{B(w,\widetilde{z})} = \sum_{k=1}^{+\infty} (-1)^{k+1} \delta_k \, (\widetilde{z})^{-\lambda-1} J_{k+\lambda}(\widetilde{z}) C_{k-1}^{\lambda+1}(w)$$

and with $\gamma_k, \delta_k \in \mathbb{C}$. We need to have that

$$\mathscr{F}^{-1}\left[\mathscr{F}[f]\right] = \mathscr{F}\left[\mathscr{F}^{-1}[f]\right] = f.$$

Using Theorem 1 and 2, this condition is equivalent with the system of equations $(k = 0, 1, \ldots)$

$$\left(\frac{\lambda}{\lambda+k}\alpha_k - \sin\alpha \frac{k}{2(\lambda+k)}\beta_k\right)\left(\frac{\lambda}{\lambda+k}\gamma_k + \sin\alpha \frac{k}{2(\lambda+k)}\delta_k\right) = (-1)^k(\Gamma(\lambda+1))^2 2^{2\lambda}$$
$$\left(\frac{\lambda}{\lambda+k}\alpha_k + \sin\alpha \frac{k+2\lambda}{2(\lambda+k)}\beta_k\right)\left(\frac{\lambda}{\lambda+k}\gamma_k - \sin\alpha \frac{k+2\lambda}{2(\lambda+k)}\delta_k\right) = (-1)^k(\Gamma(\lambda+1))^2 2^{2\lambda}.$$

Solving this system for γ_k and δ_k then yields the statement of the theorem. $\square$

5 Generalized translation operator

The classical convolution $f *_{cl} g$ plays a fundamental role in classical Fourier analysis. It is defined by

$$(f *_{cl} g)(\underline{x}) = \int_{\mathbb{R}^m} f(\underline{y}) g(\underline{x}-\underline{y}) dy = \int_{\mathbb{R}^m} f(\underline{x}-\underline{y}) g(\underline{y}) dy$$

and it depends on the translation operator $\tau_{\underline{y}} : f \to f(. - \underline{y})$. Under the classical Fourier transform $\mathscr{F}_{cl}$, $\tau_{\underline{y}}$ satisfies

$$\mathscr{F}_{cl}\left[\tau_{\underline{y}} f\right](\underline{x}) = e^{-i\langle \underline{x},\underline{y}\rangle} \mathscr{F}_{cl}[f](\underline{x}), \qquad \underline{x} \in \mathbb{R}^m.$$

We now define a generalized translation operator related to the integral transform $\mathscr{F}$ defined in Section 3.

Definition 1. Let $f \in \mathscr{S}(\mathbb{R}^m) \otimes \mathscr{C}l_{0,m}$. For $\underline{y} \in \mathbb{R}^m$ the generalized translation operator $f \longmapsto \tau_{\underline{y}} f$ is defined by

$$\mathscr{F}[\tau_{\underline{y}} f](\underline{x}) = K(\underline{y},\underline{x}) \, \mathscr{F}[f](\underline{x}), \qquad \underline{x} \in \mathbb{R}^m.$$

It can be expressed, by the inverse of $\mathscr{F}$ (see Section 4), as an integral operator

$$\tau_{\underline{y}} f(\underline{x}) = \frac{1}{(\pi(1-e^{2i\alpha}))^{m/2}} \int_{\mathbb{R}^m} \widetilde{K(\underline{\xi},\underline{x})} \, K(\underline{y},\underline{\xi}) \, \mathscr{F}[f](\underline{\xi}) \, d\xi. \qquad (5)$$

6 Four types of generalized convolution

6.1 Definitions using the generalized translation operator

Using the generalized translation of the previous section, we can define two types of convolution for functions with values in the Clifford algebra.

Definition 2. For $f, g \in \mathscr{S}(\mathbb{R}^m) \otimes \mathscr{C}l_{0,m}$, the generalized convolution $f *_L g$, resp. $f *_R g$, is defined by

$$(f *_L g)(\underline{x}) := \int_{\mathbb{R}^m} \tau_{\underline{y}} f(\underline{x})\, g(\underline{y})\, dy, \qquad \underline{x} \in \mathbb{R}^m,$$

resp.

$$(f *_R g)(\underline{x}) := \int_{\mathbb{R}^m} f(\underline{y})\, \tau_{\underline{y}} g(\underline{x})\, dy, \qquad \underline{x} \in \mathbb{R}^m.$$

Using the integral expression for the generalized translation operator (see (5)):

$$\begin{aligned}\tau_{\underline{y}} f(\underline{x}) = &\left(\pi(1-e^{2i\alpha})\right)^{-m/2} \left(\pi(1-e^{-2i\alpha})\right)^{-m/2} \\ &\times \int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \widetilde{K(\underline{u},\underline{x})} K(\underline{y},\underline{u}) K(\underline{t},\underline{u}) f(\underline{t})\, dt\, du,\end{aligned}$$

we obtain the explicit formulae for the generalized convolutions introduced above.

Proposition 1. *The generalized convolutions $f *_L g$ and $f *_R g$ take the following explicit form:*

$$(f *_L g)(\underline{x}) = c_{\alpha,m} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \widetilde{K(\underline{u},\underline{x})} K(\underline{y},\underline{u}) K(\underline{t},\underline{u}) f(\underline{t}) g(\underline{y})\, dt\, du\, dy$$

and

$$(f *_R g)(\underline{x}) = c_{\alpha,m} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} f(\underline{t}) \widetilde{K(\underline{u},\underline{x})} K(\underline{t},\underline{u}) K(\underline{y},\underline{u}) g(\underline{y})\, dy\, du\, dt.$$

with

$$c_{\alpha,m} := \left(\pi(1-e^{2i\alpha})\right)^{-m/2} \left(\pi(1-e^{-2i\alpha})\right)^{-m/2} = (2\pi)^{-m} |\sin\alpha|^{-m}.$$

6.2 Definitions based on an idea of Mustard

The definitions of two other types of convolutions are based on the observation that in the classical case the following interaction between the convolution and the Fourier transform holds:

$$\mathscr{F}_{cl}[f *_{cl} g] = (2\pi)^{m/2} \mathscr{F}_{cl}[f]\, \mathscr{F}_{cl}[g].$$

A similar idea was used by Mustard in [9] for the fractional Fourier transform.

Definition 3. For $f,g \in \mathscr{S}(\mathbb{R}^m) \otimes \mathscr{C}l_{0,m}$, the generalized convolution $f *_{C,L} g$ is defined by

$$(f *_{C,L} g)(\underline{x}) := \left(\pi(1-e^{-2i\alpha})\right)^{m/2} \mathscr{F}^{-1}\left[\mathscr{F}[f]\ \mathscr{F}[g]\right](\underline{x}), \qquad \underline{x} \in \mathbb{R}^m.$$

Similarly, the generalized convolution $f *_{C,R} g$ takes the form

$$(f *_{C,R} g)(\underline{x}) := \left(\pi(1-e^{-2i\alpha})\right)^{m/2} \mathscr{F}^{-1}\left[\mathscr{F}[g]\ \mathscr{F}[f]\right](\underline{x}), \qquad \underline{x} \in \mathbb{R}^m.$$

Taking into account the integral expression for $\mathscr{F}$ and its inverse $\mathscr{F}^{-1}$ we obtain the following explicit formulae.

Proposition 2. *The generalized convolutions $f *_{C,L} g$ and $f *_{C,R} g$ take the following explicit form:*

$$(f *_{C,L} g)(\underline{x}) = c_{\alpha,m} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \widetilde{K(\underline{u},\underline{x})} K(\underline{t},\underline{u}) f(\underline{t}) K(\underline{y},\underline{u}) g(\underline{y}) dt dy du$$

and

$$(f *_{C,R} g)(\underline{x}) = c_{\alpha,m} \int_{\mathbb{R}^m}\int_{\mathbb{R}^m}\int_{\mathbb{R}^m} \widetilde{K(\underline{u},\underline{x})} K(\underline{y},\underline{u}) g(\underline{y}) K(\underline{t},\underline{u}) f(\underline{t}) dt du dy.$$

A thorough study of the connection between and the properties of these four types of convolution is a topic of current research.

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