Branching processes, the max-plus algebra and network calculus

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Abstract. Branching processes can describe the dynamics of various queueing systems, peer-to-peer systems, delay tolerant networks, etc. In this paper we study the basic stochastic recursion of multitype branching processes, but in two non-standard contexts. First, we consider this recursion in the max-plus algebra where branching corresponds to finding the maximal offspring of the current generation. Secondly, we consider network-calculus-type deterministic bounds as introduced by Cruz, which we extend to handle branching-type processes. The paper provides both qualitative and quantitative results and introduces various applications of (max-plus) branching processes in queueing theory.

Keywords: Stochastic recursive equations, Branching processes, Max-plus algebra, Network calculus.

1 Introduction

Branching processes model the dynamics of populations over successive generations, each member of some generation independently producing offspring in the next generation in accordance with a given probability distribution. Originating from the nobility and family extinction problem, branching process theory has been applied in diverse fields including computer science and networking.

Branching processes have been frequently identified in queueing theory and the connection between branching processes and queueing theory is well established. Already in 1942, Borel showed that the numbers of customers in a busy period of an M/G/1 queue corresponds to the number of generations till extinction of a Galton-Watson branching process [8, 19]. Similarly, Crump-Mode-Jagers branching processes describe the dynamics of processor sharing queues [16], whereas multitype branching processes with immigration have been used to study retrial queues [17] and polling systems [23].

Applications of branching processes in networking are not limited to queueing theory. In [22], a multitype branching process is studied to determine the maximum stable throughput of tree algorithms with free access. Stability of the tree algorithm corresponds to criticality of the associated multitype Galton-Watson branching process. Multitype branching processes with migration can also capture the dynamics of distributing a file into a delay tolerant network with the single-hop forwarding paradigm [11]. In addition, peer-to-peer networks also provide many interesting applications of branching processes [12, 13, 21]. For example, Leskala et Al. [21] study interacting branching processes in the context of file sharing networks.

In this paper we study the basic stochastic recursion of multitype branching processes, but in two non-standard contexts. The first part of the paper introduces problems that yield similar recursions but where summation and multiplication are understood as being in another possible algebra than the standard one. In particular, we define and characterize branching processes in the max-plus algebra, for both discrete and continuous state spaces. As shown further, we obtain general characterizations of the stationary behaviour of the single-type branching process with independent migration in the max-plus algebra. In addition, multitype branching processes in the max-plus algebra are introduced and sufficient conditions are proven for stability of these processes in the presence of stationary ergodic migration.

We then introduce a deterministic framework for studying branching processes. A deterministic view on branching allows for focussing on worst-case behaviour rather then average behaviour, as advocated by literature on network calculus. When designing a network so as to meet strict bounds on quality of service, then standard probabilistic descriptions of input and output processes are no longer relevant; one has to come up with a design adapted to the worst case of the input process. Much attention has been given to producing a network calculus in which each network element has a transfer function: it maps a given description of its input process to a similar description of the output process. A complete mapping of this type allows one to dimension buffers in the network that guarantee that there are no losses [10, 20]. A more complex situation arises when the network includes feedback. The outputs are no longer functions of exogenous traffic. Computing tight bounds for feedback systems proves to be much harder. A well known example of such a system is given in part ii of [10]; the bound given there is indeed not tight and has later been improved. Other examples of explicit or implicit feedback for which obtaining tight bounds is not a simple task are presented in [2, 3] in the context of polling systems.

The deterministic framework for branching processes closely relates to arrival processes in network calculus. It concerns processes that are shaped at the entrance of the network using RED (Random Early Discard) or leaky bucket mechanisms. These processes are characterized by a bound ρ on the average rate as well as some bounded σ on the burstiness. More precisely, the output process is "$\sigma - \rho$ constrained", i.e. for any interval $[s, t)$, the output process from such a buffer is bounded by $\rho(t - s) + \sigma$. R. L. Cruz showed in [10] that standard network elements preserve this type of bound. Moreover, they imply uniform bounds on the amount of workload in the network, which allows one to dimension buffer sizes so as to guarantee no overflow as long as the input processes are $\sigma - \rho$ constrained. In this paper we study some form of feedback in which the arrival process itself depends on the output process. We show that if this dependence can be described using $\sigma - \rho$ type bounds, one still obtains uniform bounds on the workload in the system. We show that the feedback mechanism is of the same type that is used to define branching processes. Our results thus provide a motivation for investigating a deterministic type of branching processes. Finally, as for ordinary network calculus, we relax the deterministic bounds by introducing elements of stochastic network calculus [18].

2 Branching in the standard algebra

We start by recalling some basic characterization of branching in the standard algebra. The standard branching process is defined as follows. Let Y_n be the number of individuals in generation n. Starting with a fixed Y_0, we define recursively

$$Y_{n+1} = \sum_{i=1}^{Y_n} \xi_n^{(i)}$$

where $\xi_n^{(i)}$ are independent and identically distributed (i.i.d.) random variables taking non-negative integer values. Define $A_n(m) := \sum_{i=1}^{m} \xi_n^{(i)}$, we can then rewrite the above as

$$Y_{n+1} = A_n(Y_n)\,. \tag{1}$$

Given the definition of the branching process above, branching processes with immigration are defined through the recursion,

$$Y_{n+1} = A_n(Y_n) + B_n\,, \tag{2}$$

B_n being referred to as the immigration component.

Next we recall the definition of branching process on a continuous state space. We note that A_n is nonnegative and has a divisibility property: for any nonnegative integers m, m_1 and m_2 such that $m_1 + m_2 = m$, and for any n,

$$A_n(m) = A_n^{(1)}(m_1) + A_n^{(2)}(m_2)$$

where for each n, $A_n^{(1)}$ and $A_n^{(2)}$ are independent random processes, both with the same distribution as A_n. We take this property, together with the nonnegativity of A_n as the basis to define the continuous state branching processes. Noting that these properties are satisfied by Lévy processes, we define a continuous state branching process as one satisfying (1) where A_n is a nonnegative Lévy process (a subordinator).

Example 1. Consider an $M/G/1$ queue with gated repeated vacations: the arrivals are modeled by a Poisson process and the service and vacation times constitute sequences of i.i.d. random variables. Each time the server returns from vacation, it closes a gate, and the next busy period starts. The busy period consists of the service times requested by all those that are present when the gate is closed. When the busy period ends then the server leaves on vacation. The period that consists of a busy period followed by a vacation is called a cycle. Let Y_n be the number of customers present at the beginning of the nth cycle. Let $\xi_n^{(i)}$ denote the number of customer arrivals during the service time of the ith customer among those present in the queue at the beginning of cycle n and let B_n be the number of arrivals during the nth vacation. With these definitions it easily follows that Y_n satisfies (2).

Example 2. Consider the model of the previous example and Let C_n be the nth cycle time. Moreover, let $A_n(C_n)$ denote the workload that arrives during C_n (i.e. the time to serve all those that arrive during the nth cycle time) and B_n denote the length of the $n+1$st vacation. Then again (2) holds (thereby replacing Y_n by C_n).

3 Branching in the max-plus algebra

In the max-plus algebra, summation corresponds to max, and multiplication to summation. Hence, we define the (single-type) branching process in the max-plus as follows,

$$Y_{n+1} = \bigoplus_{i=1}^{Y_n} \xi_n^{(i)} ,$$

where $\bigoplus$ stands for maximization and where $\{\xi_n^{(i)}\}$ constitutes a doubly indexed sequence of i.i.d. random variables taking non-negative integer values. Thus (1) still holds but this time with,

$$A_n(m) := \bigoplus_{i=1}^{m} \xi_n^{(i)} . \tag{3}$$

When considering immigration we shall consider two forms. The first form is,

$$Y_{n+1} = A_n(Y_n) \otimes B_n , \tag{4}$$

with $\otimes$ denoting summation in the standard algebra, such that the former expression can be written in the standard algebra as,

$$\max_{i=1,\dots,Y_n} \xi_n^{(i)} + B_n .$$

Notice that we here replaced only the branching part by its max-plus version. The second form of immigration we consider is,

$$Y_{n+1} = A_n(Y_n) \oplus B_n , \tag{5}$$

which can be written in the standard algebra as,

$$Y_{n+1} = \max\left(\max_{i=1}^{Y_n} \xi_n^{(i)}, B_n \right) .$$

To define continuous branching in the max-plus algebra, we relate max-plus branching with a continuous state space to Lévy processes, just as is done for ordinary branching. The max-plus equivalent of branching in continuous state space is defined as the maximum step of a (non-decreasing) Lévy process $L_n(t)$ over an interval of length y,

$$A_n(y) = \sup_{0 \le t < y} dL_n(t) \,. \tag{6}$$

Notice that the divisibility of the branching process now holds in the max-plus algebra. For any non-negative real values y, y_1 and y_2 such that $y_1 + y_2 = y$, and for any n, we now have,

$$A_n(y) = A_n^{(1)}(y_1) \oplus A_n^{(2)}(y_2) \,,$$

A_n being defined in either (3) or (6) (in the former case, y_1 and y_2 are positive integers).

We now consider some queueing systems whose dynamics can be described by the max-plus branching processes introduced above.

Example 3. Consider a discrete-time infinite-server queue with exactly one arrival at each time slot. We consider gated service and general vacations: when the nth vacation ends, a gate is closed and the $n+1st$ busy period starts. Let Y_n denote the number of customers present when the nth busy period starts. All customers that are present are served in parallel, their service times being i.i.d. and the next vacation starts when all services are completed. Let $\xi_n^{(i)}$ be the service time of the ith customer served during the nth busy period and let B_n denote the length of the nth vacation. As there is a single arrival in each slot, the sequence Y_n satisfies (4) with A_n as defined in (3).

Example 4. Consider the same model as in previous example and let $\widehat{Y}_n$ be the number of customers at the end of the nth busy period. Retaining the notation introduced in example 3, $\widehat{Y}_n$ relates to Y_n as

$$\widehat{Y}_n = A_n(Y_n) = Y_{n+1} - B_n$$

such that,

$$\widehat{Y}_{n+1} = A_{n+1}(\widehat{Y}_n) \oplus \widehat{A}_{n+1}(B_n) \,,$$

by the divisibility of the max-plus branching process. Here $\widehat{A}_n$ is an independent copy of A_n such that $\widehat{A}_{n+1}(B_n)$ represents the maximal service time of a customer that arrives during the nth vacation. This is a branching process in the max-plus algebra of the same type as (5), the migration process being $Q_n \doteq \widehat{A}_{n+1}(B_n)$.

Example 5. We consider the same setting of the previous examples except for the arrival process. The ith arrival occurs at time τ^i and brings a service requirement of $\xi^{(i)}$ which need not be integer valued. The arriving workload is then given by

$$V(t) = \sum_i \xi^{(i)} 1\{0 \le \tau^i \le t\} \,.$$

If the arrival process is Poisson, and the service times are i.i.d. and independent of the arrival times, then $V(t)$ is a non-decreasing Lévy process. The independent increments property of Lévy processes allows us to introduce a sequence of i.i.d. Lévy processes $V_n(t)$, distributed as $V(t)$ which denote the arriving workload during the nth cycle. As before, the $n+1$st busy period is the maximum service time of all those that arrived during the nth cycle,

$$A_n(C_n) := \sup_{0\le t\le C_n} dV_n(t)\,.$$

Hence A_n is a max-branching process, see (6). As the $n+1$st cycle time equals the sum of the largest service time of a customer that arrived during the preceding cycle and the vacation length B_n, consecutive cycle times relate as in (4), B_n being the length of the nth vacation as before.

4 Solution

4.1 Discrete state space

We first consider max-plus branching with a discrete state space. For a discrete random variable r, its distribution function and probability mass function are denoted by $F_r(i) = \Pr[r \le i]$ and $p_r(i) = \Pr[r = i]$, respectively, whereas $r^*(z) = \mathrm{E}[z^r]$ denotes its probability generating function.

We first solve (5). In this case, $Y_{n+1} \le i$ if $B_n \le i$ as well as all $\xi_n^{(j)}$ for $j = 1, \dots, Y_n$; see (3). Hence, we have,

$$\Pr(Y_{n+1} \le i) = \mathrm{E}[\Pr(Y_{n+1} \le i | Y_n, B_n)] = \mathrm{E}([F_\xi(i)]^{Y_n} 1\{B_n \le i\})\,.$$

Here $1\{\cdot\}$ denotes the indicator function. As the consecutive B_n are i.i.d. and independent of A_n, this gives

$$\Pr(Y_{n+1} \le i) = Y_n^*(F_\xi(i)) \Pr(B_n \le i)\,.$$

Let π be the steady state probability vector of Y, $\pi(j) = \Pr[Y = j]$, then we get the following set of equations for π:

$$\sum_{j=0}^{i} \pi(j) = \sum_{j=0}^{\infty} \pi(j)[F_\xi(i)]^j F_B(i)\,. \tag{7}$$

Now assume that ξ_n and B_n have finite support, they are both bounded by an integer L. This implies that (7) consists of a set of at most $L+1$ linear equations which allows us to solve for the unknowns (together with the equation $\sum_i \pi(i) = 1$).

We now solve (4) for the discrete state space. In this case, $Y_{n+1} \le i$ if all $\xi_n^{(j)} \le i - B_n$ for $j = 1, \dots, Y_n$; see (3). Hence, we have,

$$\Pr(Y_{n+1} \le i) = \mathrm{E}[\Pr(Y_{n+1} \le i | Y_n, B_n)] = \sum_{\ell=0}^{i} \mathrm{E}([F_\xi(i-\ell)]^{Y_n} 1\{B_n = \ell\})$$
$$= \sum_{\ell=0}^{i} Y_n^*(F_\xi(i-\ell))\, p_B(\ell)\,.$$

As before, let π be the steady state probability vector of Y, $\pi(j) = \Pr[Y = j]$, then we get the following set of equations for π,

$$\sum_{j=0}^{i} \pi(j) = \sum_{j=0}^{\infty} \sum_{\ell=0}^{i} \pi(j) [F_\xi(i-\ell)]^j\, p_B(\ell)$$

Again assuming that ξ_n and B_n have finite support, let L denote the common upper bound, the former set (7) consists of at most L linear equations which allows us to solve for the unknowns (together with the normalization condition $\sum_i \pi(i) = 1$).

4.2 Continuous state space

We now consider max-plus branching with a continuous state space. For a continuous random variable r, let $F_r(x) = \Pr[r \le x]$ denote its distribution function and, with some abuse of notation, let $r^*(\zeta) = \mathrm{E}[\exp(-\zeta r)]$ denote its Laplace-Stieltjes transform.

We first consider (5). By conditioning on the Y_n and B_n, we find,

$$\Pr(Y_{n+1} \le x) = \mathrm{E}[\Pr(Y_{n+1} \le x | Y_n, B_n)] = \mathrm{E}(\exp(\lambda Y_n(\sigma(x) - 1)) 1\{B_n \le x\})$$
$$= Y_n^*(\lambda(1 - \sigma(x))) F_B(x)\,,$$

where $\lambda = \mathit{\Pi}[0, \infty)$ and $\sigma(x) = \mathit{\Pi}[0, x)/\lambda$ relate to the Lévy measure $\mathit{\Pi}$ of L_n. In view of the former expression, we then obtain the following integral equation,

$$Y_{n+1}^*(\zeta) = \int_0^\infty \exp(-\zeta x) d(Y_n^*(\lambda(1 - \sigma(x))) F_B(x))\,.$$

Therefore, the Laplace-Stieltjes transform of the steady state distribution of Y satisfies,

$$Y^*(\zeta) = \zeta \int_0^\infty Y^*(\lambda(1 - \sigma(x))) F_B(x) e^{-\zeta x} dx - Y^*(\lambda) F_B(0)\,.$$

We now consider (4). By conditioning on the Y_n and B_n, we find,

$$\begin{aligned}\Pr(Y_{n+1} \le x) &= \mathrm{E}[\Pr(Y_{n+1} \le x | Y_n, B_n)] \\ &= \int_0^x \mathrm{E}(\exp(\lambda Y_n(\sigma(x-y) - 1))) F_B(dy) \\ &= \int_0^x Y_n^*(\lambda(1 - \sigma(x-y))) F_B(dy)\,,\end{aligned}$$

such that the Laplace-Stieltjes transform of the steady-state distribution satisfies,

$$Y^*(\zeta) = \zeta \int_0^\infty \int_0^x Y^*(\lambda(1-\sigma(x-y)))e^{-\zeta x} F_B(dy)dx\,.$$

In general, no easy solution for these integral equations is available. One can nevertheless resort to numerical solution techniques for integral equations, see e.g. [15].

5 The multitype branching

We now turn to stability conditions for max-plus branching processes. We do this in a more general setting: (i) we consider vector-valued processes and (ii) we consider all types of processes which have the same divisibility property as branching processes. In particular, consider the $\mathbb{R}_+^K$ valued process $\{Y_n\}$ and denote the ith entry of Y_n by Y_n^i, $i = 1, ..., K$. The process Y_n satisfies the following equation in vector form:

$$Y_{n+1} = A_n(Y_n) + B_n. \tag{8}$$

The K-dimensional column vector B_n is a stationary ergodic stochastic process whose entries B_n^i, $i = 1, ..., K$ are in subsets of the nonnegative real numbers.

For each n, A_n are non-negative vector valued random fields that are non-decreasing in their arguments. A_n are i.i.d. with respect to n, and $A_n(0) = 0$.

We characterize max-branching processes by their divisibility property. That is, we assume that A_n satisfies the following. If for some k, $y = y^0 + y^1 + ... + y^k$ where y^m are vectors, then $A_n(y)$ can be represented as

$$A_n(y) = \bigoplus_{i=0}^{k} \widehat{A}_n^{(i)}(y^i) \tag{9}$$

where $\{\widehat{A}_n^{(i)}\}_{i=0,1,2,...,k}$ are identically distributed with the same distribution as $A_n(\cdot)$. In particular, for any sequence $k(n)$, $\{\widehat{A}_n^{(k(n))}\}_n$ are independent.

Remark 1. For a given n, we do not assume independence of the random variables $\{\widehat{A}_n^{(i)}\}_{i=0,1,2,...}$. In the case of ordinary multitype branching processes, this leads to a unified framework of linear difference equations and branching processes. In the case of max-branching considered here, the correspondence with max-plus-linear difference equations does not hold. Nevertheless, independence of $\{\widehat{A}_n^{(i)}\}_{i=0,1,2,...}$ is not required for proving stability and is therefore not assumed.

5.1 Examples

We first introduce some processes that satisfy the divisibility property.

Example 6. We define the discrete multitype branching process $A(y)$ as follows. Let $\xi^{(k)}(n)$, $k = 1, 2, 3, ...$, $n = 1, 2, 3, ...$ be a doubly-indexed sequence of i.i.d. random $K \times K$ matrices. The elements of these matrices take values in the nonnegative integers. Moreover, assume that for any $\ell = 1, 2, 3, ...$, $\ell' = 1, 2, 3, ...$, $k = 1, ..., K$, $i = 1, ..., K$, $m = 1, ..., K$, $j = 1, ..., K$ and $m \neq k$, $\xi_{ki}^{(\ell)}$ and $\xi_{mj}^{(\ell')}$ are independent.

Let y_j be the jth element of the vector y, the ith element of the column vector $A(y)$ is given by

$$[A(y)]_i = \bigoplus_{j=1}^{K} \bigoplus_{k=1}^{y_j} \xi_{ji}^{(k)} \,. \tag{10}$$

One easily verifies that the divisibility property holds for this process.

Example 7. As for the single-type max-branching, we express the continuous multitype max-branching in terms of Lévy processes. To this end, let $L_n(y), y \in \mathbb{R}_+^K$ be an additive Lévy field. That is, we assume that $L(y)$ can be decomposed into the sum of K independent $\mathbb{R}_+^K$ valued Lévy processes,

$$L(y) = \sum_{i=1}^{K} L_i(y_i) \,,$$

y_i being the ith element of the vector y as before. The jth element of the continuous multitype max-branching process $A(y)$ is then defined as follows,

$$[A(y)]_j = \bigoplus_{i=1}^{K} d[L_i(y_i)]_j \,,$$

where $[L_i(y_i)]_j$ is the jth element of $L_i(y_i)$. Again, one easily verifies that the divisibility property holds for this process.

5.2 Stability conditions

We shall understand below $\bigotimes_{i=n}^{k} A_i(x) = x$ whenever $k < n$, and $\bigotimes_{i=n}^{k} A_i(x) = A_k(A_{k-1}(...(A_n(x))..))$ whenever $k > n$.

In the remainder, let $\|x\|$ denote the max-norm in $\mathbb{R}^K$ and, with some abuse of notation, let $\|A_n\|$ denote the corresponding operator norm,

$$\|A_n\| = \inf\{c \geq 0 : \|A_n(y)\| \leq c\|y\|, \forall y \in \mathbb{R}^K\} \,.$$

Let $\mathcal{A} \doteq \mathrm{E}[\|A_0\|]$. Then, we have $A_n(y) \leq \|A_0\|\|y\|$, almost surely such that $\mathrm{E}[A_n(y)] \leq \mathcal{A}\|y\|$. By the independence of the consecutive branching processes, this further implies for $j > 1$,

$$\mathrm{E}\left[\left\|\left(\bigotimes_{i=1}^{j} A_i\right)(y)\right\|\right] \leq \mathcal{A}^j \|y\| \,. \tag{11}$$

We now introduce our stability conditions.

Theorem 1. *Let Y_n satisfy* (8)*, with A_n satisfying the divisibility property* (9) *and B_n stationary ergodic. We then have the following.*

(i) For $n > 0$, Y_n can be written in the form

$$Y_n = \widetilde{Y}_n + \left(\bigotimes_{i=0}^{n-1} \widehat{A}_i^{(0)} \right) (Y_0) \tag{12}$$

where

$$\widetilde{Y}_n = \sum_{j=0}^{n-1} \left(\bigotimes_{i=n-j}^{n-1} \widehat{A}_i^{(n-j)} \right) (B_{n-j-1}) \tag{13}$$

is the solution of (8) with initial condition $Y_0 = 0$.

(ii) For $\mathcal{A} < 1$ and $\mathrm{E}[\|B_0\|] < \infty$, there is a unique stationary solution Y_n^* of (8), distributed like,

$$Y_n^* =_d \sum_{j=0}^{\infty} \left(\bigotimes_{i=n-j}^{n-1} \widehat{A}_i^{(n-j)} \right) (B_{n-j-1}), \qquad n \in Z. \tag{14}$$

The sum on the right side of (14) converges absolutely almost surely. Furthermore, one can construct a probability space such that $\lim_{n\to\infty} \|Y_n - Y_n^*\| = 0$, almost surely, for any initial value Y_0.

Proof. (13) is obtained by iterating (8). Now, define the following set of stochastic recursions on the same probability space as Y_n:

$$Y_{n+1}^{[\ell]} = A_n(Y_n^{[\ell]}) + B_n, \quad m \geq -\ell, \qquad Y_{-\ell}^{[\ell]} = 0. \tag{15}$$

For each $n \geq 0$, $Y_n^{[\ell]}$ is monotonically non-decreasing in ℓ so that the limit $Y_n^* = \lim_{n\to\infty} Y_n^{[\ell]}$ is well defined. Since this is measurable on the tail σ-algebra generated by the stationary ergodic sequence $\{A_n, B_n\}$, it is either finite almost surely or infinite almost surely. The last possibility is excluded since it follows by induction that for every $\ell \geq 0$ and $n \geq -\ell$ that $\mathrm{E}[\|Y_n^{[\ell]}\|] \leq (1-\mathcal{A})^{-1}\,\mathrm{E}[\|B_0\|]$, and hence $\mathrm{E}[\|Y_n^*\|] \leq (1-\mathcal{A})^{-1}\,\mathrm{E}[\|B_0\|]$, which is finite.

By the definition of $\widehat{A}_n^{(i)}$ and by (11), we have

$$E\left[\left\| \left(\bigotimes_{i=1}^{j} \widehat{A}_i^{(0)} \right) (y) \right\| \right] = \mathcal{A}^j \|y\|,$$

which converges to zero since $\mathcal{A} < 1$. Since

$$\left\| \left(\bigotimes_{i=1}^{j} \widehat{A}_i^{(0)} \right) (y) \right\|$$

is non-negative, it then follows from Fatou's Lemma that it converges to zero almost surely. Finally, this implies that the difference

$$Y_n - Y_n^* = \left(\bigotimes_{i=1}^{j} \widehat{A}_i^{(0)} \right) (Y_0) - \left(\bigotimes_{i=1}^{j} \widehat{A}_i^{(0)} \right) (Y_0^*)$$

converges to 0 almost surely. This implies also the uniqueness of the stationary regime.

Remark 2. Recall that two forms of immigration were studied in section 3. The stability conditions of Theorem 1 also hold in the case:

$$Y_{n+1} = A_n(Y_n) \oplus B_n \,,$$

A_n and B_n as defined in the current section. To verify this, note that the inequality $\mathrm{E}[\|Y_n^{[\ell]}\|] \leq (1-\mathcal{A})^{-1}\,\mathrm{E}[\|B_0\|]$ is also valid for this modified recursion. The rest of the proof remains unaltered.

6 Deterministic Cruz type branching

We now return to ordinary branching processes and study these by means of a Cruz-type network calculus. Recall the following definition of an arrival curve in (deterministic) network calculus.

Definition 1. *An arrival process is said to satisfy the (σ, ρ) constraints for some constant ρ and σ, if it satisfies for any interval $[s,t]$, $t \geq s$:*

$$A[s,t] \doteq A(t) - A(s) \leq \rho(t-s) + \sigma \,.$$

In order to apply network calculus for branching processes, we first show how a single arrival process can be identified for a standard discrete branching processes. That is, the whole branching process can be derived from this single arrival process. We shall apply the same type of derivation to an arrival process that satisfies Cruz-type constraints and obtain a new recursive characterization of the branching process. We then study the properties of the resulting process.

Consider a discrete-time, one-dimensional branching process given by

$$y_{n+1} = \sum_{i=1}^{y_n} \xi_i^{(n)} + B_n$$

where $\xi_i^{(n)}$ are i.i.d. random variables taking values nonnegative integer numbers.

This branching process is driven by an immigration process B_n and by an infinite set $\xi^{(n)}$ of driving sequences. In making the relation between Cruz-type processes and a branching type structure, the immigration term will not play an important role, and we shall replace it for simplicity by a constant $B_n = B$. Our extension of the Cruz framework is to replace the driving processes $\xi^{(n)}$ by a single $\sigma - \rho$ constrained arrival process.

More generally, we shall define below the arrival processes for processes that satisfy the recursion

$$y_{n+1} = A_n(y_n) + B_n$$

where, in the case of standard discrete branching, we have

$$A_n(y_n) = \sum_{i=1}^{y_n} \xi_i^{(n)} \tag{16}$$

where $\xi_i^{(n)}$ are i.i.d. random variables taking values in the nonnegative integers.

Definition 2. *Let A be a monotone nonnegative random function from $\mathbb{R}$ to $\mathbb{R}$. We call it an arrival generator process (AGP).*

Given an AGP A and some t_0, we define $A_1(y)$ as $A_1(y) = A[t_0, t_0 + y]$, for $0 \leq y \leq y_1$ and define $t_1 = t_0 + y_1$. We then recursively define $t_n = y_n + t_{n-1}$ and for $n > 1$,

$$A_n(y) = A[t_n, t_n + y]. \tag{17}$$

where $0 \leq y \leq y_{n+1}$. Thus for a given AGP A, we obtain a unique sequence A_n of arrival processes. Conversely, assume that the sequence A_n is given, then (17) defines uniquely the AGP A.

Example 8. In the case of standard discrete branching, y is discrete and the AGP A is the counting function of a single infinite i.i.d. sequence ζ_n,

$$A(y) = \sum_{n=1}^{y} \zeta_n \,,$$

where ζ_n have the same distribution as $\xi_i^{(n)}$. It is now easy to check that with the definition (17), $A_n(y)$ have the same distribution as those given by (16), and in particular, the consecutive A_n are i.i.d.

Example 9. This way of describing a branching process easily extends to branching processes with a continuous state space. In particular, the AGP is now a subordinator A. The construction above then ensures that consecutive A_n are i.i.d. by the independent increment property of Lévy processes.

We now assume that the AGP is (σ, ρ) constraint which implies that all A_n are (σ, ρ) constraint as well. Before proceeding to our main results, we note that if each A_n is (σ, ρ) constraint, then the following bound is obtained by applying the recursion directly.

$$\begin{aligned} y_{n+1} &= A_{n+1}(y_n) + B \leq \rho y_n + \sigma + B \leq \rho^2 y_{n-1} + \rho(\sigma + B) + \sigma + B \\ &\leq \ldots \leq \rho^n y_1 + \frac{1 - \rho^{n+1}}{1 - \rho}(\sigma + B) \end{aligned} \tag{18}$$

We shall be mainly interested in the case $\rho < 1$ for which we get the following uniform bound,

$$y_n \leq y_1 + \frac{\sigma + B}{1 - \rho} \tag{19}$$

Finding tighter bounds is the subject of the following section.

7 Bounds on branching process

We use the construction of the process given in the preceding section, based on a AGP A. Define

$$\sigma_{s,t} = A[s,t] - \rho(t-s)$$

We can then rewrite the branching recursion as follows. The first step is,

$$y_2 = A[0, y_1] + B = \rho y_1 + \sigma_{0,t_1} + B\,,$$

whereas the nth step is,

$$y_{n+1} = A[t_n, t_n + y_n] + B = \rho y_n + \sigma_{t_n, t_{n+1}} + B\,.$$

Solving this recursion gives the following lemma.

Lemma 1. *The branching process can be written as*

$$y_{n+1} = \rho^n y_1 + \sum_{i=0}^{n-1} \rho^i (\sigma_{t_{n-i}, t_{n-i+1}}) + \frac{1-\rho^{n+1}}{1-\rho} B\,. \tag{20}$$

We shall use the following Lemma, proved in [2].

Lemma 2. *Suppose we have two sequences of real numbers, $\{V_i\}_{i=1}^n$ and $\{\zeta_i\}_{i=1}^n$, such that $0 \le \zeta_1 \le \cdots \le \zeta_n$. Then*

$$\zeta_1 V_1 + \cdots + \zeta_n V_n \;\le\; \zeta_n \max\{0, V_n, V_n + V_{n-1}, \ldots, V_n + \cdots + V_1\}\,. \tag{21}$$

Proof. The proof is by induction on n. Suppose (21) holds for n. Since the right hand side is non–negative, we can replace ζ_n with ζ_{n+1} on the right hand side, and add $\zeta_{n+1}V_{n+1}$ to both sides, thus obtaining equation (22):

$$\begin{aligned}
&\zeta_1 V_1 + \cdots + \zeta_n V_n + \zeta_{n+1} V_{n+1} \\
&\quad \le \zeta_{n+1} \left(V_{n+1} + \max\{0,\, V_n, V_n + V_{n-1}, \ldots, V_n + \cdots + V_1\}\right) \\
&\quad = \zeta_{n+1} \max\{V_{n+1}, V_{n+1} + V_n, V_{n+1} + V_n + V_{n-1}, \ldots, V_{n+1} + V_n + \cdots + V_1\} \\
&\quad \le \zeta_{n+1} \max\{0, V_{n+1}, V_{n+1} + V_n, \ldots, V_{n+1} + \cdots + V_1\}\,.
\end{aligned} \tag{22}$$

which establishes (21) for $n+1$.

By combining the preceding lemmas, we now obtain a substantial improvement over (19).

Theorem 2. *Assume that $\rho < 1$. Then we have for all n*

$$y_n \le y_1 + \sigma + \frac{B}{1-\rho}$$

Proof. The proof of the Theorem follows by combining the last two lemmas. The sequence $\hat{\zeta}_i$ in the last Lemma corresponds to the power of ρ's: $\zeta_n = \rho^0 = 1$, $\zeta_{n-1} = \rho$, $\zeta_{n-i} = \rho^i$. Also we have $V_i = \sigma_{t_i,t_{i+1}}$. All elements of the max in (21) are given by a summation of the form

$$V_n + V_{n-1} + ... + V_{n-i} = \sigma_{t_{n-i},t_{n+1}}$$

which is bounded by σ. This proves the bound.

It has been argued that deterministic bounds yield overly pessimistic performance bounds, which gave rise to various competing stochastic network calculi [18]. We here adopt the so-called traffic-amount-centric arrival curves to the branching processes considered here.

Definition 3. *An arrival process is said to satisfy the (σ, ρ) constraints probabilistically with non-increasing bounding function $f(x)$ for some constant ρ and σ, if it satisfies for any interval $[s, t]$, $t \geq s$:*

$$\Pr[A[s,t] - \rho(t-s) > \sigma + x] \leq f(x)$$

The definition above allows that the (σ, ρ) constraint is violated by the AGP, albeit with a small probability which is bounded by $f(x)$. We then obtain the following probabilistic bound for the branching process.

Theorem 3. *Assume that $\rho < 1$ and that A_n satisfies the (σ, ρ) constraints probabilistically with bounding function $f(x)$. Then we have,*

$$\Pr[y_{n+1} - \rho^n y_1 - \frac{1-\rho^{n+1}}{1-\rho} B > \sigma + x] \leq f(x)\,.$$

Proof. Following the arguments of the proof of Theorem 2, we have,

$$y_{n+1} \leq \rho^n y_1 + \sigma_{t_1,t_{n+1}} + \frac{1-\rho^{n+1}}{1-\rho} B\,,$$

or equivalently,

$$y_{n+1} - \rho^n y_1 - \frac{1-\rho^{n+1}}{1-\rho} B \leq \sigma_{t_1,t_n+1}\,,$$

This inequality then implies,

$$\Pr[y_{n+1} - \rho^n y_1 - \frac{1-\rho^{n+1}}{1-\rho} B \;>\; \sigma + x] \;\leq\; \Pr[\sigma_{t_1,t_n+1} \;>\; \sigma + x] \;\leq\; f(x)\,.$$

Here the last inequality follows from the definition of $\sigma_{t_1,t_{n+1}}$ and definition 3.

8 Conclusions

In this paper we reconsider branching processes and their use in evaluating performance of communication systems from two non-standard perspectives. First, we introduce max-plus branching, where branching corresponds to finding the maximal offspring of a member of the current generation rather then summing all offspring of members of the current generation. We show that, as for a standard branching processes, a divisibility property holds. However, in the case of max-plus branching, dividing the current generation leads to maximizing over the respective offspring. The divisibility property also allows us to define continuous-state max-branching in terms of Lévy processes, just like for ordinary branching. All max-plus branching processes are investigated in the presence of a migration component which is either added in the ordinary sense or in the max-plus sense. Various applications in queueing theory for this type of branching processes are introduced along the way.

For the single-type discrete max-branching with i.i.d. migration, we obtain a system of equations for the stationary solution. For the continuous equivalent, a functional equation is obtained for the Laplace-Stieltjes transform of the stationary solution. Finally, for multitype max-branching, we study conditions which ensures the existence of a stationary solution.

A network calculus approach to branching processes constitutes the second non-standard perspective. We show that a branching process can be created from a single arrival process and then find bounds on the growth of this branching process in terms of the deterministic constraints on this arrival process. Finally, we relax these constraints by assuming probabilistic bounds on the arrival process.

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