A new method for learning imprecise hidden Markov models

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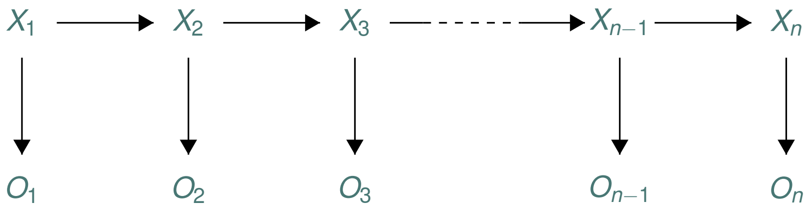
Imprecise hidden Markov models

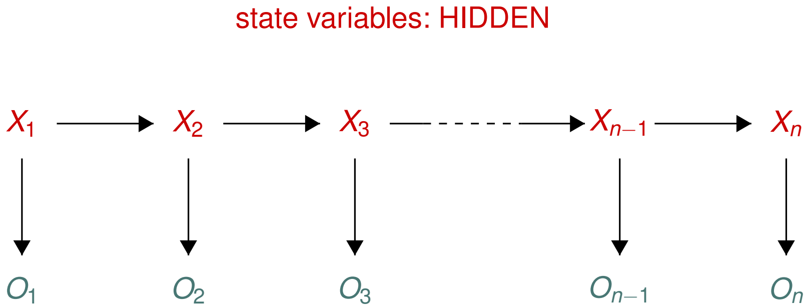
Imprecise hidden Markov model

graphical representation



Imprecise hidden Markov model

random variables



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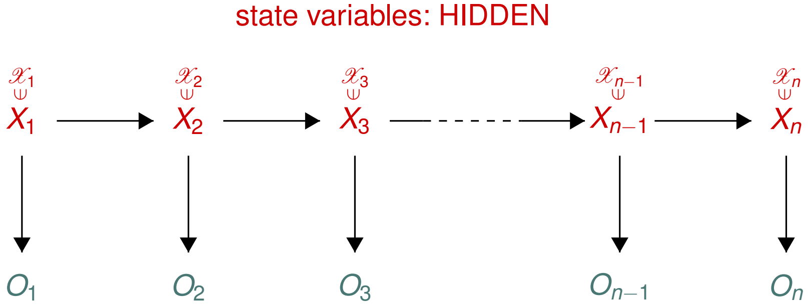

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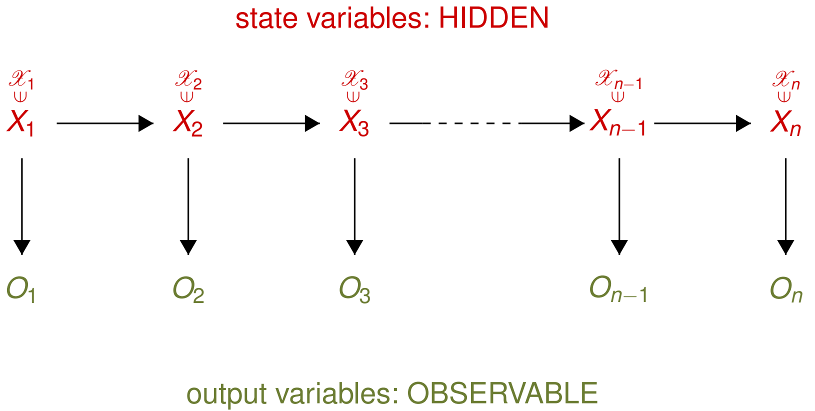

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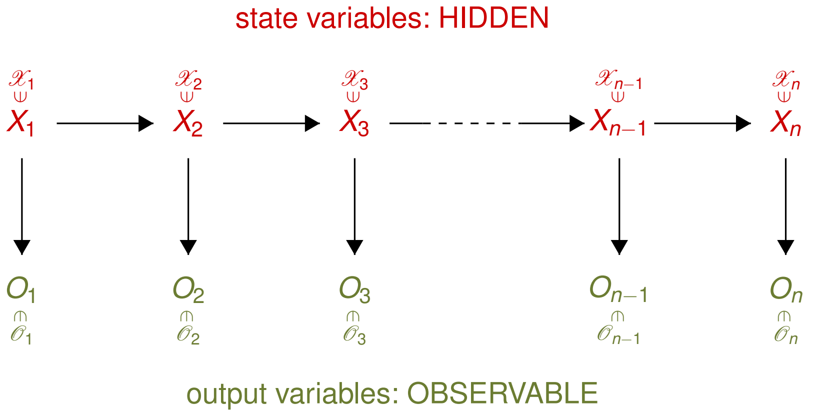
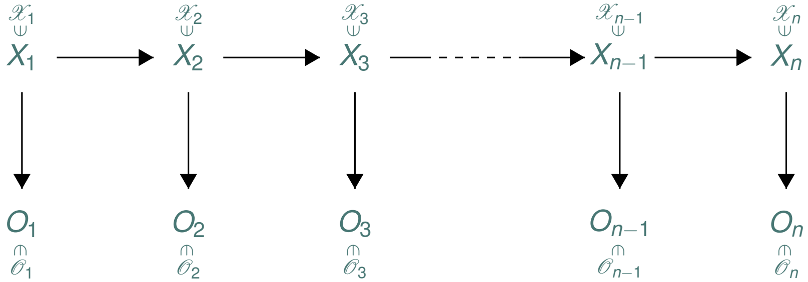


Imprecise hidden Markov model

We consider stationary imprecise hidden Markov models



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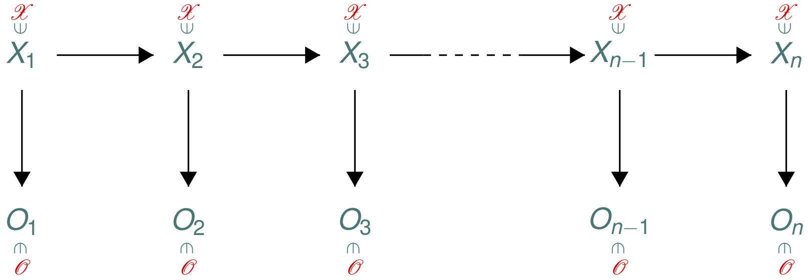

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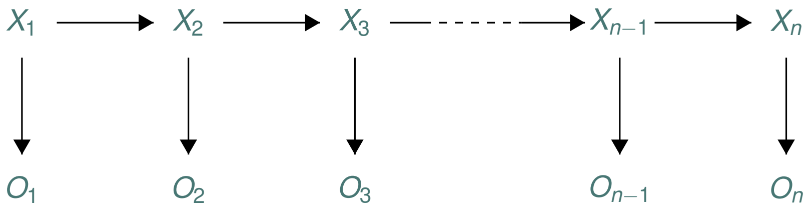
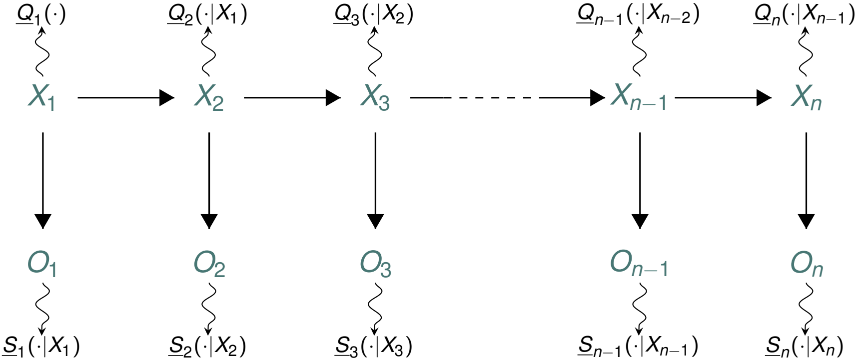


Imprecise hidden Markov model

local uncertainty models in terms of coherent lower previsions



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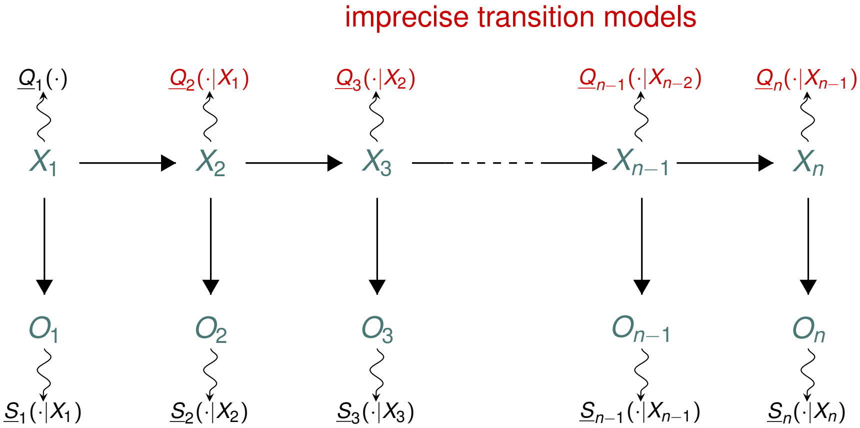

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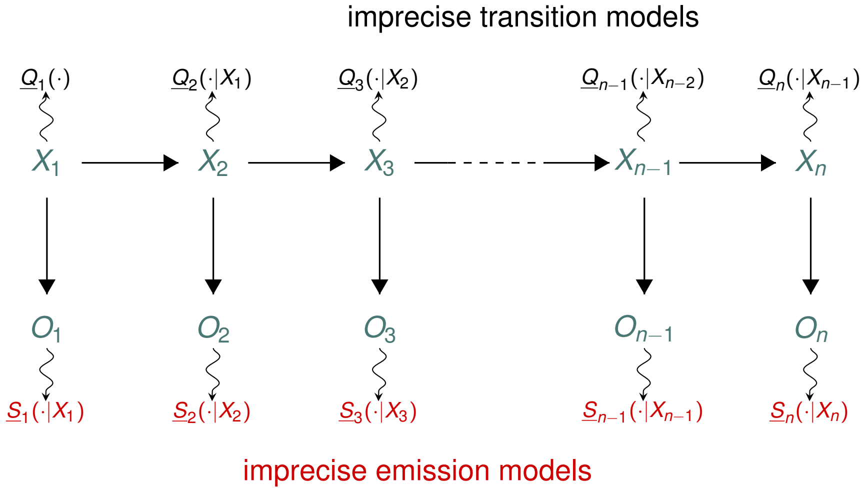

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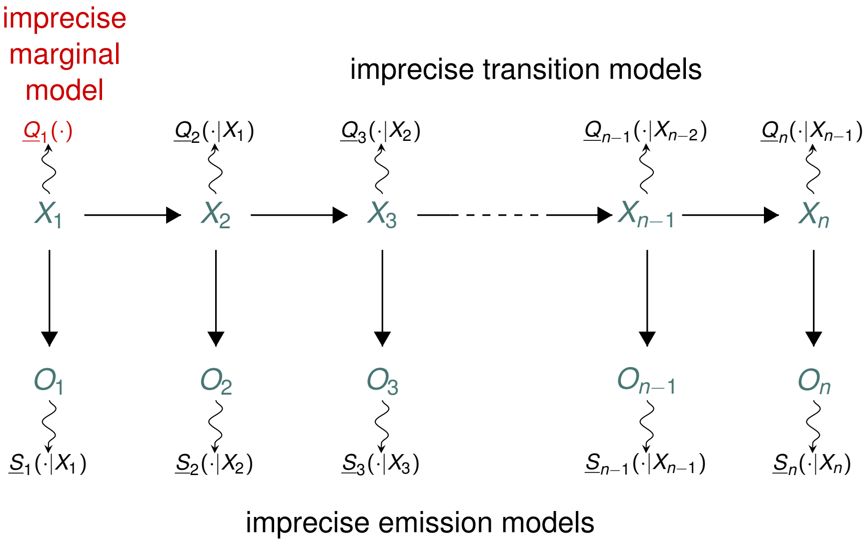

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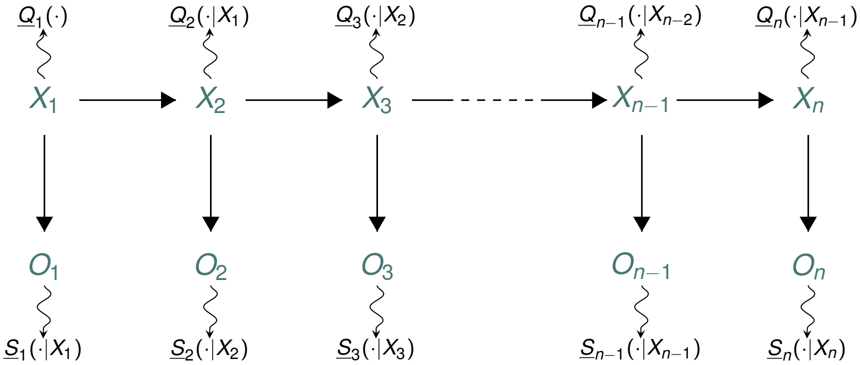

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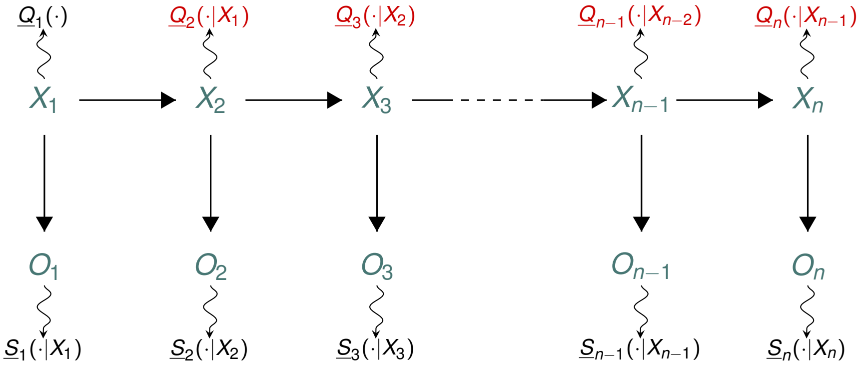

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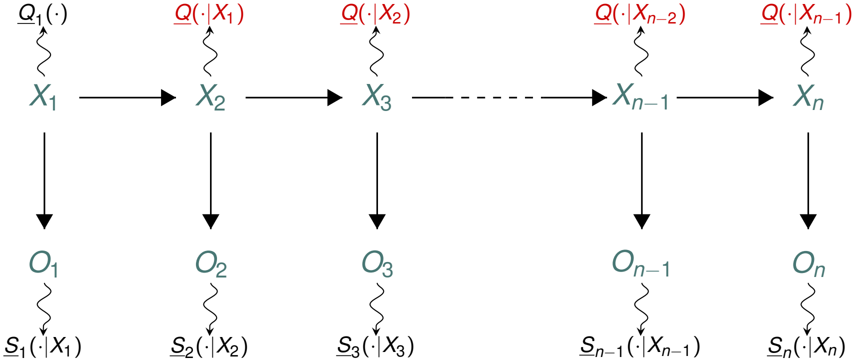

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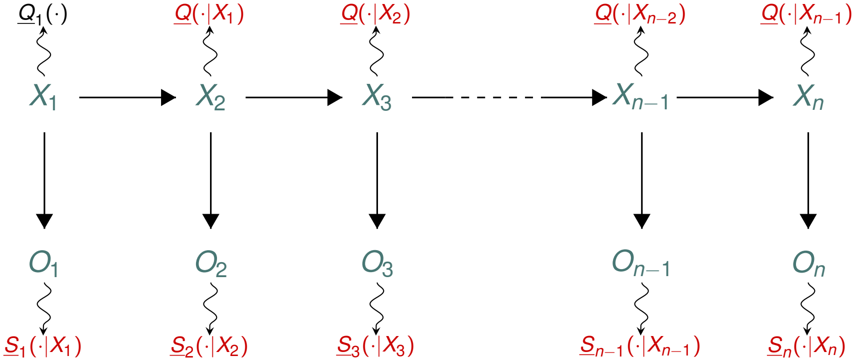

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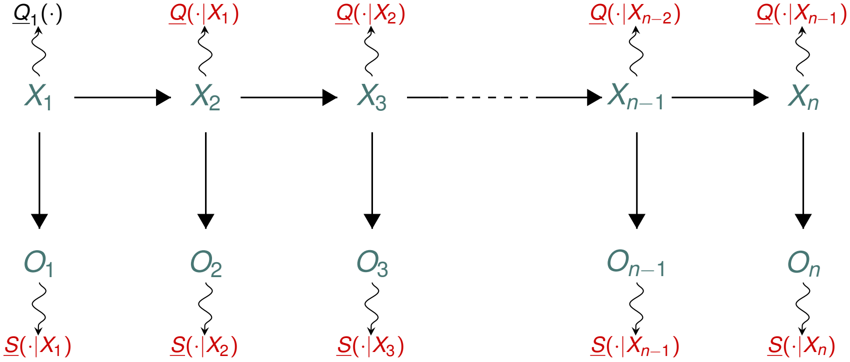
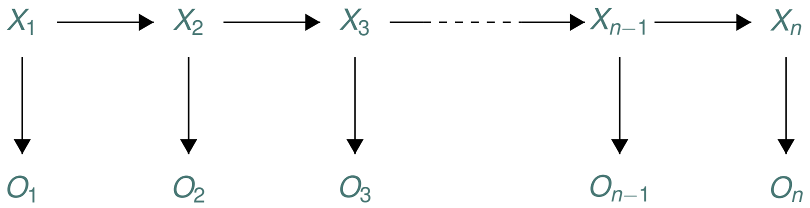


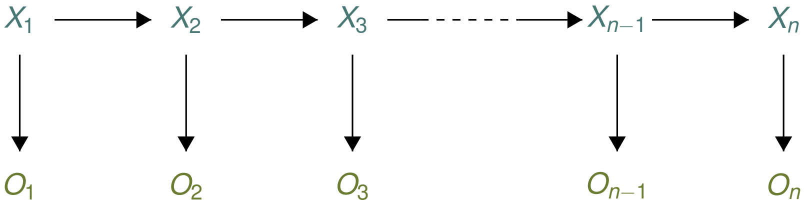
What do we want to do?

Our problem of interest



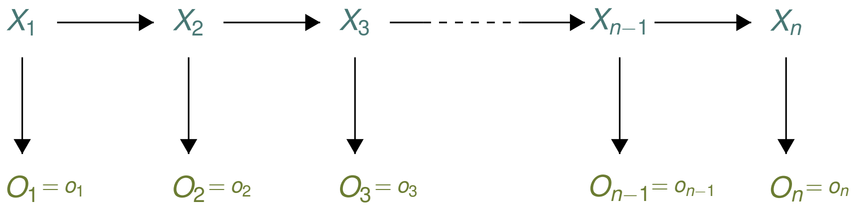
Our problem of interest

Suppose we know the output sequence



Our problem of interest

Suppose we know the output sequence: $O_{1:n} = o_{1:n} \in \mathscr{O}^n$



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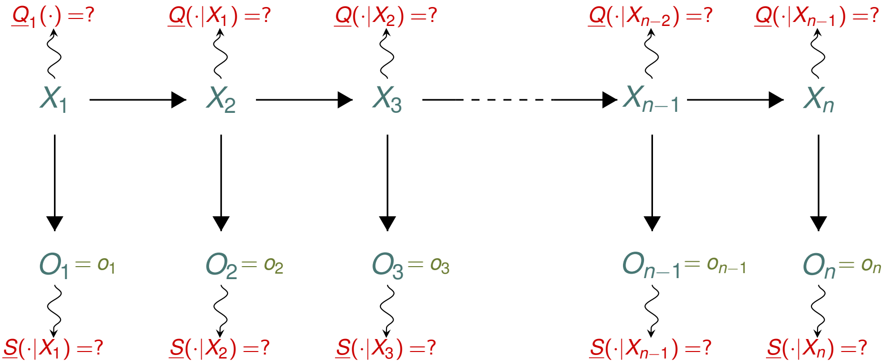
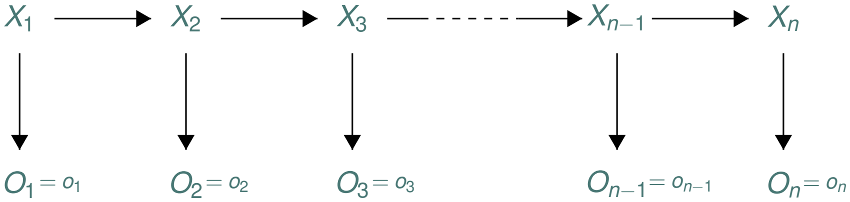
Suppose we know the output sequence: $O_{1:n} = o_{1:n} \in \mathscr{O}^n$,
we want to estimate the unknown local uncertainty models.



How could you learn the local models if the state sequence were known?

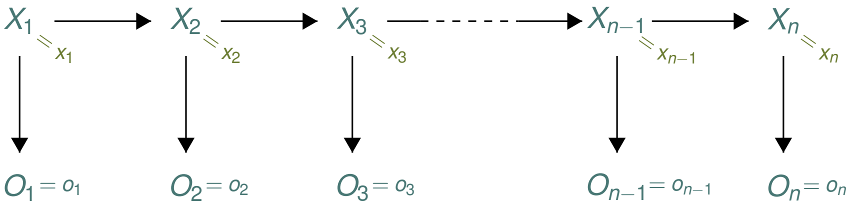
An easier problem

We know the output sequence: $O_{1:n} = o_{1:n} \in \mathscr{O}^n$.



An easier problem

We know the output sequence: $O_{1:n} = o_{1:n} \in \mathscr{O}^n$.
Suppose we know in addition also the state sequence: $X_{1:n} = x_{1:n} \in \mathscr{X}^n$



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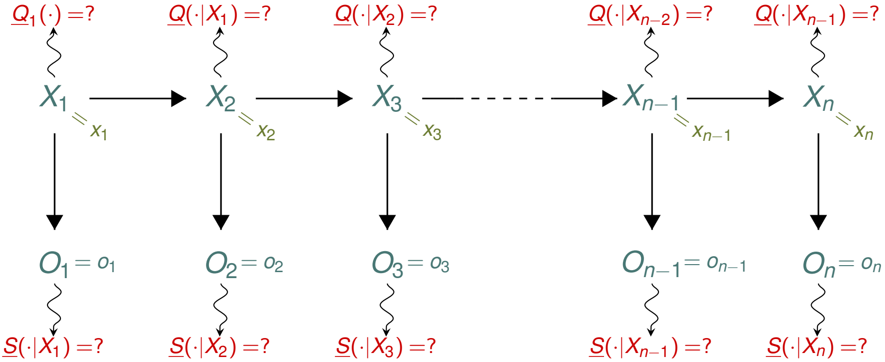
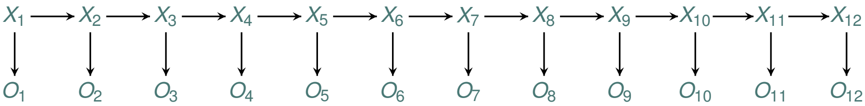
how can we learn local models now?



Solution



X_1 X_2 X_3 X_4 X_5 X_6 X_7 X_8 X_9 X_{10} X_{11} X_{12}
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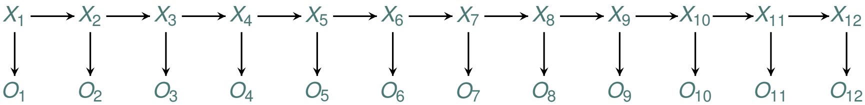

Suppose $\mathscr{X} = \{a, b\}$ and $\mathscr{O} = \{o, p, q\}$.

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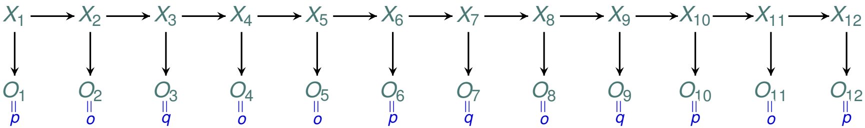

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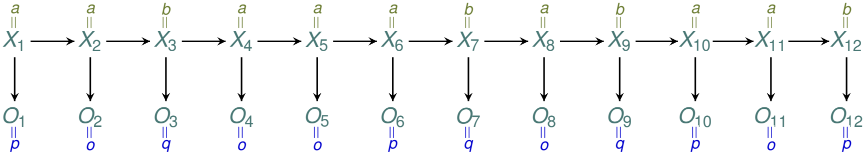

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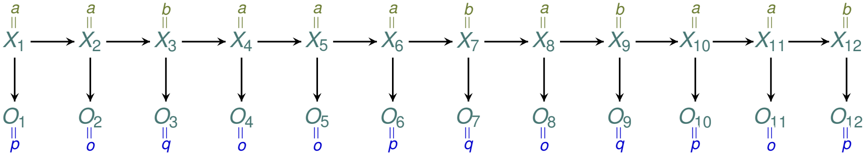

Suppose $\mathscr{X} = \{a, b\}$ and $\mathscr{O} = \{o, p, q\}$.

With (known) hidden state sequence $x_{1:n}$ and output sequence $o_{1:n}$ ($x, y \in \mathscr{X}$ and $z \in \mathscr{O}$):

n_x : number of times a state x is reached,

$n_{x,y}$: number of times a state transition from x to y takes place,

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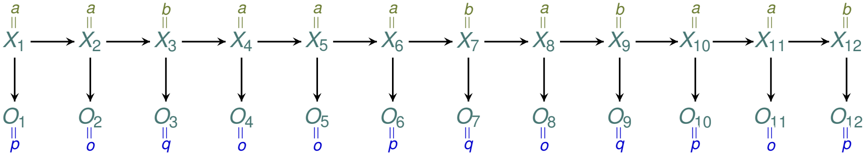

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Here:

$$\left.\begin{aligned} n_a = 8, n_b = 4, \\ n_{a,a} = 4, n_{a,b} = 4, n_{b,a} = 3, n_{b,b} = 0, \\ n_{a,o} = 5, n_{a,p} = 3, n_{a,q} = 0, \\ n_{b,o} = 0, n_{b,p} = 1, n_{b,q} = 3. \end{aligned}\right\}$$

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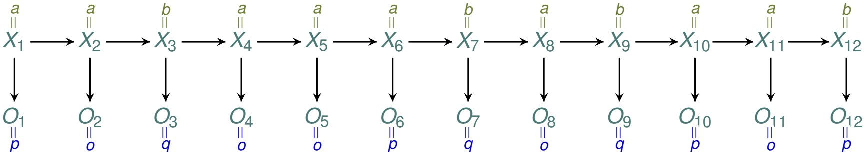

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With these counts, how can we build local models?

Imprecise Dirichlet model

We use the imprecise Dirichlet model (IDM) to compute estimates for the local models. If $n(A)$ is the number of occurences of an event A in N experiments, then the lower and upper probability of A according to an IDM are defined as

$$\underline{P}(A) = \frac{n(A)}{s+N} \quad \text{and} \quad \overline{P}(A) = \frac{s+n(A)}{s+N}.$$

$s > 0$ is the number of pseudo-counts, which is an inverse measure of the speed of convergence to a precise model.

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Now, we use the quantities n_x, $n_{x,y}$ and $n_{x,z}$ (with $x, y \in \mathscr{X}$ and $z \in \mathscr{O}$) to estimate the imprecise transition and emission models:

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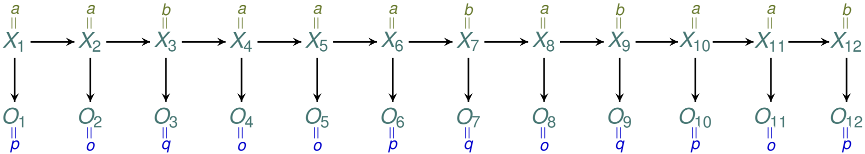
$$\underline{Q}(\{y\}|x) = \frac{n_{x,y}}{s+\sum_{y^* \in \mathscr{X}} n_{x,y^*}} \quad \text{and} \quad \overline{Q}(\{y\}|x) = \frac{s+n_{x,y}}{s+\sum_{y^* \in \mathscr{X}} n_{x,y^*}},$$

$$\underline{S}(\{z\}|x) = \frac{n_{x,z}}{s+n_x} \quad \text{and} \quad \overline{S}(\{z\}|x) = \frac{s+n_{x,z}}{s+n_x}.$$

Example



(with $x, y \in \mathscr{X}$ and $z \in \mathscr{O}$):

$$\underline{Q}(\{y\}|x) = \frac{n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}}, \overline{Q}(\{y\}|x) = \frac{s + n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}}, \underline{S}(\{z\}|x) = \frac{n_{x,z}}{s + n_x}, \overline{S}(\{z\}|x) = \frac{s + n_{x,z}}{s + n_x}.$$

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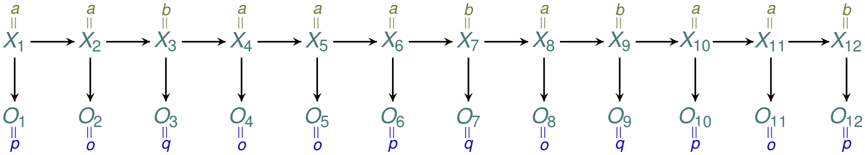

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Here, with $s = 2$:

$\underline{Q}(\{a\}|a) = 2/5$, $\overline{Q}(\{a\}|a) = 3/5$, $\underline{Q}(\{b\}|a) = 2/5$, $\overline{Q}(\{b\}|a) = 3/5$,
$\underline{Q}(\{a\}|b) = 3/5$, $\overline{Q}(\{a\}|b) = 1$, $\underline{Q}(\{b\}|b) = 0$, $\overline{Q}(\{b\}|b) = 2/5$,
$\underline{S}(\{o\}|a) = 1/2$, $\overline{S}(\{o\}|a) = 7/10$, $\underline{S}(\{o\}|b) = 0$, $\overline{S}(\{o\}|b) = 1/3$,
$\underline{S}(\{p\}|a) = 3/10$, $\overline{S}(\{p\}|a) = 1/2$, $\underline{S}(\{p\}|b) = 1/6$, $\overline{S}(\{p\}|b) = 1/2$,
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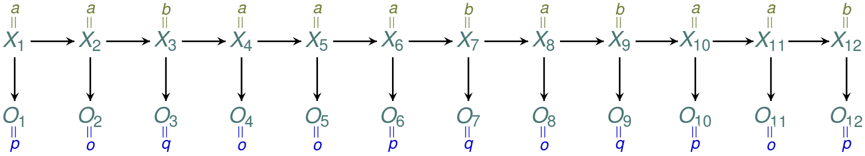
But the state sequence is hidden...

We are almost there



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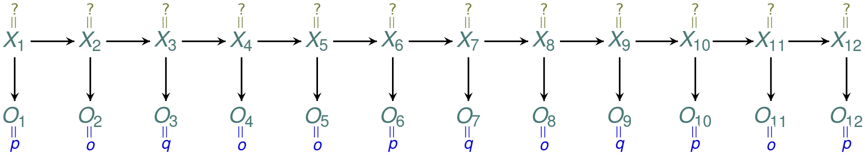

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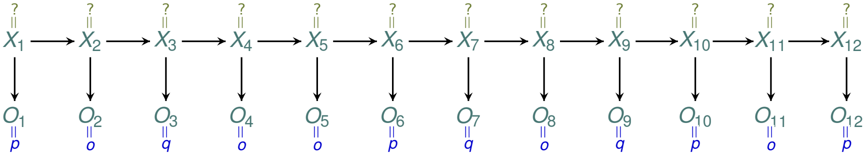

The state sequence $x_{1:n} \in \mathscr{X}^n$ is hidden, so it is a random variable $X_{1:n}$.

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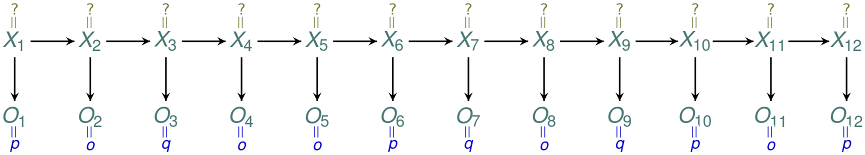

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(with $x, y \in \mathscr{X}$ and $z \in \mathscr{O}$) n_x, $n_{x,y}$ and $n_{x,z}$ are random variables N_x, $N_{x,y}$ and $N_{x,z}$.

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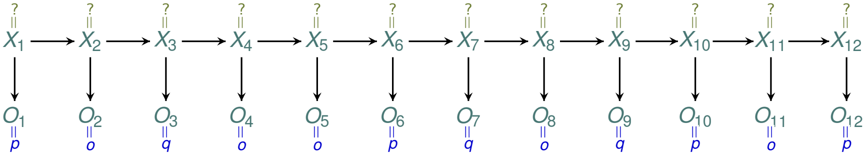

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Idea: instead of using real counts, use estimates:

$$\hat{n}_x,$$
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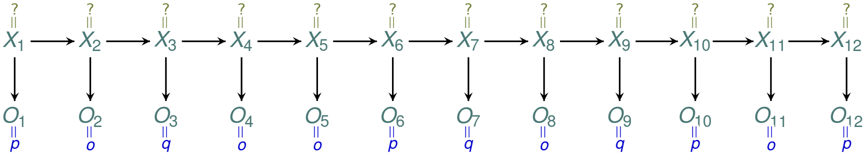

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Idea: instead of using real counts, use expected counts:

$$\hat{n}_x = E(N_x | o_{1:n}, \theta^*),$$
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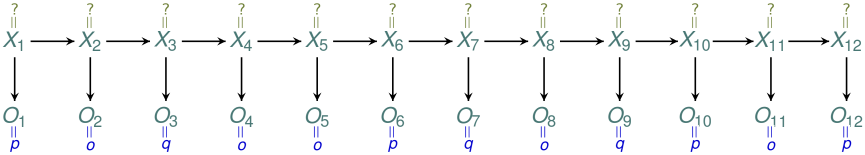

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$o_{1:n}$ is the known output sequence, and θ^* represents the model parameter.

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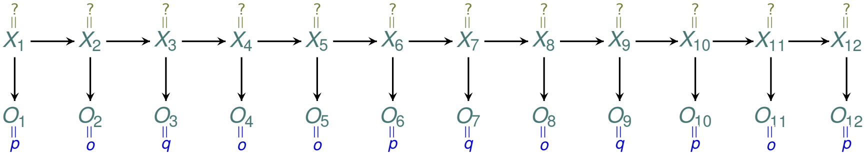

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$$\begin{aligned} \hat{n}_x &= E(N_x | o_{1:n}, \theta^*), \\ \hat{n}_{x,y} &= E(N_{x,y} | o_{1:n}, \theta^*), \\ \hat{n}_{x,z} &= E(N_{x,z} | o_{1:n}, \theta^*). \end{aligned}$$

$o_{1:n}$ is the known output sequence, and θ^* represents the model parameter. We can calculate θ^* with the Baum–Welch algorithm, so the idea makes sense.

Estimated local models

With known state sequence $x_{1:n}$ ($x, y \in \mathscr{X}$ and $z \in \mathscr{O}$) :

$$\underline{Q}(\{y\}|x) = \frac{n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}} \text{ and } \overline{Q}(\{y\}|x) = \frac{s + n_{x,y}}{s + \sum_{y^* \in \mathscr{X}} n_{x,y^*}},$$

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We want to predict future earthquake rates, based on number of earthquakes in previous years.

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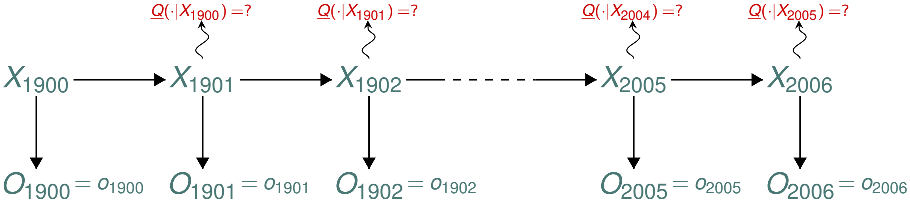
Our observation: number of earthquakes from 1900 to 2006

Example: learned model



Based on the data, we learn the (imprecise) transition model.

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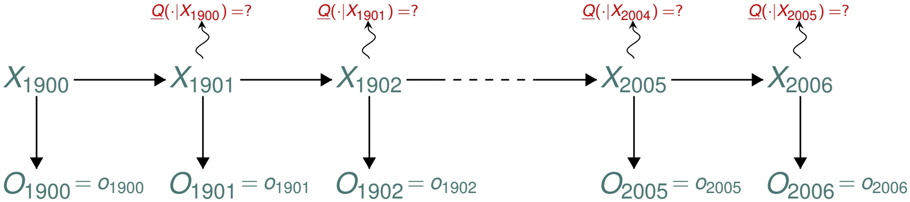

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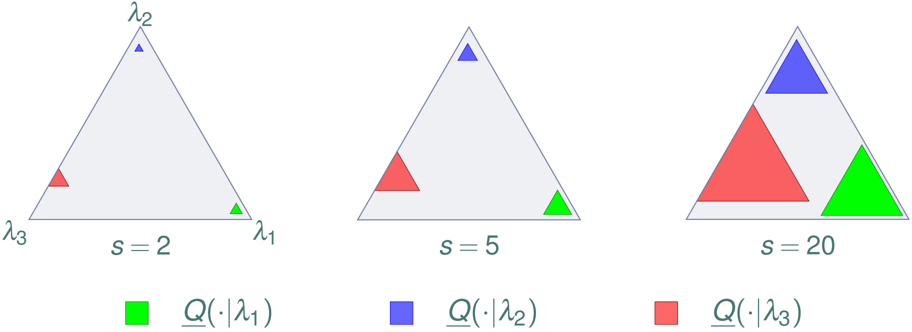

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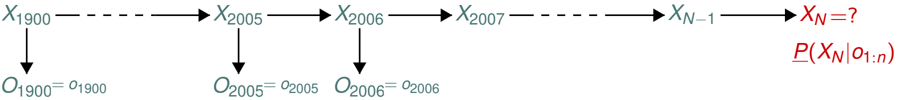

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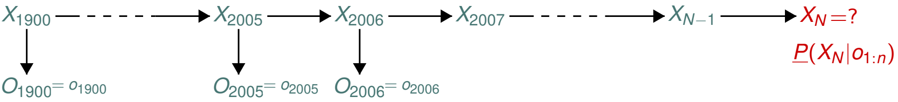

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