The residuation principle for intuitionistic fuzzy t-norms

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Intuitionistic fuzzy sets defined by Atanassov in 1983 [1] form an extension of fuzzy sets. While fuzzy sets give only a degree of membership, and the degree of non-membership equals one minus the degree of membership, intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership that are more or less independent: the only condition is that the sum of the two degrees is smaller than or equal to 1. Formally, an intuitionistic fuzzy set A in a universe U is defined as $A = \{(u, \mu_A(u), \nu_A(u)) \mid u \in U\}$, where μ_A and ν_A are $U - [0,1]$ mappings giving the membership degree and non-membership degree of u in A respectively, and where $\mu_A(u) + \nu_A(u) \leq 1$, for all $u \in U$.

Deschrijver and Kerre [4] have shown that intuitionistic fuzzy sets can also be seen as L-fuzzy sets in the sense of Goguen [6]. Consider the set L^* and the operation $\leq_{L^*}$ defined by :

$$L^* = \{(x_1, x_2) \mid (x_1, x_2) \in [0,1]^2 \text{ and } x_1 + x_2 \leq 1\},$$
$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \quad \forall (x_1, x_2), (y_1, y_2) \in L^*.$$

Then $(L^*, \leq_{L^*})$ is a complete lattice [4]. We denote its units by $0_{L^*} = (0,1)$ and $1_{L^*} = (1,0)$. From now on, we will assume that if $x \in L^*$, then x_1 and x_2 denote respectively the first and second component of x, i.e. $x = (x_1, x_2)$. It is easily seen that with every intuitionistic fuzzy set A corresponds an L^*-fuzzy set, i.e. a mapping $A : U \to L^* : u \mapsto (\mu_A(u), \nu_A(u))$. We will also use in the sequel the set $D = \{x \mid x \in L^* \text{ and } x_1 + x_2 = 1\}$.

Using the lattice $(L^*, \leq_{L^*})$, Deschrijver, Cornelis and Kerre have extended the notion of triangular norm to the intuitionistic fuzzy case [2, 3]. An intuitionistic fuzzy triangular norm is a commutative, associative, increasing $(L^*)^2 - L^*$ mapping $\mathcal{T}$ satisfying $\mathcal{T}(1_{L^*}, x) = x$, for all $x \in L^*$. Intuitionistic fuzzy t-norms can be constructed using t-norms and t-conorms on $[0,1]$ in the following way. Let T be a t-norm and S a t-conorm, then the dual t-norm S^* of S is defined by $S^*(a,b) = 1 - S(1-a, 1-b)$, for all $a, b \in [0,1]$. If for all $a, b \in [0,1]$, $T(a,b) \leq S^*(a,b)$, then the mapping $\mathcal{T}$ defined by $\mathcal{T}(x,y) = (T(x_1,y_1), S(x_2,y_2))$, for all $x, y \in L^*$, is an intuitionistic fuzzy t-norm. We call an intuitionistic fuzzy t-norm $\mathcal{T}$ for which such a t-norm T and t-conorm S exist t-representable. Not all intuitionistic fuzzy t-norms are t-representable, e.g. $\mathcal{T}_W(x,y) = (\max(0, x_1 + y_1 - 1), \min(1, x_2 + 1 - y_1, y_2 + 1 - x_1))$ is not t-representable.

An intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies the residuation principle if and only if, for all $x, y, z \in L^*$, $\mathcal{T}(x,y) \leq_{L^*} z \Leftrightarrow y \leq_{L^*} I_{\mathcal{T}}(x,z)$, where $I_{\mathcal{T}}$ denotes the residual implicator generated by $\mathcal{T}$, defined as, for $x, y \in L^*$, $I_{\mathcal{T}}(x,y) = \sup\{\gamma \mid \gamma \in L^* \text{ and } \mathcal{T}(x,\gamma) \leq_{L^*} y\}$.

In the fuzzy case, the residuation principle is equivalent to left-continuity of the t-norm[5]. The intuitionistic fuzzy counterpart of left-continuity is intuitionistic fuzzy left-continuity, defined as follows. Let F be an arbitrary $L^* - L^*$ mapping and $a \in L^*$, then F is called intuitionistic fuzzy left-continuous in a iff

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x \in L^*)((d(a,x) < \delta \text{ and } x \leq_{L^*} a) \Rightarrow d(F(x),F(a)) < \varepsilon),$$

where d denotes the Euclidean or Hamming distance of $\mathbb{R}^2$ restricted to L^*.

Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm. Then $\mathcal{T}$ satisfies the residuation principle if and only if $\sup_{z \in Z} \mathcal{T}(x,z) = \mathcal{T}(x, \sup_{z \in Z} z)$, for all $x \in L^*$ and all $\varnothing \subset Z \subseteq L^*$. Only in the case of t-representable intuitionistic fuzzy t-norms the last property is equivalent to intuitionistic fuzzy left-continuity. So we have that a t-representable intuitionistic fuzzy t-norm $\mathcal{T}$ satisfies the residuation principle if and only if $\mathcal{T}$ is intuitionistic fuzzy left-continuous, but in general we only have that if $\mathcal{T}$ satisfies the residuation principle then $\mathcal{T}$ is intuitionistic fuzzy left-continuous [2].

In general a characterization of intuitionistic fuzzy t-norms satisfying the residuation principle has not yet been established. However, we have the following cases.

For the first representation theorem we will use the following possible properties of an intuitionistic fuzzy t-norm $\mathcal{T}$:

(P.1) $\mathcal{T}(x,x) <_{L^*} x$, for all $x \in L^* \setminus \{0_{L^*}, 1_{L^*}\}$;

(P.2) there exist $x, y \in L^*$ such that x_1 and y_1 are non-zero and such that $\mathcal{T}(x,y) = 0_{L^*}$.

Deschrijver, Cornelis and Kerre have proven that if $\mathcal{T}$ is an $(L^*)^2 - L^*$ mapping, then the following are equivalent [2]:

(i) $\mathcal{T}$ is a continuous intuitionistic fuzzy t-norm satisfying the residuation principle, the properties (P.1) and (P.2), $I_{\mathcal{T}}(D,D) \subseteq D$ and $\mathcal{T}((0,0),(0,0)) = 0_{L^*}$;

(ii) there exists a continuous increasing permutation φ of $[0,1]$ such that, for all $x, y \in L^*$,

$$\begin{aligned} \mathcal{T}(x,y) &= (\varphi^{-1}(\max(0, \varphi(x_1) + \varphi(y_1) - 1)), \\ &\quad 1 - \varphi^{-1}(\max(0, \varphi(x_1) + \varphi(1-y_2) - 1, \varphi(y_1) + \varphi(1-x_2) - 1))); \end{aligned}$$

(iii) there exists a continuous increasing permutation Φ of L^* such that $\mathcal{T} = \Phi^{-1} \circ \mathcal{T}_W \circ (\Phi \times \Phi)$.

A more general class of intuitionistic fuzzy t-norms that satisfy the residuation principle is the following. Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm such that, for all $x \in D$, $y_2 \in [0,1]$, $pr_2 \mathcal{T}(x,(0,y_2)) = pr_2 \mathcal{T}(x,(1-y_2,y_2))$. Then $\mathcal{T}$ satisfies the residuation principle if and only if there exist two left-continuous t-norms T_1 and T_2 on $[0,1]$ such that, for all $x, y \in L^*$,

$$\begin{aligned} \mathcal{T}(x,y) &= (T_1(x_1,y_1), \min\{1 - T_2(1 - pr_2 \mathcal{T}((0,0),(0,0)), \\ &\quad T_2(1-x_2, 1-y_2)), 1 - T_2(x_1, 1-y_2), 1 - T_2(y_1, 1-x_2)\}), \end{aligned}$$

and $T_2(x_1,y_1) = T_1(x_1,y_1)$ as soon as $T_2(x_1,y_1) > T_2(1 - pr_2 \mathcal{T}((0,0),(0,0)), T_2(x_1,y_1))$, and $T_1(x_1,y_1) \leq T_2(x_1,y_1)$ else, for all $x_1, y_1 \in [0,1]$.

In the case that $\mathcal{T}(D,D) \subseteq D$, we have the following. Let $\mathcal{T}$ be an intuitionistic fuzzy t-norm satisfying the residuation principle such that $\mathcal{T}(D,D) \subseteq D$, T_1 be the $[0,1]^2 - [0,1]$ mapping defined by $T_1(x_1,y_1) = pr_1 \mathcal{T}((x_1, 1-x_1),(y_1, 1-y_1))$, for all $x_1, y_1 \in [0,1]$, and $N_1(x_1) = \sup\{y_1 \mid y_1 \in [0,1] \text{ and } T_1(x_1,y_1) = 0\}$. Assume that range$(N_1) = [0,1]$, and

$$pr_2 \mathcal{T}((0,0),(y_1, 1-y_1)) = 1 \Leftrightarrow y_1 = 0, \quad \forall y_1 \in [0,1].$$

Then, for all $x, y \in L^*$,

$$\begin{aligned}\mathcal{T}(x,y) &= (T_1(x_1,y_1), \min\{1-T_1(1-pr_2\mathcal{T}((0,0),(0,0)), T_1(1-x_2,1-y_2)),\\ &\quad 1-T_1(1-y_2,x_1), 1-T_1(1-x_2,y_1)\}).\end{aligned}$$

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