Accurate and Conforming Mixed Discretization of the Chiral Müller Equation

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Abstract—**Scattering of time-harmonic fields by chiral objects can be modeled by a second kind boundary integral equation, similar to Müller's equation for scattering by nonchiral penetrable objects. In this contribution, a mixed discretization scheme for the chiral Müller equation is introduced using both Rao-Wilton-Glisson and Buffa-Christiansen funtions. It is shown that this mixed discretization yields more accurate solutions than classical discretizations, and that they can be computed in a limited number of iterations using Krylov type solvers.**

I. INTRODUCTION

Scattering by homogeneous objects often is modeled using boundary integral equations (BIEs), which can be solved numerically using the boundary element method.

Until recently, the numerical solution of second-kind BIE's, such as the magnetic field integral equation (MFIE) and the Müller equation, was plagued by an inaccurate discretization of the unit operator. This problem has been solved by a mixed discretization scheme using curl-conforming Buffa-Christiansen (BC) testing functions [1]. This scheme greatly increases the accuracy of both the MFIE [2] and the Müller equation [3].

In this contribution, the mixed discretization scheme is applied to the chiral Müller equation [4] describing scattering by chiral media. Numerical experiments show that the resulting system is well-conditioned, and that the accuracy of the solution is comparable to that of the chiral PMCHWT equation.

II. DISCRETIZATION OF THE MÜLLER EQUATION

A. The Chiral Müller equation

Electromagnetic fields in homogeneous chiral media are modeled by the Maxwell equations

$$\nabla \times \boldsymbol{e} = -j\omega\mu\boldsymbol{h} + \kappa\omega\sqrt{\epsilon\mu}\boldsymbol{e}, \quad (1a)$$

$$\nabla \times \boldsymbol{h} = j\omega\epsilon\boldsymbol{e} + \kappa\omega\sqrt{\epsilon\mu}\boldsymbol{h}, \quad (1b)$$

where ϵ and μ are the usual permittivity and permeability, respectively, and κ is the dimensionless chirality parameter. Scattering by homogeneous chiral bodies can be described by the chiral extensions of either the PMCHWT equation [5] or the Müller equation [4]. The latter is obtained by combining the internal and external representation formulas in such a way that all hypersingular operators cancel out. This results in an equation of the following form:

$$(\mathcal{U} + \mathcal{C})\begin{pmatrix} -\hat{\boldsymbol{n}} \times \boldsymbol{e} \\ \hat{\boldsymbol{n}} \times \boldsymbol{h} \end{pmatrix} = \begin{pmatrix} \epsilon_0 & 0 \\ 0 & \mu_0 \end{pmatrix}\begin{pmatrix} -\hat{\boldsymbol{n}} \times \boldsymbol{e}^i \\ \hat{\boldsymbol{n}} \times \boldsymbol{h}^i \end{pmatrix}, \quad (2)$$

where

$$\mathcal{U} = \frac{1}{2}\begin{pmatrix} \epsilon_0 + \epsilon & j\sqrt{\epsilon\mu}\kappa \\ -j\sqrt{\epsilon\mu}\kappa & \mu_0 + \mu \end{pmatrix},$$

$\hat{\boldsymbol{n}}$ is the exterior normal vector and $\mathcal{C}$ is a compact operator. The integral operator on the left hand side of (2) therefore is of the second kind if $\det \mathcal{U} \neq 0$, which holds true for realistic materials. In this sense, the chiral Müller operator has the same spectral properties as the nonchiral Müller operator and the MFIE operator.

B. Discretization

In order to obtain accurate results, the unknown tangential traces $\hat{\boldsymbol{n}} \times \boldsymbol{e}$ and $\hat{\boldsymbol{n}} \times \boldsymbol{h}$ must be expanded in divergence-conforming basis functions, such as the Rao-Wilton-Glisson (RWG) functions $\boldsymbol{f}_i$. Then, the BIE must be tested using curl-conforming basis functions such as the rotated RWG functions $\hat{\boldsymbol{n}} \times \boldsymbol{f}_i$.

When applied to the unit operator, this approach yields an ill-conditioned system matrix $O_{ij} = \left(\hat{\boldsymbol{n}} \times \boldsymbol{f}_i, \boldsymbol{f}_j\right)$. A well-conditioned system can be obtained by testing with div-conforming RWG functions $\boldsymbol{f}_i$, but this non-conforming scheme produces inaccurate results.
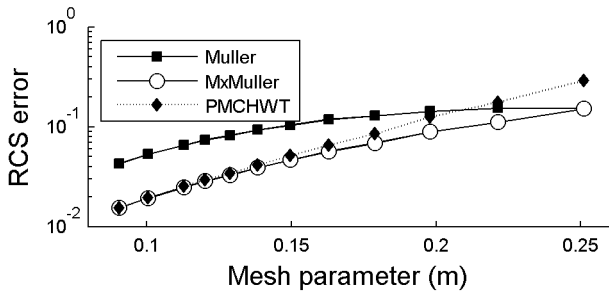
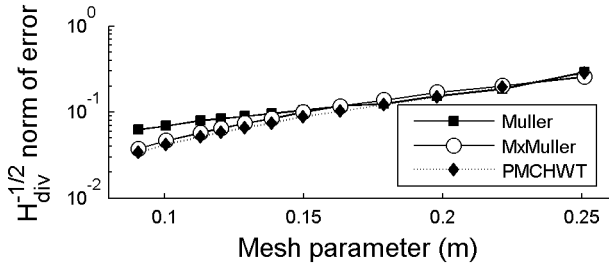Fig. 1. Infinity norm of the error of the RCS w.r.t. the Mie series.

Fig. 2. Relative $H_{div}^{-1/2}$ norm of the error of the surface currents w.r.t. the projection of the Mie series onto the RWG basis.

However, the curl-conforming rotated Buffa-Christiansen (BC) functions $\hat{n} \times g_i$ [1] can be used as testing functions to obtain a well-conditioned overlap matrix $O_{ij} = \left(\hat{n} \times g_i, f_j \right)$. This mixed discretization scheme greatly improves the accuracy of both the MFIE [2] and the nonchiral Müller equation [3].

As the chiral Müller operator has the same structure as its nonchiral counterpart, it is to be expected that the accuracy of its numerical solution will also be improved by the mixed discretization.

III. Numerical Results

Consider a chiral sphere of radius 66 cm, permittivity $\epsilon = 2\epsilon_0$, permeability $\mu = \mu_0$ and chirality parameter $\kappa = 0.5$, illuminated by a linearly polarized plane wave at a frequency of 150 MHz. The scattered fields are computed numerically using the following simulation techniques:

- *PMCHWT*: the chiral PMCHWT equation, tested using curl-conforming rotated RWG functions;
- *Muller*: the chiral Müller equation, tested using div-conforming RWG functions;
- *MxMuller*: the chiral Müller equation, tested using curl-conforming rotated BC functions.

For every simulation type, the mesh parameter (i.e. the minimum edge length) is varied between 9 and 25 cm.

The accuracy of the radar cross section (RCS) is assessed by means of the infinity norm of the error
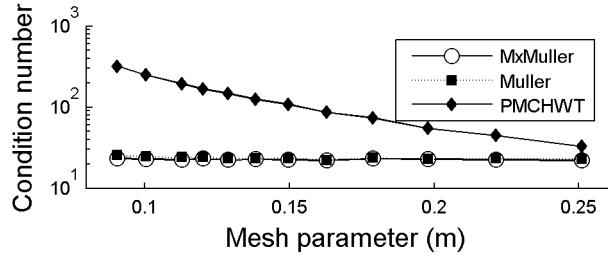

Fig. 3. Condition number of the system matrix.

with respect to the RCS predicted by the Mie series [6] (Fig. 1). The accuracy of the surface currents ($\hat{n} \times e$ and $\hat{n} \times h$) is measured using the $H_{div}^{-1/2}$ norm of the error with respect to the projection of the currents predicted by the Mie series onto the RWG basis (Fig. 2).

The mixed discretization scheme significantly reduces the discretization error of the chiral Müller equation. Indeed, in terms of accuracy *MxMuller* is comparable to *PMCHWT*, and superior to *Muller*. In contrast to *PMCHWT*, which requires preconditioning at small mesh parameters [5], *MxMuller* yields well-conditioned system matrices (Fig. 3) and can be solved efficiently using Krylov iterative methods.

IV. Conclusion

A mixed discretization scheme has been applied to the chiral Müller equation. For any given mesh, the solution yielded by this scheme is as accurate as the chiral PMCHWT equation's solution. In contrast, it does not suffer from dense discretization breakdown and therefore does not require preconditioning.

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