Analysis and Discretization of the Yukawa-Calderon Preconditioned CFIE

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Abstract: This work focuses on the spectral analysis of Loop, Star, Tree, and related Gram matrices and on their effects on the conditioning of the decomposed EFIE and second kind operator matrices. A theoretical explanation of the slower convergence of Loop-Star decompositions with respect to Loop-Tree decompositions will be provided together with a regularization to be used in conjunction with the Loop-Star decomposed Calderón EFIE.

Keywords: Integral Equations, CFIE, Calderón Preconditioning, Quasi-Helmholtz decompositions

1. Introduction

Loop-Star and Loop-Tree decompositions have been widely used to cure the low-frequency breakdown of the Electric Field Integral Equation (EFIE) which is due to an unfavorable scaling of the solenoidal and non solenoidal branches of the EFIE spectrum. Loop-Star and Loop-Tree decompositions solve the breakdown by enabling the rescaling of the EFIE spectrum branches [1-4]. Despite this, these decompositions are not able to change the shape of each branch [5]. EFIE ill-conditioning that can be traced back to the shape of EFIE spectrum branches can be referred as "dense discretization breakdown". In addition to these two breakdowns, when dealing with Loop-Star and Loop-Tree decompositions, there is an additional problem pertaining the conditioning of Loop, Star, and Tree associated Gram matrices (a Gram matrix of a basis function set $\{\boldsymbol{f}_i, i = 1, \ldots, N\}$ is the matrix $\bar{\bar{\mathbf{G}}}$ so that $(\bar{\bar{\mathbf{G}}})_{i,j} = \langle \boldsymbol{f}_i, \boldsymbol{f}_j \rangle = \int \boldsymbol{f}_i \cdot \boldsymbol{f}_j \mathrm{d}r$, where the integral is computed over the functions' domain). The Loop, Star, and Tree Gram matrices condition number is contributing to the overall condition number of the (decomposed) EFIE. Moreover when the Loop-Star,-Tree bases are used in conjunction with second kind integral equations (such as the Magnetic Field Integral Equation, MFIE, or the Calderón preconditioned EFIE), the Gram matrix condition number is the *sole* responsible for the overall condition number of the decomposed equation. This work will analyze the Gram matrix conditioning of Loop, Star, and Tree bases and it will quantify the effects on the conditioning of the decomposed EFIE and second kind operator matrices. The analysis will give a theoretical explanation of the slower convergence of Loop-Star decompositions with respect to Loop-Tree decompositions which has been observed in literature [2, 4, 6]. In addition the theoretical framework will provide a strategy for inverting the Loop-Star decomposition and for regularizing the Loop-Star Gram matrix. From this, a regularization to be used in conjunction with the Loop-Star decomposed Calderon EFIE [6] will be obtained.

Among several integral equations for analyzing radiation and scattering by perfect electrically conducting (PEC) objects, the Electric Field Integral Equation (EFIE) and the Magnetic Field Integral Equation (MFIE) are the most prominent ones.

The EFIE is ill-conditioned when either the frequency or the discretization density goes to zero leading to the dense-discretization or to the low-frequency breakdown respectively. The high condition number results in a high number of iterations when the discretized EFIE is solved iteratively that renders the EFIE solution difficult in several real case scenarios. These breakdowns can be solved by Calderon preconditioning.

The MFIE, instead, is traditionally known to give rise to less accurate solutions, when compared to the EFIE, and recent studies have established that the reason for that resides in the inaccuracy of traditional MFIE discretizations. A solution to this problem has been recently proposed and leverages on the use of mixed discretizations, i.e. a judicious choice of source and testing discretization spaces that accurately matches the operator mappings of the continuous MFIE.

When the structure of the scatterer under consideration is closed, both the EFIE and MFIE suffers from internal resonances, i.e. spurious null-spaces appearing when the wavenumber corresponds to a resonance of the internal problem. The presence of these resonances can severely jeopardize accuracy and solution time of both EFIE and MFIE. A solution to this problem is the use of the Combined Field Integral Equation (CFIE). The CFIE is a linear combination of EFIE and MFIE and can be proved to be resonance free. On the other hand, being a linear combination of EFIE and MFIE, inherits part of the ill-conditioning of the first and part of the inaccuracy of the second.

To solve the ill conditioning of the CFIE, the equation can be modified into the Yukawa-Calderón CFIE (YC-CFIE), proposed in [7, 8], i.e. a linear combination of the MFIE and of a modified Calderón preconditioned EFIE. However, with standard discretizations of the YC-CFIE, the inaccuracy is still inherited by the MFIE.

This work will solve this problem, by presenting a mixed discretized YC-CFIE that is stable and regularized till very low frequencies and independent of the discretization density. In addition we will prove the resonance free behavior of the equation and we will present a complete low-frequency stability and preconditioning analysis.

Numerical results will show the effectiveness of the proposed scheme and its applicability on real case scenarios. Preliminary results on this topic have been presented in [9].

2. Formulation

Consider the surface Γ of an orientable PEC object residing in a space of permittivity ϵ and permeability μ. Denote with $\hat{\boldsymbol{n}}_{\boldsymbol{r}}$ its outward pointing unit normal at $\boldsymbol{r}$. Denote with Ω^+ and Ω^- the exterior and interior regions of Γ respectively. An incident electromagnetic field $\left(\boldsymbol{E}^i(\boldsymbol{r}), \boldsymbol{H}^i(\boldsymbol{r})\right)$ is impinging on Γ inducing a surface current density $\boldsymbol{J}(\boldsymbol{r})$. The current $\boldsymbol{J}(\boldsymbol{r})$ can be retrieved by solving the Electric Field Integral Equation (EFIE)

$$\mathcal{T}\left(\boldsymbol{J}\right) = -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^i \tag{1}$$

where $\mathcal{T}(\boldsymbol{J}) = ik\,\mathcal{T}_s(\boldsymbol{J}) + \frac{1}{ik}\,\mathcal{T}_h(\boldsymbol{J})$ with

$$\mathcal{T}_s(\boldsymbol{J}) = \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \int_\Gamma \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}', \tag{2}$$

$$\mathcal{T}_h(\boldsymbol{J}) = -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \nabla \int_\Gamma \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi\,|\boldsymbol{r}-\boldsymbol{r}'|} \nabla_s \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}', \tag{3}$$

and $k = 2\pi/\lambda = \omega\sqrt{\epsilon\mu}$. Alternatively, if Γ is closed, $\boldsymbol{J}(\boldsymbol{r})$ can be retrieved by solving the Magnetic Field Integral Equation (MFIE)

$$\left(\frac{\mathcal{I}}{2} + \mathcal{K}\right)(\boldsymbol{J}) = \eta\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{H}^i\right) \tag{4}$$

where

$$\mathcal{K}(\boldsymbol{J}) = -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \int_\Gamma \nabla \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|} \times \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \tag{5}$$

and $\eta = \sqrt{\mu/\epsilon}$. Note that with these definitions the current $\boldsymbol{J}(\boldsymbol{r})$ represents the jump over Γ of the total magnetic field multiplied by the medium characteristic impedance.

Both the EFIE and the MFIE can be solved by using boundary elements, Γ is approximated by a mesh of planar triangles with average edge length h, and $\boldsymbol{J}(\boldsymbol{r})$ is approximated as

$$\boldsymbol{J}(\boldsymbol{r}) \approx \sum_{n=1}^{N} I_n \boldsymbol{f}_n(\boldsymbol{r}) \tag{6}$$

where $\boldsymbol{f}_n(\boldsymbol{r})$, $n = 1, \ldots, N$ are Rao-Wilton-Glisson (RWG) div-conforming basis functions defined on the mesh's N internal edges [10]. To simplify the notation in what follows the RWGs are taken without the edge length normalization. To solve the EFIE, (6) is substituted into (1) and the resulting equation is tested with the functions $\boldsymbol{f}_n$ yielding the $N \times N$ discretized EFIE system

$$\bar{\bar{\mathbf{T}}}\,\bar{\mathbf{I}} = \bar{\mathbf{V}} \tag{7}$$

where $(\bar{\bar{\mathbf{T}}})_{i,j} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \mathcal{T}(\boldsymbol{f}_j)\rangle$, $(\bar{\mathbf{V}})_i = -\langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^i\rangle$, and $(\bar{\mathbf{I}})_j = I_j$. Here and in the following $\langle a, b\rangle = \int_\Gamma a \cdot b\,\mathrm{d}\Gamma$. By solving (7), $\bar{\mathbf{I}}$ is recovered and an approximation of $\boldsymbol{J}$ is obtained via (6).

In a standard approach, the MFIE can be solved in a similar way, (6) is substituted into (1) and the resulting equation is tested with the functions $\boldsymbol{f}_n$ yielding the $N \times N$ discretized MFIE system

$$\left(\frac{\bar{\bar{\mathbf{G}}}}{2} + \bar{\bar{\mathbf{K}}}\right)\bar{\mathbf{I}} = \bar{\mathbf{V}}_H \tag{8}$$

where $\bar{\bar{\mathbf{K}}} = \langle \boldsymbol{f}_i, \mathcal{K}(\boldsymbol{f}_j)\rangle$, $(\bar{\mathbf{V}}_H)_i = -\langle \boldsymbol{f}_i, \eta\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{H}^i\right)\rangle$, and

$$\left(\bar{\bar{\mathbf{G}}}\right)_{i,j} = \langle \boldsymbol{f}_i, \mathcal{I}(\boldsymbol{f}_j)\rangle = \langle \boldsymbol{f}_i, \boldsymbol{f}_j\rangle \tag{9}$$

is the Gram matrix of the RWG basis. The MFIE is the sum of the identity and of an operator ($\mathcal{K}$) which is compact when Γ is smooth. Operators that are the sum of the identity and of a compact operator are called *second kind operators* and matrices arising from their discretization are well-conditioned. For this reason the condition number of the linear system matrix in (8) is usually much better than the conditioning of the linear system matrix in (7). A left multiplication of (1) by $\mathcal{T}$ gives the Calderón preconditioned EFIE

$$\mathcal{T}^2(\boldsymbol{J}) = \mathcal{T}\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^i\right). \tag{10}$$

The Calderón identity

$$-\mathcal{T}^2 = \frac{\mathcal{I}}{4} - \mathcal{K}^2 \tag{11}$$

and the fact that also $\mathcal{K}^2$ is a compact operator, show that the Calderón preconditioned EFIE operator is a second kind operator and thus, upon discretization, gives rise to well-conditioned linear system matrices.

When the surface Γ is closed and when k corresponds to an internal resonance, i.e. to an eigenvalue of the Laplacian on Γ's interior region, both the EFIE and MFIE are not univocally solvable [11]. When k is approaching such a resonant value, the linear systems arizing from the discretization of the EFIE and of the MFIE become almost singular and difficult to solve. A similar problem is plaguing the Calderón preconditioned EFIE. A classical solution is represented by the use of the Combined Field Integral Equation (CFIE) [12] which is a linear combination of EFIE and MFIE

$$\alpha\mathcal{T}\left(\boldsymbol{J}\right)+\left(\frac{\mathcal{I}}{2}+\mathcal{K}\right)\left(\boldsymbol{J}\right)=-\alpha\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{E}^i+\eta\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{H}^i\right) \tag{12}$$

where α is real and $\alpha\neq 0$. The CFIE can be proved to be uniquely solvable for any value of k. On the other hand, the presence of the EFIE operator $\mathcal{T}$ results in an ill-conditioning that, althoght milder than the ill-conditioning of the EFIE, still renders the equation difficult to be solved in many real case scenarios. One could think that, to obtain a resonance-free *and* well-conditioned equation, one could linearly combine the MFIE and the Calderón preconditioned EFIE

$$\alpha\mathcal{T}^2\left(\boldsymbol{J}\right)+\left(\frac{\mathcal{I}}{2}+\mathcal{K}\right)\left(\boldsymbol{J}\right)=-\alpha\mathcal{T}\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{E}^i\right)+\eta\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{H}^i\right) \tag{13}$$

Unfortunately this choice does not lead to a resonance-free equation since

$$\mathcal{T}^2=-\frac{\mathcal{I}}{2}+\mathcal{K}^2=-\left(\frac{\mathcal{I}}{2}-\mathcal{K}\right)\left(\frac{\mathcal{I}}{2}+\mathcal{K}\right) \tag{14}$$

so that the null space of $\mathcal{T}^2$ contains the null space of $\left(\frac{\mathcal{I}}{2}+\mathcal{K}\right)$ and so does (13). A way to solve this problem is to precondition the EFIE operator $\mathcal{T}$ not with the operator $\mathcal{T}$ itself, but with a localized counterpart of it. This localization can be obtained either by a space windowing of the Green's function or by using, in the leftmost operator $\mathcal{T}$, a purely complex wavenumber obtaining

$$\mathcal{C}\left(\boldsymbol{J}\right)=-\alpha\mathcal{T}^{loc}\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{E}^i\right)+\eta\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{H}^i\right) \tag{15}$$

where

$$\mathcal{C}\left(\boldsymbol{J}\right)=\alpha\mathcal{T}^{loc}\mathcal{T}\left(\boldsymbol{J}\right)+\left(\frac{\mathcal{I}}{2}+\mathcal{K}\right)\left(\boldsymbol{J}\right) \tag{16}$$

$\mathcal{T}^{loc}(\boldsymbol{J})=-k\,\mathcal{T}_s^{loc}(\boldsymbol{J})-\frac{1}{k}\,\mathcal{T}_h^{loc}(\boldsymbol{J})$ with

$$\mathcal{T}_s^{loc}(\boldsymbol{J})=\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\int_\Gamma\frac{e^{-k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi\left|\boldsymbol{r}-\boldsymbol{r}'\right|}\boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \tag{17}$$

$$\mathcal{T}_h^{loc}(\boldsymbol{J})=-\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\nabla\int_\Gamma\frac{e^{-k|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi\left|\boldsymbol{r}-\boldsymbol{r}'\right|}\nabla_s\cdot\boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}' \tag{18}$$

The remaining of the paper will be devoted to the analysis the properties of (15) and of its stable and conforming discretization.

3. Mapping properties and uniqueness of the solution

Denoting by $H^{m}_{div}(\Gamma)$ the divergence Sobolev space on Γ of regularity m [13], for the operators $\mathcal{T}$ and $\mathcal{K}$ we have

$$\mathcal{T} : H^{m-\frac{1}{2}}_{div} \to H^{m-\frac{1}{2}}_{div} \quad \text{and} \quad \mathcal{K} : H^{m-\frac{1}{2}}_{div} \to H^{m+\frac{1}{2}}_{div} \tag{19}$$

[13]. Similarly

$$\mathcal{T}^{loc} : H^{m-\frac{1}{2}}_{div} \to H^{m-\frac{1}{2}}_{div} \tag{20}$$

then

$$\mathcal{C} : H^{m-\frac{1}{2}}_{div} \to H^{m-\frac{1}{2}}_{div} \tag{21}$$

We will now prove the uniqueness of the solution of (15). For doing this we will adapt to our case the demonstrative approach of Bruno et al. [14].

Given an operator $\mathcal{A}$, call $\mathcal{A}^T$ the adjoint operator with respect to $\langle .,. \rangle$. In other words, $\mathcal{A}^T$ is the operator such that

$$\langle \boldsymbol{I}, \mathcal{A}(\boldsymbol{J}) \rangle = \left\langle \mathcal{A}^T(\boldsymbol{I}), \boldsymbol{J} \right\rangle \quad \forall \boldsymbol{I}, \boldsymbol{J} \in H^{-\frac{1}{2}}_{div}(\Gamma) \tag{22}$$

Since the operators $\mathcal{T}$, $\mathcal{T}^{loc}$, $\frac{\mathcal{I}}{2} + \mathcal{K}$ are Fredholm operators of order 0 [11], we get that so is $\mathcal{C}$ and that, by the Fredholm alternative and the non degeneracy of $\langle .,. \rangle$, $\mathcal{C}$ is invertible if and only if $\mathcal{C}^T$ is is injective, i.e. if

$$\mathcal{C}^T(\boldsymbol{J}) = 0 \Rightarrow \boldsymbol{J} = 0 \tag{23}$$

The following properties

$$\mathcal{T}^T = \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times) \tag{24}$$

$$\left(\mathcal{T}^{loc}\right)^T = \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times) \tag{25}$$

$$\left(\frac{\mathcal{I}}{2} + \mathcal{K}\right)^T = -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \left(\frac{\mathcal{I}}{2} - \mathcal{K}\right)(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times) \tag{26}$$

holds [13]. Then

$$\left(\mathcal{T}^{loc}\mathcal{T}\right)^T = \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times)\right) = -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}\mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times) \tag{27}$$

and

$$\mathcal{C}^T(\boldsymbol{J}) = \left[-\alpha \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}\mathcal{T}^{loc} - \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \left(\frac{\mathcal{I}}{2} - \mathcal{K}\right)\right](\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J}). \tag{28}$$

Define the operators

$$\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \tilde{\mathcal{K}}(\boldsymbol{J}) = \nabla \times \int_\Gamma \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi |\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{J}(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}' \quad \boldsymbol{r} \in \Omega^+ \cup \Omega^- \tag{29}$$

and

$$\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \tilde{\mathcal{T}}(\boldsymbol{J}) = -ik \int_\Gamma \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi |\boldsymbol{r}-\boldsymbol{r}'|} \boldsymbol{J}(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}' + \frac{1}{ik} \nabla \int_\Gamma \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi |\boldsymbol{r}-\boldsymbol{r}'|} \nabla_s \cdot \boldsymbol{J}(\boldsymbol{r}')\, \mathrm{d}\boldsymbol{r}' \quad \boldsymbol{r} \in \Omega^+ \cup \Omega^- \tag{30}$$

By denoting the exterior and interior trace operators [11] by τ_+ and τ_- and defining

$$\tilde{\mathcal{C}}^T(\boldsymbol{J}) = \left[-\alpha\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \tilde{\mathcal{T}}\right)\mathcal{T}^{loc} + \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \tilde{\mathcal{K}}\right](\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J}) \tag{31}$$

we have that

$$\mathcal{C}^T(\boldsymbol{J}) = \tau_+\left(\tilde{\mathcal{C}}^T(\boldsymbol{J})\right) \tag{32}$$

so that the condition $\mathcal{C}^T(\boldsymbol{J}) = 0$ and the fact that $\tilde{\mathcal{C}}^T$ solves the Helmholtz equation with outgoing radiation condition we get

$$\mathcal{C}^T(\boldsymbol{J}) = 0 \Rightarrow \tau^+\tilde{\mathcal{C}}^T = 0 \Rightarrow \tilde{\mathcal{C}}^T = 0 \quad \forall \boldsymbol{r} \in \Omega^+ \Rightarrow \tau^+\left(\nabla \times \tilde{\mathcal{C}}^T\right) = 0 \tag{33}$$

From the jump relationships of single and double layers we get

$$\tau^-\tilde{\mathcal{C}}^T(\boldsymbol{J}) = \tau^+\tilde{\mathcal{C}}^T(\boldsymbol{J}) - \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J} = \boldsymbol{J} \tag{34}$$

$$\tau^-\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \nabla \times \tilde{\mathcal{C}}^T(\boldsymbol{J})\right) = \tau^+\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \nabla \times \tilde{\mathcal{C}}^T(\boldsymbol{J})\right) + ik\alpha\mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J}) = ik\alpha\mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J}) \tag{35}$$

Thus

$$\begin{aligned} ik\alpha\int_\Gamma \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J}^* \cdot \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J})\, \mathrm{d}\boldsymbol{r}' &= ik\alpha\int_\Gamma \boldsymbol{J}^* \cdot \mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J})\, \mathrm{d}\boldsymbol{r}' \\ &= \int_\Gamma \left(\mathcal{C}^T\right)^* \cdot \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \nabla \times \mathcal{C}^T\, \mathrm{d}\boldsymbol{r}' \\ &= -\int_\Gamma \left(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{C}^T\right)^* \cdot \nabla \times \mathcal{C}^T\, \mathrm{d}\boldsymbol{r}' \\ &= \int_\Gamma k^2\left|\mathcal{C}^T\right|^2 - \left|\nabla \times \mathcal{C}^T\right|^2\, \mathrm{d}\boldsymbol{r}' \end{aligned} \tag{36}$$

Since $\int_\Gamma \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J}^* \cdot \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J})\, \mathrm{d}\boldsymbol{r}'$ is purely real, it follows that $ik\alpha\int_\Gamma \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J}^* \cdot \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J})\, \mathrm{d}\boldsymbol{r}'$ is purely imaginary, while $\int_\Gamma k^2\left|\mathcal{C}^T\right|^2 - \left|\nabla \times \mathcal{C}^T\right|^2 \mathrm{d}\boldsymbol{r}'$ is purely real. Then the equality in (36) is satisfied only if

$$\int_\Gamma \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J}^* \cdot \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \mathcal{T}^{loc}(\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{J})\, \mathrm{d}\boldsymbol{r}' = 0 \tag{37}$$

from which $\boldsymbol{J} = 0$, which terminates the proof.

4. Mixed discretization

It has been shown that the standard discretization (8) of the MFIE leads to incorrect results at low frequencies [15]. The standard discretization of the MFIE uses the div-conforming RWGs both as source and testing functions. On the other hand, the mapping properties of the MFIE operator (equation (19)) suggests that the MFIE testing functions should be curl-conforming basis functions. For this reason the mixed discretized MFIE has been proposed in [16] and further analyzed in [17, 18, 19, 20]. In the mixed discretized MFIE, the testing functions are the Buffa-Christiansen (BC) or the Chen-Wilton (CW) basis functions defined on the barycentric refinement. In [19] and [20], in particular, it has been shown that that the mixed MFIE does not suffer from the low-frequencies inaccuracies observed in [15] for the standard MFIE.

At very low frequencies, the inaccuracies of a standardly discretized MFIE could jeopardize also the accuracy of the YC-CFIE. However, since the mapping properties of the Yukawa-Calderón CFIE are the same of the MFIE (equation (21)), this suggests that a mixed discretization will solve the inaccuracy problems of the equation.

The mixed discretized YC-CFIE that we propose reads

$$\left(\alpha\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}}+\frac{\bar{\bar{\mathbf{G}}}_{mix}}{2}+\bar{\bar{\mathbf{K}}}_{mix}\right)\bar{\mathbf{I}}=\alpha\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\mathbf{V}}+\bar{\mathbf{V}}_{H}^{mix} \tag{38}$$

where

$$(\bar{\bar{\mathbf{G}}}_{mix})_{i,j}=\left\langle\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{f}_i,\boldsymbol{f}_j^{BC}\right\rangle \tag{39}$$

$$\left(\bar{\bar{\mathbf{K}}}_{mix}\right)_{i,j}=\left\langle\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{f}_i^{BC},\mathcal{K}(\boldsymbol{f}_j)\right\rangle \tag{40}$$

$$(\bar{\mathbf{V}}_H)_i=-\left\langle\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{f}_i^{BC},\eta\left(\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{H}^i\right)\right\rangle \tag{41}$$

$$\bar{\bar{\mathbf{T}}}^{loc}=\bar{\bar{\mathbf{T}}}_s^{loc}+\bar{\bar{\mathbf{T}}}_h^{loc} \tag{42}$$

and

$$\left(\bar{\bar{\mathbf{T}}}_s^{loc}\right)_{i,j}=\left\langle\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{f}_i^{BC},\mathcal{T}_s^{loc}(\boldsymbol{f}_j^{BC})\right\rangle\quad\left(\bar{\bar{\mathbf{T}}}_s^{loc}\right)_{i,j}=\left\langle\hat{\boldsymbol{n}}_{\boldsymbol{r}}\times\boldsymbol{f}_i^{BC},\mathcal{T}_h^{loc}(\boldsymbol{f}_j^{BC})\right\rangle. \tag{43}$$

The functions $\boldsymbol{f}_i^{BC}$ are the Buffa-Christiansen basis functions [21] defined on the barycentric refinement. These functions, similarly to the RWGs, are div-conforming and defined on the mesh's N internal edges, however they are also quasi-curl-conforming in the sense that the mixed gram matrix is well-conditioned. The reader may also refer to [5] for further details on the shape and definition of these functions.

The low frequency behavior of this equation can be analyzed by using a Loop-Star decomposition [3]. Denote with $\bar{\bar{\mathbf{\Lambda}}}$ and $\bar{\bar{\mathbf{\Sigma}}}$ the Loop and Star-to-RWGs transformation matrices respectively [3]. Denote with $\bar{\bar{\mathbf{H}}}=\left[\bar{\bar{\mathbf{\Lambda}}},\bar{\bar{\mathbf{\Sigma}}}\right]$ the Loop-Star-to-RWG transformation matrix. The Loop-Star decomposed MFIE presents the following block dependence of the frequency [19]

$$\bar{\bar{\mathbf{H}}}_D^T\left(\frac{\bar{\bar{\mathbf{G}}}_{mix}}{2}+\bar{\bar{\mathbf{K}}}_{mix}\right)\bar{\bar{\mathbf{H}}}=\begin{pmatrix}\bar{\bar{\mathbf{\Sigma}}}^T\left(\frac{\bar{\bar{\mathbf{G}}}_{mix}}{2}+\bar{\bar{\mathbf{K}}}_{mix}\right)\bar{\bar{\mathbf{\Lambda}}} & \bar{\bar{\mathbf{\Sigma}}}^T\left(\frac{\bar{\bar{\mathbf{G}}}_{mix}}{2}+\bar{\bar{\mathbf{K}}}_{mix}\right)\bar{\bar{\mathbf{\Sigma}}}\\ \bar{\bar{\mathbf{\Lambda}}}^T\left(\frac{\bar{\bar{\mathbf{G}}}_{mix}}{2}+\bar{\bar{\mathbf{K}}}_{mix}\right)\bar{\bar{\mathbf{\Lambda}}} & \bar{\bar{\mathbf{\Lambda}}}^T\left(\frac{\bar{\bar{\mathbf{G}}}_{mix}}{2}+\bar{\bar{\mathbf{K}}}_{mix}\right)\bar{\bar{\mathbf{\Sigma}}}\end{pmatrix} \tag{44}$$

$$=\begin{pmatrix}O(\omega^2) & O(1)\\ O(1) & O(1)\end{pmatrix} \tag{45}$$

A similar behavior is valid for the matrices $\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}}$, in fact

$$\bar{\bar{\mathbf{H}}}_D^T\left(\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}}\right)\bar{\bar{\mathbf{H}}}=\begin{pmatrix}\bar{\bar{\mathbf{\Sigma}}}^T\left(\bar{\bar{\mathbf{T}}}_s^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}}_s\right)\bar{\bar{\mathbf{\Lambda}}} & \bar{\bar{\mathbf{\Sigma}}}^T\left(\bar{\bar{\mathbf{T}}}_s^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}}\right)\bar{\bar{\mathbf{\Sigma}}}\\ \bar{\bar{\mathbf{\Lambda}}}^T\left(\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}}_s\right)\bar{\bar{\mathbf{\Lambda}}} & \bar{\bar{\mathbf{\Lambda}}}^T\left(\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}}\right)\bar{\bar{\mathbf{\Sigma}}}\end{pmatrix} \tag{46}$$

$$=\begin{pmatrix}O(\omega^2) & O(1)\\ O(1) & O(1)\end{pmatrix} \tag{47}$$

It can be shown that

$$\bar{\bar{\mathbf{H}}}_D^T\bar{\mathbf{V}}_H^{mix}=\begin{pmatrix}O(\omega)\\ O(1)\end{pmatrix} \tag{48}$$

and that

$$\bar{\bar{\mathbf{H}}}^T\bar{\mathbf{V}} = \begin{pmatrix} O(\omega) \\ O(1) \end{pmatrix} \tag{49}$$

[19]. Moreover, notice that

$$\bar{\bar{\mathbf{H}}}_D^T\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\mathbf{V}} = \bar{\bar{\mathbf{H}}}_D^T\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{H}}}_D\left(\bar{\bar{\mathbf{H}}}^T\bar{\bar{\mathbf{G}}}_{mix}\bar{\bar{\mathbf{H}}}_D\right)^{-1}\bar{\bar{\mathbf{H}}}^T\bar{\mathbf{V}} \tag{50}$$

$$= \begin{pmatrix} O(\omega) & O(\omega) \\ O(\omega) & O(1) \end{pmatrix}\begin{pmatrix} O(1) & \bar{\bar{\mathbf{0}}} \\ O(1) & O(1) \end{pmatrix}\begin{pmatrix} O(\omega) \\ O(1) \end{pmatrix} \tag{51}$$

$$= \begin{pmatrix} O(\omega) \\ O(1) \end{pmatrix} \tag{52}$$

Finally, the equation

$$\bar{\bar{\mathbf{H}}}_D^T\left(\alpha\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}} + \frac{\bar{\bar{\mathbf{G}}}_{mix}}{2} + \bar{\bar{\mathbf{K}}}_{mix}\right)\bar{\bar{\mathbf{H}}}\left(\bar{\bar{\mathbf{H}}}^{-1}\bar{\mathbf{I}}\right) = \bar{\bar{\mathbf{H}}}_D^T\left(\alpha\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\mathbf{V}} + \bar{\mathbf{V}}_H^{mix}\right) \tag{53}$$

has the structure

$$\begin{pmatrix} O(\omega^2) & O(1) \\ O(1) & O(1) \end{pmatrix}\left(\bar{\bar{\mathbf{H}}}^{-1}\bar{\mathbf{I}}\right) = \begin{pmatrix} O(\omega) \\ O(1) \end{pmatrix} \tag{54}$$

from which

$$\left(\bar{\bar{\mathbf{H}}}^{-1}\bar{\mathbf{I}}\right) = \begin{pmatrix} O(1) \\ O(\omega) \end{pmatrix} \tag{55}$$

which is a physical scaling in accordance to the continuity equation

$$\nabla \cdot \boldsymbol{J}(\boldsymbol{r}) = O(\omega). \tag{56}$$

Preconditioning of the Loop-Star decomposed YC-CFIE

Equations (53) and (54) shows that the mixed discretized YC-CFIE (38) is stable till statics when infinite precision is used. In practice, however, finite precision requires that (53) instead (38) is actually solved at very low-frequencies. Unfortunately, the use of the Loop-Star decomposition introduces an additional h-dependenent ill-conditioning that needs to be regularized [22]. Since up to compact operators

$$\left(\alpha\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}} + \frac{\bar{\bar{\mathbf{G}}}_{mix}}{2} + \bar{\bar{\mathbf{K}}}_{mix}\right) \asymp \bar{\bar{\mathbf{G}}}_{mix} \tag{57}$$

we get that

$$\bar{\bar{\mathbf{H}}}_D^T\left(\alpha\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}} + \frac{\bar{\bar{\mathbf{G}}}_{mix}}{2} + \bar{\bar{\mathbf{K}}}_{mix}\right)\bar{\bar{\mathbf{H}}} \asymp \bar{\bar{\mathbf{H}}}^T\bar{\bar{\mathbf{G}}}_{mix}\bar{\bar{\mathbf{H}}} \tag{58}$$

So that we can use as a left regularizer for the Loop-Star decomposed YC-CFIE the matrix

$$\bar{\bar{\mathbf{P}}} = \bar{\bar{\mathbf{H}}}_D^{-1} \left(\bar{\bar{\mathbf{H}}}^T\right)^{-1} = \left[\bar{\bar{\boldsymbol{\Sigma}}}, \bar{\bar{\boldsymbol{\Lambda}}}\right]^{-1} \left[\begin{array}{c} \bar{\bar{\boldsymbol{\Lambda}}} \\ \bar{\bar{\boldsymbol{\Sigma}}} \end{array}\right]^{-1} \tag{59}$$

$$= \left[\begin{array}{c} \left(\bar{\bar{\boldsymbol{\Sigma}}}^T\bar{\bar{\boldsymbol{\Sigma}}}\right)^{-1} \bar{\bar{\boldsymbol{\Sigma}}}^T \\ \left(\bar{\bar{\boldsymbol{\Lambda}}}^T\bar{\bar{\boldsymbol{\Lambda}}}\right)^{-1} \bar{\bar{\boldsymbol{\Lambda}}}^T \end{array}\right] \left[\left(\bar{\bar{\boldsymbol{\Lambda}}}^T\bar{\bar{\boldsymbol{\Lambda}}}\right)^{-1} \bar{\bar{\boldsymbol{\Lambda}}}^T, \left(\bar{\bar{\boldsymbol{\Sigma}}}^T\bar{\bar{\boldsymbol{\Sigma}}}\right)^{-1} \bar{\bar{\boldsymbol{\Sigma}}}^T\right] \tag{60}$$

$$= \left(\begin{array}{cc} \bar{\bar{\mathbf{0}}} & \left(\bar{\bar{\boldsymbol{\Sigma}}}^T\bar{\bar{\boldsymbol{\Sigma}}}\right)^{-1} \\ \left(\bar{\bar{\boldsymbol{\Lambda}}}^T\bar{\bar{\boldsymbol{\Lambda}}}\right)^{-1} & \bar{\bar{\mathbf{0}}} \end{array}\right) \tag{61}$$

Note that the inverses required by the definition of $\bar{\bar{\mathbf{P}}}$ can be obtained by using a preconditioned scheme for inverting the Laplacian (matrix $\bar{\bar{\boldsymbol{\Lambda}}}^T\bar{\bar{\boldsymbol{\Lambda}}}$) and a direct inversion scheme (matrix $\bar{\bar{\boldsymbol{\Sigma}}}^T\bar{\bar{\boldsymbol{\Sigma}}}$) as explained in [22]. Finally, the preconditioned Loop-Star decomposed YC-CFIE reads

$$\bar{\bar{\mathbf{P}}}\bar{\bar{\mathbf{H}}}_D^T \left(\alpha\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{T}}} + \frac{\bar{\bar{\mathbf{G}}}_{mix}}{2} + \bar{\bar{\mathbf{K}}}_{mix}\right)\bar{\bar{\mathbf{H}}}\left(\bar{\bar{\mathbf{H}}}^{-1}\bar{\mathbf{I}}\right) = \bar{\bar{\mathbf{P}}}\bar{\bar{\mathbf{H}}}_D^T \left(\alpha\bar{\bar{\mathbf{T}}}^{loc}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\mathbf{V}} + \bar{\mathbf{V}}_H^{mix}\right) \tag{62}$$

to be solved in the unknown $\left(\bar{\bar{\mathbf{H}}}^{-1}\bar{\mathbf{I}}\right)$.

Numerical results

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