

# Hypohamiltonian and Almost Hypohamiltonian Graphs

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*To my mother*



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# Contributions

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## Sections 2.1.1–2.1.3

M. Jooyandeh, B. D. McKay, P. R. J. Östergård, V. H. Pettersson, and C. T. Zamfirescu. Planar hypohamiltonian graphs on 40 vertices. To appear in: *J. Graph Theory*. arXiv:1302.2698 [math.CO]

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## Section 2.2

C. T. Zamfirescu. Hypohamiltonian graphs and their crossing number. *Electron. J. Combin.* **19** (2012) #P12.

## Sections 4.1–4.2

A. Shabbir, C. T. Zamfirescu, and T. I. Zamfirescu. Intersecting longest paths and longest cycles: A survey. *Electron. J. Graph Theory Appl.* **1** (2013) 56–76.

## Section 4.3

B. Schauerte and C. T. Zamfirescu. Regular graphs in which every pair of points is missed by some longest cycle. *Ann. Univ. Craiova, Math. Comp. Sci. Ser.* **33** (2006) 154–173.

## Chapter 5

C. T. Zamfirescu. On non-hamiltonian graphs for which every vertex-deleted subgraph is traceable. Submitted.



# Summary

This Dissertation is structured as follows. In Chapter 1, we give a short historical overview and define fundamental concepts. Chapter 2 contains a clear narrative of the progress made towards finding the smallest planar hypohamiltonian graph, with all of the necessary theoretical tools and techniques—especially Grinberg’s Criterion. Consequences of this progress are distributed over all sections and form the leitmotif of this Dissertation. Chapter 2 also treats girth restrictions and hypohamiltonian graphs in the context of crossing numbers. Chapter 3 is a thorough discussion of the newly introduced almost hypohamiltonian graphs and their connection to hypohamiltonian graphs. Once more, the planar case plays an exceptional role. At the end of the chapter, we study almost hypotraceable graphs and Gallai’s problem on longest paths. The latter leads to Chapter 4, wherein the connection between hypohamiltonicity and various problems related to longest paths and longest cycles are presented. Chapter 5 introduces and studies non-hamiltonian graphs in which every vertex-deleted subgraph is traceable, a class encompassing hypohamiltonian and hypotraceable graphs. We end with an outlook in Chapter 6, where we present a selection of open problems enriched with comments and partial results.



# Chapter 1

## Introduction

Throughout this Dissertation, all graphs are undirected, finite, connected, and contain neither loops nor multiple edges, unless explicitly stated otherwise. A graph  $G$  is *hamiltonian* (*traceable*) if it contains a *hamiltonian cycle* (*hamiltonian path*), i.e. a cycle (path) visiting every vertex of the graph. (More generally, a subgraph of  $G$  which contains all vertices of  $G$  will be called *spanning*.) The term is named after Sir William Rowan Hamilton, who presented in 1857 a puzzle in which, by travelling along the edges of a dodecahedron, a path was sought that visits every vertex of said dodecahedron precisely once, and ends in the same vertex it began: the “Icosian game”. However, Hamilton certainly was not the first to study spanning cycles in graphs—Leonhard Euler [37] had already treated the problem in the (closed variant of the) “knight’s tour” problem in 1766. Even concerning spanning cycles in polyhedra, this had already been done by Thomas P. Kirkman in 1856, see [86]. For more on the history of graph theory in the period 1736–1936, see the monograph by Biggs, Lloyd, and Wilson [11].

More recent material on hamiltonian cycles (and related concepts) in graphs and hypergraphs can be found in Gould’s comprehensive three-part survey [48–50]. For an overview of results on hamiltonicity in directed graphs, see [90] by Kühn and Osthus. From a complexity standpoint, determining hamiltonicity is difficult and one of Karp’s famous 21 NP-complete problems [83]. It even remains NP-complete if one restricts the problem to cubic polyhedra [44].

We introduce the central notion of this Dissertation. A graph  $G$  is *hypohamiltonian* if  $G$  does not contain a hamiltonian cycle but for any  $v \in V(G)$  the graph  $G - v$  does contain a hamiltonian cycle. A graph is *hypotraceable* if we replace in the preceding sentence every instance of “cycle” by “path”. The smallest hypohamiltonian graph is the famous Petersen graph [107] on 10 vertices, shown in Figs. 1 and 45. For a proof that it is indeed the smallest such graph, see e.g. [67]. Petersen published this graph in 1898, but it appears in the literature (at least) as early as 1886, in a semi-mathematical, semi-philosophical article of Kempe [85]. The seven smallest hypohamiltonian graphs are shown in Fig. 10.

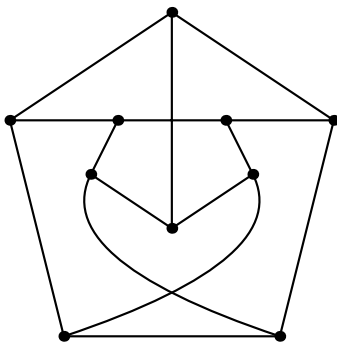


Fig. 1: Petersen’s graph drawn with two edge crossings—that it cannot be drawn with fewer is proven in Section 2.2. The Petersen graph has order 10 and is the smallest hypohamiltonian graph.

The study of hypohamiltonian graphs was initiated in the early sixties by René Sousselier [10, 119]. He stated a mathematical problem of recreational nature in [119] entitled “Le Cercle des Irascibles”. His paper is in French, but a translation can be found in Holton and Sheehan’s survey on hypohamiltonian graphs [71]. A solution to Sousselier’s problem was given by Gaudin, Herz, and Rossi [46] in 1964. Further early work on the subject was done by Busacker and Saaty [18], Lindgren [93], and Herz, Duby, and Vigué [67].

The topic was quickly picked up by many researchers. It was extensively studied by Thomassen [125–129]—Thomassen’s body of work is an important influence for this Dissertation, both concerning techniques and style. Further sig-

nificant early contributions include work of Bondy [12], Chvátal [25], Doyen and van Diest [33], Harary and Thomassen [63], and Collier and Schmeichel [28, 29]. For further details, see the survey by Holton and Sheehan [71]. Not included therein are papers published in the past two decades, for instance work of Katerinis [84], Máčajová and Škoviera [94, 95], Araya and Wiener [7, 141], McKay [96] and publications by the author and his collaborators [47, 80, 143–146] discussed here. Ozeki and Vrána [103] recently used hypohamiltonicity to show that there exist infinitely many graphs which are 2-hamiltonian<sup>1</sup> but not 2-edge-hamiltonian-connected<sup>2</sup>.

Regarding combinatorial optimisation, Grötschel [52, 53], and Grötschel and Wakabayashi [55–58] extensively discuss hypohamiltonian (as well as hypotraceable) graphs in the context of the travelling salesman polytope. The computational complexity of determining whether a graph is hypohamiltonian is unknown, but is believed to be high, see Grötschel’s paper [53]. Recent applications of concepts closely related to hypohamiltonicity, e.g. fault-tolerant networks, can be found in [104].

Substantial work has also been done investigating *directed* graphs which are hypohamiltonian, but due to the extensive literature, here we can only select certain articles and invite the reader to follow the references found there. Early work includes Thomassen’s [128] and an article by Fouquet and Jolivet [41], a paper by Grötschel, Thomassen, and Wakabayashi [54], and work of Penn and Witte [105]. Using a result from [105], Thomassen [130] disproves the old conjecture of Adám [3] that any digraph containing a directed cycle has an arc whose reversal decreases the total number of directed cycles. For articles published recently—these also provide overviews—see [1, 2].

Likewise, we will not treat the interesting family of *infinite* hypohamiltonian graphs. For results therein, see for instance Thomassen’s paper [127] and

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<sup>1</sup>Let  $k \geq 1$ . A graph  $G$  is *k-hamiltonian* if for any  $S \subset V(G)$  with  $|S| \leq k$  we have that  $G - S$  is hamiltonian.

<sup>2</sup>A graph  $G$  is *2-edge-hamiltonian-connected* if for any  $X \subset \{x_1x_2 : x_1, x_2 \in V(G)\}$  with  $1 \leq |X| \leq 2$ ,  $G \cup X$  has a hamiltonian cycle containing all edges in  $X$ , where  $G \cup X$  is the graph obtained from  $G$  by adding all edges in  $X$ .

Schmidt-Steup’s article [112], where the latter settles the question raised in the former whether there exists an infinite hypohamiltonian graph which is locally finite. (In fact, Schmidt-Steup showed that such graphs exist even with the additional condition of planarity imposed.)

Lastly, we mention hypohamiltonicity in connection with snarks. We shall call a *reducible snark* a bridgeless cubic graph which has chromatic index<sup>3</sup> 4 and girth at least 5. A *snark* is a cyclically 4-edge-connected<sup>4</sup> reducible snark. Early work on the subject includes Fiorini’s paper [39] from 1983. Fiorini showed that there exist infinitely many hypohamiltonian snarks [39]. (Later it was discovered that this had already been done by Gutt [60].) Steffen [121] proved that there exist hypohamiltonian snarks of order  $n$  for every even  $n \geq 92$  (and certain  $n < 92$ ). For more material, see e.g. [14, 94, 116, 122]. Hypohamiltonian snarks have been studied in connection with the famous Cycle Double Cover Conjecture, see [14] for more details, and Sabidussi’s Compatibility Conjecture [40].

The number of hypohamiltonian snarks on  $n \leq 36$  vertices can be found in [14, Table 2], see also sequence A218880 in Sloane’s On-Line Encyclopedia of Integer Sequences [118]. Many snarks are hypohamiltonian, but not all—consider for instance the reducible snark shown in [150, Fig. 12] and discussed in Section 3.6. The smallest snark is the Petersen graph shown in Figs. 1 and 45. Steffen [120] also showed that every cubic hypohamiltonian graph with chromatic index 4 is bicritical, i.e. the graph itself is not 3-edge-colourable but the removal of any two distinct vertices yields a 3-edge-colourable graph. Nedela and Škoviera [101] showed that every cubic bicritical graph is cyclically 4-edge-connected and has girth at least 5. Therefore, every cubic hypohamiltonian graph with chromatic index 4 must be a snark. In 2015, Steffen [122] published a conjecture on hypohamiltonian snarks—see Problem 3 in Chapter 6. In the recent not yet pub-

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<sup>3</sup>The *chromatic index* of a graph is the smallest number of colours necessary to colour the edges of the graph such that any two edges sharing an end-point do not have the same colour.

<sup>4</sup>In a graph  $G$ ,  $M \subset E(G)$  is *cycle-separating* if  $G - M$  is disconnected, and at least two of its components contain cycles. (Note that there exist graphs with no cycle-separating sets, for instance  $K_{3,3}$ .) A graph  $G$  which contains disjoint cycles is *cyclically  $k$ -edge-connected* if no set of fewer than  $k$  edges is cycle-separating in  $G$ . The *cyclic edge-connectivity* of  $G$  is the maximum  $k$  such that  $G$  is cyclically  $k$ -edge-connected.

lished manuscript [42], de Freitas, Nunes da Silva, and Lucchesi show that every hypohamiltonian snark has a 5-flow<sup>5</sup>, thus answering a question proposed by Cavicchioli, Murgolo, Ruini, and Spaggiari in 2003 [20].

We now provide definitions and notation used throughout this Dissertation—for notions not defined here, please see Diestel’s book [32]. For a graph  $G$ , we will denote by  $V(G)$  its vertex set and by  $E(G)$  its edge set. For a set  $S$ ,  $|S|$  shall be the cardinality of  $S$ .  $|V(G)|$  is the *order* of  $G$  and  $|E(G)|$  its *size*. Unless stated otherwise, we put  $n = |V(G)|$  and  $m = |E(G)|$ , and when we describe a graph as the “smallest” or “smaller than”, we always refer to the order of the graph.

For a vertex  $v$ , we denote by  $N(v)$  the set of vertices which are joined to  $v$  by an edge, and put  $N[v] = N(v) \cup \{v\}$ . We call  $N(v)$  ( $N[v]$ ) the *open neighbourhood* (*closed neighbourhood*) of  $v$ .  $|N(v)|$  is the *degree* of  $v$ . A vertex is *cubic* if its degree is 3. A graph, a cycle (in a graph), or a face (in a planar graph) is *cubic* if all of its vertices are cubic. Denote by  $P_k$  ( $C_k$ ) a path (cycle) on  $k$  vertices. A cycle on  $k$  vertices will also be called a *k-cycle*. An edge between the vertices  $v$  and  $w$  will be denoted by  $vw$ . For a path  $P$  with  $V(P) = \{a_1, \dots, a_k\}$ ,  $k \geq 3$ , and  $E(P) = \{a_i a_{i+1}\}_{i=1}^{k-1}$  we write  $a_1 \dots a_k$ . Let  $\kappa(G)$ ,  $\delta(G)$ ,  $\Delta(G)$ , and  $\lambda(G)$  denote the vertex-connectivity, minimum degree, maximum degree, and edge-connectivity of  $G$ , respectively. In the following, when we speak simply of connectivity, we always refer to vertex-connectivity.

A graph  $G$  is *bipartite* if there exist disjoint sets  $A, B \subset V(G)$  such that  $A \cup B = V(G)$ , and every edge of  $G$  is of the form  $ab : a \in A, b \in B$ . We call  $(A, B)$  a *bipartition* of  $G$ .  $G$  is *balanced* if there exists a bipartition  $(A, B)$  of  $G$  such that  $|A| = |B|$ , and *unbalanced* if it is not balanced. For  $S \subset V(G)$  we denote by  $G[S]$  the subgraph of  $G$  induced by  $S$ .

We call a graph *polyhedral* if it is planar and 3-connected.  $G$  is *almost hypohamiltonian* if  $G$  is non-hamiltonian, and there exists a vertex  $w$ , which we will call *exceptional*, such that  $G - w$  is non-hamiltonian, yet for any vertex  $v \neq w$  the graph  $G - v$  is hamiltonian. We will denote the family of all hypohamiltonian

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<sup>5</sup>Consider a graph  $G$ , an integer  $k > 1$ , an orientation  $D$  of  $G$ , and let  $\varphi : E(G) \rightarrow \{1, \dots, k-1\}$ . The pair  $(D, \varphi)$  is a (*nowhere-zero*) *k-flow* of  $G$  if for every  $v \in V(G)$ , the sum of the values of all edges leaving  $v$  equals the sum of the values of all edges entering  $v$ .

(almost hypohamiltonian) graphs by  $\mathcal{H}$  ( $\mathcal{H}_1$ ). These families restricted to planar graphs will be denoted by  $\overline{\mathcal{H}}$  and  $\overline{\mathcal{H}}_1$ , respectively.



## Chapter 2

# Hypohamiltonian and Hypotraceable Graphs

### 2.1 The Planar Case

In the early seventies, Chvátal [25–27] raised the problem whether there exist *planar* hypohamiltonian graphs, and offered \$5 for its solution [27, Problem 19]. Grünbaum conjectured that no such graph exists [59, p. 37]. In 1976, Thomassen [127] constructed infinitely many planar hypohamiltonian graphs. (I do not know whether he received the \$5.) The smallest among them has order 105, and is shown in Fig. 2. In analogy to hypohamiltonian graphs in general, the following natural question arose. What is the order of the smallest planar hypohamiltonian graph?

In the general case this was solved in 1967 by Herz, Duby, and Vigué [67], and the answer is that Petersen’s graph of order 10, shown in Fig. 1 and Fig. 45 in two different drawings, is the smallest (both in terms of order and size, in fact). In the planar case, the question is as of now unanswered, but significant progress has been made.

In 1979, Hatzel [64] found a smaller planar hypohamiltonian graph than the 105-vertex example given by Thomassen, and substantially improved the upper bound for the smallest planar hypohamiltonian graph; Hatzel’s graph has only

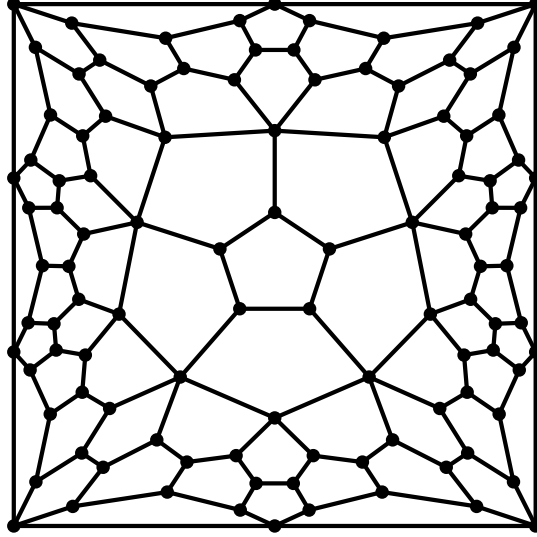


Fig. 2: Thomassen's plane hypohamiltonian graph—the first of its kind.  
It has order 105.

57 vertices. This was then improved to 48 by the author and T. Zamfirescu [146], to 42 by Araya and Wiener [141], and most recently to 40 vertices by Jooyandeh, McKay, Östergård, Pettersson, and the author [80]. The latter three graphs are shown in Fig. 3. The 40-vertex graph is the smallest example known so far, together with 24 other graphs of the same order [80], shown in Fig. 6.

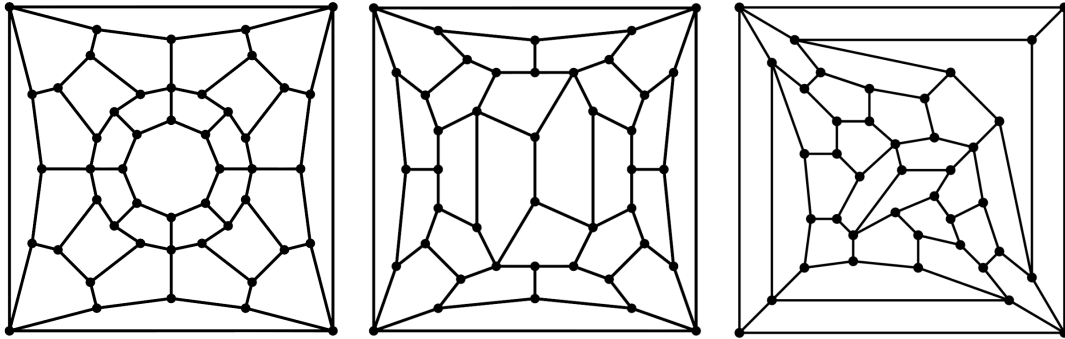


Fig. 3: Plane hypohamiltonian graphs of order 48, 42, and 40, resp. They held or hold the record of smallest known planar hypohamiltonian graph for the periods 2007–09, 2009–12, and since 2012, resp.

We will tacitly use the fact that for  $G \in \overline{\mathcal{H}}$ , we have  $\kappa(G) = \lambda(G) = \delta(G) = 3$ , due to the following argument. Since the deletion of any vertex gives a hamiltonian graph, we have  $\kappa(G) \geq 3$ . Thomassen [128] showed that  $G$  must contain a cubic vertex, so  $\delta(G) \leq 3$ . The equalities now follow from the fact that  $\kappa(G) \leq \lambda(G) \leq \delta(G)$ .

### 2.1.1 Grinberg's Criterion

In a plane graph, we call a face of size  $k$  a  $k$ -face. An essential ingredient for the progress that has been made towards finding smaller planar hypohamiltonian graphs in the past decades is the following useful result of Grinberg.

**Theorem 2.1: Grinberg's Criterion** (Grinberg, 1968 [51]) *Given a plane graph with a hamiltonian cycle  $\mathfrak{h}$  and  $f_k$  ( $f'_k$ )  $k$ -faces inside (outside) of  $\mathfrak{h}$ , we have*

$$\sum_{k \geq 3} (k-2)(f_k - f'_k) = 0. \quad (\dagger)$$

Instead of giving here a proof of this theorem central to our efforts, we present two proofs in Section 2.5.

For  $j \in \{0, 1, 2\}$ , define  $\mathcal{A}_j = \mathcal{A}_j(G)$  to be the set of faces of  $G$  with size congruent to  $j$  modulo 3. We call a graph *grinbergian* if it is 3-connected, plane and all of its faces but one belong to  $\mathcal{A}_2$ . We call the face not lying in  $\mathcal{A}_2$  *exceptional*.

A few examples: Thomassen's graph from Fig. 2 is grinbergian since all of its faces are pentagons, with the exception of the unbounded face, which is a 10-gon, and  $10 \equiv 1 \pmod{3}$ . Two of the graphs shown in Fig. 3 (left and centre), and the graph depicted in Fig. 12 on the right are grinbergian as well, while the Herschel graph given in Fig. 4 is not grinbergian, since all of its faces are quadrilaterals. However, it can be shown to be non-hamiltonian making use of Grinberg's Criterion, see the caption of Fig. 4.

By Grinberg's Criterion, grinbergian graphs are non-hamiltonian—their face sizes are such that the left-hand side of  $(\dagger)$  cannot possibly vanish. Thus, they are good candidates for hypohamiltonian graphs. Notice the difference between

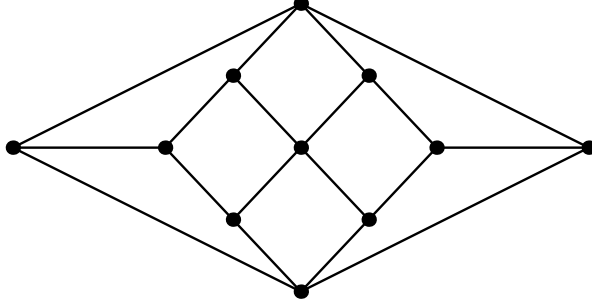


Fig. 4: Herschel's graph. Grinberg's Criterion yields that Herschel's graph is non-hamiltonian, since it only contains quadrilaterals and we have  $f_4 + f'_4 = 9$ , so the left side of  $(\dagger)$  becomes  $(4 - 2)(f_4 - f'_4)$ , which cannot vanish. In fact, Herschel's graph is (one of) the smallest non-hamiltonian polyhedral graphs [9].

our definition and that of Zaks [142], who defines *non-Grinbergian* graphs to be graphs with every face in  $\mathcal{A}_2$ .

**Theorem 2.2** (Jooyandeh, McKay, Östergård, Pettersson, Zamfirescu, 2016 [80]) *Every grinbergian hypohamiltonian graph has the property that its exceptional face belongs to  $\mathcal{A}_1$  and that its order is a multiple of 3.*

*Proof.* Consider  $G \in \mathcal{H}$  to be grinbergian, let the  $j$ -face  $F$  be the exceptional face of  $G$  (so  $F \notin \mathcal{A}_2$ ), and let  $v$  be a vertex of  $F$ .  $v$  belongs to  $F$  and to several, say  $h$ , faces in  $\mathcal{A}_2$ . The face of  $G - v$  containing  $v$  in its interior has size  $j - 2 \pmod{3}$ , while all other faces have size  $2 \pmod{3}$ . Since  $G \in \mathcal{H}$ ,  $G - v$  must be hamiltonian. Thus  $G - v$  cannot be a grinbergian graph, so  $j - 2 \equiv 2 \pmod{3}$ , which implies  $F \in \mathcal{A}_1$ .

Counting the edges twice we get  $2m \equiv 2(f - 1) + 1 \pmod{3}$ , where  $f$  is the number of faces in  $G$ . Together with Euler's formula [35, 36] this gives

$$2n = 2m - 2f + 4 \equiv 2f - 1 - 2f + 4 \equiv 0 \pmod{3},$$

so  $n$  is a multiple of 3. □

**Lemma 2.3** [80] *In a grinbergian hypohamiltonian graph  $G$ , all vertices of the exceptional face have degree at least 4.*

*Proof.* Denote the exceptional face by  $F$ . Now assume that there is a cubic vertex  $v \in V(F)$ , and consider the vertex  $w \in N(v) \setminus V(F)$ . (Note that  $N(v) \setminus V(F) \neq \emptyset$ , because  $G$  is 3-connected.) Let  $k$  be the degree of  $w$ , and denote by  $N_1, \dots, N_k$  the sizes of the faces of  $G$  that contain  $w$ . We have  $N_i \equiv 2 \pmod{3}$  for all  $i$ . Put  $G' = G - w$ . The size of the face of  $G'$  which in  $G$  contained  $w$  in its interior is  $\ell = \sum_i (N_i - 2) \equiv 0 \pmod{3}$ . Since  $G$  is hypohamiltonian,  $G'$  is hamiltonian. The graph  $G'$  contains only faces in  $\mathcal{A}_2$  except for one face in  $\mathcal{A}_1$  and one in  $\mathcal{A}_0$ . The face in  $\mathcal{A}_1$  and the face in  $\mathcal{A}_0$  are on different sides of any hamiltonian cycle in  $G'$ , as the cycle must pass through  $v$ . Since  $\ell - 2 \equiv 1 \pmod{3}$ , ( $\dagger$ ) modulo 3 gives  $1 + 1 \equiv 0 \pmod{3}$  or  $2 + 2 \equiv 0 \pmod{3}$ , which are both false.  $\square$

In what follows we will use these properties to show that the smallest grinbergian hypohamiltonian graph has 42 vertices. This shows that, restricted to this particular family, the result of Araya and Wiener published in [141] is best-possible.

### 2.1.2 Generation of 4-face Deflatable Hypohamiltonian Graphs

We define the operation *4-face deflater* denoted by  $\mathcal{FD}_4$  which squeezes a 4-face of a plane graph into a path of length 2, see Fig. 5. The inverse of this operation is called *2-path inflater* which expands a path of length 2 into a 4-face and is denoted by  $\mathcal{PI}_2$ . In Fig. 5, each half-line connected to a vertex designates an edge incident to the vertex at that position; a small triangle allows zero or more incident edges at that position. For example  $v_3$  has degree at least 3 and at least 4 on the left-hand side and right-hand side of Fig. 5, respectively. The set of all graphs obtained by applying  $\mathcal{PI}_2$  and  $\mathcal{FD}_4$  to a graph  $G$  is denoted by  $\mathcal{PI}_2(G)$  and  $\mathcal{FD}_4(G)$ , respectively.

Let  $\mathcal{D}_5(f)$  be the set of all (simple connected) plane graphs with  $f$  faces and minimum degree at least 5. This class of graphs can be generated using the

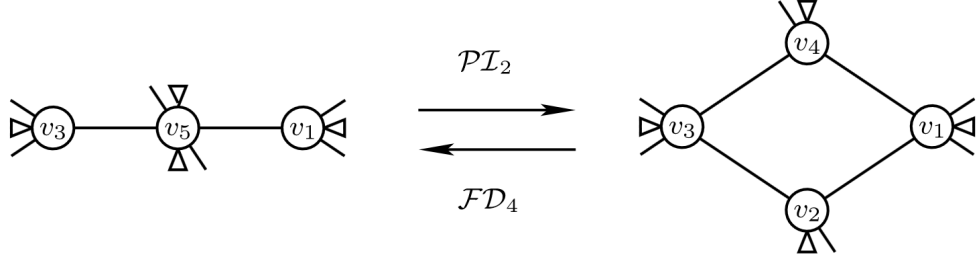


Fig. 5: Operations  $\mathcal{FD}_4$  and  $\mathcal{PI}_2$ .

program *plantri* [15, 16], written and maintained by Brinkmann and McKay. Let us denote the dual of a plane graph  $G$  by  $G^*$ . We define the family of *4-face deflatable graphs* (which are not necessarily simple) with  $f$  4-faces and  $n$  vertices, denoted by  $\mathcal{M}_f^4(n)$ , recursively as

$$\mathcal{M}_f^4(n) = \begin{cases} \{G^* : G \in \mathcal{D}_5(n)\}, & f = 0, \\ \bigcup_{G \in \mathcal{M}_{f-1}^4(n-1)} \mathcal{PI}_2(G), & f > 0. \end{cases}$$

Note that applying  $\mathcal{PI}_2$  to a graph increases the number of both vertices and 4-faces by exactly one. We now filter  $\mathcal{M}_f^4$  for possible hypohamiltonian graphs and put

$$\mathcal{H}_f^4(n) = \mathcal{M}_f^4(n) \cap \mathcal{H}.$$

The set  $\mathcal{H}_f^4(n)$  can be defined for all non-negative  $n$ , but is non-empty only for  $n \geq 20$  because the minimum face count for a simple plane 5-regular graph is 20 (icosahedron). Also it is straightforward to check that  $f \leq n - 20$  because  $\mathcal{H}_f^4(n)$  is defined based on  $\mathcal{H}_{f-1}^4(n-1)$  for  $f > 0$ .

To test hamiltonicity of graphs, we use depth-first search with the following pruning rule. If there is a vertex that does not belong to the current partial cycle, and has fewer than two neighbours that either do not belong to the current partial cycle or are an end-vertex of the partial cycle, the search can be pruned. This approach can be implemented efficiently with careful bookkeeping of the number of neighbours that do not belong to the current partial cycle for each vertex. It turns out to be reasonably fast for small planar graphs.

Finally, we define the set of *4-face deflatable hypohamiltonian graphs* denoted by  $\mathcal{H}^4(n)$  as

$$\mathcal{H}^4(n) = \bigcup_{f=0}^{n-20} \mathcal{H}_f^4(n).$$

The graphs found on 105 vertices by Thomassen [127], 57 by Hatzel [64], 48 by the author and T. Zamfirescu [146], and 42 by Araya and Wiener [141] are all 4-face deflatable and belong to  $\mathcal{H}_0^4(105)$ ,  $\mathcal{H}_1^4(57)$ ,  $\mathcal{H}_1^4(48)$ , and  $\mathcal{H}_1^4(42)$ , respectively. But, using this definition, we are also able to find many hypohamiltonian graphs which were not discovered so far.

Jooyandeh and McKay have generated  $\mathcal{H}_f^4(n)$  exhaustively for  $20 \leq n \leq 39$  and all possible  $f$ , but no graph was found, which means that for all  $n < 40$  we have  $\mathcal{H}^4(n) = \emptyset$ . For  $n > 39$  they were not able to finish the computation for all  $f$  due to the amount of required time. For  $n = 40, 41, 42, 43$  they finished the computation up to  $f = 12, 12, 11, 10$ , respectively. The only values of  $n$  and  $f$  for which  $\mathcal{H}_f^4(n)$  was non-empty were  $\mathcal{H}_5^4(40)$ ,  $\mathcal{H}_1^4(42)$ ,  $\mathcal{H}_7^4(42)$ ,  $\mathcal{H}_4^4(43)$ , and  $\mathcal{H}_5^4(43)$ . More details about these families are provided in Tables 1–3 in the Appendix. Based on the computations we obtain Theorem 2.4. The complete list of graphs generated is available for download at Jooyandeh’s repository [78]. They can also be obtained from the House of Graphs [13] by searching for the keywords “planar hypohamiltonian graph”.

**Theorem 2.4** [80]

- *There is no planar 4-face deflatable hypohamiltonian graph of order less than 40.*
- *There are at least 25 planar 4-face deflatable hypohamiltonian graphs on 40 vertices.*
- *There are at least 179 planar 4-face deflatable hypohamiltonian graphs on 42 vertices.*
- *There are at least 497 planar 4-face deflatable hypohamiltonian graphs on 43 vertices.*

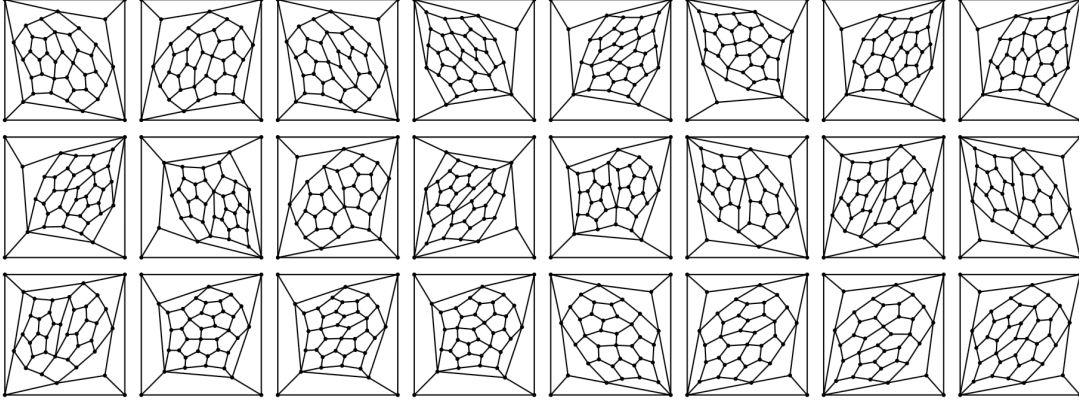


Fig. 6: In Figs. 3 and 8, we present the same plane hypohamiltonian graph on 40 vertices. Here we show the remaining 24 plane hypohamiltonian graphs of order 40 mentioned in Theorem 2.4.

**Lemma 2.5** [80] *Let  $G$  be a grinbergian hypohamiltonian graph whose faces have size at least 5 with one exception, which has size 4. Then both (possibly isomorphic) graphs in  $\mathcal{FD}_4(G)$  have a simple dual.*

*Proof.* As  $G$  is a simple 3-connected graph, the dual  $G^*$  of  $G$  is simple, too. Let  $G' \in \mathcal{FD}_4(G)$  and assume to the contrary that  $G'^*$  is not simple.

If  $G'^*$  contains multiedges, then the fact that  $G^*$  is simple implies that either the two faces incident with  $v_1v_5$  or with  $v_3v_5$  in Fig. 7 (b) (we assume the first by symmetry) have a common edge  $v_8v_9$  in addition to  $v_1v_5$ . Let  $v_1v_6$  and  $v_1v_7$  be the edges adjacent to  $v_1v_5$  in the cyclic order of  $v_1$ . Note that  $v_6 \neq v_7$  because the degree of  $v_1$  in  $G'$  is at least 3 by Lemma 2.3. If  $v_1$  and  $v_8$  were the same vertex, then  $v_1$  would be a cut-vertex in  $G$  considering the closed walk  $v_1v_6\dots v_8(=v_1)$ . But this is impossible as  $G$  is 3-connected, so  $v_1 \neq v_8$ . Now we can see that  $\{v_1, v_8\}$  is a 2-cut for  $G$  considering the closed walk  $v_1v_6\dots v_8\dots v_7v_1$ .

Also, if  $G'^*$  has a loop, with the same discussion, we can assume that the two faces incident with  $v_1v_5$  are the same. However, then  $v_1$  would be a cut-vertex for  $G$ . Therefore, both having multi-edges or having loops violate the fact that  $G$  is 3-connected. So the assumption that  $G'^*$  is not simple is incorrect, which completes the proof.  $\square$



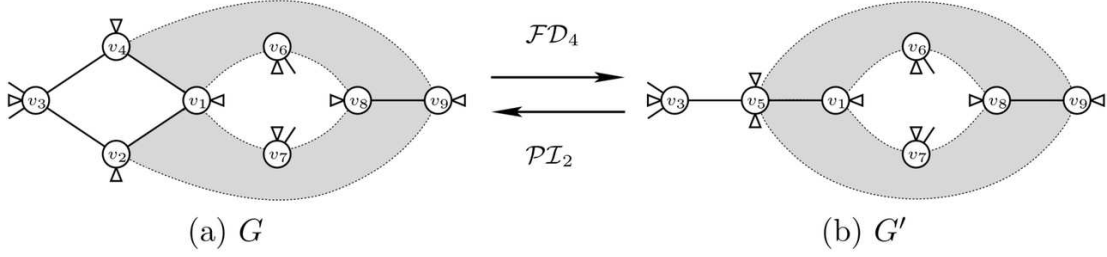


Fig. 7: Showing that  $\mathcal{FD}_4(G)$  has a simple dual.

**Theorem 2.6** [80] *Every grinbergian hypohamiltonian graph is 4-face deflatable. More precisely, every grinbergian hypohamiltonian graph of order  $n$  is in  $\mathcal{H}_0^4(n) \cup \mathcal{H}_1^4(n)$ .*

*Proof.* Let  $G \in \mathcal{H}$  be a grinbergian graph with  $n$  vertices. By Theorem 2.2, the exceptional face belongs to  $\mathcal{A}_1$  so its size is 4 or greater. If the exceptional face is a 4-face, then by Lemma 2.3 the 4-face has two non-adjacent vertices of degree at least 4. So we can apply  $\mathcal{FD}_4$  to obtain a graph  $G'$  which has no face of size less than 5. So  $\delta(G'^*) \geq 5$  and  $G'^*$  is a simple plane graph by Lemma 2.5. Thus  $G'^* \in \bigcup_f \mathcal{D}_5(f)$  and as a result of the definition of  $\mathcal{M}_f^4$ ,  $G'^{**} = G' \in \mathcal{M}_0^4(n-1)$ . Furthermore,  $G \in \mathcal{M}_1^4(n)$  because  $G \in \mathcal{PI}_2(G')$  and as  $G \in \mathcal{H}$ , we have  $G \in \mathcal{H}_1^4(n)$ .

But if the exceptional face is not a 4-face, then by the fact that it is 3-connected and simple,  $G^*$  is simple as well and as the minimum face size of  $G$  is 5, we have  $\delta(G^*) \geq 5$ , which implies that  $G \in \mathcal{M}_0^4(n)$  and so  $G \in \mathcal{H}_0^4(n)$ .  $\square$

**Corollary 2.7** [80] *The smallest grinbergian hypohamiltonian graph has 42 vertices and there are exactly seven such graphs on 42 vertices.*

*Proof.* By Theorem 2.6 every grinbergian graph belongs to  $\mathcal{H}_0^4(n) \cup \mathcal{H}_1^4(n)$  but according to the results presented in the paragraph preceding Theorem 2.4, we have  $\mathcal{H}_0^4(n) \cup \mathcal{H}_1^4(n) = \emptyset$  for all  $n < 42$ . So there is no such graph of order less than 42. On the other hand, we have  $\mathcal{H}_0^4(42) = \emptyset$  and  $|\mathcal{H}_1^4(42)| = 7$  which completes the proof.  $\square$

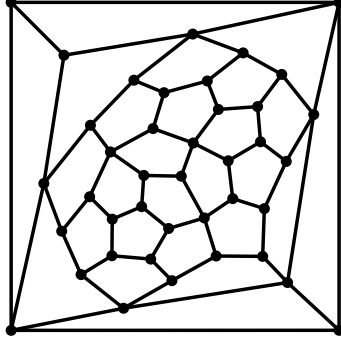


Fig. 8: A plane hypohamiltonian graph of order 40.  
(It is isomorphic to the right-most graph from Fig. 3, but drawn differently.)

**Theorem 2.8** [80] *The graph shown in Fig. 8 is hypohamiltonian.*

*Proof.* This was already tested by computer but we also give a manual proof. We first show that the graph is non-hamiltonian. Assume to the contrary that the graph contains a hamiltonian cycle, which must then satisfy Grinberg's Criterion. The graph in Fig. 8 contains five 4-faces and 22 5-faces. Then (†) becomes

$$\sum_{i \geq 3} (i - 2)(f_i - f'_i) \equiv 2(f_4 - f'_4) \equiv 0 \pmod{3},$$

where  $f_4 + f'_4 = 5$ . So w.l.o.g.  $f'_4 = 1$  and  $f_4 = 4$ . Let  $Q$  be the 4-face on a different side of the hamiltonian cycle than the other four 4-faces.

Note that an edge belongs to a hamiltonian cycle if and only if the two faces it belongs to are on different sides of the cycle. Since all edges of the outer face of the embedding in Fig. 8 have a 4-face on the other side, and not all of its edges can lie in a hamiltonian cycle, that face cannot be  $Q$ .

If  $Q$  is one of the other 4-faces, then the only edge of the outer face in the embedding in Fig. 8 that belongs to a hamiltonian cycle is the edge belonging to  $Q$  and the outer face. The two vertices of the outer face that are not endpoints of that edge have degrees 3 and 4, and we arrive at a contradiction as we know that two of the edges incident to the vertex with degree 3 are not part of the hamiltonian cycle. Thus, the graph is non-hamiltonian.

To end the proof, Fig. 9 shows for each vertex of the graph a cycle omitting that vertex. □

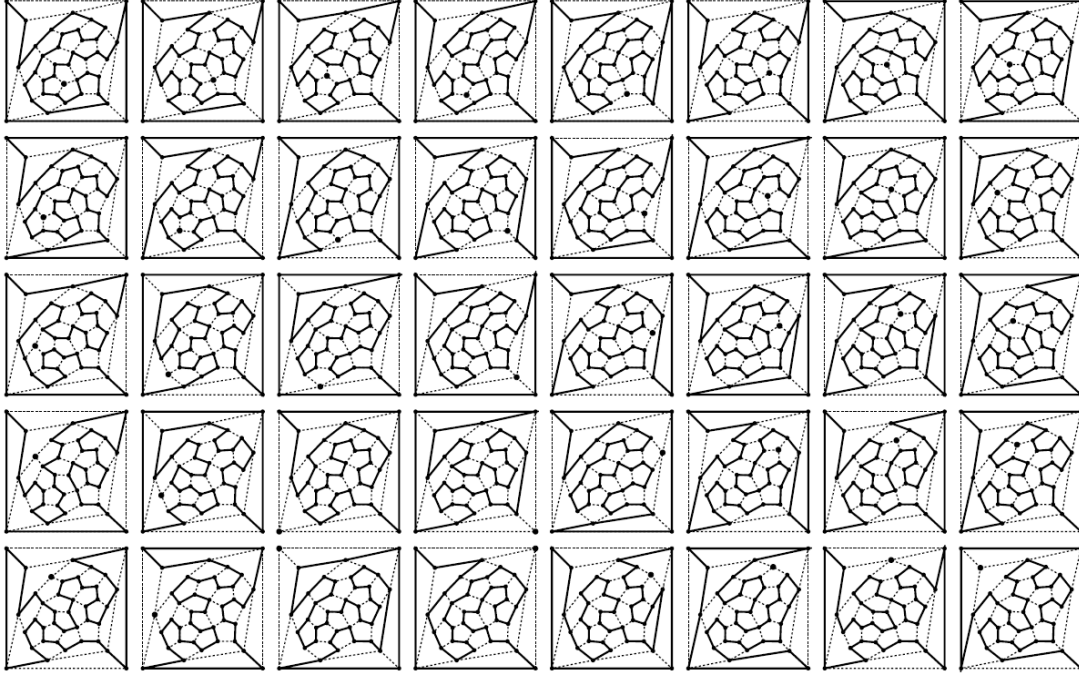


Fig. 9: For each vertex  $v$  of the plane hypohamiltonian graph  $G$  of order 40 shown in Fig. 8, we depict here a cycle of length  $|V(G)| - 1$  avoiding  $v$ .

### 2.1.3 Possible Orders of (Planar) Hypohamiltonian Graphs

In 1973, Chvátal showed [25] that if we choose  $n$  to be sufficiently large, then there exists a hypohamiltonian graph of order  $n$ . We now know that for every  $n \geq 18$  there exists such a graph of order  $n$ , and that 18 is optimal, since Aldred, McKay and Wormald showed that there is no hypohamiltonian graph on 17 vertices [5]. Their paper determined for all orders whether hypohamiltonian graphs exist or do not exist. (For more details, see the survey of Holton and Sheehan [71].)

They also provide a complete list of hypohamiltonian graphs with at most 17 vertices—there is exactly one such graph for each of the orders 10 (the Petersen graph, see Fig. 1), 13, and 15 (these were found by computer searches of Herz [66]), four of order 16, among them Sousselier’s graph, and none of order 17. These seven hypohamiltonian graphs on fewer than 18 vertices are shown

in Fig. 10. There exist at least thirteen 18-vertex hypohamiltonian graphs—the exact number is unknown. (See also sequence A141150 in Sloane’s On-Line Encyclopedia of Integer Sequences [117].) In [97], McKay lists all known hypohamiltonian graphs and cubic hypohamiltonian graphs up to 26 vertices (where for the general case the enumerations on 18 vertices and higher may be incomplete), as well as for 28 and 30 vertices the cubic hypohamiltonian graphs with girth at least 5 and girth at least 6, respectively.

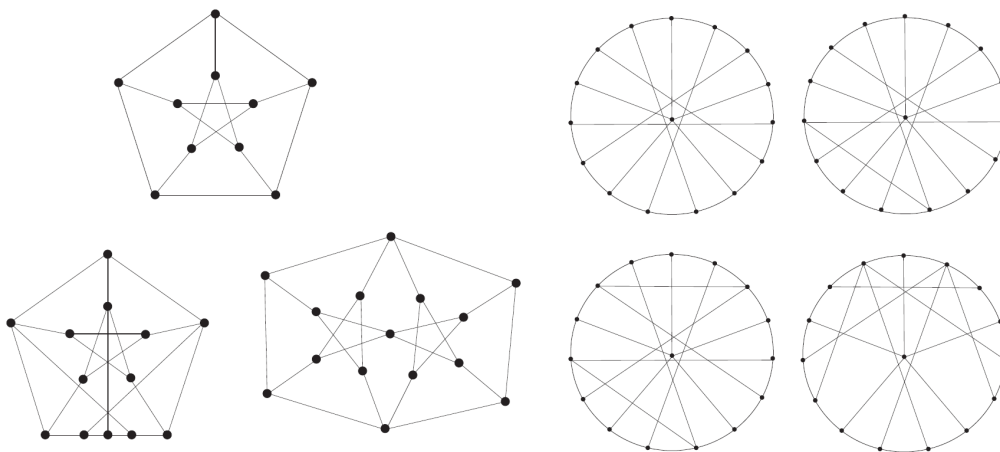


Fig. 10: The seven smallest hypohamiltonian graphs. Their orders are 10, 13, 15, 16, 16, 16, and 16, resp. Of the four 16-vertex graphs, the bottom-left one is called Sousselier’s graph.

The same question as the one discussed in the above paragraph is of course also interesting for planar hypohamiltonian graphs (see for instance [71]): is there an  $n_0$  such that there exists a planar hypohamiltonian graph of order  $n$  for all  $n \geq n_0$ ? In 2011, Araya and Wiener settled this question affirmatively.

**Theorem 2.9** (Araya and Wiener, 2011 [141]) *There exists a planar hypohamiltonian graph on  $n$  vertices for every  $n \geq 76$ .*

The bound for  $n_0$  was improved by the author to 48 (unpublished). We now present the result that gives the currently best bound for  $n_0$ . But first, we need an operation introduced by Thomassen [129] (and called by Araya and Wiener [141])

the “Thomassen operation”) with which he showed that there exist infinitely many planar cubic hypohamiltonian graphs. Let  $G$  be a graph containing a 4-cycle  $v_1v_2v_3v_4 = C$ , and consider vertices  $v'_1, v'_2, v'_3, v'_4 \notin V(G)$ . We denote by  $\text{Th}(G_C)$  the graph obtained from  $G$  by deleting the edges  $v_1v_2, v_3v_4$  and adding a new 4-cycle  $v'_1v'_2v'_3v'_4$  and the edges  $v_iv'_i$ ,  $1 \leq i \leq 4$ . Abusing notation, when we speak of “the graph  $\text{Th}(G_C)$ ” and  $C$  is a not further specified 4-cycle, we refer to (an arbitrary but fixed) one of the two (possibly isomorphic) graphs obtained when applying  $\text{Th}$ . In the following, if not stated otherwise, we will use the notation for  $C$  and the vertices added to  $G$  as introduced in this paragraph. The operation  $\text{Th}$  is illustrated in Fig. 11 and preserves polyhedrality and 3-regularity.

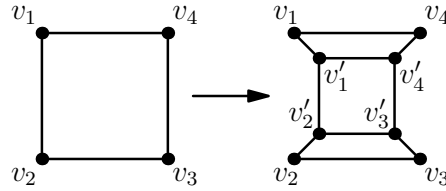


Fig. 11: The operation  $\text{Th}$ .

The following lemma is essentially due to Thomassen, who gives it (without proof) in [129]. A detailed proof for the planar case can be found in [141].

**Lemma 2.10** (Thomassen, 1981 [129]) *Let  $G \in \mathcal{H}$  contain a cubic 4-cycle  $C$ . Then  $\text{Th}(G_C) \in \mathcal{H}$ . If  $G$  is planar or cubic (possibly both), then so is  $\text{Th}(G_C)$ .*

We now extend Lemma 2.10.

**Proposition 2.11** *Let  $G \in \mathcal{H}$  contain a 4-cycle  $C$ , and let  $V(C)$  contain two adjacent cubic vertices (and no condition is imposed on the other two vertices). Then  $\text{Th}(G_C) \in \mathcal{H}$ . If  $G$  is planar or cubic (possibly both), then so is  $\text{Th}(G_C)$ .*

*Proof.* It is shown in [141, Lemma 4.3] that  $\text{Th}(G_C)$  is non-hamiltonian—there  $G$  is supposed to be planar, but this is not used in the proof of the non-hamiltonicity of  $\text{Th}(G_C)$ . It remains to show that  $\text{Th}(G_C) - v$  is hamiltonian for every  $v \in V(\text{Th}(G_C))$ . We consider  $G - \{v_1v_2, v_3v_4\}$  to be a subgraph of  $\text{Th}(G_C)$ .

Claim. Consider  $v \in V(G)$  and let  $\mathfrak{h}$  be a hamiltonian cycle in  $G - v$ . If  $E(C) \cap E(\mathfrak{h}) \neq \emptyset$ , then there exists a hamiltonian cycle in  $\text{Th}(G_C) - v$ .

*Proof of the Claim.* Let  $e \in E(C) \cap E(\mathfrak{h})$ . If  $e = v_1v_2$  and  $v_3v_4 \notin E(\mathfrak{h})$ , then  $(\mathfrak{h} - v_1v_2) \cup v_1v'_1v'_4v'_3v'_2v_2$  is a hamiltonian cycle in  $\text{Th}(G_C) - v$ . If  $e = v_1v_2$  and  $v_3v_4 \in E(\mathfrak{h})$ , then  $(\mathfrak{h} - \{v_1v_2, v_3v_4\}) \cup v_1v'_1v'_2v_2 \cup v_3v'_3v'_4v_4$  is a hamiltonian cycle in  $\text{Th}(G_C) - v$ . If  $e = v_1v_4$ , then  $(\mathfrak{h} - v_1v_4) \cup v_1v'_1v'_2v'_3v'_4v_4$  is a hamiltonian cycle in  $\text{Th}(G_C) - v$ .

Consider  $v \in V(G)$  and a hamiltonian cycle  $\mathfrak{h}$  in  $G - v$ . Since there are at least two cubic vertices in  $V(C)$ , there is at least one cubic vertex in  $V(\mathfrak{h}) \cap V(C)$ . Thus  $E(C) \cap E(\mathfrak{h}) \neq \emptyset$ , and the Claim implies that there exists a hamiltonian cycle in  $\text{Th}(G_C) - v$ . It remains to show that  $\text{Th}(G_C) - v'_i$  is hamiltonian for all  $i$ . W.l.o.g. let  $v_1$  be cubic. By replacing in the hamiltonian cycle in  $G - v_4$  the path  $v_1v_2$  with  $v_1v_4v'_4v'_3v'_2v_2$  ( $v_1v_4v'_4v'_1v'_2v_2$ ), we obtain a hamiltonian cycle in  $\text{Th}(G_C) - v'_1$  ( $\text{Th}(G_C) - v'_3$ ). If  $v_4$  is cubic, then by replacing in the hamiltonian cycle in  $G - v_1$  the path  $v_3v_4$  with  $v_3v'_3v'_4v'_1v_4$  ( $v_3v'_3v'_2v'_1v_4$ ), we obtain a hamiltonian cycle in  $\text{Th}(G_C) - v'_2$  ( $\text{Th}(G_C) - v'_4$ ). If  $v_2$  is cubic, the treatment is very similar.  $\square$

**Theorem 2.12** (Jooyandeh, McKay, Östergård, Pettersson, and Zamfirescu, 2016 [80]) *There exists a planar hypohamiltonian graph of order  $n$  for every  $n \geq 42$ .*

*Proof.* Figs. 3 and 12 show plane hypohamiltonian graphs on 40, 42, 43, and 45 vertices. It is not difficult to check that applying the operation  $\text{Th}$  to the outer face of these graphs gives planar hypohamiltonian graphs with 44, 46, 47, and 49 vertices. Indeed, for the 44-vertex example (consider for instance the graph from Fig. 8) and the 47-vertex example (see Fig. 12 (left)) this follows from Prop. 2.11, but we skip here the details of this part of the proof, since a detailed account of this step is given in the proof of Lemma 3.6 and Fig. 31. By the construction, these graphs will have a cubic 4-face, so the statement now follows from repeated application of Lemma 2.10.  $\square$

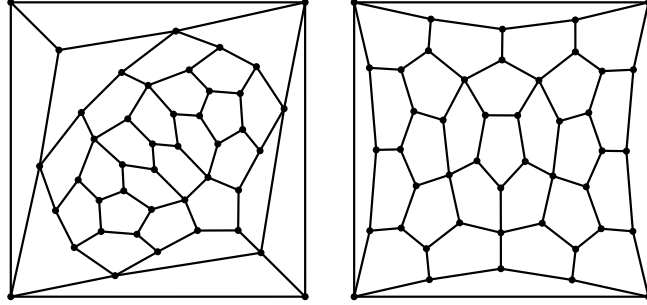


Fig. 12: Plane hypohamiltonian graphs of order 43 and 45, resp.

We have seen that planar hypohamiltonian graphs exist on 40 vertices, and of order  $n$  for every  $n \geq 42$ . However, whether such graphs on fewer than 40 vertices exist, or whether such a graph of order 41 exists, remains open.

#### 2.1.4 A Planar Counter-Example to a Conjecture of Chvátal

Chvátal [25] conjectured that if the deletion of an edge  $e$  from a hypohamiltonian graph  $G$  does not create a vertex of degree 2, then  $G - e$  is hypohamiltonian. Thomassen [126] gave numerous counter-examples to this conjecture, yet none of them is planar. We now provide a planar counter-example.

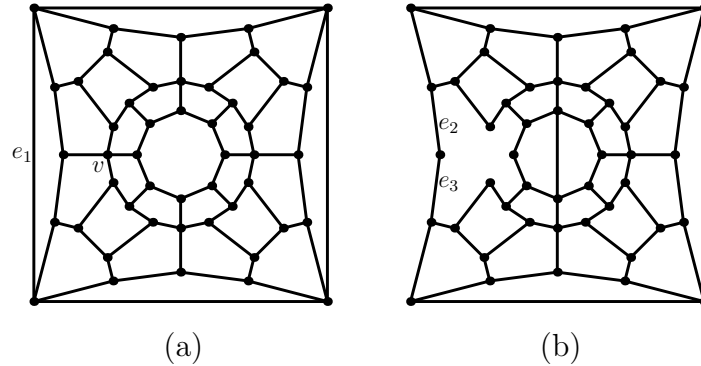


Fig. 13

Consider the plane graph  $G$  from Fig. 13 (a) which has 48 vertices, the edge denoted by  $e_1$ , and the vertex denoted by  $v$ .  $G$  is hypohamiltonian, as shown

in [146]. Using Grinberg's Criterion, it is clear that  $G$  remains hypohamiltonian if we add an edge such that the octagon becomes two pentagons. Call this graph  $G'$ . Notice that  $G' - e_1 = G''$  has no vertices of degree 2. In  $G'' - v$ , shown in Fig. 13 (b), there is exactly one heptagon (the unbounded face) and exactly one dodecagon, and all other faces are pentagons. Assume that  $G'' - v$  contains a hamiltonian cycle  $\mathfrak{h}$ . Then Grinberg's Criterion yields

$$3(f_5 - f'_5) - 5 \pm 10 = 0,$$

where as before  $f_5$  ( $f'_5$ ) is the number of pentagons inside (outside) of  $\mathfrak{h}$ . This can only hold if the ambiguous sign is “−”, which implies that the dodecagon, like the heptagon, lies on the outside of  $\mathfrak{h}$ . But as the edges  $e_2$  and  $e_3$  (see Fig. 13 (b)) both must lie in  $\mathfrak{h}$ , a contradiction is obtained. So  $G'' - v$  is not hamiltonian, whence,  $G''$  is not hypohamiltonian. As both  $G'$  and  $G''$  are obviously planar, we are done.

Inspired by Chvátal's conjecture, we make here the following two observations.

1. Let  $G$  be a hypohamiltonian graph. If there exists an edge  $e \in E(G)$  such that there is exactly one vertex  $w \in V(G)$  with the property that for every hamiltonian cycle  $\mathfrak{h}$  in  $G - w$  we have  $e \in E(\mathfrak{h})$ , then  $G - e$  is almost hypohamiltonian with exceptional vertex  $w$ . Almost hypohamiltonian graphs will be discussed in Chapter 3.
2. Although for arbitrary  $e \in E(G)$  the graph  $G - e$  is not necessarily hypohamiltonian, the graph  $G - e$  does have the property that (i) it is non-hamiltonian and (ii) for every vertex  $v \in V(G)$  the graph  $G - e - v$  is traceable. This will be studied further in Chapter 5.

### 2.1.5 Planar Cubic Hypohamiltonian Graphs

We now turn our attention to the family of planar cubic hypohamiltonian graphs. A brief motivation follows. Hamiltonian paths and cycles in planar cubic graphs have been investigated extensively since in the early 20th century, Tait tried to prove the Four Colour Conjecture (which is now a theorem) based on the



conjecture that every planar 3-connected cubic graph is hamiltonian. However, Tutte [132] provided a counter-example in 1946.

Before 1968, when Grinberg proved his hamiltonicity criterion [51], such graphs were quite difficult to find. Since then, several planar 3-connected cubic graphs which are non-hamiltonian have been constructed. However, for the smallest example, the Lederberg-Bosák-Barnette graph on 38 vertices (see Fig. 14), the proof does not use Grinberg's Criterion.

In 1988, Holton and McKay [68] (finalising the efforts of several authors) showed that all planar 3-connected cubic graphs on fewer than 38 vertices are hamiltonian. So certainly the smallest planar cubic hypohamiltonian graph has order at least 38.

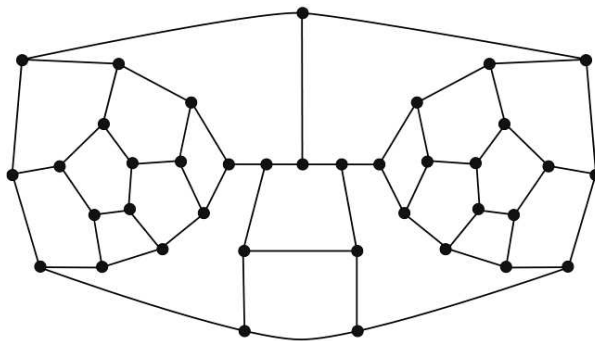


Fig. 14: The Lederberg-Bosák-Barnette graph. It has order 38 and is the smallest planar 3-connected cubic non-hamiltonian graph [68].

In 1973, Chvátal asked whether planar cubic hypohamiltonian graphs exist [26], see also [127, Problem 6.2]. Eight years later, Thomassen [129] showed not only that planar cubic hypohamiltonian graphs exist, but that there is an infinite family of such graphs. Using this, he constructs infinitely many planar cubic hypotractable graphs and gives a simple proof of the result of Collier and Schmeichel [28] that every bipartite graph is the induced subgraph of some hypohamiltonian graph. We will come back to this result in Section 3.6.

The smallest member of the family of planar cubic hypohamiltonian (planar cubic hypotractable) graphs given by Thomassen has order 94 (460). Currently, the smallest known planar cubic hypohamiltonian (planar cubic hypotractable)

graph has 70 (340) vertices. Both were constructed by Araya and Wiener [7] in 2011—the 70-vertex graph is shown in Fig. 15. Six more planar cubic hypohamiltonian graphs of order 70 were found by Jooyandeh and McKay [79]. It is unknown whether smaller such graphs exist, see Problem 5 in Chapter 6.

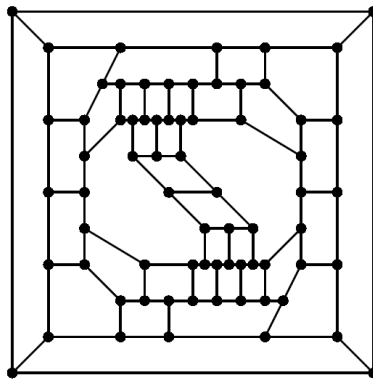


Fig. 15: The plane cubic hypohamiltonian graph of order 70 discovered by Araya and Wiener [7]. No smaller example of its kind is known.

This paragraph's discussion on the lower bound for the order of the smallest planar cubic hypohamiltonian graph was already given by Araya and Wiener in [7]. We have already mentioned that in 1988 it was shown that all planar 3-connected cubic graphs on fewer than 38 vertices are hamiltonian. In 2000, Aldred, Bau, Holton, and McKay [4] proved that there is no planar cubic hypohamiltonian graph on 42 or fewer vertices. They showed that every 3-connected, cyclically 4-edge-connected cubic planar graph has at least 42 vertices and presented all such graphs on exactly 42 vertices.

Hypohamiltonian graphs are easily seen to be 3-connected. We now show a further structural property, which implies that hypohamiltonian graphs are cyclically 4-edge-connected.

**Proposition 2.13** *Let  $M$  be a 3-edge-cut in a hypohamiltonian graph  $G$ . Then  $G - M$  contains exactly two components  $A_1$  and  $A_2$  with  $A_1 = K_1$  and  $A_2 \neq K_1$ .*

*Proof.* Let  $G - M$  contain components  $A_1, \dots, A_k$ ,  $k \geq 2$ , and assume that  $A_1 \neq K_1$  and  $A_2 \neq K_1$ . We put  $M = \{a_1b_1, a_2b_2, a_3b_3\}$ , where  $a_i \in V(A_1)$ ,  $b_i \in V(A_2)$  for

all  $i$ . Since  $G$  is 3-connected, the elements of the set  $\{a_1, a_2, a_3, b_1, b_2, b_3\}$  are pairwise distinct. Since  $G$  is hypohamiltonian,  $G - b_3$  is hamiltonian, so there is a hamiltonian path in  $A_1$  with end-vertices  $a_1$  and  $a_2$ . As  $G - a_3$  is hamiltonian, there is a hamiltonian path in  $A_2$  with end-vertices  $b_1$  and  $b_2$ . These paths together with  $a_1b_1$  and  $a_2b_2$  yield a hamiltonian cycle in  $G$ , a contradiction. Note that if at least one of the components of  $G - M$  is isomorphic to  $K_1$ , then  $k = 2$ . This completes the proof.  $\square$

As discussed above, planar hypohamiltonian graphs have at least 42 vertices in the cubic case. Moreover, all 42-vertex graphs presented in [4] have exactly one face of size not congruent to 2 modulo 3 (i.e. an exceptional face as defined in Section 2.1.1), and it was observed by Thomassen [125] that cubic graphs with this property cannot be hypohamiltonian. The order of the smallest cubic planar hypohamiltonian graph is therefore at least 44 and at most 70. Araya and Wiener [7] proposed ways to improve the lower bound, but these have not yet been put into practice.

In order to improve a Theorem of Araya and Wiener from [7], we require the following result of Thomassen.

**Lemma 2.14** (Thomassen, 1981 [129]) *Let  $G$  be a plane cubic graph containing a quadrilateral adjacent to four heptagons, and suppose furthermore that the size of every other face is congruent to 2 modulo 3. Then  $G$  is non-hamiltonian.*

We will show in Lemma 2.16 the existence of a 76-vertex planar cubic hypohamiltonian graph, which we call  $Q$  (see Fig. 16), with which we strengthen the main result of Araya and Wiener in [7]; they showed the following.

**Lemma 2.15** (Araya and Wiener, 2011 [7]) *There exist planar cubic hypohamiltonian graphs on  $70 + 4k$  vertices for every  $k \geq 0$ , and on  $n$  vertices for every even  $n \geq 86$ .*

Note that the result of Araya and Wiener answers a question of Holton and Sheehan [71], who asked if there exists an integer  $n$  such that a planar cubic hypohamiltonian graph exists for every even integer greater than or equal to  $n$ .

**Lemma 2.16** (2015 [144]) *There exist planar cubic hypohamiltonian graphs on  $76 + 4k$  vertices for every  $k \geq 0$ .*

*Proof.* Fig. 16 shows the graph  $Q$ , which is obviously planar and cubic.  $Q$  contains precisely one quadrilateral surrounded by four heptagons, while all other faces are pentagons, octagons or 11-gons. By Lemma 2.14,  $Q$  is non-hamiltonian. In the Appendix we provide for each vertex of  $Q$  a cycle of length 75 avoiding it. By applying Lemma 2.10 successively, the proof is complete.  $\square$

**Theorem 2.17** [144] *There exist planar cubic hypohamiltonian graphs on 70 vertices and on  $n$  vertices for every even  $n \geq 74$ .*

*Proof.* Combining Lemmas 2.15 and 2.16, the statement is verified.  $\square$

Summarising, Theorem 2.17 improves the bound linked to the question of Holton and Sheehan mentioned on the previous page from 86 to 74, which is currently the best bound. It is unknown whether a planar cubic hypohamiltonian graph of order 72 exists. The order of the smallest planar cubic hypohamiltonian graph is at least 44 and at most 70. We have not discussed here what occurs if we additionally restrict the girth of the graph—we shall do so in Section 2.4.

## 2.2 Crossing Numbers

In this section we study the crossing number of certain hypohamiltonian graphs. Historically, much effort has gone into the analysis of *planar* hypohamiltonian graphs (i.e., in the context of this section, with crossing number 0), but few distinctions were made between non-planar hypohamiltonian graphs, and little seems to be known. The main result of this section is extending the scope of Theorem 2.12 (which states that there exist planar hypohamiltonian graphs of order  $n$  for every  $n \geq 42$ ) beyond the planar case. We conclude by providing

lower and upper bounds on the order of the smallest hypohamiltonian graphs of fixed crossing number  $k$  for  $k \leq 3$ .

A *drawing* of a graph  $G$  is an injective mapping  $f$  that assigns to each vertex a point in the plane and to each edge  $uv$  a Jordan arc (i.e. a homeomorphic image of a closed interval) connecting  $f(u)$  and  $f(v)$ , not passing through the image of any other vertex. For simplicity, the arc assigned to  $uv$  is called an *edge* of the drawing, and if this leads to no confusion, it is also denoted by  $uv$ . We assume that no three edges have an interior point in common, and if two edges share an interior point  $p$ , then we say that they cross at  $p$ . We also assume that any two edges of a drawing have only a finite number of crossings (i.e. common interior points). A common endpoint of two edges does not count as a crossing. The *crossing number* of  $G$ , denoted by  $\text{cr}(G)$ , is the minimum number of edge crossings for all possible drawings of  $G$ . An overview of results concerning the crossing number can be found in Székely's paper [123] and the survey [109] by Richter and Salazar.

We shall use the family of *generalised Petersen graphs*  $\text{GP}(n, k)$  introduced by Coxeter [30]. A graph in  $\text{GP}(n, k)$  has vertex set  $\{u_0, \dots, u_{n-1}, v_0, \dots, v_{n-1}\}$  and edge set  $\bigcup_{i=0}^{n-1} \{u_i u_{i+1}, u_i v_i, v_i v_{i+k}\}$ , the indices being taken modulo  $n$ , and  $k < n/2$ . Alspach [6] showed that  $\text{GP}(n, k)$  is non-hamiltonian if and only if  $n \equiv 5 \pmod{6}$  and  $k = 2$ . We call this particular subfamily  $\mathcal{F}$ . Bondy [12] showed that  $\mathcal{F} \subset \mathcal{H}$ .

The Petersen graph  $P = \text{GP}(5, 2) \in \mathcal{F}$  has crossing number 2. The proof of this well-known fact is short. Firstly, draw  $P$  with two crossings, see Fig. 1. Secondly, prove that it is impossible to draw the Petersen graph with less than two crossings. One can do this with the following inequality based on Euler's formula (see for instance [123]). Denote the girth, size, order of a graph  $G$  by  $g$ ,  $m$ ,  $n$ , respectively. Then we have

$$\text{cr}(G) \geq \left\lceil m - \frac{g(n-2)}{g-2} \right\rceil. \quad (\star)$$

For  $P$ ,  $g = 5$ ,  $m = 15$ ,  $n = 10$ ; thus  $(\star)$  yields  $\text{cr}(P) \geq 2$ . Exoo, Harary, and Kabell [38] showed that  $\text{cr}(\text{GP}(n, 2)) = 3$  when  $n$  is odd and at least 7. Therefore, all other members of  $\mathcal{F}$  have crossing number 3.

In 2011, it was shown by Araya and Wiener [141] that there exist planar hypohamiltonian graphs of order  $n$  for every  $n \geq 76$ . In Theorem 2.12, we decreased this to 42. The smallest hypohamiltonian graph is the Petersen graph, which has crossing number 2, see Fig. 1. All non-planar hypohamiltonian graphs constructed (explicitly) in the literature have crossing number at least 2, and in fact for few of them the crossing number has been computed explicitly in the sense that it is certainly at least 2, but the exact crossing number is unknown.

We require a construction method introduced by Thomassen [129]. Let  $G$  be a hypohamiltonian graph containing a 4-cycle  $abcd = C$ . We delete the edges  $ab$  and  $cd$ , add two new vertices  $a'$  and  $d'$ , and add the edges  $a'd'$ ,  $aa'$ ,  $dd'$ ,  $a'c$ ,  $d'b$ . Denote the resulting graph by  $G_C^*$ . Whenever the choice of  $abcd = C$  is clear, we simply write  $G^*$ . For an illustration of a case where  $G$  is plane, see Fig. 16. Thomassen mentions in [129] the first part of Lemma 2.18, but gives no proof. Therefore, we choose to prove here both parts.

**Lemma 2.18** (2012 [143]) *If  $G \in \mathcal{H}$  contains a cubic 4-cycle  $C$ , then  $G_C^* \in \mathcal{H}$ . If  $G$  is planar, then  $\text{cr}(G_C^*) = 1$ .*

*Proof.* Notice that the 4-cycle  $abcd = C$  is chordless, as it is cubic and  $G$  is 3-connected. If  $G$  is planar, then  $abcd$  is a facial cycle. Assume  $G^*$  contains a hamiltonian cycle  $\mathfrak{h}$ . There are two essentially different possibilities for  $\mathfrak{h}$  to visit  $a'$  and  $d'$ : either  $a'd' \in E(\mathfrak{h})$  or  $a'd' \notin E(\mathfrak{h})$ .

For every edge  $xy \in E(C)$  there exists a path  $xvwy$  in  $G^*$  such that  $\{v, w\} = \{a', d'\}$ , and vice-versa for every such path in  $G^*$  there exists the corresponding edge in  $G$ . This yields a contradiction for the case  $a'd' \in E(\mathfrak{h})$ , since we obtain for each of the four ways  $\mathfrak{h}$  might visit  $a'$  and  $d'$ , a hamiltonian cycle in  $G$ .

In the latter case, i.e.  $a'd' \notin E(\mathfrak{h})$ , we have that  $ca'a \subset \mathfrak{h}$  and  $bd'd \subset \mathfrak{h}$ . Then  $\mathfrak{h} - (ca'a \cup bd'd)$  is the union of two paths which, together with either  $ab$  and  $cd$  or  $ad$  and  $bc$ , form a hamiltonian cycle of  $G$ . This contradicts the fact that  $G$  is hypohamiltonian.

Hence  $G^*$  is non-hamiltonian. Put  $n = |V(G)|$ , so  $|V(G^*)| = n + 2$ . We now show that for each  $v \in V(G^*)$  there exists an  $(n + 1)$ -cycle  $\mathfrak{h}_v^*$  in  $G^* - v$ . Consider  $v \in V(G)$  and an  $(n - 1)$ -cycle  $\mathfrak{h}_v$  in  $G - v$ . Since  $C$  is cubic,  $\mathfrak{h}_v$  must contain an

edge of  $C$ . By the observation from the preceding paragraph, we have obtained hamiltonian cycles  $\mathfrak{h}_w^*$  for all  $w \in V(G^*) \setminus \{a', d'\}$ . For  $\mathfrak{h}_{a'}^*$ , consider  $\mathfrak{h}_d$  and replace  $ab$  with  $add'b$ . Analogously we get  $\mathfrak{h}_{d'}^*$ . This shows that  $G^*$  is hypohamiltonian.

Now suppose  $G$  is planar. In order to see that  $G^*$  contains a graph  $K$  homeomorphic to  $K_{3,3}$ , where  $V(K) = \{a, c, d', a', b, d\}$ , it suffices to prove that there exist two disjoint paths in  $G^*$ , one with end-vertices  $a$  and  $b$ , and the other with end-vertices  $c$  and  $d$ . Consider the graph  $\overline{G} = G - \{ab, cd\}$ , which is a subgraph of  $G^*$ . Assume  $\overline{G}$  has connectivity 1 with cut-vertex  $v$ . Since  $a, b, c$ , and  $d$  are cubic, and a triangle in a hypohamiltonian graph has no cubic vertex,  $\{a, v, d\}$  and  $\{b, v, c\}$  are 3-cuts in  $\overline{G}$ . As  $K_{3,3}$  is non-planar and  $\overline{G}$  is planar, for every 3-cut  $X$  in  $\overline{G}$ ,  $\overline{G} - X$  has exactly two components. Let  $Y'$  and  $Y''$  be the components of  $\overline{G} - \{a, v, d\}$  such that  $b \in V(Y'')$ . Put  $G' = \overline{G}[V(Y') \cup \{a, v, d\}]$  and  $G'' = \overline{G}[V(Y'') \cup \{v\}]$ . Since  $G - a$  is hamiltonian, there exists a hamiltonian path  $\mathfrak{p}''$  in  $G''$  with end-vertices  $v$  and  $c$ . As  $G - b$  is hamiltonian, there exists a hamiltonian path  $\mathfrak{p}'$  in  $G'$  with end-vertices  $v$  and  $d$ . Now  $\mathfrak{p}' \cup \mathfrak{p}'' \cup cd$  is a hamiltonian cycle in  $G$ . We have obtained a contradiction. Hence  $\overline{G}$  is 2-connected and the two paths exist due to Menger's Theorem [99].

Finally, consider the face in  $\overline{G}$  which contains the vertices  $a, b, c, d$ . Its boundary minus the edges  $ad$  and  $bc$  yields the two desired disjoint paths. Thus, by Kuratowski's Theorem [91],  $G^*$  is non-planar. It is easy to draw  $G^*$  in the plane with exactly one crossing—an example is given in Fig. 16.  $\square$

**Lemma 2.19** [143] *There exists a hypohamiltonian graph with crossing number 1.*

*Proof.* Apply Lemma 2.18 to a plane cubic hypohamiltonian graph containing a quadrilateral face. For instance we can take the 76-vertex graph  $Q$  from Fig. 16. The resulting graph  $Q^*$  is shown in Fig. 16. Using the second part of Lemma 2.18, we obtain the statement.  $\square$

Consider graphs  $G$  and  $G'$  containing cubic vertices  $x$  and  $x'$ , respectively. Denote by  $G_x G'_{x'}$ , one of the graphs obtained from  $G - x$  and  $G' - x'$  by identifying the vertices in  $N(x)$  with those in  $N(x')$  using a bijection. We continue our preparations for the main result of this section.

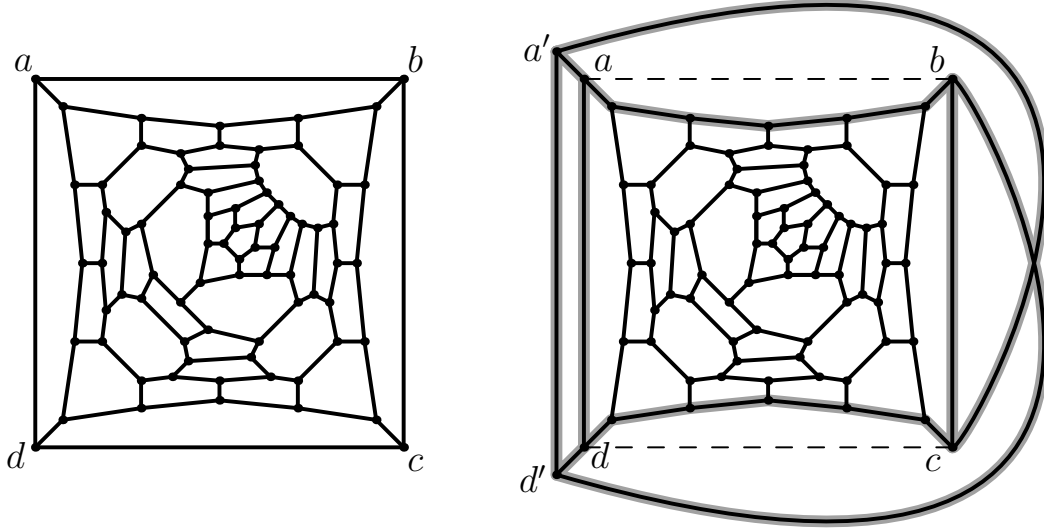


Fig. 16: On the left, the plane cubic hypohamiltonian graph  $Q$  is shown, and on the right,  $Q^*$ .  $Q^*$  has crossing number 1. The grey edges in  $Q^*$  show a subgraph homeomorphic to  $K_{3,3}$ .

**Lemma 2.20** (Thomassen, 1974 [125]) *Let  $G, G' \in \mathcal{H}$  each contain a cubic vertex, say  $x$  and  $x'$ , respectively. Then  $G_x G'_{x'} \in \mathcal{H}$ .*

**Lemma 2.21** *If a graph  $G$  contains  $k$  pairwise edge-disjoint copies of subgraphs homeomorphic to  $K_{3,3}$ , then  $\text{cr}(G) \geq k$ .*

*Proof.* Let us assume that  $\text{cr}(G) = \ell < k$ . In  $G$ , we can delete (at most)  $\ell$  edges such that the (possibly disconnected) graph we obtain is planar. Since  $\ell < k$ , in at least one copy of  $K_{3,3}$  no edge was deleted—a contradiction, since  $K_{3,3}$  is non-planar.  $\square$

**Lemma 2.22** (2012 [143]) *For any  $k \geq 0$  there exists a hypohamiltonian graph which has crossing number  $k$ .*

*Proof.* For  $k = 0$ , consider a planar cubic hypohamiltonian graph such as the graph  $Q$  of Fig. 16. Put  $Q^1 = Q^*$  (see Fig. 16). In  $Q$  minus the edges  $ab$  and  $cd$ , denote the shortest path between  $a$  and  $b$  by  $S$  and the shortest path between  $c$



and  $d$  by  $T$ . Consider

$$J = Q^*[\{a, a', b, c, d, d'\}] \cup S \cup T,$$

marked with grey edges in Fig. 16, and a cubic vertex  $v \in V(Q^*)$  with  $N(v) \cap J = \emptyset$ . Take two copies  $Q', Q''$  of  $Q^*$ , the corresponding subgraphs  $J', J''$ , and the corresponding vertices  $v', v''$ . Consider  $Q^2 = Q'_{v'} Q''_{v''}$ . This graph contains two graphs homeomorphic to  $K_{3,3}$ , namely  $J'$  and  $J''$ , and  $J' \cap J'' = \emptyset$ . By Lemma 2.21, we have  $\text{cr}(Q^2) \geq 2$ . Drawing as we did in Fig. 16, it is clear that in fact  $\text{cr}(Q^2) = 2$ . (Think of  $Q^2$  as drawn on a sphere, with  $Q' - v'$  on the northern hemisphere,  $Q'' - v''$  on the southern hemisphere, and the three vertices resulting from identification on the equator.) We have shown the statement for  $k \leq 2$ . Now assume that we have constructed in this manner a graph  $Q^{k-1}$ , which is planar, cubic, hypohamiltonian, and has crossing number  $k-1$ . The graph  $Q^k = Q_w^{k-1} Q_v^1$  (where  $w \in V(Q^{k-1})$  is a cubic vertex such that  $N(w)$  does not meet any subgraph isomorphic to  $J$ ) then has crossing number  $k$ , and the lemma is proven.  $\square$

We state the main result of this section.

**Theorem 2.23** [143] *For every  $k \geq 0$  there is an integer  $n_0(k)$  such that, for every  $n \geq n_0$ , there exists a hypohamiltonian graph which has order  $n$  and crossing number  $k$ .*

*Proof.* Let  $Q^k$  be the hypohamiltonian graph with crossing number  $k$  constructed in the proof of Lemma 2.22. It has plenty of cubic vertices. Now choose a cubic vertex  $v \in V(Q^k)$  no neighbour of which belongs to any of the used subgraphs isomorphic to  $J$ . Theorem 2.12 states that for every  $n \geq 42$  there exists a planar hypohamiltonian graph  $H^n$  of order  $n$ . By a result of Thomassen [128], every planar hypohamiltonian graph contains a cubic vertex. Thus, for every  $n$  the graph  $H^n$  contains a cubic vertex  $w$ . We apply Lemmas 2.20 and 2.21 to  $Q^k$  and  $H^n$ , obtaining the graph  $Q_v^k H_w^n$ . Drawing in the same manner as in Fig. 16 and providing the family  $\{Q_v^k H_w^n\}_{n=42}^\infty$  yields the statement.  $\square$

Let  $\mathfrak{o}_k$  be the order of the smallest hypohamiltonian graph with crossing number  $k$ , and let  $\mathfrak{o}'_k$  denote the minimum number such that there exists a hypohamiltonian graph of order  $n$  and crossing number  $k$  for every  $n \geq \mathfrak{o}'_k$ .

An upper bound for  $\mathfrak{o}_0$  is a direct consequence of Theorem 2.8, where it is shown that there exists a planar hypohamiltonian graph of order 40. We now prove that  $\mathfrak{o}_1 \leq 46$ . By applying Thomassen's operation Th defined in Section 2.1.3 and Prop. 2.11, we obtain from the plane hypohamiltonian graph of order 40 shown in Fig. 8 a plane hypohamiltonian graph  $W$  of order 44 containing a cubic 4-face. By Lemma 2.18,  $W^*$  is a hypohamiltonian graph of order 46 and crossing number 1. We recall that the Petersen graph  $P$  has crossing number 2 (see Fig. 1), so  $\mathfrak{o}_2 = 10$  (as  $P$  is the smallest hypohamiltonian graph, see e.g. [67]). Moreover,  $\text{cr}(\text{GP}(11, 2)) = 3$ , see [38]. In summary, we have

$$18 \leq \mathfrak{o}_0 \leq 40, \quad 18 \leq \mathfrak{o}_1 \leq 46, \quad \mathfrak{o}_2 = 10, \quad \text{and} \quad 13 \leq \mathfrak{o}_3 \leq 16,$$

where the lower bounds on  $\mathfrak{o}_0$  and  $\mathfrak{o}_1$  can be computed by using the list of small hypohamiltonian graphs provided by Aldred, McKay and Wormald [5] and applying ( $\star$ ): all seven hypohamiltonian graphs on fewer than 18 vertices (see Fig. 10) have crossing number at least 2. The upper bound for  $\mathfrak{o}_3$  is proven by Fig. 17. This embedding was found using Markus Chimani's tool accessible at <http://crossings.uos.de/>. It is described in detail in [23, 24]. In fact, the tool also outputs that the hypohamiltonian graphs on 13 and 15 vertices each have crossing number exactly 4, but we await an independent implementation.

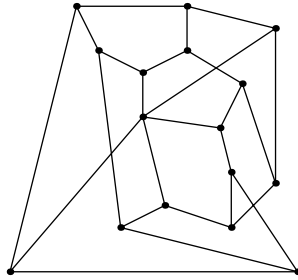


Fig. 17: A hypohamiltonian graph of order 16 and crossing number at most 3. It is isomorphic to the top-left of the four 16-vertex graphs depicted in Fig. 10.

By Theorem 2.12, we have  $\mathfrak{o}'_0 \leq 42$ . For crossing number 1, denote by  $H^n$  the hypohamiltonian graph of order  $n$  used to show this result. Once again, by applying the operation  $\text{Th}$  to  $H^n$  we obtain a plane hypohamiltonian graph  $\text{Th}(H^n)$  of order  $n + 4$  with a cubic quadrilateral face (see the proof of Theorem 2.12). Via Lemma 2.18 we obtain the family  $\{(\text{Th}(H^n))^*\}_{n=42}^\infty$  with  $|V((\text{Th}(H^n))^*)| = n + 6$ . This shows that

$$\mathfrak{o}'_0 \leq 42 \quad \text{and} \quad \mathfrak{o}'_1 \leq 48.$$

In Theorem 2.17, we showed that there exists a planar cubic hypohamiltonian graph  $L^n$  of order  $n$  for  $n = 70$  and every even  $n \geq 74$ . Each  $L^n$  contains a quadrilateral face, so by Lemma 2.18, each member of the family  $\{(L^{70})^*\} \cup \{(L^n)^*\}_{n=74}^\infty$  is hypohamiltonian and has crossing number 1. Therefore, for  $n = 72$  and every even  $n \geq 76$  there exists a cubic hypohamiltonian graph of order  $n$  and crossing number 1.

## 2.3 Hypotraceable Graphs

First of all, note that hypotraceable graphs, in contrast to hypohamiltonian graphs, may have connectivity 2. Kapoor, Kronk, and Lick [82] asked in 1968 whether hypotraceable graphs exist—see also Kronk’s note [89]. (It is left to the reader to verify with a simple program that no such graphs exist on 10 or fewer vertices.) This question was answered when a hypotraceable graph was subsequently found by Horton [72]. It has 40 vertices and is 3-connected (but not planar) and will prove to be very useful when discussing almost hypohamiltonian graphs in Chapter 3.

Thomassen [125, 127] showed that there exists a hypotraceable graph with  $n$  vertices for  $n \in \{34, 37\}$  and every  $n \geq 39$ , but we emphasise that Horton’s graph is 3-connected, whereas some of Thomassen’s graphs have connectivity 2, others 3. Since 1976, this list has been neither expanded—in particular, no

hypotraceable graph of order  $< 34$  is known—, nor has it been shown to be complete.

All constructions of hypotraceable graphs we know of rely on hypohamiltonian graphs as “building blocks”, as Wiener observes in [140]. The smallest known hypotraceable graph was found by Thomassen [125] and has order 34, see Fig. 18.

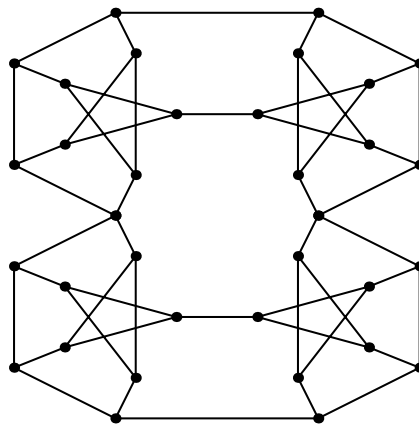


Fig. 18: The smallest known hypotraceable graph.

We now focus on the planar case. In 2011, Araya and Wiener [141] proved that there exist planar hypotraceable graphs on  $162 + 4k$  vertices for every  $k \geq 0$ , and on  $n$  vertices for every  $n \geq 180$ . To improve this result, we make use of the following theorem, which is essentially due to Thomassen.

**Theorem 2.24** (Thomassen) *Let  $G_1, G_2, G_3, G_4 \in \overline{\mathcal{H}}$ . Then there is a planar hypotraceable graph of order  $|V(G_1)| + |V(G_2)| + |V(G_3)| + |V(G_4)| - 6$ .*

*Proof.* The statement follows from Thomassen’s result [128] that every  $G_i$  must contain a cubic vertex, the proof of [125, Lemma 1], and the fact that the construction used in that proof (which does not address planarity) can be carried out to obtain a planar graph when all graphs  $G_i$  are planar—a detailed description of Thomassen’s method is given in Section 3.1.  $\square$

Applying Theorem 2.24 to Theorems 2.8 and 2.12, we obtain the following.

**Theorem 2.25** (Jooyandeh, McKay, Östergård, Pettersson, and Zamfirescu, 2016 [80]) *There exist planar hypotractable graphs on 154 vertices, and on  $n$  vertices for every  $n \geq 156$ .*

Concerning polyhedral hypotractable graphs, Thomassen was the first to show that such graphs exist [127]. He used a different method of construction than the one presented in Section 3.1 (which is also due to him—it yielded Theorem 2.24), the details of which are presented in the proof of Theorem 2.27. Thomassen’s smallest example of a polyhedral hypotractable graph has 515 vertices. Currently, the smallest known such graph has order 190. It is obtained by applying Thomassen’s method given in the proof of Theorem 2.27 to the 40-vertex plane hypohamiltonian graph from Fig. 8.

Now we discuss planar cubic hypotractable graphs. In 1993, Holton and Sheehan [71] asked if there exists an integer  $n$  such that a planar cubic hypotractable graph exists for every even integer  $\geq n$ . Araya and Wiener [7] settled the question with  $n = 356$ —in fact, their graphs are not only planar, but polyhedral! Until recently, the strongest result concerning the question for which orders such graphs exist was given in 2011 by Araya and Wiener [7] and is as follows.

**Theorem 2.26** (Araya and Wiener, 2011 [7]) *There exists a polyhedral cubic hypotractable graph on 340 vertices and on  $n$  vertices for every even  $n \geq 356$ .*

With Theorem 2.17 we can improve Theorem 2.26, and thus the bound linked to the question of Holton and Sheehan mentioned above. The short proof of this improvement is essentially identical to the proof of Araya and Wiener given in [7], but still we choose to present it, since it uses an interesting construction method introduced by Thomassen [127].

**Theorem 2.27** *There exists a polyhedral cubic hypotractable graph on 340 vertices and on  $n$  vertices for every even  $n \geq 344$ .*

*Proof.* We require a result of Thomassen [127], described in the following. For  $1 \leq i \leq 5$ , let  $T_i$  be pairwise disjoint hypohamiltonian graphs and let  $x_i$  and  $y_i$  be

adjacent cubic vertices, both lying in  $T_i$ . Let the neighbours of  $x_i$  ( $y_i$ ), different from  $y_i$  ( $x_i$ ) be  $a_i$  and  $b_i$  ( $c_i$  and  $d_i$ ). Consider the disjoint union of the graphs  $T_i - \{x_i, y_i\}$  and add the edges  $c_1a_2, c_2a_3, c_3a_4, c_4a_5, c_5a_1, d_1b_2, d_2b_3, d_3b_4, d_4b_5$ , and  $d_5b_1$ . The resulting graph  $T$  is hypotraceable by a lemma of Thomassen [127]. He notes [127] that  $T$  is 3-connected. Furthermore, if each  $T_i$  is cubic, then  $T$  is cubic, as well, and if each  $T_i$  is planar, then the labels can be chosen such that  $T$  is planar.

Now let each  $T_i$  be isomorphic to the planar cubic hypohamiltonian graph on 70 vertices constructed by Araya and Wiener [7] and shown in Fig. 15. Then we obtain a polyhedral cubic hypotraceable graph on 340 vertices. By Theorem 2.17, there exists a planar cubic hypohamiltonian graph  $T^n$  of order  $n$  for every even  $n \geq 74$ . By choosing  $T_1, \dots, T_4$  each to be the 70-vertex graph mentioned earlier, and  $T_5 \in \{T^n\}_{n \geq 74, n \text{ even}}$ , the statement is shown.  $\square$

We will come back to concepts closely related to hypotraceability in Section 3.5, where *almost* hypotraceable graphs are discussed, and in Section 3.6 and Chapter 4, in which a famous problem of Gallai concerning the intersection of all longest paths in a given graph will play a central role.

## 2.4 Girth Restrictions

There have been interesting questions posed concerning the girth of a hypohamiltonian graph. We begin with a simple remark. Collier and Schmeichel [29, p. 196] observed that the vertices of a triangle in a hypohamiltonian graph have degree at least 4. (This is true for hypotraceable graphs, as well.) It seems a priori strange that a hypohamiltonian graph may contain a triangle, i.e. have girth 3. In 1967, Herz, Duby and Vigué went so far as to conjecture [67] that every hypohamiltonian graph has girth  $\geq 5$ . But this was disproved by Thomassen, who constructed in [126] hypohamiltonian graphs of girth 3, as well as hypohamiltonian

nian graphs of girth 4. (With his approach, infinite families of such graphs can be constructed.) His smallest example of girth 3, based on six copies of the Petersen graph, is shown in Fig. 19.

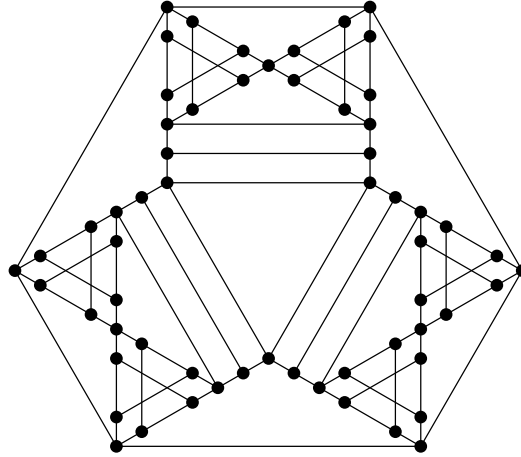


Fig. 19: Thomassen's hypohamiltonian graph of girth 3, published in 1974 and the first of its kind. It has order 60.

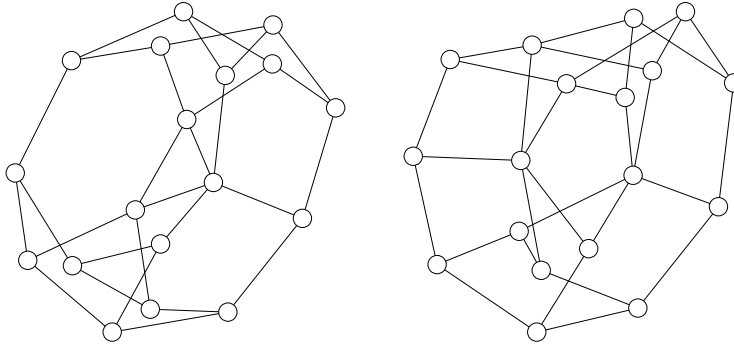


Fig. 20: The smallest hypohamiltonian graph of girth 3, and one of the five smallest hypohamiltonian graphs of girth 4, resp. Each has 18 vertices. The former was first given by Collier and Schmeichel [28] in 1977, while the latter was found recently by Goedgebeur and the author [47], and earlier and independently, by McKay [97] (but not published).

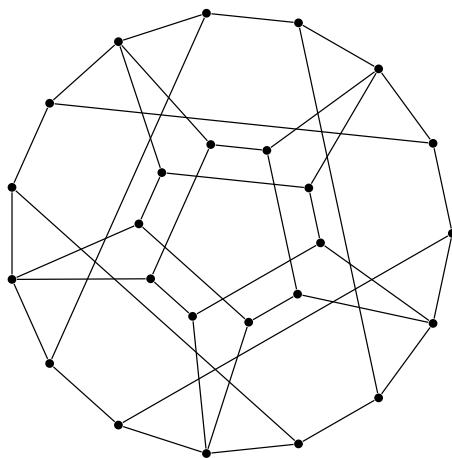


Fig. 21: The smallest hypohamiltonian graph of girth 6. It has order 25 and its automorphism group has order 80. It is due to Jan Goedgebeur and the author [47]. The illustration shown here is due to Nico Van Cleemput.

The smallest hypohamiltonian graphs for a given girth have the following orders:

- Girth 3: 18 (due to Collier and Schmeichel [28] and shown in Fig. 20)
- Girth 4: 18 (constructed by Goedgebeur and the author [47], as well as earlier and independently, by McKay [97], and depicted in Fig. 20)
- Girth 5: 10 (Petersen's graph shown in Figs. 1 and 45)
- Girth 6: 25 (due to Goedgebeur and the author [47] and shown in Fig. 21)
- Girth 7: 28 (Coxeter's graph [31], see Fig. 24)

The proof that the graph of girth 3 is minimal follows from the list of all hypohamiltonian graphs of order  $\leq 17$  given by Aldred, McKay, and Wormald [5]. It is noteworthy that Collier and Schmeichel [28] showed that there exists a hypohamiltonian graph of girth 3 and order  $n$  for every  $n \geq 18$  with the possible exceptions of 19, 20, 22, and 25. In [47] it is proven that the graphs of girth 4 and 6 are of smallest order. Petersen's graph is in fact the smallest hypohamiltonian graph (as proven in e.g. [71]), while the result that Coxeter's graph is the smallest



hypohamiltonian graph of girth 7 is shown at the end of this section. We note here that it is unknown whether hypohamiltonian graphs of girth greater than 7 exist.

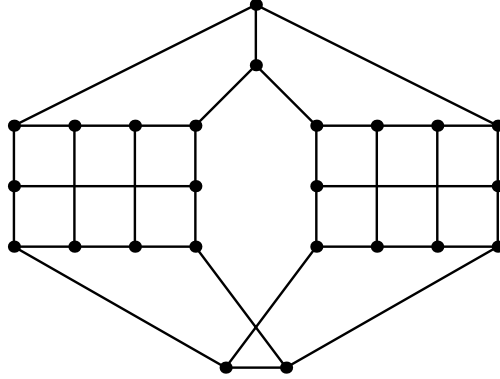


Fig. 22: A cubic hypohamiltonian graph of girth 4 constructed by Thomassen.  
It is the smallest such graph and has order 24.

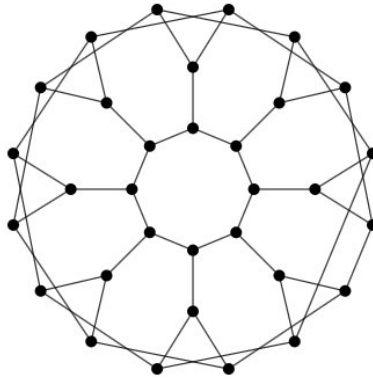


Fig. 23: Isaacs' flower snark  $J_7$ , a cubic hypohamiltonian graph of girth 6.  
It is the smallest such graph and has order 28.

We briefly summarise the situation in the cubic case. It is easy to see that every hypohamiltonian cubic graph has girth at least 4, since, as remarked above, a triangle in a hypohamiltonian graph cannot contain cubic vertices. Infinite classes of cubic hypohamiltonian graphs with girth 4, 5, and 6 are known, see [94, 95] for details and further references. However, there was only one known example of

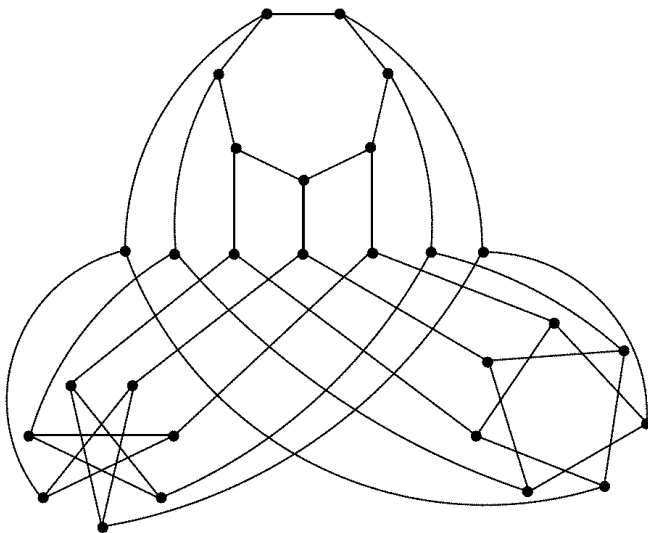


Fig. 24: Coxeter's graph, a cubic hypohamiltonian graph of girth 7 and order 28. It is the smallest hypohamiltonian graph of girth 7.

a hypohamiltonian cubic graph of girth 7, the Coxeter graph on 28 vertices (see Fig. 24), and no examples of girth greater than 7. Máčajová and Škoviera [95] proved that there exist infinitely many hypohamiltonian cubic graphs of girth 7 and cyclic connectivity 6. The existence of cyclically 7-edge-connected hypohamiltonian cubic graphs other than the Coxeter graph, however, remains open.

Through an exhaustive computer-search, McKay was able to determine the order of the smallest cubic hypohamiltonian graph of girth 4, 5, 6, and 7, establishing that certain such graphs previously constructed turned out to be the smallest of a fixed girth, see [97]. (Note that McKay does not state this explicitly. These results were verified independently by J. Goedgebeur.) These orders are presented in the table at the end of this section.

We now discuss the planar case. By the following theorem we know that any planar hypohamiltonian graphs improving on the upper bound for the order of the smallest planar hypohamiltonian graph—which currently stands at 40, see Theorem 2.8 and Fig. 8—must have girth either 3 or 4. A planar hypohamiltonian graph can have girth at most 5, since such a graph has a simple dual, and the average degree of a plane graph is less than 6. Note that, perhaps surprisingly,

there exist planar hypohamiltonian graphs of girth 3. Such graphs have not yet appeared in the literature, but by applying Thomassen's method from [126] to six copies of the plane hypohamiltonian 40-vertex graph presented in Fig. 8, one obtains such a graph on 240 vertices. This is the smallest known example.

**Theorem 2.28** (Jooyandeh, McKay, Östergård, Pettersson, and Zamfirescu, 2016 [80]) *There are no planar hypohamiltonian graphs with girth 5 on fewer than 45 vertices, and there is exactly one such graph on 45 vertices.*

*Proof.* The program *plantri* [15, 16] can be used to construct all planar graphs with a simple dual, girth 5, and up to 45 vertices. By checking these graphs, it turns out that only a single graph of order 45 is hypohamiltonian. That graph, which has an automorphism group of order 4, is shown in Fig. 25.  $\square$

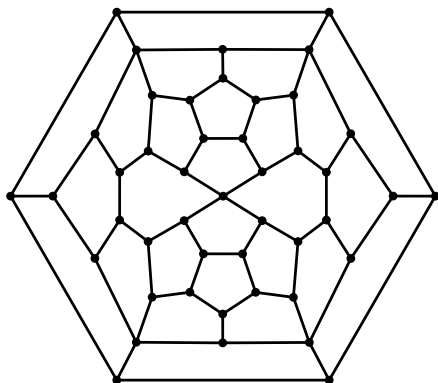


Fig. 25: A plane hypohamiltonian graph of girth 5 and order 45.

This is the unique graph with the aforementioned properties, and there is no smaller such graph.

The smallest known planar hypohamiltonian graphs of girth 3, 4, and 5 have order 240 (application of a construction of Thomassen [126]), 40 ([80]; see Theorem 2.8 and Fig. 8), and 45 ([80]; see Theorem 2.28 and Fig. 25), respectively. The only graph of which we know to be the smallest with the advertised property is the 45-vertex graph, see Theorem 2.28.

As we have seen, infinitely many planar cubic hypohamiltonian graphs have been described, starting with work of Thomassen [129] in 1981. However, *all* of these graphs had girth 4, so until recently it was unknown whether planar cubic hypohamiltonian graphs of girth 5 exist.

**Theorem 2.29** (McKay, 2016 [96]) *There exist planar cubic hypohamiltonian graphs of girth 5. There exist exactly three such graphs of order 76, and none on fewer vertices.*

McKay recently proved the above result. Each of the three graphs provided has only the identity as automorphism. One of them is shown in Fig. 26. Subsequently, Goedgebeur showed by computer search that there exists a planar cubic hypohamiltonian graph of girth 5 and order 78, which has a non-trivial automorphism group, see Fig. 27. This may prove to be useful in tackling the following natural question raised by McKay in [96]: are there infinitely many planar cubic hypohamiltonian graphs of girth 5?

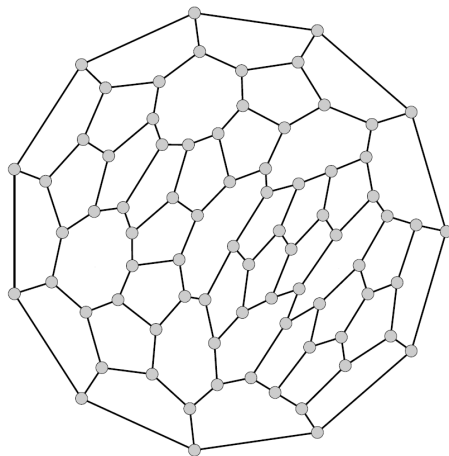


Fig. 26: A plane cubic hypohamiltonian graph of girth 5. It has order 76.

McKay showed that there is no smaller such graph [96].

We collect the findings of this section in the following table. The entries show the order of the smallest known graph with the given properties, listed by girth. The symbol “—” designates an impossible combination of properties, and

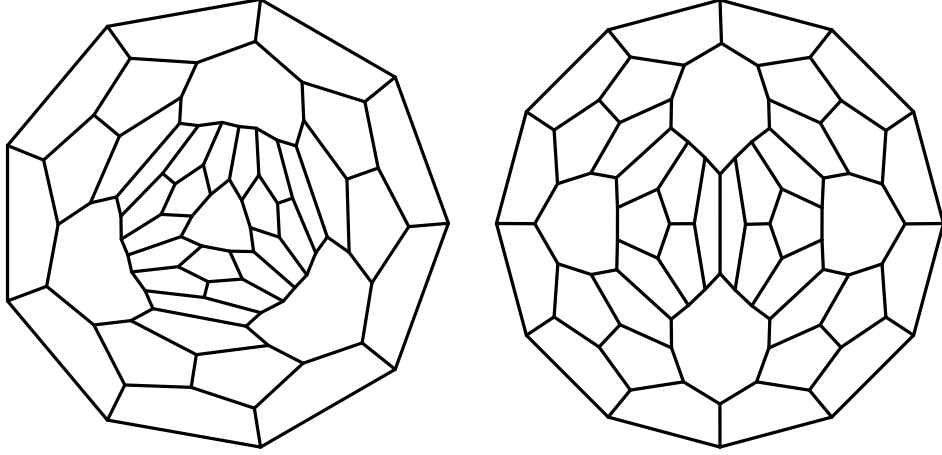


Fig. 27: Both figures show different symmetries of the same graph constructed by J. Goedgebeur. It is the smallest planar cubic hypohamiltonian graph of girth 5 with a non-trivial automorphism group. It has 78 vertices and  $D_{3h}$  symmetry (as an abstract group, this is the dihedral group of order 12).

underlined entries have been shown to be optimal, i.e. there exists no graph of order  $k$  satisfying the given properties with  $k$  strictly smaller than the value given in the table.

Little is known concerning the *lower* bounds for the orders of the smallest planar hypohamiltonian graphs discussed in this section, excluding the case when the girth of the graph is 5. In [5], Aldred, McKay, and Wormald gave a complete list of hypohamiltonian graphs of order at most 17.

Finally, let us give a simple argument for the fact that there are no hypohamiltonian graphs of girth 7 on fewer than 28 vertices: assume there was such a graph  $G$ . It would have to be non-cubic, since we know that the smallest cubic hypohamiltonian graph has order 28. Let  $v \in V(G)$  have degree at least 4 and denote by  $d(v, w)$  the *distance* between  $v$  and  $w$ , i.e. the length of a shortest path between  $v$  and  $w$ . Since the girth of  $G$  is 7, we have that all elements in

$$\{w \in V(G) : d(v, w) \leq 3\} \subset V(G)$$

are pairwise distinct, so  $|V(G)| \geq 29$ , which yields a contradiction.

<b>girth</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>
hypohamiltonian	<u>18</u>	<u>18</u>	<u>10</u>	<u>25</u>	<u>28</u>
cubic hypohamiltonian	–	<u>24</u>	<u>10</u>	<u>28</u>	<u>28</u>
planar hypohamiltonian	240	40	<u>45</u>	–	–
planar cubic hypohamiltonian	–	70	<u>76</u>	–	–

## 2.5 Two Proofs of Grinberg’s Criterion

We give Grinberg’s Theorem once more and then provide two proofs. We also remark that in order to prove (and indeed formulate) Grinberg’s Criterion, we tacitly make use of the Jordan curve theorem—for Jordan’s original work, see [81, pp. 587–594], and for an undisputed proof by Thomassen, [131].

**Theorem 2.1: Grinberg’s Criterion** (Grinberg, 1968 [51]) *Given a plane graph  $G$  with a hamiltonian cycle  $\mathfrak{h}$  and  $f_k$  ( $f'_k$ )  $k$ -faces inside (outside) of  $\mathfrak{h}$ , we have*

$$\sum_{k \geq 3} (k - 2)(f_k - f'_k) = 0. \quad (\dagger)$$

For the first proof, we require the following proposition. A graph  $G$  is *outerplanar* if it is planar and contains a face  $F$  such that all vertices of  $G$  lie on the boundary of  $F$ . An outerplanar graph is hamiltonian if and only if it is 2-connected.

**Proposition 2.30** *Let  $G$  be a 2-connected outerplanar graph of order  $n$ , and denote by  $f_k$  the number of  $k$ -faces in  $G$ . Then we have*

$$\sum_{k \geq 3} (k - 2)f_k = 2(n - 2).$$

*Proof.* If  $G$  is an  $n$ -cycle, then the statement is trivially true, so we may assume that the  $n$ -face  $F$  of  $G$  is unique. Let  $E' = E(G) \setminus E(F)$  and  $m' = |E'|$ . We have

$$\sum_{3 \leq k \leq n-1} f_k = m' + 1,$$

and by taking  $F$  into account,

$$\sum_{k \geq 3} f_k = m' + 2. \quad (\diamond)$$

Each edge in  $E'$  lies in the boundary of two faces both of which are not  $F$ . Therefore

$$\sum_{3 \leq k \leq n-1} k f_k = 2m' + n, \quad \text{whence} \quad \sum_{k \geq 3} k f_k = 2(m' + n).$$

Using  $(\diamond)$  we obtain the statement.  $\square$

*First Proof of Grinberg's Criterion.* Let  $G$  be a plane graph of order  $n$  with a hamiltonian cycle  $\mathfrak{h}$ .  $G$  is composed of two 2-connected outerplanar graphs  $H$  and  $H'$ , the intersection of which is  $\mathfrak{h}$ , where  $H$  shall lie inside of  $\mathfrak{h}$  and  $H'$  outside of  $\mathfrak{h}$ . We denote by  $h_k$  ( $h'_k$ ) the number of  $k$ -faces in  $H$  ( $H'$ ). We apply Prop. 2.30 to  $H$  and  $H'$  and obtain

$$\sum_{k \geq 3} (k-2)h_k = 2(n-2) \quad \text{and} \quad \sum_{k \geq 3} (k-2)h'_k = 2(n-2).$$

In  $G$ , denote by  $f_k$  ( $f'_k$ ) the number of  $k$ -faces inside (outside) of  $\mathfrak{h}$ . Then, since

$$f_k = \begin{cases} h_k & \text{if } k < n \\ h_k - 1 & \text{if } k = n \end{cases} \quad \text{and} \quad f'_k = \begin{cases} h'_k & \text{if } k < n \\ h'_k - 1 & \text{if } k = n \end{cases}$$

hold, we have

$$\sum_{k \geq 3} (k-2)f_k = n-2 \quad \text{and} \quad \sum_{k \geq 3} (k-2)f'_k = n-2,$$

which completes the proof.  $\square$

*Second Proof of Grinberg's Criterion.* Consider a cycle  $\mathfrak{h}$  of length  $|V(G)|$  embedded in the plane. By successively adding edges to  $\mathfrak{h}$ , we can construct  $G$  in a finite number of steps such that each intermediary graph contains neither loops, nor multiple edges, and is planar, as well.

We use induction.  $(\dagger)$  clearly holds for  $\mathfrak{h}$ . Now let  $(\dagger)$  hold for an intermediary graph  $G'$ . By adding an edge  $e$  to  $G'$ , where w.l.o.g. we consider  $e$  to lie in the interior of  $\mathfrak{h}$ , we split a  $j$ -face into a  $(j - \ell)$ -face and an  $(\ell + 2)$ -face, and nothing else changes, i.e. in the interior of  $\mathfrak{h}$  we have one  $j$ -face less, but one  $(j - \ell)$ -face and one  $(\ell + 2)$ -face more. Since  $G'$  satisfies  $(\dagger)$ , it remains to see that  $\ell + 2 - 2 + j - \ell - 2 - j + 2 = 0$ .  $\square$



# Chapter 3

## Almost Hypohamiltonian Graphs

### 3.1 Introduction

In this Chapter we discuss a new family of graphs, introduced and baptised by the author. We repeat the definition given in the introduction: a graph  $G$  is *almost hypohamiltonian* if  $G$  is non-hamiltonian, there exists a vertex  $w$ , which we will call *exceptional*, such that  $G - w$  is non-hamiltonian, and for every vertex  $v \neq w$  the graph  $G - v$  is hamiltonian. Note that this does not coincide with Thomassen's concept of an almost hypohamiltonian graph mentioned in [127] and used in [126]. The motivation for introducing almost hypohamiltonian graphs is three-fold: (i) we acquire new methods to construct hypohamiltonian graphs, while also (ii) obtaining a result related to an old conjecture of Thomassen, and (iii) working towards a more general theory of non-hamiltonian graphs in which some, but not all vertex-deleted subgraphs are hamiltonian, and a method how to deal with these exceptions.

For a graph  $G$ , define  $G^w$  as  $G$  to which we add a vertex  $w$  and edges  $vw$  for all  $v \in V(G)$ . (This is the *join* of  $G$  and  $(\{w\}, \emptyset)$  and we will write  $G + (\{w\}, \emptyset)$ .) Let  $S \subset V(G)$  satisfy the following property. For each  $v \in V(G)$ , there is a hamiltonian path in  $G - v$  with end-vertices in  $S$ . Call such a set  $S$  a *set of ends*, and write  $G^{w,S} = (V(G) \cup \{w\}, E(G) \cup \{vw : v \in S\})$ . Thus  $G^{w,V(G)} = G^w$  if  $V(G)$  is a set of ends.

**Lemma 3.1** (2015 [144]) *Let  $G$  be a hypotraceable graph, and  $S$  a set of ends. Then  $G^{w,S}$  is almost hypohamiltonian with exceptional vertex  $w$ .*

*Proof.*  $G$  is hypotraceable and therefore non-traceable, so  $G$  and  $G^{w,S}$  are non-hamiltonian. Consider  $v \in V(G)$ . In  $G - v$  there exists a hamiltonian path  $\mathbf{p}$  the end-vertices  $u$  and  $u'$  of which belong to  $S$ . If we add to  $\mathbf{p}$  the edges  $uw$  and  $wu'$ , we obtain a hamiltonian cycle in  $G^{w,S} - v$ .  $\square$

Thomassen [125] introduced the following method to construct hypotraceable graphs from known hypohamiltonian graphs—note that this is the same technique as in Theorem 2.24 but differs from the approach used to prove Theorem 2.27. Consider four hypohamiltonian graphs  $G_1, \dots, G_4$ , and assume there exist cubic vertices  $v_i \in V(G_i)$ . Put  $N(v_i) = \{v_{i1}, v_{i2}, v_{i3}\}$ . Take the four vertex-disjoint graphs  $G_i - v_i$ . Therein, identify  $v_{11}$  with  $v_{21}$  and  $v_{31}$  with  $v_{41}$ , and add the edges  $v_{12}v_{32}, v_{22}v_{42}, v_{13}v_{33}, v_{23}v_{43}$ . This operation preserves planarity. Thomassen [125] showed that the resulting graph is hypotraceable. Call  $T$  the 34-vertex hypotraceable graph constructed by Thomassen [125] by applying above method to four copies of the Petersen graph. The graph  $T$  is shown in Fig. 18.  $T$  is the smallest known hypotraceable graph.

**Corollary 3.2** [144]  *$T^w$  is an almost hypohamiltonian graph of order 35.*

We recall Thomassen's [128] question from 1978 whether hypohamiltonian graphs with minimum degree 4 exist. Demanding even more, in [128] he also asked if 4-connected such graphs exist—this is one of the central open problems in the theory of hypohamiltonian graphs.  $T^w$  is almost hypohamiltonian and has minimum degree 4. But we obtain an even more surprising result if we apply Lemma 3.1 to Horton's hypotraceable graph  $H$  from [72] (with  $S = V(H)$ ). As  $H$  is 3-connected, this yields a 4-connected almost hypohamiltonian graph of order 41, shown in Fig. 28. (Adding planarity as a condition is futile due to Tutte's famous result that planar 4-connected graphs are hamiltonian [133].) After Horton's discovery, Thomassen [127] generalised his construction and showed that there exist infinitely many 3-connected hypotraceable graphs. This immediately yields infinitely many 4-connected almost hypohamiltonian graphs.

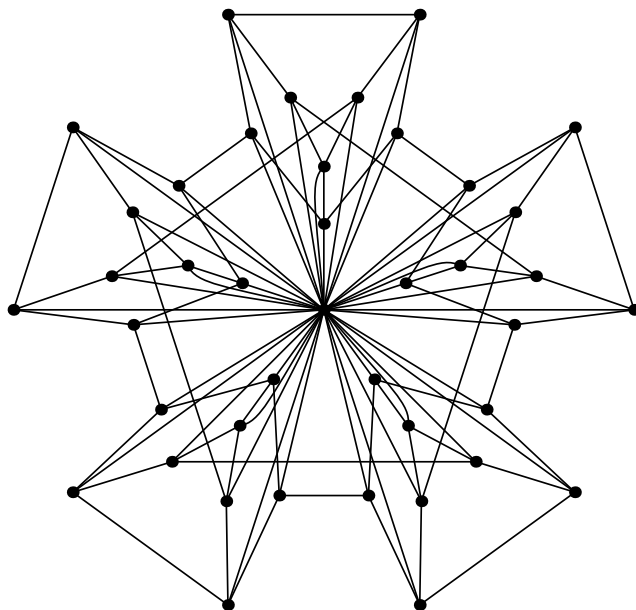


Fig. 28: A 4-connected almost hypohamiltonian graph of order 41.  
It is the smallest known such graph.

In reverse order, if we take an almost hypohamiltonian graph and delete its exceptional vertex, we are only guaranteed to obtain a non-hamiltonian graph which is traceable if an arbitrary vertex is deleted—the family of such graphs is of interest as it contains both the family of all hypotraceable graphs and the family of all hypohamiltonian graphs, but is not their union. It will be discussed in Chapter 5.

## 3.2 The Planar Case

**Lemma 3.3** [144] *The 39-vertex graph shown in Fig. 29 is planar and almost hypohamiltonian.*

*Proof.* We denote the graph from Fig. 29 by  $G$ .

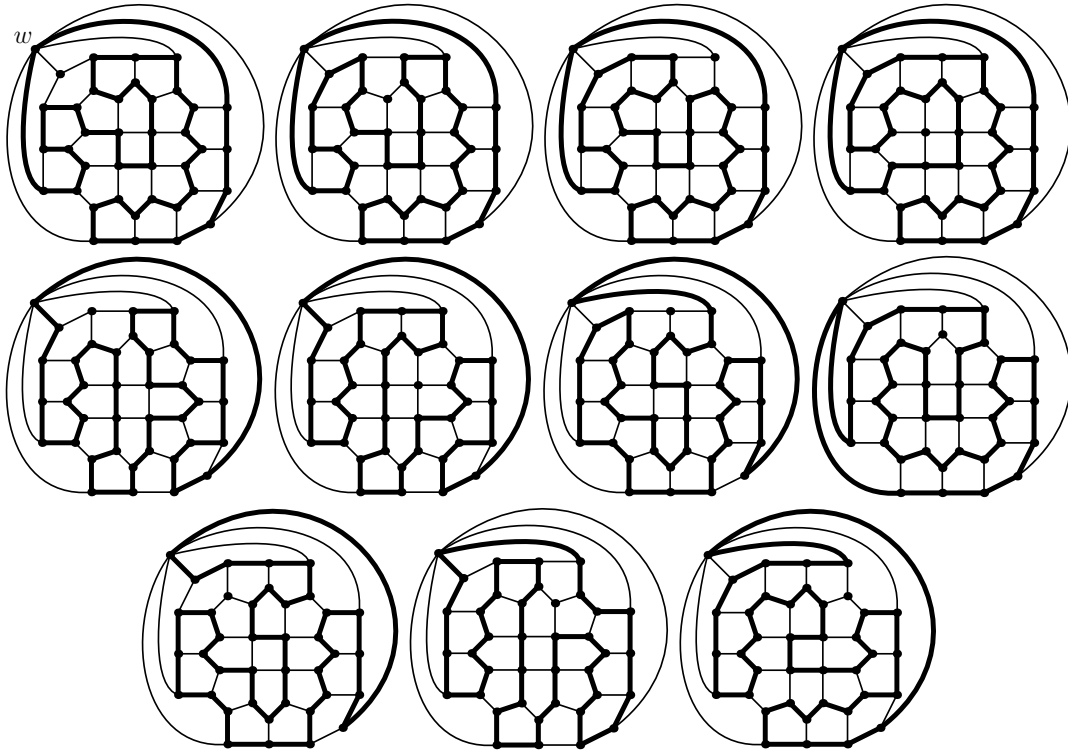


Fig. 29: After taking symmetries into account,  
a hamiltonian cycle in  $G - v$  for every  $v \neq w$  is shown.

$G$  is obviously planar. By Grinberg's Criterion,  $G$  is non-hamiltonian. Fig. 29 shows that for all  $v \in V(G) \setminus \{w\}$  the graph  $G - v$  is hamiltonian. It remains to prove that  $G - w$  is non-hamiltonian. Assume the contrary, i.e.  $G - w$  contains a hamiltonian cycle  $\mathfrak{h}$ . Let  $C$  be the cycle of length 12 in the dual of  $G - w$  depicted as a dotted cycle in Fig. 30. We denote by  $\mathcal{A}$  the union of all faces of  $G$  lying on the same side of  $\mathfrak{h}$  as the unbounded face, and by  $\mathcal{B}$  the union of all faces not lying in  $\mathcal{A}$ . Denote by  $f_5$  ( $f'_5$ ) the number of pentagons inside (outside) of  $\mathfrak{h}$ . ( $\dagger$ ) is now

$$\pm 2 + 3(f_5 - f'_5) - 16 = 0,$$

which implies that the (unique) quadrilateral lies outside of  $\mathfrak{h}$ , i.e. in  $\mathcal{A}$ . In Fig. 30, we have labeled faces  $\mathcal{A}$  or  $\mathcal{B}$  if they obviously belong to one of them. Edges are bold if they certainly lie in  $\mathfrak{h}$ .

Claim. *Of every two faces corresponding to adjacent vertices in  $C$ , one must lie in  $\mathcal{A}$  and the other must lie in  $\mathcal{B}$ .*

*Proof of the Claim.* For  $v \in V(C)$ , denote by  $F(v)$  the face in  $G - w$  corresponding to  $v$ . Assume there exist adjacent vertices  $v_1, v_2 \in V(C)$  such that  $F(v_1)$  and  $F(v_2)$  belong to the same component  $\mathcal{X} \in \{\mathcal{A}, \mathcal{B}\}$ . We can immediately observe that  $\mathcal{X} \neq \mathcal{B}$ , since otherwise we would have a vertex surrounded exclusively by faces from  $\mathcal{B}$ , see Fig. 30. With the argument concerning Grinberg's Criterion given above, we have that among the twelve pentagons corresponding to the vertices in  $C$ , six lie in  $\mathcal{A}$  and six lie in  $\mathcal{B}$ . Thus, the Claim is proven.

By the Claim, either the quadrilateral shares no edge with a face in  $\mathcal{A}$ , or  $\mathcal{A}$  is not connected. So  $G - w$  does not contain a hamiltonian cycle and the lemma is proven.  $\square$

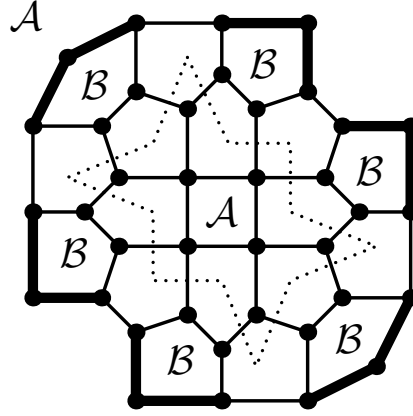


Fig. 30

Consider graphs  $G$  and  $H$  containing cubic vertices  $x \in V(G)$  and  $y \in V(H)$ . Denote by  $G_x H_y$  one of the graphs obtained from  $G - x$  and  $H - y$  by identifying the vertices in  $N(x)$  with those in  $N(y)$  using a bijection. As stated in Lemma 2.20, Thomassen [125] showed that if  $G, H \in \mathcal{H}$ , then  $G_x H_y \in \mathcal{H}$ . Recall that if  $G \in \mathcal{H}$ , then  $G$  contains no triangle with a cubic vertex.

**Lemma 3.4** [144] *Let  $G \in \mathcal{H}_1$  contain a cubic vertex  $x$  different from the exceptional vertex  $w$  of  $G$ , and  $H \in \mathcal{H}$  contain a cubic vertex  $y$ . Then  $G_x H_y \in \mathcal{H}_1$  with exceptional vertex  $w$ . If  $G$  and  $H$  are planar, then so is  $G_x H_y$ .*

*Proof.* We treat  $G - x$  and  $H - y$  as subgraphs of  $G_x H_y$ . Let  $N(x) = N(y) = \{z_1, z_2, z_3\}$  in  $G_x H_y$ . Abusing notation, we also denote by  $z_i$  the corresponding vertices in  $G$  and  $H$ , i.e. by  $\{z_1, z_2, z_3\}$  the set of neighbours of  $x$  in  $G$ , and by  $\{z_1, z_2, z_3\}$  the set of neighbours of  $y$  in  $H$ . Assume  $G_x H_y$  contains a hamiltonian cycle  $\mathfrak{h}$ . If  $\mathfrak{h} \cap G$  is connected, then it is a path. W.l.o.g. this path has end-vertices  $z_2, z_3$ . Then  $(\mathfrak{h} \cap G) \cup z_2 x z_3$  is a hamiltonian cycle in  $G$ , a contradiction, as  $G \in \mathcal{H}_1$ . If  $\mathfrak{h} \cap G$  consists of two components, w.l.o.g. a path with end-vertices  $z_2, z_3$  and the isolated vertex  $z_1$ , then  $(\mathfrak{h} \cap H) \cup z_2 y z_3$  is a hamiltonian cycle in  $H$ , again a contradiction.

Next, we prove that  $G_x H_y - w$  is non-hamiltonian. Assume the contrary, and let  $\mathfrak{h}$  be a hamiltonian cycle in  $G_x H_y - w$ . Suppose that  $w \notin N(x)$ . W.l.o.g. let  $z_1$  be a vertex satisfying either  $\{e \in E(\mathfrak{h}) : e \text{ is incident to } z_1\} \subset E(G)$  or  $\{e \in E(\mathfrak{h}) : e \text{ is incident to } z_1\} \subset E(H)$ . In the former case, by adding the edges  $xz_2$  and  $xz_3$  to  $\mathfrak{h} \cap (G - x)$  we obtain a hamiltonian cycle in  $G - w$ , a contradiction, as  $w$  is an exceptional vertex of  $G$ . In the latter case,  $(\mathfrak{h} \cap (H - y)) \cup z_2 y z_3$  is a hamiltonian cycle in  $H$ , a contradiction. Now say  $w = z_1$ . Once more, let  $\mathfrak{h}$  be a hamiltonian cycle in  $G_x H_y - w$ . By adding to  $\mathfrak{h} \cap (G - x)$  the edges  $xz_2$  and  $xz_3$  we obtain a hamiltonian cycle in  $G - w$ , again a contradiction.

It remains to show that  $G_x H_y - v$  is hamiltonian for all  $v \neq w$ . Let  $v \in V(G) \setminus N(x)$ . Then there exists a hamiltonian cycle  $\mathfrak{h}$  in  $G - v$ . Assume w.l.o.g. that  $z_2 x z_3 \subset \mathfrak{h}$ . Put  $\mathfrak{p}_G = \mathfrak{h} - x$ . Let  $\mathfrak{p}_H$  be the path in  $H$  obtained by taking the hamiltonian cycle in  $H - z_1$  minus  $y$ .  $\mathfrak{p}_G \cup \mathfrak{p}_H$  is the desired hamiltonian cycle in  $G_x H_y - v$ . What if  $v \in N(x)$ , say  $v = z_1$ ? Then certainly  $z_2 x z_3 \subset \mathfrak{h}$ ,  $\mathfrak{h}$  being a hamiltonian cycle in  $G - z_1$ . We are once more in the situation discussed above. For  $v \in V(H)$  the treatment is very similar.  $\square$

In Section 2.1.3, we defined the operation  $\text{Th}$ , which was originally introduced by Thomassen [129]. The statement of the following lemma is a slight modification of a claim of Thomassen from [129]; as mentioned in the proof of Prop. 2.11, a detailed proof of this lemma was given by Araya and Wiener in [141].

**Lemma 3.5** (Thomassen, 1981 [129]) *Let  $G$  be a planar non-hamiltonian graph containing a 4-face bounded by the cycle  $C$ . Then  $\text{Th}(G_C)$  is planar and non-hamiltonian.*

We now prove a modified version of Thomassen's Lemma 2.10.

**Lemma 3.6** (2015 [144]) *Let  $G \in \overline{\mathcal{H}}_1$  have exceptional vertex  $w$  and contain a cubic 4-face bounded by the cycle  $C$ ,  $w \notin V(C)$ . Then  $\text{Th}(G_C) \in \overline{\mathcal{H}}_1$ . If  $G$  is cubic, then so is  $\text{Th}(G_C)$ .*

*Proof.* Let  $C = v_1v_2v_3v_4$ . By Lemma 3.5, both  $\text{Th}(G_C)$  and  $\text{Th}((G - w)_C) = \text{Th}(G_C) - w$  are planar and non-hamiltonian. We first show that  $\text{Th}(G_C) - v'_i$  is hamiltonian. Let  $\mathfrak{h}$  be a hamiltonian cycle in  $G - v_2$ . Clearly,  $P = v_1v_4v_3 \subset \mathfrak{h}$ . In  $\text{Th}(G_C)$ , transform  $P$  into  $v_1v_4v'_4v'_3v'_2v_2v_3$ . We obtain a hamiltonian cycle in  $\text{Th}(G_C) - v'_1$ . The cases  $v'_2$ ,  $v'_3$ , and  $v'_4$  work in the same way. We construct the remaining hamiltonian cycles in  $\text{Th}(G_C) - v$ , where  $v \neq w$ , by modifying cycles of length  $|V(G)| - 1$  in  $G - v$ , see Fig. 31.  $\square$

**Theorem 3.7** [144] *There exists a planar almost hypohamiltonian graph of order  $n$  for every  $n \geq 76$ .*

*Proof.* Theorem 2.12 states that for every  $n \geq 42$  there exists a planar hypohamiltonian graph  $H$  of order  $n$ . By Lemma 3.4, if  $G$  is the 39-vertex planar almost hypohamiltonian graph given in Fig. 29 and  $x \in V(G)$ ,  $y \in V(H)$  are cubic (note that  $x \neq w$  since  $w$  is not cubic), then  $G_xH_y$  provides a suitable graph of order  $n + 34$ .  $\square$

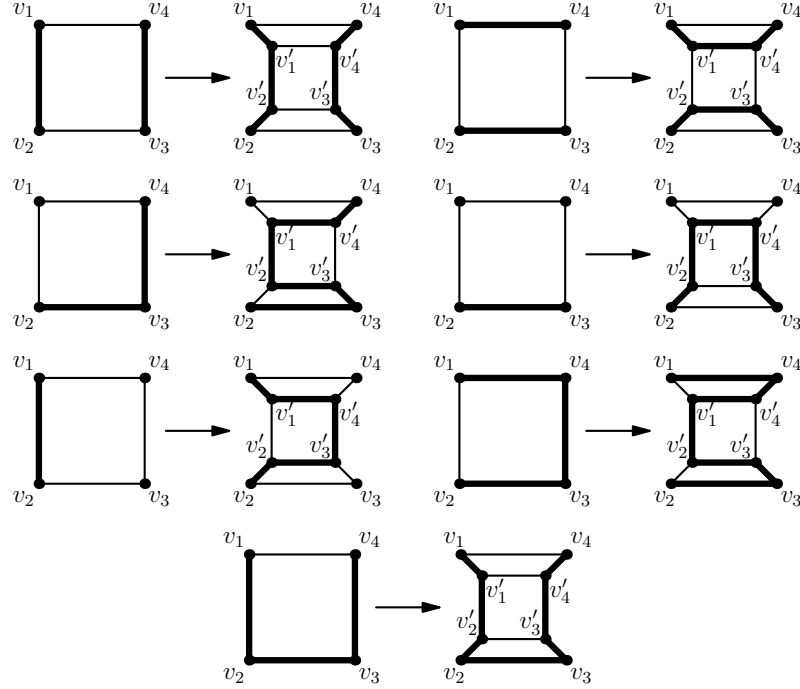


Fig. 31: On the left-hand side of each of the seven diagrams the bold edges show the subset of edges of the 4-cycle  $C$  contained in an  $(n-1)$ -cycle  $\mathfrak{h}'$  in  $G$ ; on the right-hand side it is shown what the modified  $\mathfrak{h}'$  looks like in  $\text{Th}(G_C)$ .

### 3.3 The General Case

The following operation was already defined for hypohamiltonian graphs in Section 2.2. Let  $G \in \mathcal{H}_1$  contain a 4-cycle  $v_1v_2v_3v_4 = C$ . We delete the edges  $v_1v_2$  and  $v_3v_4$ , add two new vertices  $v'_1$  and  $v'_4$ , and add the edges  $v'_1v'_4$ ,  $v_1v'_1$ ,  $v_4v'_4$ ,  $v'_1v_3$  and  $v'_4v_2$ . Denote the resulting graph by  $G_C^*$ . The proof of Lemma 3.8 essentially coincides with the proof of Lemma 2.18, and is omitted here.

**Lemma 3.8** [144] *Consider  $G \in \mathcal{H}_1$  with exceptional vertex  $w$ , and let  $G$  include a cubic 4-cycle  $C$  not containing  $w$ . Then  $G_C^* \in \mathcal{H}_1$ . If  $G$  is cubic, then so is  $G_C^*$ . If  $G$  is planar, then  $\text{cr}(G_C^*) = 1$ .*



Notice that  $\text{Th}(G_C) = (G_C^*)_{C'}^*$ , where  $C' = v_1v'_1v'_4v_4$  and for the second iteration of  $\star$  we delete  $v_1v'_1$  and  $v_4v'_4$ . So in a certain sense, this describes “half” of an application of the operation  $\text{Th}$  introduced in Section 2.1.3.

In order to give a good upper bound on the smallest integer  $n_0$  for which there exists an almost hypohamiltonian graph of order  $n$  for every  $n \geq n_0$ , we prove a simple yet useful gluing lemma that transforms two hypohamiltonian graphs into an almost hypohamiltonian one.

**Lemma 3.9** [144] *Let  $G$  and  $H$  be hypohamiltonian graphs containing cubic vertices  $w \in V(G)$  and  $w' \in V(H)$ , and let  $u, v \in N(w)$  and  $u', v' \in N(w')$ . If we identify  $u$  with  $u'$ ,  $v$  with  $v'$  and  $w$  with  $w'$ , we obtain an almost hypohamiltonian graph  $\Gamma$  with exceptional vertex  $w = w'$ . This operation preserves planarity.*

*Proof.* Note that as  $G$  and  $H$  are hypohamiltonian and  $w$  and  $w'$  are cubic, we have  $uv \notin E(G)$  and  $u'v' \notin E(H)$ . Assume there exists a hamiltonian cycle  $\mathfrak{h}$  in  $\Gamma$ . By abuse of notation  $u, v, w$  shall also denote the vertices in  $\Gamma$  obtained when identifying  $u$  with  $u'$ ,  $v$  with  $v'$  and  $w$  with  $w'$ , respectively. Let  $xvy \subset \mathfrak{h}$ . There are three cases to study.

(a)  $x, y \in V(G) \setminus \{w\}$ . Then  $(\mathfrak{h} \cap G) \cup uw$  is a hamiltonian cycle in  $G$ , a contradiction.

(b)  $x \in V(G) \setminus \{w\}$  and  $y \in V(H) \setminus \{w\}$ . Now either  $(\mathfrak{h} \cap G) \cup vw$  is a hamiltonian cycle in  $G$  or  $(\mathfrak{h} \cap H) \cup vw$  is a hamiltonian cycle in  $H$ . In both cases a contradiction is obtained.

(c)  $x = w$  and (w.l.o.g.)  $y \in V(G)$ . Thus  $vw \in E(\mathfrak{h})$ . But then  $(\mathfrak{h} \cap G) \cup uw$  is a hamiltonian cycle in  $G$ , once more a contradiction. Hence,  $\Gamma$  is non-hamiltonian.

We now show that  $\Gamma - w$  is non-hamiltonian. Again, assume the contrary, and let  $\mathfrak{h}$  be a hamiltonian cycle in  $\Gamma - w$ . Then  $(\mathfrak{h} \cap G) \cup vw$  yields a hamiltonian cycle in  $G$ , a contradiction.

Finally, we prove that  $\Gamma - x$  is hamiltonian for every  $x \neq w$ . There are two cases.

(a)  $x \in \{u, v\}$ , say  $x = u$ . As  $w$  has degree 3 in  $G$ , a hamiltonian cycle  $\mathfrak{h}$  in  $G - u$  contains the edge  $vw$ . Similarly, a hamiltonian cycle  $\mathfrak{h}'$  in  $H - u'$  uses the edge  $v'w'$ . Now  $(\mathfrak{h} - vw) \cup (\mathfrak{h}' - v'w')$  yields a hamiltonian cycle in  $\Gamma - u$ .

(b)  $x \notin \{u, v\}$ . Let  $x \in V(G)$ . Consider a hamiltonian cycle  $\mathfrak{h}$  in  $G - x$ .  $\mathfrak{h}$  contains  $wu$  or  $vw$  (possibly both), say  $vw$ . Let  $\mathfrak{h}'$  be a hamiltonian cycle in  $H - u'$ . As before,  $\mathfrak{h}'$  contains  $v'w'$ . Now  $(\mathfrak{h} - vw) \cup (\mathfrak{h}' - v'w')$  is a hamiltonian cycle in  $\Gamma - x$ .  $\square$

Observe that if  $\Gamma$  is the graph from the statement of Lemma 3.9, it follows from its proof that  $\Gamma - vw$  is also almost hypohamiltonian.

**Theorem 3.10** [144] *There exists an almost hypohamiltonian graph of order  $n$  for every  $n \geq 17$  with the possible exception of 18, 19, 21, and 24.*

*Proof.* It is known (see e.g. [5] for details) that there exist hypohamiltonian graphs of order  $n$  if and only if  $n \in \{10, 13, 15, 16\}$  or  $n \geq 18$ , to which we apply Lemma 3.9. (Note that no hypohamiltonian graph with minimum degree at least 4 is known.) The equation  $x + y - 3 = n$  has solutions  $x, y \in \{10, 13, 15, 16, 18, 19, 20, \dots\}$  for every  $n \geq 17$  except  $n = 18, 19, 21, 24$ .  $\square$

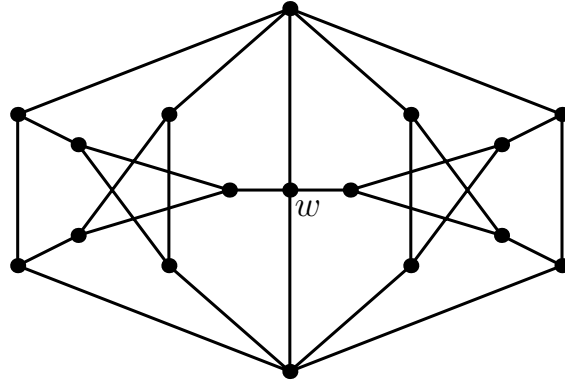


Fig. 32: The smallest known almost hypohamiltonian graph. Its exceptional vertex is marked  $w$ . It has order 17 and is obtained by applying Lemma 3.9 to two copies of the Petersen graph.

Next, we present a method of transforming two almost hypohamiltonian graphs into a hypohamiltonian one.

**Theorem 3.11** [144] *Consider  $G, H \in \mathcal{H}_1$  with cubic exceptional vertices  $w$  and  $w'$ , respectively. Then  $G_w H_{w'} \in \mathcal{H}$ . If  $G$  and  $H$  are planar, then so is  $G_w H_{w'}$ .*

*Proof.* We denote by  $x, y, z$  the vertices in  $G_w H_{w'}$  obtained when identifying  $N(w)$  with  $N(w')$ . Abusing notation, we also write  $N(w) = \{x, y, z\}$  in  $G$  and  $N(w') = \{x, y, z\}$  in  $H$ , where  $x$  in  $G_w H_{w'}$  is the vertex obtained when identifying  $x$  in  $G$  with  $x$  in  $H$ , and analogously for  $y$  and  $z$ .

First we show that  $G_w H_{w'} - x$  is hamiltonian. Let  $\mathfrak{h}_G$  be a hamiltonian cycle in  $G - x$ , and  $\mathfrak{h}_H$  a hamiltonian cycle in  $H - x$ . By deleting from  $\mathfrak{h}_G$  the edges  $yw$  and  $wz$  we obtain a path  $\mathfrak{p}_G$  in  $G$  which avoids  $x$  and  $w$  and has end-vertices  $y$  and  $z$ . From  $\mathfrak{h}_H$  we delete  $yw'$  and  $w'z$  and obtain a path  $\mathfrak{p}_H$  which avoids  $x$  and  $w'$  and has end-vertices  $y$  and  $z$ . Now  $\mathfrak{p}_G \cup \mathfrak{p}_H$  is a cycle of length  $|V(G_w H_{w'})| - 1$  avoiding  $x$ , as wished. Exactly in the same manner one shows that  $G_w H_{w'} - y$  and  $G_w H_{w'} - z$  are hamiltonian

Now we prove that  $G_w H_{w'} - v$  is hamiltonian, where  $v \in V(G_w H_{w'}) \setminus N(w)$ ; w.l.o.g.  $v \in V(G) \setminus \{w\}$ . Consider a hamiltonian cycle  $\mathfrak{h}_G$  in  $G - v$ . Assume w.l.o.g. that  $yw, wz \in E(\mathfrak{h}_G)$ . Now consider a hamiltonian cycle  $\mathfrak{h}_H$  in  $H - x$ . Delete from  $\mathfrak{h}_G$  the edges  $yw$  and  $wz$ , thus obtaining a path  $\mathfrak{p}_G$ , and delete from  $\mathfrak{h}_H$  the edges  $yw'$  and  $w'z$ , thereby obtaining a path  $\mathfrak{p}_H$ . Now  $\mathfrak{p}_G \cup \mathfrak{p}_H$  yields the desired cycle.

Finally, we prove that  $G_w H_{w'}$  is not hamiltonian. Indeed, if  $G_w H_{w'}$  is hamiltonian, either  $G - w$  or  $H - w'$  has a hamiltonian path joining two vertices in  $\{x, y, z\}$ , which can be immediately extended to a hamiltonian cycle in  $G$  or  $H$ , contrary to the hypothesis.  $\square$

Theorem 3.11 warrants the question whether there exist almost hypohamiltonian graphs whose exceptional vertex is cubic. In fact, we have already seen such graphs taking the observation below the proof Lemma 3.9 into account. What happens in the planar case?

**Theorem 3.12** [144] *There exists a planar almost hypohamiltonian graph of order  $n$  whose exceptional vertex is cubic for  $n = 47$  and for every  $n \geq 84$ .*

*Proof.* Consider the graph  $G$  of order 47 from Fig. 33, and denote by  $w$  the (unique) cubic vertex surrounded by quadrilaterals.

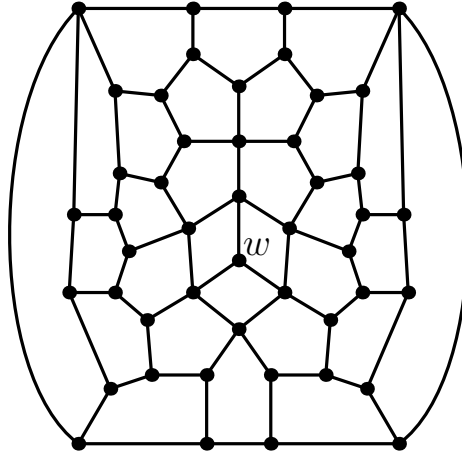


Fig. 33: A plane almost hypohamiltonian graph with cubic exceptional vertex  $w$ .

By Grinberg's Criterion,  $G$  and  $G - w$  are non-hamiltonian. The proof that for every  $v \in V(G) \setminus \{w\}$  the graph  $G - v$  is hamiltonian can be found in the Appendix. In order to obtain an infinite family, as in the proof of Theorem 3.7, we use Theorem 2.12: for every  $n \geq 42$  there exists a planar hypohamiltonian graph of order  $n$ . To  $G$  and each of these graphs we apply Lemma 3.4. (Note that in order to apply Lemma 3.4, each of the graphs obtained by Theorem 2.12 is required to contain a cubic vertex—but this is guaranteed by a theorem of Thomassen which states that every planar hypohamiltonian graph contains a cubic vertex [128].)  $\square$

Very recently, B. D. McKay announced (personal communication) that he has shown the following.

**Theorem 3.13** (McKay) *There exist three planar cubic almost hypohamiltonian graphs on 68 vertices, and 25 such graphs of order 74.*

### 3.4 Beyond Almost Hypohamiltonicity

Consider a 2-connected graph  $G$  of circumference<sup>1</sup>  $|V(G)| - 1$  and let  $W \subset V(G)$  be the (possibly empty) inclusion-maximal set of vertices such that for all  $w \in W$  the graph  $G - w$  is non-hamiltonian. (And thus, for all  $v \in V(G) \setminus W$ , the graph  $G - v$  is hamiltonian.) We call  $|W|$  the *hypohamiltonicity* of  $G$  and say that  $G$  is  $|W|$ -*hypohamiltonian*. A vertex from  $W$  is called *exceptional*. Denote the family of all  $k$ -hypohamiltonian graphs by  $\mathcal{H}_k$ .  $\mathcal{H}_0 = \mathcal{H}$  is the family of all hypohamiltonian graphs, and  $\mathcal{H}_1$  the family of all almost hypohamiltonian graphs.  $\bigcup_{k \geq 0} \mathcal{H}_k$  is a disjoint union and constitutes the family of all 2-connected graphs  $G$  of circumference  $|V(G)| - 1$ .

Somewhat surprisingly, it turns out that for  $k \geq 2$  it is easy to construct very small  $k$ -hypohamiltonian graphs, even if one adds the condition of planarity: consider a 4-cycle  $v_1v_2v_3v_4 = C$ . For  $k \geq 4$ , add to  $C$  the path  $v_2w_1w_2 \dots w_{k-3}w_{k-2}v_4$ . The graph one obtains is  $k$ -hypohamiltonian with  $v_2, v_4, w_1, \dots, w_{k-2}$  as exceptional vertices. For  $k = 2$  take  $K_{2,3}$ , and for  $k = 3$  consider the construction for  $k = 5$  to which the edge  $v_1w_2$  is added.

Summarising, if we define  $\alpha_k$  ( $\bar{\alpha}_k$ ) as the order of the smallest (smallest planar)  $k$ -hypohamiltonian graph, then

$$\alpha_0 = 10, \quad \alpha_1 \leq 17, \quad \alpha_2 = \bar{\alpha}_2 = 5, \quad \alpha_3 = \bar{\alpha}_3 = 7, \quad \alpha_4 = \bar{\alpha}_4 = 6, \quad \alpha_5 = \bar{\alpha}_5 = 7,$$

$$\text{and } \alpha_\ell = \bar{\alpha}_\ell = \ell + 1 \text{ for all } \ell \geq 6,$$

where for  $\alpha_2$  and  $\bar{\alpha}_2$  the equalities follow from the fact that all three 2-connected graphs on fewer than 5 vertices are hamiltonian. The equalities  $\alpha_3 = \bar{\alpha}_3 = \alpha_5 = \bar{\alpha}_5 = 7$  can be verified using a simple program. Concerning  $\alpha_4$  and  $\bar{\alpha}_4$ , the four 2-connected non-isomorphic spanning subgraphs of  $K_5$  with at least eight edges are hamiltonian. Among the three with seven edges, two are hamiltonian, while the third one is 2-hypohamiltonian. Among the two with six edges, one is hamiltonian, the other one—which is  $K_{2,3}$ —is 2-hypohamiltonian. The only one with five edges is the 5-cycle. No other spanning subgraphs are 2-connected. To justify the last equality, let  $\ell \geq 6$ . Between two fixed vertices take three

---

<sup>1</sup>The *circumference* of a graph  $G$  is the length of a longest cycle in  $G$ .

paths, one of length 2, one of length 3, and one of length  $\ell - 3$ . This graph is  $\ell$ -hypohamiltonian and has order  $\ell + 1$ . By Theorem 2.8 and Lemma 3.3, we have

$$\bar{\alpha}_0 \leq 40 \quad \text{and} \quad \bar{\alpha}_1 \leq 39,$$

noticing a striking discrepancy between the cases  $k = 1$  and  $k = 2$ .

Let  $H$  be a 2-connected graph containing three vertices  $v_1, v_2, v_3$ , and write  $\{v_1, v_2, v_3\} = \partial H$ . Additionally, for any  $i, j$  with  $i \neq j$ , there exists a hamiltonian path between  $v_i$  and  $v_j$ . We call such a graph  $H$  *nice*.

**Theorem 3.14** (2015 [144]) *Let  $G \in \mathcal{H}_j$ ,  $j \geq 0$ ,  $v$  be a cubic vertex in  $G$  with no exceptional vertex in  $N(v)$ , and let  $H$  be a nice graph. Join by three edges the vertices of  $\partial H$  to those of  $N(v)$ , according to a bijection, and delete  $v$ . Then for the resulting graph  $\Gamma$  we have*

$$\Gamma \in \begin{cases} \mathcal{H}_{j+|V(H)|-1} & \text{if } v \text{ is exceptional in } G \\ \mathcal{H}_{j+|V(H)|} & \text{otherwise.} \end{cases}$$

*If  $G$  and  $H$  are planar and  $\partial H$  lies in the boundary of a face, then the edges between  $\partial H$  and  $N(v)$  can be chosen such that  $\Gamma$  is planar. If all vertices in  $V(G)$  and  $V(H) \setminus \partial H$  are cubic, and all vertices in  $\partial H$  have degree 2, then  $\Gamma$  is cubic.*

*Proof.* Let  $\partial H = \{v_1, v_2, v_3\}$ ,  $N(v) = \{v'_1, v'_2, v'_3\}$  with  $v_i v'_i \in E(\Gamma)$  for all  $i \in \{1, 2, 3\}$ . We consider  $G - v$  and  $H$  to be subgraphs of  $\Gamma$ .

First we show that  $\Gamma$  is non-hamiltonian and that  $\Gamma - x$  is non-hamiltonian for all  $x \in V(H)$ . Assume the contrary. A hamiltonian cycle of  $\Gamma$  or  $\Gamma - x$  intersects  $G - v$  (which we here consider as a subgraph of  $\Gamma$  or  $\Gamma - x$ , respectively) along a hamiltonian path  $\mathfrak{p}$ . W.l.o.g. suppose  $v'_2, v'_3$  are the end-vertices of  $\mathfrak{p}$ . In  $G$ ,  $\mathfrak{p} \cup v'_3 v v'_2$  is a hamiltonian cycle, a contradiction.

Consider the set  $W$  of exceptional vertices in  $G$  and  $w \in W$ . Assume there exists a hamiltonian cycle  $\mathfrak{h}$  in  $\Gamma - w$ . W.l.o.g.  $v'_1 v_1 \notin E(\mathfrak{h})$ . Now  $(\mathfrak{h} \cap G) \cup v'_2 v v'_3$  is a hamiltonian cycle in  $G - w$ , a contradiction, as  $w$  is exceptional in  $G$ . By construction,  $v \notin V(\Gamma)$ , so  $\Gamma \in \mathcal{H}_{j+|V(H)|-1}$  if  $v$  is exceptional and  $\Gamma \in \mathcal{H}_{j+|V(H)|}$  otherwise.

Finally, we show that  $\Gamma - z$  is hamiltonian if  $z \in V(G) \setminus (\{v\} \cup W)$ . Let  $\mathfrak{h}'$  be a hamiltonian cycle in  $G - z$ , which exists, as  $z$  is non-exceptional. W.l.o.g.  $v'_1 v v'_2 \subset \mathfrak{h}'$ . Put  $\mathfrak{p}' = \mathfrak{h}' \cap (G - v)$ . There exists a hamiltonian path  $\mathfrak{p}''$  between  $v_1$  and  $v_2$  in  $H$  since  $H$  is nice. Now  $\mathfrak{p}' \cup v'_1 v_1 \cup v'_2 v_2 \cup \mathfrak{p}''$  is the desired hamiltonian cycle in  $\Gamma - z$ .  $\square$

Actually, the above operation can be applied simultaneously to  $k$  cubic vertices. Recall that there exist infinitely many cubic hypohamiltonian graphs [12], even if one adds planarity as condition [129].

A strengthening of Lemma 3.4 follows.

**Theorem 3.15** [144] *Let  $i, j \geq 0$ ,  $G \in \mathcal{H}_i$  have the set of exceptional vertices  $W$ , and  $H \in \mathcal{H}_j$  have the set of exceptional vertices  $W'$ . Let  $x \in V(G)$  and  $y \in V(H)$  be cubic vertices with the property that  $N[x] \cap W = \emptyset$  and  $N[y] \cap W' = \emptyset$ . Then  $G_x H_y \in \mathcal{H}_{i+j}$  with  $W \cup W'$  as set of exceptional vertices. If  $G$  and  $H$  are planar, then so is  $G_x H_y$ .*

*Proof.* Let  $i \leq j$ . The case  $i = j = 0$  coincides with Thomassen's Lemma 2.20, while  $i = 0$  and  $j = 1$  is Lemma 3.4. When  $i = 0$  and  $j \geq 2$  the proof is very similar to the proof of Lemma 3.4, so we skip it and assume in the following  $i \geq 1$ . We denote by  $z_1, z_2, z_3$  the vertices in  $G_x H_y$  obtained when identifying  $N(x)$  with  $N(y)$ . Abusing notation, we also write  $N(x) = \{z_1, z_2, z_3\}$  in  $G$  and  $N(y) = \{z_1, z_2, z_3\}$  in  $H$ , where  $z_k$  in  $G_x H_y$  is the vertex obtained when identifying  $z_k$  in  $G$  with  $z_k$  in  $H$ , for all  $k \in \{1, 2, 3\}$ .

We first show that  $G_x H_y$  is non-hamiltonian. Assume  $G_x H_y$  contains a hamiltonian cycle  $\mathfrak{h}$ . W.l.o.g. both edges in  $\mathfrak{h}$  incident with  $z_1$  lie in  $E(G - x)$ . But then  $(\mathfrak{h} \cap G) \cup z_2 x z_3$  yields a hamiltonian cycle in  $G$ , a contradiction.

Now we show that for all  $v \in W \cup W'$ , the graph  $G_x H_y - v$  is non-hamiltonian. W.l.o.g. let  $v \in V(G) \setminus \{x\}$ . Assume there exists a hamiltonian cycle  $\mathfrak{h}$  in  $G_x H_y - v$ . Among the vertices in  $\{z_1, z_2, z_3\}$  there exists exactly one, say  $z_1$ , for which either both edges in  $\mathfrak{h}$  incident with  $z_1$  lie in (a)  $E(G - x)$  or (b)  $E(H - y)$ . If (a) holds, then  $(\mathfrak{h} \cap G) \cup z_2 x z_3$  is an  $(n - 1)$ -cycle in  $G$  avoiding  $v$ , a contradiction, as  $v$  is an

exceptional vertex of  $G$ . If (b) holds, then  $(\mathfrak{h} \cap H) \cup z_2 y z_3$  yields a hamiltonian cycle in  $H$ , once more a contradiction.

Next we prove that  $G_x H_y - z_1$  is hamiltonian. Let  $\mathfrak{h}_G$  be a hamiltonian cycle in  $G - z_1$ , and  $\mathfrak{h}_H$  a hamiltonian cycle in  $H - z_1$ ; these exist as  $z_1$  is non-exceptional in both  $G$  and  $H$ . Put  $\mathfrak{p}_G = \mathfrak{h}_G - x$ .  $\mathfrak{p}_G$  avoids  $z_1$  and has end-vertices  $z_2$  and  $z_3$ . Similarly we obtain  $\mathfrak{p}_H$ , which avoids  $z_1$  and has end-vertices  $z_2$  and  $z_3$ . Now  $\mathfrak{p}_G \cup \mathfrak{p}_H$  is a cycle of length  $|V(G_x H_y)| - 1$  avoiding  $z_1$ . Analogously,  $G_x H_y - z_2$  and  $G_x H_y - z_3$  are hamiltonian.

Finally we show that  $G_x H_y - u$  is hamiltonian, for all  $u \in V(G_x H_y) \setminus (W \cup W' \cup \{z_1, z_2, z_3\})$ ; w.l.o.g.  $u \in V(G)$ . Consider a hamiltonian cycle  $\mathfrak{h}_G$  in  $G - u$ . Assume w.l.o.g. that  $z_2 x, x z_3 \in E(\mathfrak{h}_G)$ . Now consider a hamiltonian cycle  $\mathfrak{h}_H$  in  $H - z_1$ . Delete from  $\mathfrak{h}_G$  the vertex  $x$  (and edges incident to  $x$ ), thus obtaining a path  $\mathfrak{p}_G$ , and delete from  $\mathfrak{h}_H$  the vertex  $y$  (and edges incident to  $y$ ), thereby obtaining a path  $\mathfrak{p}_H$ . Now  $\mathfrak{p}_G \cup \mathfrak{p}_H$  is the desired cycle.  $\square$

Consider  $k \geq 0$ . Let  $n_k$  be the smallest integer such that for every  $n \geq n_k$  there exists a planar  $k$ -hypohamiltonian graph of order  $n$ .

**Corollary 3.16** [144] *For every  $k \geq 0$  we have  $n_k < \infty$ .*

*Proof.* In Theorem 2.12 we showed that  $n_0 \leq 42$ , and in Theorem 3.7 we proved  $n_1 \leq 76$ . For every  $n \geq 76$ , let  $G_n$  denote the graph of order  $n$  constructed in the proof of Theorem 3.7, and put  $\{G_n\}_{n \geq 76} = \mathcal{G}_1$ . Due to the nature of Lemma 3.6, it is clear that each  $G_n$  contains many cubic vertices. By applying Theorem 3.15 to  $G_{76}$  and every graph  $G \in \mathcal{G}_1$ , we obtain an infinite family  $\mathcal{G}_2$  of graphs proving  $n_2 \leq 147$ . (Note that in Theorem 3.15,  $|V(G_x H_y)| = |V(G)| + |V(H)| - 5$ .) Now apply Theorem 3.15 to  $G_{76}$  and every  $G \in \mathcal{G}_2$ , whence,  $n_3 \leq 218$ . This can be continued ad infinitum. We obtain  $n_p \leq n_{p-1} + 71$ , for every  $p \geq 2$ .  $\square$

Finally, Theorem 3.17 is a natural strengthening of Lemma 3.8. Its proof is analogous to the proof of Lemma 2.18, so we skip it.



**Theorem 3.17** [144] *Let  $G \in \mathcal{H}_k$  with the set  $W$  of exceptional vertices contain a cubic 4-cycle  $C$  with  $W \cap V(C) = \emptyset$ . Then  $G_C^* \in \mathcal{H}_k$ . If  $G$  is cubic, then so is  $G_C^*$ . If  $G$  is planar, then  $\text{cr}(G_C^*) = 1$ .*

### 3.5 Almost Hypotraceable Graphs

In the previous section, we defined hypohamiltonicity. In analogy to the definition given for cycles, consider a graph  $G$  in which the length of a longest path is  $|V(G)| - 2$  and let  $W \subset V(G)$  be the (possibly empty) inclusion-maximal set such that for all  $w \in W$ , the graph  $G - w$  is non-traceable. (And thus, for all  $v \in V(G) \setminus W$ , the graph  $G - v$  is traceable.) We call  $|W|$  the *hypotraceability*  $\mathfrak{t}(G) = \mathfrak{t}_G$  of  $G$  and call  $G$   $\mathfrak{t}_G$ -*hypotraceable*.

It is easy to check that in any graph, two longest paths meet (see for instance [102]). Gallai [43] asked in 1966 whether *all* longest paths intersect. (Which is reminiscent of *Helly's property*: a collection of sets satisfies it, if any subcollection of pairwise intersecting sets has a nonempty intersection.) We follow Ehrenmüller, Fernandes, and Heise [34], and call a vertex present in all longest paths of a given graph a *Gallai vertex*, and the set of all Gallai vertices the *Gallai set*.

It turns out that, in general, the answer to Gallai's question is negative. Walther [137] was the first to show that there exists a graph in which the intersection of all longest paths is empty, i.e. with empty Gallai set. A few years later, a significantly smaller example—of order 12—was independently found by Walther and T. Zamfirescu, see [59, 136, 150]. It is shown in Fig. 34. Brinkmann and Van Cleemput [17] proved (using computers) that in fact there is no smaller example.

The smallest known example of a *planar* graph in which all longest paths have empty intersection has order 17 and is due to Schmitz [113], see Fig. 34. More

problems in the same spirit as Gallai’s problem were discussed, for instance by asking larger sets of arbitrary vertices to be missed, by asking for a particular connectivity, or by posing the question for graphs embeddable in various lattices. (For an overview, see the survey [114].) Some of these variations will be discussed in Chapter 4.

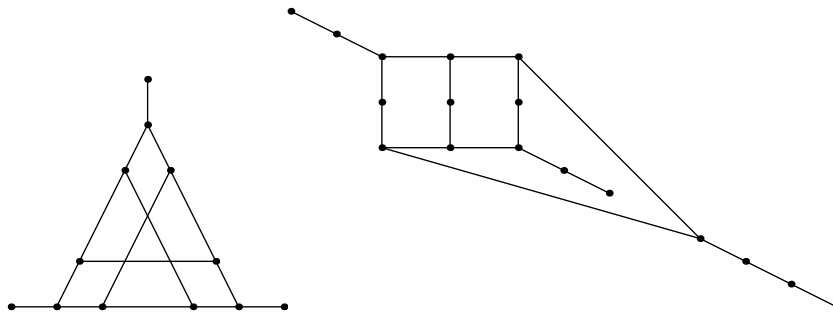


Fig. 34: On the left, a graph found independently by Walther and T. Zamfirescu, and on the right, Schmitz’ planar graph. In both graphs, the intersection of all longest paths is empty.

Let us emphasise the connection between hypotraceability and Gallai’s problem: consider a graph  $G$  in which the longest path has length  $|V(G)| - 2$ . This is extremal in the sense that it is the greatest length of a longest path for which Gallai’s question is interesting. Then  $t_G$ , the hypotraceability of  $G$ , is precisely the cardinality of the Gallai set. Hence, in the (extremal) family of graphs  $G$  with longest paths of length  $|V(G)| - 2$ , the answer to Gallai’s question is positive if and only if  $t_G \neq 0$ . Why is this interesting?

Gallai’s question has drawn much attention; we can give here only a small selection of results. One central direction of research was—since with Walther’s result, in general, Gallai’s question has a negative answer—to study in which families of graphs Gallai’s question had a *positive* answer. We call such graphs, ad hoc, *good*. Trees, for instance, abide, since in every tree each longest path contains its centre<sup>2</sup>. Klavžar and Petkovšek [87] proved that if every block of  $G$

<sup>2</sup>The *eccentricity* of a vertex  $v$  is defined as  $\max_{w \in V(G)} d(v, w)$ , where  $d(v, w)$  denotes the distance between two vertices  $v, w$ , i.e. the number of edges in a shortest path which has end-vertices  $v, w$ . The *centre* of a graph is the set of all vertices of minimum eccentricity.

is hamiltonian-connected<sup>3</sup>, then  $G$  is good. Balister, Györi, Lehel, and Schelp [8] showed that circular arc graphs<sup>4</sup> are good, as well. (Note that this includes interval graphs. According to Rautenbach and Sereni [108], there is a gap in the proof of Balister et al.—this alleged gap was closed by Joos [77].) A recent breakthrough was announced by Chen and Wu [22], who claim to have shown that chordal graphs<sup>5</sup> are also good.

As Kapoor, Kronk, and Lick [82], we denote by  $\partial(G)$  the length of a longest path in a given graph  $G$ . This section is devoted to the family of good graphs  $G$  which are “extremal” in two senses: first,  $\partial(G) = |V(G)| - 2$  (the greatest length of a longest path for which Gallai’s question is interesting), and second,  $\mathbf{t}_G = 1$  (the smallest hypotraceability for which the answer to Gallai’s question is positive), while in Section 3.6, we present families of graphs with empty Gallai set satisfying additional conditions, for instance specifying the difference between the order of the graph and the length of a longest path therein, or we ask the graphs to be snarks.

In the beginning of this chapter, the author studied *almost hypohamiltonian* graphs, which are graphs of hypohamiltonicity 1. In analogy thereto, we here define  $G$  to be *almost hypotraceable* if  $\mathbf{t}_G = 1$ , i.e. if  $G$  is non-traceable, there exists a vertex  $w$  such that  $G - w$  is non-traceable, and for any vertex  $v \neq w$  the graph  $G - v$  is traceable. As before, we call  $w$  the *exceptional vertex* of  $G$ . When handling simultaneously  $k$ -hypotraceable and  $\ell$ -hypohamiltonian graphs, we will call their respective exceptional vertices *t-exceptional* and *h-exceptional*. Where confusion is impossible, we suppress the prefix.

Observe that an almost hypohamiltonian graph  $G$  cannot be almost hypotraceable, since for every non-h-exceptional vertex  $v \in V(G)$  we have that  $G - v$  contains a hamiltonian cycle, which immediately yields a hamiltonian path in  $G$ .

---

<sup>3</sup>A graph is *hamiltonian-connected*, if any two of its vertices are the end-vertices of a hamiltonian path.

<sup>4</sup>A graph  $G$  is a *circular arc graph* if there exists a mapping  $\alpha$  of its vertex set  $V(G)$  into a collection of arcs of a circle such that, for every  $v, w \in V(G)$ ,  $vw$  is an edge of  $G$  if and only if  $\alpha(v) \cap \alpha(w) \neq \emptyset$ .

<sup>5</sup>A graph is *chordal* if every cycle of length at least 4 has a chord.

Hypotraceable graphs have no vertices of degree 1 or 2 (trivially, a hypotraceable cannot have connectivity 1, and the in-existence of vertices of degree 2 is proven as in Prop. 3.21). For an almost hypotraceable graph  $G$  the latter holds, see Prop. 3.21, while the former is not necessarily true:  $G$  may contain a vertex  $v$  of degree 1, but then  $v$  must be a neighbour of the exceptional vertex in  $G$ . Do such graphs exist? One of the graphs answering this question is the smallest almost hypotraceable graph and surprisingly minute: the “claw”  $K_{1,3}$ . But the claw plays a special role, which is discussed in the next proposition—its proof is a consequence of the paragraph following it, which treats the issue more generally.

**Proposition 3.18** *With the exception of  $K_{1,3}$ , every almost hypotraceable graph contains at most one vertex of degree 1.*

We now discuss almost hypotraceable graphs of connectivity 1. Let  $G$  be such a graph, and consider  $w \in V(G)$ .  $w$  is exceptional if and only if  $G - w$  is disconnected, so  $G$  contains exactly one 1-cut, namely  $\{w\}$ . As for all  $v \in V(G) \setminus \{w\}$  there is a hamiltonian path in  $G - v$ , and such a path has exactly two end-vertices, we have that  $G - w$  consists of at most three components. If  $G - w$  consists of exactly three components, then each component must be  $K_1$ —assume to the contrary that there is a component  $X \neq K_1$ , and let  $v \in V(X)$ . Then a hamiltonian path in  $G - v$  would have to traverse  $w$  at least twice, absurd. So  $G = K_{1,3}$ .  $G - w$  consisting of exactly two components  $X_1 \neq K_1, X_2 \neq K_1$ , is impossible: for  $x_i \in V(X_i)$  there exists a path  $\mathbf{p}_i$  visiting all vertices in  $G - x_i$ . But then

$$(\mathbf{p}_1 \cap (G[V(X_2) \cup \{w\}])) \cup (\mathbf{p}_2 \cap G[V(X_1) \cup \{w\}])$$

is a hamiltonian path in  $G$ , a contradiction.  $X_1 = X_2 = K_1$  is impossible, so if  $G - w$  consists of exactly two components  $X_1, X_2$ , then  $X_1 \neq K_1$  and  $X_2 = K_1$ .

**Proposition 3.19** *Let  $G \neq K_{1,3}$  be an almost hypotraceable graph with exceptional vertex  $w$  and of connectivity 1. Then the following hold.*

- (i)  *$G$  has exactly one 1-cut, namely  $\{w\}$ .*
- (ii)  *$G - w$  consists of exactly two components  $X_1, X_2$ , with  $X_1 \neq K_1$  and  $X_2 = K_1$ . We put  $V(X_2) = \{z\}$ .*
- (iii)  *$G[V(X_1) \cup \{w\}] = H$  is 2-connected and for every 2-cut  $C$  in  $H$ ,  $w \notin C$ .*
- (iv)  *$G$  has no vertices of degree 2 and exactly one vertex of degree 1, namely  $z$ .*
- (v)  *$H$  is traceable and for every hamiltonian path  $\mathbf{p}$  in  $H$ ,  $w$  is not an end-vertex of  $\mathbf{p}$ .*
- (vi) *For every  $v \in V(X_1)$ , there is a hamiltonian path in  $H - v$  which has  $w$  as one of its end-vertices.*
- (vii) *For every  $\varphi \in \text{Aut}(G)$ ,  $\varphi(z) = z$  and  $\varphi(w) = w$ .*
- (viii) *The vertices of a triangle in  $G$  have degree at least 4.*

*Proof.* We give here the proofs of the properties not discussed in the paragraphs preceding the statement of the proposition. Let us show (iii): assume there is a 2-cut  $\{x, y\}$  in  $H$ . Due to an argument very similar to the one given after Prop. 3.18, since  $G$  is almost hypotraceable,  $H - \{x, y\}$  contains exactly two components  $C_1, C_2$ . Assume that  $w = x$ . Then the fact that  $G - y$  contains a hamiltonian path of which  $z$  certainly is an end-vertex implies that  $C_1$  and  $C_2$  are isomorphic to  $K_1$ . But then  $G$  is traceable, a contradiction. Thus  $w \notin \{x, y\}$ .

For (iv), we use Prop. 3.21 for the first part, and (i) and (ii) for the second part. For (vii), combine (iv) with the arguments used for proving Prop. 3.22.  $\square$

We now settle the natural question whether, in addition to the claw, almost hypotraceable graphs exist.

**Theorem 3.20**

- *There exists an almost hypotraceable graph of order  $n$  for  $n \in \{4, 36, 39\}$  and every  $n \geq 41$ .*
- *For every  $d \geq 40$ , there exists an almost hypotraceable graph of maximum degree  $d$ .*

*Proof.* For order 4, consider  $K_{1,3}$ . Now let  $H$  be a hypotraceable graph and  $w \notin V(H)$ . Consider  $H' = H + (\{w\}, \emptyset)$ . By Lemma 3.1,  $H'$  is almost hypohamiltonian with exceptional vertex  $w$ . Let  $v \notin V(H')$ . Add to  $H'$  the edge  $vw$ . We obtain a graph  $G$ . First assume that  $G$  contains a hamiltonian path  $\mathbf{p}$ . Then  $H \cap \mathbf{p}$ , where we consider  $H$  to be a subgraph of  $G$ , is a hamiltonian path in  $H$ , which contradicts the fact that  $H$  is hypotraceable. Hence  $G$  is non-traceable. Furthermore, since  $G - w$  is disconnected, it is trivially non-traceable.

Consider a vertex  $u \in V(G) \setminus \{v, w\}$ . Then there exists a hamiltonian path  $\mathbf{p}'$  in  $H - u = G - \{u, v, w\}$ , since  $H$  is hypotraceable, and let  $z$  be an end-vertex of  $\mathbf{p}'$ . Now  $\mathbf{p}' \cup zwv$  is a hamiltonian path in  $G - u$ . Similarly,  $\mathbf{p}' \cup zwu$  is a hamiltonian path in  $G - v$ . Therefore  $G$  is almost hypotraceable with exceptional vertex  $w$ . Using two theorems of Thomassen [125, 127] stating that there exists a hypotraceable graph of order  $n$  for  $n \in \{34, 37\}$  and every  $n \geq 39$ , the proof is complete.  $\square$

Although Theorem 3.20 does provide an infinite family of almost hypotraceable graphs, every member of this family is of connectivity 1, which is unsatisfactory. In the following we shall see that infinitely many such graphs of connectivity 2 as well as 3 exist. But first we give two structural propositions. Note that Prop. 3.21 also holds for hypotraceable, hypohamiltonian, and almost hypohamiltonian graphs.

**Proposition 3.21** *An almost hypotraceable graph  $G$  does not contain vertices of degree 2. Furthermore, the vertices of each triangle in  $G$  have degree at least 4.*

*Proof.* Let  $G$  be an almost hypotraceable graph with exceptional vertex  $w$ , and let  $v \in V(G)$  be a vertex of degree 2 with neighbours  $v'$  and  $v''$ . There exists a

hamiltonian path  $\mathbf{p}$  in  $G - v'$ , so  $\mathbf{p} + vv'$  is a hamiltonian path in  $G$ , a contradiction. If  $v' = w$  we argue with  $G - v''$ .

For the second part, consider a triangle  $T$  in  $G$  with  $V(T) = \{v_1, v_2, v_3\}$ , where  $v_3$  shall be cubic. Since  $G$  is almost hypotraceable, at least one of  $G - v_1$  and  $G - v_2$  must be traceable, say  $G - v_1$ . Let  $\mathbf{p}$  be a hamiltonian path in  $G - v_1$ . If  $v_3$  is an end-vertex of  $\mathbf{p}$ , then  $\mathbf{p} + v_3v_1$  is a hamiltonian path in  $G$ , a contradiction. If  $v_3$  is not an end-vertex of  $\mathbf{p}$ , then  $v_3v_2 \in E(\mathbf{p})$ . Consider  $\mathbf{p}$  and replace  $v_3v_2$  with  $v_3v_1v_2$ . We obtain a hamiltonian path in  $G$ , once more a contradiction.  $\square$

**Proposition 3.22** *Let  $G$  be an almost hypohamiltonian or almost hypotraceable graph  $G$  with exceptional vertex  $w$ . Then the stabiliser of  $w$  must be all of  $\text{Aut}(G)$  and the orbit of  $w$  is  $\{w\}$ . In other words, for every  $\varphi \in \text{Aut}(G)$ ,  $w$  is a fix-point of  $\varphi$ .*

*Proof.* The *stabiliser* of  $w$  is defined as the set of all automorphisms  $\varphi$  of  $G$  such that  $\varphi(w) = w$ . Assume there exists a  $\psi \in \text{Aut}(G)$  such that  $\psi(w) \neq w$ . Then, since  $G - v$  is traceable for every vertex  $v \neq w$ , there exists a hamiltonian path  $\mathbf{p}$  in  $G - \psi(w)$ .  $\psi$  preserves vertex adjacency—recall that a graph isomorphism is always also an isometry, and vice versa. Thus  $\psi^{-1}(\mathbf{p})$  is a hamiltonian path in  $G - w$ , a contradiction. The *orbit* of a vertex  $v$  is defined as  $\{\varphi(v) : \varphi \in \text{Aut}(G)\}$ . Since for every  $\varphi \in \text{Aut}(G)$  we have  $\varphi(w) = w$ , the orbit of  $w$  is  $\{w\}$ .  $\square$

We now generalise an idea used in the proof of Theorem 3.20.

**Proposition 3.23** *Let  $k \geq 0$ , consider  $G$  to be a  $k$ -hypotraceable graph with  $t$ -exceptional vertices  $W$ , and let  $w' \notin V(G)$ . Then  $G + (\{w'\}, \emptyset)$  is a  $(k + 1)$ -hypohamiltonian graph with  $h$ -exceptional vertices  $W \cup \{w'\}$ .*

*Proof.* Put  $G + (\{w'\}, \emptyset) = G'$ . We consider  $G$  as a subgraph of  $G'$ . Assume  $G'$  is hamiltonian. Then  $G' - w'$  is traceable, a contradiction. Since  $G' - w' = G$  is non-traceable, it is non-hamiltonian. Now consider  $w \in W$  and assume that  $G' - w$  is hamiltonian. This implies that  $G' - w - w' = G - w$  is traceable, which is false, since  $w$  is  $t$ -exceptional in  $G$ .

Consider  $v \in V(G') \setminus (W \cup \{w'\})$ . Since  $G$  is almost hypotraceable,  $G - v$  contains a hamiltonian path  $\mathbf{p}$  with end-vertices  $x, y$ . Now  $\mathbf{p} \cup xw'y$  is a hamiltonian cycle in  $G' - v$ .  $\square$

We showed earlier in this chapter that 4-connected almost hypohamiltonian graphs exist—such a graph is depicted in Fig. 28. The most blunt approach to achieve this was to consider a 3-connected hypotraceable graph  $T$  (for instance the graph of Horton constructed in [72]), an extra vertex  $w$ , and connect  $w$  to every  $v \in V(T)$ . This yields the join of  $K_1$  and  $T$ , i.e.  $T + K_1$ .  $T + K_1$  is an almost hypohamiltonian graph, as proven in Lemma 3.1. What would be the analogous procedure for almost hypotraceable graphs, i.e. which properties must a (not necessarily connected) graph  $H$  have in order for the join of  $H$  and  $K_1$  to be almost hypotraceable?

Two conditions must be satisfied: at least three pairwise disjoint paths, each of which may consist of a single vertex, are required to span  $H$  (since if two or fewer paths suffice, then  $H + K_1$  would be traceable) and for any  $v \in V(H)$ , at most two disjoint paths suffice to span  $H - v$  (since if three or more paths are required,  $(H + K_1) - v$  cannot be traceable). It is easy to see that this can immediately be rewritten in the following manner, where  $\text{sp}(G)$  denotes the minimum number of pairwise disjoint paths required to span a graph  $G$ :

$$(\star) \text{ sp}(H) = 3 \quad \text{and} \quad (\star\star) \text{ for every } v \in V(H): \text{ sp}(H - v) = 2.$$

Let us call such graphs—which we explicitly allow to be disconnected—*pre-almost hypotraceable*. Clearly, if the connectivity of a pre-almost hypotraceable graph  $G$  is  $k \geq 0$ , then the connectivity of the almost hypotraceable graph  $G + K_1$  is  $k + 1$ . One might be inclined to think that no pre-almost hypotraceable graphs exist, but this would be false: take for instance  $3K_1$ , i.e. the disjoint union of three isolated vertices. Now  $3K_1 + K_1 = K_{1,3}$ , which is almost hypotraceable. But this is a special case: assume we have a pre-almost hypotraceable graph consisting of three components, not all of which are  $K_1$ ; say  $A \neq K_1$ . Then for every  $v \in V(A)$  we have that  $\text{sp}(A - v) \geq 1$ , so  $\text{sp}(G - v) \geq 3$ , which contradicts  $(\star\star)$ . Due to



( $\star\star$ ) it is also impossible that a pre-almost hypotraceable graph contains more than three components.

What if we consider two components  $A$  and  $B$ ? Clearly, at least one of them, say  $A$ , must be  $\neq K_1$ . If we consider for a moment that  $B = K_1$ , then  $\text{sp}(A) = 2$  and for any vertex  $v \in V(A)$ , the graph  $A - v$  must be traceable. Thus,  $A$  is hypotraceable. (We are in the situation of the proof of Theorem 3.20. Since there are infinitely many hypotraceable graphs as shown by Thomassen [125], there exist infinitely many pre-almost hypotraceable graphs consisting of two components.) Now let both  $A$  and  $B$  contain more than one vertex. Due to ( $\star\star$ ), since  $\text{sp}(A - v) \geq 1$ ,  $v \in V(A)$ ,  $B$  must be traceable. The same holds for  $A$ , so both  $A$  and  $B$  must be traceable. But then ( $\star$ ) is not satisfied.

It remains to study connected pre-almost hypotraceable graphs. Due to very similar arguments as the ones presented above, if we consider two disjoint hypotraceable graphs  $T, T'$  and  $v \in V(T)$ ,  $v' \in V(T')$ , identifying  $v$  with  $v'$  yields a connected pre-almost hypotraceable graph. As above, we may construct in this manner infinitely many connected pre-almost hypotraceable graphs, which yields infinitely many almost hypotraceable graphs of connectivity 2. At this point, the most interesting further discovery would be a 2-connected pre-almost hypotraceable graph. Unfortunately, we were unable to prove the (in-)existence of such graphs.

Taking the preceding discussion into account, we have the following.

**Theorem 3.24** *Consider hypotraceable graphs of order  $n$  and  $n'$ . Then there exists an almost hypotraceable graph of order  $n + 2$  and of order  $n + n'$ .*

In the next proposition, we adapt a technique of Thomassen [125] to devise a second way to construct almost hypotraceable graphs of connectivity 2. Thomassen's approach provides the smallest known hypotraceable graph, which has order 34, by using four copies of Petersen's graph (see Fig. 18)—contrasting this, we shall present in Theorem 3.26 a method which provides smaller almost hypotraceable graphs with higher connectivity than the ones derivable from Prop. 3.25.

**Proposition 3.25** *Consider  $G_1 \in \mathcal{H}_1$  with exceptional vertex  $w$ , and  $G_2, G_3, G_4 \in \mathcal{H}$ . Let these graphs be pairwise disjoint. Assume furthermore that each  $G_i$  contains a cubic vertex  $x_i$  with  $N(x_i) = \{x_i^1, x_i^2, x_i^3\}$ ,  $1 \leq i \leq 4$ , such that  $w \notin N[x_1]$ . Now take the graphs  $G_i - x_i$ ,  $1 \leq i \leq 4$ , and identify  $x_1^3, x_2^3$  into a vertex  $y_1$  and  $x_3^3, x_4^3$  into a vertex  $y_2$ . Also add the edges  $x_1^1 x_3^1, x_1^2 x_3^2, x_2^1 x_4^1, x_2^2 x_4^2$ . The graph we obtain is almost hypotraceable with exceptional vertex  $w$ .*

*Proof.* Denote the graph constructed in the statement by  $G$ . Above assumptions provide all ingredients in order to apply Thomassen's proof of [125, Lemma 3.1] that  $G$  is non-traceable, and that  $G - v$  is traceable for every  $v \in V(G) \setminus \{w\}$ .

It remains to prove that  $G - w$  is non-traceable. Assume the opposite and let  $\mathbf{p}$  be a hamiltonian path in  $G - w$ . We will treat  $H_i = G_i - x_i$  as subgraphs of  $G$ . In  $G$ , put  $Y = \{v \in V(H_1) : vy_1 \in E(\mathbf{p})\}$ ,  $e = x_1^1 x_3^1$ , and  $e' = x_1^2 x_3^2$ .

Case 1:  $e, e' \notin E(\mathbf{p})$ . Then there is a hamiltonian path  $\mathbf{q}$  in  $H_2$  with end-vertices  $x_2^3$  and  $z$ , where  $z$  is either  $x_2^1$  or  $x_2^2$ . But then  $\mathbf{q} \cup zx_2x_2^3$  is a hamiltonian cycle in  $G_2$ , a contradiction.

Case 2:  $e \in E(\mathbf{p}), e' \notin E(\mathbf{p})$ . If  $|Y| \in \{0, 2\}$ , then at least one of the end-vertices of  $\mathbf{p}$  lies in  $H_1$ . But then  $(\mathbf{p} \cap H_3) \cup x_3^1 x_3 x_3^3$  is a hamiltonian cycle in  $G_3$ , a contradiction. If  $|Y| = 1$ , then  $(\mathbf{p} \cap H_1) \cup x_1^3 x_1 x_1^1$  is a hamiltonian cycle in  $G_1 - w$ , once more a contradiction as  $G_1$  is almost hypohamiltonian with exceptional vertex  $w$ .

Case 3:  $e, e' \in E(\mathbf{p})$ . For  $Y = \emptyset$ , one of the end-vertices of  $\mathbf{p}$  lies in  $H_3$ , since one neighbour of  $y_2$  on  $\mathbf{p}$  must be in  $H_3$ , and the other neighbour in  $H_4$  (for otherwise we cannot visit any vertices in  $H_4 - y_2$ ). The other end-vertex of  $\mathbf{p}$  lies either in  $H_2$  or  $H_4$ . In the former case, we are led to the hamiltonicity of  $G_4$  (since either  $(\mathbf{p} \cap H_4) \cup x_4^3 x_4 x_4^2$  or  $(\mathbf{p} \cap H_4) \cup x_4^3 x_4 x_4^1$  is a hamiltonian cycle in  $G_4$ ), while in the latter case, we obtain that  $G_2$  is hamiltonian, again a contradiction. If  $|Y| = 1$ , at least one of the end-vertices of  $\mathbf{p}$  must lie in  $H_1$ . If both lie in  $H_1$ , then  $y_1$  must be one of the end-vertices of  $\mathbf{p}$ . However, in this case  $\mathbf{p}$  cannot contain any vertices in  $H_2 - y_1$ , in contradiction to the fact that  $\mathbf{p}$  is a hamiltonian path in  $G - w$ . So assume the end-vertex of  $\mathbf{p}$  which is not in  $H_1$ , lies in  $H_2$ . Then either  $\{v \in V(G) : vy_2 \in E(\mathbf{p})\} \subset V(H_3)$ , in which case  $G_3$  is

hamiltonian, or  $\{v \in V(G) : vy_2 \in E(\mathfrak{p})\} \subset V(H_4)$  and hence  $G_4$  is hamiltonian (the hamiltonicity is proven as above); in both cases a contradiction is obtained. We are left with the case when one end-vertex of  $\mathfrak{p}$  lies in  $H_1$  and the other end-vertex lies neither in  $H_1$  nor in  $H_2$ . Then  $y_1$  cannot be an end-vertex of  $\mathfrak{p}$ , since then due to  $|Y| = 1$ , both end-vertices of  $\mathfrak{p}$  would lie in  $H_1$ , which was treated above, or  $G_2$  is hamiltonian. Finally, if  $|Y| = 2$ , once again we have a contradiction, since  $(\mathfrak{p} \cap H_1) \cup x_1^1 x_1 x_1^2$  is a hamiltonian cycle in  $G_1 - w$ .  $\square$

Let  $H$  and  $G$  be graphs each containing a vertex of degree  $k$ , say  $v$  and  $w$ , respectively. We say that we *replace  $v$  with  $G - w$*  if we delete  $v$  (and all incident edges) from  $H$  and connect the neighbours of  $v$  in  $H$  to the neighbours of  $w$  in  $G - w$  using a bijection. The next theorem provides a powerful tool to construct 3-connected almost hypotraceable graphs, allowing for planar and cubic constructions as well depending on the input graphs. An example of its application is given in Fig. 35.

**Theorem 3.26** *Let  $G_1, G_2, G_3 \in \mathcal{H}$  be pairwise disjoint, each graph containing a cubic vertex  $x_1, x_2, x_3$ , respectively. Consider  $K_4$  and put  $V(K_4) = \{v_1, \dots, v_4\}$ . By replacing  $v_i$  with  $G_i - x_i$ ,  $1 \leq i \leq 3$ , a 3-connected almost hypotraceable graph  $G$  is obtained.*

*Proof.* We will treat  $G_i - x_i = H_i$  as subgraphs of  $G$ , and  $v_4$  as a vertex in  $G$ . Assume  $G$  contains a hamiltonian path  $\mathfrak{p}$ . Since  $\mathfrak{p}$  has two end-vertices, there exists a  $G_i$ , say  $G_1$ , such that  $\mathfrak{p}$  contains a subpath  $\mathfrak{q}$  which has end-vertices  $y, z \in N(x_1) \subset V(G_1)$ , and is a hamiltonian path in  $G_1 - x_1$ .  $\mathfrak{q} \cup yx_1z$  is a hamiltonian cycle in  $G_1$ , a contradiction. The same argument yields that  $G - v_4$  is non-traceable.

Consider  $v \in V(H_1)$ . Since  $G_1$  is hypohamiltonian, there exists a path  $\mathfrak{q}_1$  in  $H_1 - v$  with end-vertices  $y, z \in N(x_1)$  which visits every vertex in  $H_1 - v$ .

Case 1:  $\{y, z\} \cap N(v_4) = \emptyset$ . W.l.o.g. let the neighbour of  $y$  ( $z$ ) not lying in  $G_1$  lie in  $H_2$  ( $H_3$ ). We denote this vertex by  $y_1$  ( $z_1$ ). Put  $\{y_2\} = N(v_4) \cap V(H_2)$ ,  $\{z_2\} = N(v_4) \cap V(H_3)$ ,  $N(x_2) = \{y_1, y_2, y_3\}$ , and  $N(x_3) = \{z_1, z_2, z_3\}$ . Since  $G_2$  ( $G_3$ ) is hypohamiltonian, there is a path  $\mathfrak{q}_2$  ( $\mathfrak{q}_3$ ) in  $H_2 - y_2$  ( $H_3 - z_3$ ) with

end-vertices  $y_1$  and  $y_3$  ( $z_1$  and  $z_2$ ) which visits all vertices in  $H_2 - y_2$  ( $H_3 - z_3$ ).  
Now

$$y_2v_4z_2 \cup \mathbf{q}_3 \cup z_1z \cup \mathbf{q}_1 \cup yy_1 \cup \mathbf{q}_2 \cup y_3z_3$$

is a hamiltonian path in  $G - v$ .

Case 2:  $\{y, z\} \cap N(v_4) \neq \emptyset$ . W.l.o.g. let  $y \in N(v_4)$ .

Claim. *Let  $G \in \mathcal{H}$  and  $v \in V(G)$ . Then there exists a hamiltonian path in  $G - v$  which has  $w \in N(v)$  as an end-vertex.*

*Proof of the Claim.* Let  $\mathbf{h}$  be a hamiltonian cycle in  $G - v$  and  $wz \in E(\mathbf{h})$ . Now  $\mathbf{h} - wz$  is the desired hamiltonian path.

W.l.o.g. let the neighbour of  $z$  which does not lie in  $G_1$  lie in  $G_3$ . We denote this vertex by  $z_1$ . Let  $\{y_2\} = N(v_4) \cap V(G_2)$ . By above Claim, since  $G_2$  ( $G_3$ ) is hypohamiltonian, there is a path  $\mathbf{q}'_2$  ( $\mathbf{q}'_3$ ) in  $H_2$  ( $H_3$ ) which has  $y_2$  ( $z_1$ ) as an end-vertex and which visits all vertices in  $H_2$  ( $H_3$ ). Now

$$\mathbf{q}'_2 \cup y_2v_4y \cup \mathbf{q}_1 \cup zz_1 \cup \mathbf{q}'_3$$

is a hamiltonian path in  $G - v$ .

For a vertex in  $G_2$ ,  $G_3$ , or  $G_4$  we argue in exactly the same manner.  $\square$

If, in Theorem 3.26, each  $G_i$  is planar (cubic), then the resulting graph is planar (cubic), as well.

### Corollary 3.27

- *There is a unique 3-connected cubic almost hypotraceable graph on 28 vertices, and no smaller such graph.*
- *There exists a 3-connected almost hypotraceable graph on  $n$  vertices for every  $n \geq 28$  with the possible exceptions of 29, 30, 32, and 35.*
- *There exists a polyhedral almost hypotraceable graph on  $n$  vertices for  $n = 118$  and every  $n \geq 120$ .*
- *There exists a polyhedral cubic almost hypotraceable graph on  $n$  vertices for  $n = 208$  and every even  $n \geq 212$ .*

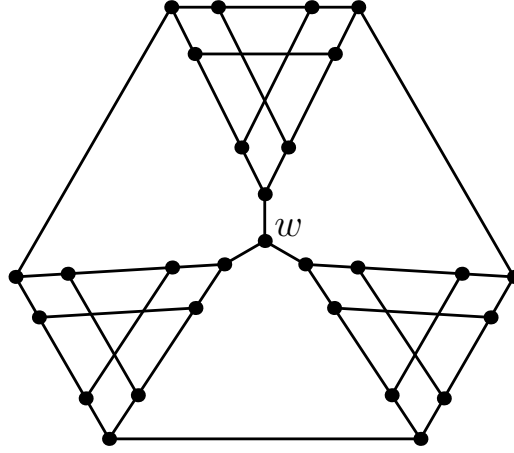


Fig. 35: By letting  $G_1, G_2, G_3$  each be the Petersen graph, we obtain above graph when applying Theorem 3.26. It is the smallest 3-connected cubic almost hypotraceable graph. Its exceptional vertex is  $w$ . Excluding  $K_{1,3}$ , it is the smallest known almost hypotraceable graph.

*Proof.* McKay [98] has shown that all bridgeless cubic graphs up to 26 vertices are traceable. Recently, he generated the 40,157,414,176 bridgeless cubic graphs of order 28 and found that 10 of them are non-traceable. Of these 10, only one is 3-connected. It is the graph shown in Fig. 35. Using in Theorem 3.26 three copies of the Petersen graph, we obtain that the aforementioned 28-vertex graph is almost hypotraceable.

We recall that every hypohamiltonian graph is 3-connected, and that every known hypohamiltonian graph has minimum degree 3. Theorem 3.26 implies that if there exist hypohamiltonian graphs of order  $k, k', k''$ , each containing a cubic vertex, then there exists a 3-connected almost hypotraceable graph of order  $k + k' + k'' - 2$ . The statement now follows from the result that there exists a hypohamiltonian graph of order  $n$  for  $n \in \{10, 13, 15, 16\}$  and every  $n \geq 18$ , see [5].

In Theorems 2.4 and 2.12 we proved that there exists a planar hypohamiltonian graph of order 40 and of order  $n$  for every  $n \geq 42$ ; denote this family of graphs by  $\mathcal{F}$ . In contrast to the general case, for which it is unknown whether hypohamiltonian graphs of minimum degree greater than 3 exist, Thomassen

showed that every planar hypohamiltonian graph contains a cubic vertex [128]. By applying Theorem 3.26 to three copies of the aforementioned 40-vertex graph, we obtain a polyhedral hypotraceable graph of order 118. The full statement follows by applying Theorem 3.26 to all triples of graphs from  $\mathcal{F}$ , as above.

There exists a planar cubic hypohamiltonian graph of order 70 due to a result of Araya and Wiener [7]—this graph is shown in Fig. 15—and furthermore, there are such graphs of order  $n$  for every even  $n \geq 74$  by Theorem 2.17. By applying Theorem 3.26, we obtain the advertised statement.  $\square$

## 3.6 Graphs with Empty Gallai Set

Inspired by the approach used by Walther and T. Zamfirescu to construct the graph shown in Fig. 34 (left), we prove the following.

**Theorem 3.28** *Let  $G$  be a hypohamiltonian graph of order  $n$  and  $v \in V(G)$  of degree  $d$ . Then there exists a graph  $H$  of order  $n + d - 1$  with  $\partial(H) = n - 1$  such that the Gallai set of  $H$  is empty. If  $G$  is planar, then so is  $H$ .*

*Proof.*  $H$  is obtained in the following way. Put  $N(v) = \{v_1, \dots, v_d\}$ . Note that  $d \geq 3$  since  $G$  is hypohamiltonian and thus 3-connected. Add to  $G - v$  new vertices  $v'_1, \dots, v'_d$  and the edges  $v_i v'_i$  for all  $i \in \{1, \dots, d\}$ . (We treat  $G - v$  as a subgraph of  $H$ .) Obviously,  $H$  has order  $n + d - 1$ .

By deleting  $v$  from a hamiltonian cycle in  $G - v_i$ , we obtain a path  $Q$  in  $H$ , which has end-vertices  $v_j, v_k$ , where  $i, j, k$  are pairwise different. Now  $Q + v_j v'_j + v_k v'_k$  is a path of length  $n - 1$  in  $H$  avoiding  $v_i$  and  $v'_i$ . Thus  $\partial(H) \geq n - 1$ . We now prove that  $\partial(H) \leq n - 1$ . Consider a longest path  $P$  in  $H$ . As every vertex in  $\{v'_1, \dots, v'_d\}$  has degree 1, at most two such vertices lie in  $V(P)$ . If no such vertex lies in  $P$ , then  $|E(P)| < n - 1$ , which contradicts the fact that  $P$  is a longest path. If exactly one such vertex lies in  $P$ , then  $|E(P)| = n - 1$  if and

only if  $P$  visits all vertices in  $V(G) \setminus \{v\}$ , and otherwise  $|E(P)| < n - 1$ . If  $P$  has, w.l.o.g., end-vertices  $v'_1$  and  $v'_2$ , then in order for  $|E(P)| > n - 1$  to hold,  $P$  would have to visit all vertices in  $V(G) \setminus \{v\}$ . If this were the case  $(P - \{v'_1, v'_2\}) \cup v_1 v v_2$  would yield a hamiltonian cycle in  $G$ , a contradiction. Thus, we have proven that  $\partial(H) = n - 1$ .

We showed above that for every longest path  $P$  in  $H$ , we have that  $|V(P) \cap \{v'_1, \dots, v'_d\}| \in \{1, 2\}$ , which we will use tacitly in the remainder of this proof. Let  $w \in V(G) \setminus \{v\}$ . Due to the hypohamiltonicity of  $G$ , there exists a hamiltonian cycle  $\mathfrak{h}$  in  $G - w$ .  $\mathfrak{h} - v$  is a hamiltonian path  $\mathfrak{p}$  in  $G - \{v, w\}$  with end-vertices  $v_j, v_k \in N(v)$ . Now  $\mathfrak{p} + v_j v'_j + v_k v'_k$  is a longest path in  $H$  avoiding  $w$ .

In the second paragraph we proved that there exists, for every  $i$ , a longest path avoiding  $v'_i$ . Thus, for every longest path  $P$  in  $H$  there exists a vertex  $w \in V(H)$  such that  $w \notin V(P)$ , so the intersection of all longest paths must be empty.  $\square$

We now discuss consequences of Theorem 3.28, especially a relaxation of hypotraceability which arises naturally: what if we wish, in the spirit of Gallai's problem, all longest paths to have empty intersection, and want to study the difference  $\delta$  between the order of the graph and the length of a longest path? Taking the cardinality of the set of avoided vertices into account, Grünbaum defines in [59] the set  $\Pi(j, k)$  to be the family of all graphs  $G$  in which  $\delta = k + 1$  and for every  $S \subset V(G)$  with  $|S| = j$  there exists a longest path  $P$  such that  $V(P) \cap S = \emptyset$ . Clearly,  $j \leq k$ . We focus here on the case  $j = 1$ .

Certainly, the smallest value for  $\delta$  is 2. This is achieved by a graph if and only if it is hypotraceable, i.e. the family  $\Pi(1, 1)$ . Are there graphs with  $\delta = 3$ ? What about larger values of  $\delta$ ? Walther [137] proved that  $\Pi(1, k) \neq \emptyset$  for all  $k \geq 4$ . Since then, much progress has been made—here we restrict ourselves to point out one of the strongest results. As Wiener writes in [140], combining results of Thomassen [127] and T. Zamfirescu [150], we obtain planar 3-connected graphs belonging to  $\Pi(1, k)$  for every  $k \geq 1$ .

In the third part of the following corollary, by using Theorem 3.28 and techniques of Thomassen, we give a different approach to prove that  $\Pi(1, k) \neq \emptyset$  for all  $k \geq 2$ .

### Corollary 3.29

- *The graph shown in Fig. 34 (left) has empty Gallai set.*
- *There exist infinitely many planar graphs with empty Gallai set in which the difference between the order of the graph and the length of a longest path is 3.*
- *For every  $\delta \geq 3$ , there exists a graph of order  $n$  with empty Gallai set and in which the length of a longest path is  $n - \delta$ .*

*Proof.* For the first part, let the graph  $G$  from Theorem 3.28 be the Petersen graph, and any vertex in  $G$  may be chosen as  $v$ .

For the second part, it was shown by Thomassen that there exist infinitely many planar hypohamiltonian graphs [127]. He also showed [128] that every such graph contains a cubic vertex, which we choose to be our vertex  $v$ . Now we apply Theorem 3.28.

For the third part, we prove that for every  $k \geq 0$ , and fixed and arbitrary  $d \in \{3, \dots, 2k+3\}$ , there exists a hypohamiltonian graph of order  $24+4k$  containing a vertex of degree  $d$ . Thomassen [129] observed that given a cubic hypohamiltonian graph  $G$  containing a 4-cycle  $C = v_1v_2v_3v_4$ , the graph  $\text{Th}(G_C)$  (we defined the operation  $\text{Th}$  in Section 2.1.3) is a cubic hypohamiltonian graph as well—in Prop. 2.11 a more general result is proven. In [129, Fig. 2], Thomassen presents the cubic hypohamiltonian graph  $G_0$  of order 24 and girth 4. We reproduced this graph in Fig. 22. Applying above observation iteratively to  $G_0$ , where  $v_1v_2v_3v_4$  shall be a 4-cycle in  $G_0$ , we obtain a cubic hypohamiltonian graph of order  $24+4k$  for every  $k \geq 0$ . We call the family of these graphs  $\mathcal{G}$ .

We now use another argument of Thomassen [129]—he presents it very succinctly, so we give here a more detailed account. For every  $G \in \mathcal{G}$ , where we consider  $G_0 - \{v_1v_2, v_3v_4\}$  to be a subgraph of  $G$ ,

$$S = V(G) \setminus [V(G_0) \setminus \{v_1, \dots, v_4\}]$$

induces a balanced bipartite subgraph of  $G$ , so we may colour each element of  $S$  either red or blue such that no two red vertices are adjacent and no two blue vertices are adjacent.



Claim. *If we add to  $G$  any set of edges in which every edge joins a red vertex to a blue vertex, then the resulting graph  $G'$  is hypohamiltonian.*

*Proof of the Claim.*  $G' - v$  is hamiltonian for every  $v \in V(G')$ , since  $G$  is hypohamiltonian and we have  $V(G) = V(G')$  and  $E(G) \subset E(G')$ . Assume  $G'$  contains a hamiltonian cycle  $\mathfrak{h}$ . Put  $P = \mathfrak{h} \cap G'[S]$ . Since  $\mathfrak{h}$  is a hamiltonian cycle in  $G'$  and  $\{v_1, v_2, v_3, v_4\}$  is a cut in  $G'$ ,  $P$  consists of either one or two components.

Case 1.  $P$  consists of one component, i.e.  $P$  is a path. Obviously, the end-vertices of  $P$  lie in  $\{v_1, \dots, v_4\}$ . W.l.o.g. let  $P$  have  $v_1$  as an end-vertex and let  $v_1$  be a red vertex. If  $v_3$  is the other end-vertex of  $P$ , then  $P$  begins in a red vertex and ends in a red vertex. But this is impossible, since  $G'[S]$  is a balanced bipartite graph and  $P$  a hamiltonian path therein. So either  $v_2$  or  $v_4$  is the other end-vertex of  $P$ . Say it is  $v_2$ . Then by replacing in  $\mathfrak{h}$  the path  $P$  with  $v_1v_4v_3v_2$ , we obtain a hamiltonian cycle in  $G_0$ —a contradiction. If  $v_4$  is the other end-vertex of  $P$ , then we replace in  $\mathfrak{h}$  the path  $P$  with  $v_1v_2v_3v_4$ .

Case 2.  $P$  consists of two components  $P', P''$ . Then  $P' \cap P'' = \emptyset$  and each of  $P$  and  $P'$  is a path with end-vertices lying in  $\{v_1, \dots, v_4\}$ . W.l.o.g. let  $v_1$  be one of the end-vertices of  $P'$ . If  $v_2$  is the other end-vertex of  $P'$ , then the end-vertices of  $P''$  are  $v_3$  and  $v_4$ . By replacing in  $\mathfrak{h}$  the path  $P'$  with the edge  $v_1v_2$  and  $P''$  with  $v_3v_4$ , we obtain a hamiltonian cycle in  $G_0$ . The arguments for  $v_4$ , as well as if  $P'$  has end-vertices  $v_1$  and  $v_3$ , are very similar. This completes the proof of the Claim.

$G'[S]$  contains  $2 + 2k$  red vertices and  $2 + 2k$  blue vertices, where  $k \geq 0$  is the number of times we have applied Thomassen's operation Th to  $G_0$ . Since  $G_0 - \{v_1v_2, v_3v_4\}$  is a subgraph of every  $G \in \mathcal{G}$ , we may consider  $v_1$  to lie in  $G$ . Let  $v_1$  be a red vertex. Then we may join  $v_1$  to  $\ell$  blue vertices (excluding the two blue vertices it is already neighbouring), for every non-negative integer  $\ell \leq 2k$ . Thus, by adding edges to  $G$  appropriately, we can make the degree of  $v_1$  take any integer value  $d$  with  $3 \leq d \leq 2k + 3$ .  $\square$

**Theorem 3.30** *Let  $G_i$ ,  $1 \leq i \leq 4$ , be pairwise disjoint hypohamiltonian snarks, and  $x_i \in V(G_i)$ . Consider  $K_4$  and put  $V(K_4) = \{v_1, \dots, v_4\}$ . By replacing  $v_i$  with  $G_i - x_i$  for all  $i$ , we obtain a reducible snark  $G$  with empty Gallai set.*

*Proof.*  $G$  is obviously a connected bridgeless cubic graph of girth  $\min_i g(G_i)$ , where  $g(G_i)$  denotes the girth of  $G_i$ , so certainly at least 5.  $G$  is a *reducible* snark, since although by Prop. 2.13 hypohamiltonian graphs are cyclically 4-edge-connected,  $G$  is not. Assume  $G$  is 3-edge-colourable. Consider the 3-edge-cut  $M$  in  $G$  separating  $G_1 - x_1$  (where we treat each  $H_i = G_i - x_i$  as a subgraph of  $G$ ) from the remainder of the graph. Isaacs' Parity Lemma [74] implies that the edges of  $M$  must have pairwise different colours. Let  $M = \{w_1w'_1, w_2w'_2, w_3w'_3\}$ , where  $w_i \in V(G_1) \setminus \{x_1\}$ ,  $1 \leq i \leq 3$ . In  $G_1 - x_1$  together with  $M$  and the vertices  $w'_1, w'_2, w'_3$ , identify  $w'_1, w'_2, w'_3$  into one vertex. We obtain a 3-edge-coloured graph isomorphic to  $G_1$ —a contradiction. Due to this observation and the fact that  $G$  has maximum degree 3, by Vizing's Theorem [135],  $G$  is 4-edge-colourable. (Note that due to the existence of the 3-edge-cut  $M$ ,  $G$  is not hypohamiltonian by Prop. 2.13.)

It remains to prove that the Gallai set of  $G$  is empty. Put  $n = |V(G)|$  and assume that  $G$  contains a path  $P$  of length  $n-1$  or  $n-2$ . Then there exists at least one  $H_i$ , say  $H_1$ , such that  $P$  visits all vertices of  $H_1$  and  $P \cap H_1$  is connected. Let  $P \cap H_1$  have end-vertices  $a, b$ , which necessarily lie in  $N(x_1)$ . Now  $(P \cap H_1) \cup ax_1b$  is a hamiltonian cycle in  $G_1$ , a contradiction. We have shown that  $\partial(G) \leq n-3$ .

Let  $v \in V(H_1)$ . We now prove that there exists a path of length  $n-3$  in  $G-v$ . We contract  $H_2$  to a single vertex  $w$  and obtain the graph  $G'$ . Theorem 3.26 states that there is a hamiltonian path  $\mathbf{p}$  in  $G' - v$ .  $w$  certainly is not an end-vertex of  $\mathbf{p}$ , since by Theorem 3.26,  $G' - w$  is non-traceable. Denote by  $y_1$  and  $y_2$  the neighbours of  $w$  on  $\mathbf{p}$  in  $G'$ . We expand  $w$  back into  $H_2$ , which renders  $\mathbf{p} - w$  into two components  $P_1$  and  $P_2$ . Let  $\{y'_1\} = V(H_2) \cap N(y_1)$  and  $\{y'_2\} = V(H_2) \cap N(y_2)$ . Note that  $y_1y'_1, y_2y'_2 \in E(\mathbf{p})$ . Since  $G_2$  is hypohamiltonian, we can join  $y_1$  and  $y_2$  with a path in  $H_2$  of length  $|V(H_2)| - 2$ . We have constructed a path of length  $n-3$  avoiding  $v$ , which completes the proof.  $\square$

Note that T. Zamfirescu’s [150, Theorem 10] is a special case of Theorem 3.30: it states that if each  $G_i$  is the Petersen graph, then the resulting graph  $S$  has empty Gallai set. (There is no proof of this fact in [150], which warrants the inclusion of the second part of the proof of Theorem 3.30.)  $S$  is a reducible snark<sup>6</sup> on 36 vertices and the smallest reducible snark with empty Gallai set mentioned in Corollary 3.31, see below.

Steffen [121] showed that there exists a hypohamiltonian snark of order  $n$  for every even  $n \geq 92$  (and certain  $n < 92$ ). We show here a similar result, where we replace “snark” with “reducible snark” (i.e. we allow the snark to not be cyclically 4-connected) and “hypohamiltonian” is replaced with “has empty Gallai set”—it follows directly from Theorem 3.30 and Steffen’s [121, Theorem 2.5], where we consider all sums of the form  $n_1 + n_2 + n_3 + n_4 - 4$  and each  $n_i$  is the order of some hypohamiltonian snark.

**Corollary 3.31** *There exists a reducible  $n$ -vertex snark with empty Gallai set for every even  $n \geq 36$  with the possible exception of 38, 40, 42, 48, 50, and 58.*

Assume there exists a snark  $G$  with a hamiltonian cycle  $\mathfrak{h}$ , the edges of which we colour alternatingly. Then the graph  $(V(G), E(G) \setminus E(\mathfrak{h}))$  contains a 1-factor, which may be coloured using a third colour. Therefore, we have a 3-edge-colouring of  $G$ , a contradiction. We have proven that every snark is non-hamiltonian. Is every snark non-traceable? Surely not, as the Petersen graph gives a counter-example. In fact, every hypohamiltonian snark—there are infinitely many such graphs [39]—provides a traceable snark, while Corollary 3.31 seems to give the first detailed account of non-traceable reducible snarks.

In the following table, we present the best currently available upper bounds for the order of the smallest graph  $G$  with the stated properties. The symbol “—” stands for an impossible combination of properties. If a number  $k$  is underlined, this means that there is a graph on  $k$  vertices and no graph on fewer than  $k$  vertices with the given properties.

---

<sup>6</sup> $S$  is reducible since it is not cyclically 4-edge-connected.

	$\kappa(G) = 1$		$\kappa(G) = 2$		$\kappa(G) = 3$	
	planar		planar		planar	
empty Gallai set	<u>12</u>	17	26	32	36	156
hypotraceable	–	–	34	154	40	190
almost hypotraceable	<u>4</u>	<u>4</u>	41	153	28	118

We discuss the graphs responsible for the first row of the table. The numbers are due to Walther and T. Zamfirescu [59, 138, 150], Schmitz [113], Skupień [115], T. Zamfirescu [150], T. Zamfirescu [150], and Theorem 4.1, respectively. The first two numbers follow from the graphs shown in Fig. 34. The bounds from the second row follow from work of Thomassen [125], Theorem 2.25, a result of Horton [72], and by applying Thomassen’s method given in the proof of Theorem 2.27 to the plane 40-vertex hypohamiltonian graph from Fig. 8, respectively. In the third row, the first two numbers are due to the claw  $K_{1,3}$ , for the third bound apply Prop. 3.25 to three copies of the Petersen graph and the almost hypohamiltonian graph given in Fig. 32, for the fourth number apply Prop. 3.25 to three copies of the plane 40-vertex hypohamiltonian graph from Fig. 8 and the plane almost hypohamiltonian graph shown in Fig. 29, while the last two bounds stem from applying Theorem 3.26 to three copies of the Petersen graph (see Fig. 35 for the resulting graph) or three copies of a 40-vertex planar hypohamiltonian graph such as the one shown in Fig. 8.

In the next chapter, we shall present variations of Gallai’s problem.

## Chapter 4

# Intersecting Longest Paths and Longest Cycles

In Sections 3.5 and 3.6, we discussed Gallai's famous problem on intersecting longest paths. Many variations of Gallai's problem have been studied in the literature. Important foundational articles concerning intersecting longest paths and longest cycles are Grünbaum's [59] and T. Zamfirescu's [150]. The survey [114] by Shabbir, the author, and T. Zamfirescu revolves around Gallai's problem and other questions in the same spirit, involving conditions on connectivity, planarity or embeddability on various lattices. It also considers the analogous problem for cycles instead of paths. We will present the problems neither as deeply nor as broadly as aforementioned survey, but will mention here certain important results that were obtained, especially consequences of our work and our collaborators' work on hypohamiltonian graphs.

Denote by  $C_k^j(P_k^j)$   $\left[\overline{C}_k^j(\overline{P}_k^j)\right]$  the smallest order of a [planar]  $k$ -connected graph in which any  $j$  vertices are avoided by some longest cycle (path). We focus in the following on the cases  $k \in \{1, 2, 3\}$  and  $j \in \{1, 2\}$ .

## 4.1 Empty Intersection of all Longest Paths

We call P1 the question: “Do there exist graphs in which any vertex is missed by some longest path?”, or equivalently: “Are there graphs with empty Gallai set?”—see Section 3.5 for the definition of the Gallai set. Certain results in the following paragraphs were briefly mentioned in Sections 3.5 or 3.6, but we choose to repeat them here in order to give more details.

Walther was the first to have found a graph with empty Gallai set in 1969 [137]. A few years later, an example with only 12 vertices was (independently) found by Walther and T. Zamfirescu, see [59, 138, 150]. It is shown in Fig. 34. The P1 problem restricted to planar graphs generated several examples, each smaller than the previous one. Walther’s first example with 25 vertices was planar, but the smallest so far was found by Schmitz [113] in 1975 and has 17 vertices, see Fig. 34. It was conjectured [148] that the orders 12 and 17 are smallest possible for the problem P1 in the general and planar case, respectively. Brinkmann and Van Cleemput proved (using computers) that in the general case order 12 is minimal.

The examples become naturally larger if higher connectivity is requested. The first 2-connected example was found in 1972 for P1 and had 82 vertices and was planar [147]. Currently, the smallest known 2-connected graph answering P1 has 26 vertices and was constructed by Skupień [115] in 1996, while the smallest known planar example has order 32 and was found by T. Zamfirescu [150] in 1976. For these two graphs, see Fig. 36.

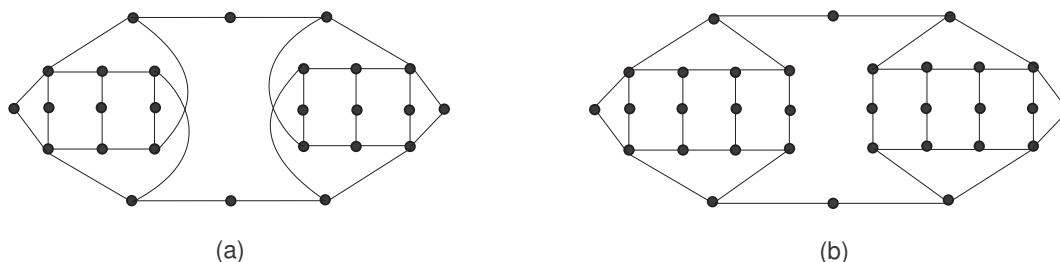


Fig. 36

The same problem for 3-connected graphs received its first answer in 1974 through Grünbaum's example with 484 vertices [59]. But the best answer so far, provided by T. Zamfirescu, as already mentioned in Section 3.6, is a graph with 36 vertices obtained in the following way. (We say that we *insert*  $G$  into  $H$ , if we replace every vertex of  $H$  with copies of  $G - w$ , where replacement is used as defined in Section 3.5.) Insert Petersen's graph into  $K_4$ . In summary, we have

$$P_1^1 = 12, \quad \overline{P}_1^1 \leq 17, \quad P_2^1 \leq 26, \quad \overline{P}_2^1 \leq 32, \quad \text{and} \quad P_3^1 \leq 36.$$

This idea of “insertion” will be used extensively in the proof of the next theorem.

Concerning the problem P1 for 3-connected planar graphs, significant progress was made by using hypohamiltonian graphs. The techniques used can be deployed for similar longest paths and longest cycles problems. We summarise these findings in the following briefly. Tables I and II on page 79 in Voss' book [136] show the following inequalities.

$$\overline{C}_3^1 \leq 57, \quad \overline{P}_3^1 \leq 224, \quad \overline{C}_3^2 \leq 6758, \quad \text{and} \quad \overline{P}_3^2 \leq 26378.$$

Based on the progress towards finding smaller planar hypohamiltonian graphs, there were numerous occasions on which these bounds were improved. We now present only the currently best bounds, while the proofs given here will contain only the major steps, which are identical to the ones introduced and described in detail by T. Zamfirescu in [150]. For a different presentation of the same approach, see [7] by Araya and Wiener.

**Theorem 4.1** (Jooyandeh, McKay, Östergård, Pettersson, Zamfirescu, 2016 [80])  
*We have*

$$\overline{C}_3^1 \leq 40, \quad \overline{P}_3^1 \leq 156, \quad \overline{C}_3^2 \leq 2625, \quad \text{and} \quad \overline{P}_3^2 \leq 10350.$$

*Proof.* The first of the four inequalities follows immediately from Theorem 2.8, which shows that there exists a planar hypohamiltonian graph of order 40. In the following, let  $G$  be such a graph.

In order to prove the second inequality, insert  $G$  into  $K_4$ . We obtain a graph in which every vertex is avoided by a path of maximal length. The details of this

proof are given in the second and third paragraph of the proof of Theorem 3.30. We obtain  $\overline{P}_3^1 \leq 4 \cdot (40 - 1) = 156$ .

For the third inequality, insert  $G$  into the 70-vertex planar cubic hypohamiltonian graph—which we will call here  $\Gamma$ —constructed by Araya and Wiener in [7] and shown in Fig. 15. ( $\Gamma$  is the smallest known planar cubic hypohamiltonian graph.) This means that each vertex of  $\Gamma$  is replaced by  $G$  minus some vertex of degree 3. We denote the resulting graph by  $G'$ . Araya and Wiener proved [7] (using a computer) that every pair of edges in  $\Gamma$  is missed by a longest cycle. Combining this fact with the hypohamiltonicity of  $G$  and  $\Gamma$ , we obtain that in  $G'$  any pair of vertices is avoided by a longest cycle. This property is not lost if all edges originally belonging to  $\Gamma$  are contracted. By construction, the order of  $G'$  is  $(40 - 1) \cdot 70 = 2730$ . Since  $|E(\Gamma)| = 105$ , after contracting all edges originally belonging to  $\Gamma$ , we obtain  $\overline{C}_3^2 \leq 2730 - 105 = 2625$ .

For the last inequality, consider the graph  $\Gamma$  from above and insert  $\Gamma$  into  $K_4$  to obtain  $H$ . Now insert  $G$  into  $H$ . Finally, contract all edges which originally belonged to  $H$ . Since  $|V(H)| = (70 - 1) \cdot 4 = 276$  and  $H$  is cubic, we have that  $|E(H)| = 414$ . Then  $\overline{P}_3^2 \leq (40 - 1) \cdot 276 - 414 = 10350$ .  $\square$

For a more detailed proof of the latter two bounds (i.e.  $\overline{C}_3^2 \leq 2625$  and  $\overline{P}_3^2 \leq 10350$ ), replace in [7, Corollary 3.6] the 42-vertex planar hypohamiltonian graph with a 40-vertex planar hypohamiltonian graph (for instance the one depicted in Fig. 8).

Research impulses from fault-tolerant designs in computer networks—for fault-tolerance problems in graph theory see Hayes' paper [65] and, for more recent developments, [21, 73, 104]—led to considering P1 in various lattices. Such embeddability problems (also concerning longest cycles) are beyond the scope of this Dissertation, but an overview can be found in [114].



## 4.2 Empty Intersection of all Longest Cycles

The  $Ck$  problem, i.e. the existence of graphs in which any  $k$  vertices are missed by some longest cycle, was completely solved, in the sense that the provided example has the smallest possible number of vertices (namely  $3k + 3$ ), by Thomassen [128] (see Fig. 37).

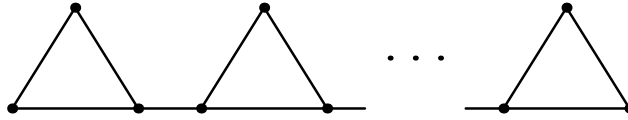


Fig. 37

However, one might argue that the appropriate frame while working with longest cycles demands connectivity at least 2, and in that case the smallest known example for the C1 problem remains, as for connectivity 3, Petersen's graph (see Fig. 1 or 45). That this graph is indeed the smallest among all 2-connected graphs was verified by Brinkmann and Van Cleemput [17]. For the C1 problem and 2-connected planar graphs, Thomassen found an example with 15 vertices, see [150] and Fig. 42 (c). Again, Brinkmann and Van Cleemput proved its optimality [17].

For 3-connected planar graphs, the first example was found by Grünbaum [59] in 1974 and had 124 vertices. Once more through the work on planar hypohamiltonian graphs, this was improved by Thomassen [127] to 105 (see Fig. 2), then by Hatzel [64] to 57. The author and T. Zamfirescu [146] decreased this to 48, and Araya and Wiener [141] further improved it to 42; for the latter two improvements, see Fig. 3. The hypohamiltonian planar graph with 40 vertices recently found by Jooyandeh, McKay, Östergård, Pettersson, and the author [80] (see Theorem 2.8 and Fig. 8), is the smallest example known to date. Due to Tutte's result that every planar 4-connected graph is hamiltonian [133], and summarising, we have for all  $j \geq 1$

$$C_1^j = \overline{C}_1^j = 3j + 3, \quad C_2^1 = C_3^1 = 10, \quad \overline{C}_2^1 = 15, \quad \overline{C}_3^1 \leq 40, \quad \text{and} \quad \overline{C}_4^j = \overline{C}_5^j = \infty.$$

Nothing is known concerning  $C_4^1$ . Note that this is a relaxation of an old problem of Thomassen whether 4-connected hypohamiltonian graphs exist [128]: see Problem 1 in Chapter 6.

### 4.3 Avoiding Arbitrary Pairs of Points

The problem P2 of finding graphs in which every pair of vertices is missed by some longest path was solved by Grünbaum in 1974, see [59]. His graph is 3-connected and has 324 vertices. Now the smallest such examples are a graph with 93 vertices and a 3-connected graph with 270 vertices, found by T. Zamfirescu [150] in 1976. Concerning planar graphs, a graph of order 308 and a 2-connected graph of order 914, found in 1975 by T. Zamfirescu [149] are still the smallest known. For 3-connected graphs, the first example with 57838 vertices was given by T. Zamfirescu [150] in 1976. This was improved step-by-step to the result given in Theorem 4.1, which is currently the best bound we have, so

$$P_1^2 \leq 93, \quad \overline{P}_1^2 \leq 308, \quad \overline{P}_2^2 \leq 914, \quad P_2^2 \leq P_3^2 \leq 270, \quad \text{and} \quad \overline{P}_3^2 \leq 10350.$$

The C2 problem, received—for 2-connected graphs—a positive answer as well. The first 2-connected example was presented already in 1969 by Walther [137] and has 220 vertices. The chronologically first 3-connected example was found in 1974 by Grünbaum [59]. Currently, the smallest such example known was given by T. Zamfirescu in 1976 and has 75 vertices [150]. It is obtained by inserting the Petersen graph  $P$  into a copy  $P'$  of  $P$ , and eventually contracting all original edges of  $P'$ . Note that T. Zamfirescu's 75-vertex graph is also the record-holder for the 2-connected case.

Concerning planar 2-connected graphs, an example constructed by T. Zamfirescu [148] in 1975 is still the smallest known. It has 135 vertices. If the graph should be polyhedral, the first example, with 14818 vertices, was found by T. Zamfirescu [150] in 1976. By providing and using smaller and smaller planar

hypohamiltonian graphs, the results improved. The best we currently have is based on work of Jooyandeh, McKay, Östergård, Pettersson, and the author, and has order 2625, as mentioned in Theorem 4.1. (Note that none of these graphs is regular.) We have

$$C_1^2 = \overline{C}_1^2 = 9, \quad \overline{C}_2^2 \leq 135, \quad C_2^2 \leq C_3^2 \leq 75, \quad \text{and} \quad \overline{C}_3^2 \leq 2625.$$

Table 4 in the Appendix summarises the findings of this section up to this point.

In 1981, Thomassen presented an infinite family of planar cubic hypohamiltonian graphs [128]. Schauerte and the author [111] have shown that there also exist polyhedral cubic graphs satisfying C2. In [111], an example of order 8742 was provided. (Subsequently, Araya and Wiener [7] found an example of order 4830 only.) We present now the argument given in [111].

**Theorem 4.2** (Schauerte and Zamfirescu, 2006 [111]) *There exists a planar 2-connected cubic graph on 30 vertices such that any vertex is missed by some longest cycle.*

*Proof.* The graph of Fig. 38, being a straightforward modification of an example due to Thomassen (see Fig. 42 (c)) enjoys the required properties—Thomassen’s graph shares all properties except for 3-regularity, which is why we have modified his graph.  $\square$

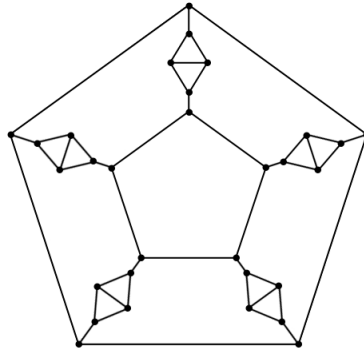


Fig. 38

**Theorem 4.3** [111] *There exists a planar 2-connected cubic graph on 280 vertices such that any pair of vertices is missed by some longest cycle.*

*Proof.* We construct the graph  $G$  in the following way. First, consider the pentagonal prism  $G_1$  shown in Fig. 39:

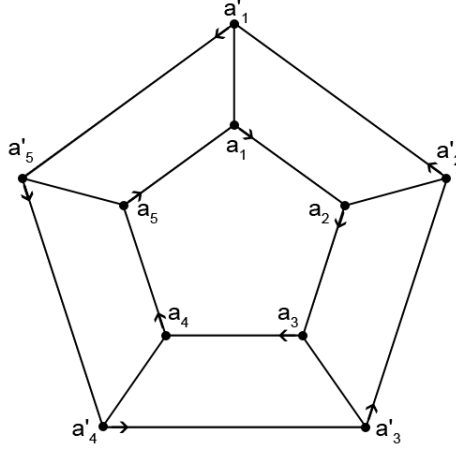


Fig. 39: The graph  $G_1$ .

Each of its vertices will be replaced by a graph  $G_2$  (replacement as defined in Section 3.5; note that  $G_2$  already has the necessary “dangling” edges), see Fig. 40, respecting the location of the arrow-marked edges.

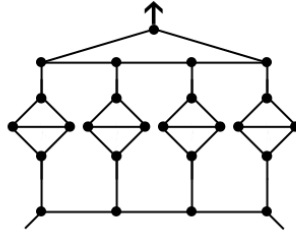


Fig. 40: The graph  $G_2$ .

We obtain a graph  $G_3$ . The intersection of any longest cycle of  $G_3$  with a  $G_2$ -copy is a path with 20 vertices if the arrow-marked edge of that copy is used, or with 24 vertices if the other two dangling edges of the  $G_2$ -copy are used.

Suppose we insert between  $a_i$  and  $a_{i+1}$  ( $1 \leq i \leq 4$ ) and  $a_5$  and  $a_1$ , and between  $a'_i$  and  $a'_{i+1}$  ( $1 \leq i \leq 4$ ) and  $a'_5$  and  $a'_1$  isomorphic copies of a graph with

$m$  vertices, and between  $a_i$  and  $a'_i$  ( $1 \leq i \leq 5$ ) isomorphic copies of a graph with  $n$  vertices, thus obtaining  $G$ .

Then each cycle of the type  $a_1a_2a'_2a'_3a_3a_4a'_4a'_5a'_1$  in  $G_1$  induces in  $G$  a cycle of length  $5 \cdot 20 + 4 \cdot 24 + 5m + 4n$ . Each cycle of the type  $a_1a_2a_3a_4a_5a'_5a'_4a'_3a'_2a'_1$  in  $G_1$  induces in  $G$  a cycle of length  $8 \cdot 20 + 2 \cdot 24 + 8m + 2n$ . Both types of cycles of  $G$  must be longest cycles. So we must have

$$5 \cdot 20 + 4 \cdot 24 + 5m + 4n = 8 \cdot 20 + 2 \cdot 24 + 8m + 2n,$$

which yields  $2n = 12 + 3m$ .

To choose a small example, we consider  $m = 0$  and  $n = 6$ , so we intercalate nothing between  $a_i$  and  $a_{i+1}$  ( $1 \leq i \leq 4$ ) and  $a_5$  and  $a_1$ , nothing between  $a'_i$  and  $a'_{i+1}$  ( $1 \leq i \leq 4$ ) and  $a'_5$  and  $a'_1$ , while between  $a_i$  and  $a'_i$  ( $1 \leq i \leq 5$ ) we intercalate the graph of Fig. 41:

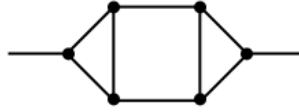


Fig. 41

Since the graph from Fig. 40 has order 25 and the graph from Fig. 41 has order 6, the resulting 2-connected graph has  $25 \cdot 10 + 5 \cdot 6 = 280$  vertices, verifies  $C2$ , and is both planar and cubic.  $\square$

**Theorem 4.4** [111] *There exists a polyhedral cubic graph on 8742 vertices such that any pair of vertices is missed by some longest cycle.*

*Proof.* Consider Thomassen's planar cubic hypohamiltonian 94-vertex graph  $T$  published in [129], open it up at some vertex, and introduce it at every vertex of  $T$ ; recall the insertion procedure used in the proof of Theorem 4.1. We have to prove that every pair of edges in  $T$  is avoided by some longest cycle of  $T$ . This turned out to be a tedious task, and it was performed by a computer. For details,

please see [111]: at the end of that paper, a (very large) table is provided which associates to every pair of edges a longest cycle omitting it.  $\square$

With the same approach, but replacing  $T$  with the 70-vertex graph shown in Fig. 15, Araya and Wiener were able to dramatically improve this result to the following, which is the best bound we have.

**Theorem 4.5** (Araya and Wiener, 2011 [7]) *There exists a polyhedral cubic graph on 4830 vertices such that any pair of vertices is missed by some longest cycle.*

## Chapter 5

# On Non-Hamiltonian Graphs for Which Every Vertex-Deleted Subgraph is Traceable

### 5.1 Introduction

We study in this chapter a new class of graphs, closely related to the families of hypohamiltonian, hypotraceable, and almost hypohamiltonian graphs discussed in Chapters 2 and 3, defined as follows. A graph  $G$  is called a *platypus*—an egg-laying mammal—if  $G$  is non-hamiltonian, yet for any vertex  $v \in V(G)$ , the graph  $G - v$  is traceable. We will denote the family of all platypuses of connectivity  $\kappa$  by  $\mathcal{P}_\kappa$  and put  $\mathcal{P} = \bigcup_\kappa \mathcal{P}_\kappa$ .

Every hypohamiltonian and every hypotraceable graph is a platypus, but there exist platypuses which are neither hypohamiltonian nor hypotraceable. Investigating platypuses was suggested to the author by Kenta Ozeki in a conversation in 2012. In order to draw connections between results proven here and Gábor Wiener’s work [140] presented in Bordeaux in 2014, we require the following. Let  $G$  be a graph and  $\mathcal{T}(G)$  the set of all spanning trees of  $G$ . Denote by  $\ell(T)$  the number of leaves of a tree  $T$ . The *minimum leaf number*  $\text{ml}(G)$  of a graph  $G$  is defined as

$$\text{ml}(G) = \begin{cases} \min_{T \in \mathcal{T}(G)} \ell(T) & \text{if } G \text{ is not hamiltonian,} \\ 1 & \text{if } G \text{ is hamiltonian.} \end{cases}$$

Note that for a 2-connected graph  $G$  we have  $\text{ml}(G - v) \geq \text{ml}(G) - 1$  for all  $v \in V(G)$ . Consider an integer  $\ell \geq 2$ . A 2-connected graph  $G$  with  $\text{ml}(G) = \ell$  is called  *$\ell$ -leaf-critical* if  $\text{ml}(G - v) = \ell - 1$  for every  $v \in V(G)$ , and  *$\ell$ -leaf-stable* if  $\text{ml}(G - v) = \ell$  for every  $v \in V(G)$ .

The family of all 2-leaf-critical graphs (3-leaf-critical graphs) coincides with the family of all hypohamiltonian (hypotraceable) graphs. Wiener shows that  $\ell$ -leaf-stable and  $\ell$ -leaf-critical graphs exist for every  $\ell \geq 2$ , and studies these graphs under the additional condition of planarity. Using these results, he solves affirmatively the open problem of Gargano, Hammar, Hell, Stacho, and Vaccaro [45, p. 93] whether non-traceable non-hypotraceable arachnoid graphs (defined in [45]) exist.

Related to the present work, for a platypus  $G$  we have  $\text{ml}(G) \in \{2, 3\}$ . It is 2 if and only if  $G$  is traceable and 3 if and only if  $G$  is hypotraceable. Furthermore,  $\text{ml}(G - v)$  in general depends on  $v \in V(G)$ , and is either 1 or 2. (If in a platypus  $G$  we have  $\text{ml}(G) = 3$ , then necessarily  $\text{ml}(G - v) = 2$  for all  $v \in V(G)$ , whence  $G$  is a 3-critical graph, i.e. hypotraceable.)  $\mathcal{P}$  contains all 2-leaf-critical, 3-leaf-critical, and 2-leaf-stable graphs, and no other leaf-critical or leaf-stable graphs. But  $\mathcal{P}$  is larger than the three aforementioned families—a polyhedral platypus not belonging to any of the three families will be discussed in Section 5.5.

Let  $G$  be a graph. We call a path  $P \subset G$  with end-vertices  $v, w$  an *ear* if  $\{v, w\}$  is a cut in  $G$  and every vertex in  $V(P) \setminus \{v, w\}$  has degree 2 in  $G$ . An ear on  $k$  vertices will be called a  *$k$ -ear*. Furthermore, we will require in the remainder of this Dissertation an ear to not contain any super-ears, i.e. for every ear  $D$  there exists no ear  $D'$  such that  $D \subsetneq D'$ . We call  $v \in V(G)$  *naughty* if  $N(v)$  contains (at least) two vertices of degree 2. For non-adjacent vertices  $v, w$  we will write  $G + vw$  for the graph  $G$  to which we add the edge  $vw$ .

Let  $G$  be a graph of connectivity 2 and  $X = \{v, w\}$  a cut in  $G$ . Let  $A$  be a connected component of  $G - X$ , and put  $H = G[A \cup \{v, w\}]$ . Consider a graph



$J$  and  $x, y \in V(J)$ . We *replace*  $(H, v, w)$  with  $(J, x, y)$  (or simply  $H$  with  $J$ ) if in  $G - A$  and  $J$  we identify  $v$  with  $x$  and  $w$  with  $y$ . In case  $H$  or  $J$  are paths,  $v, w$  and  $x, y$  are required to be their respective end-vertices.

## 5.2 Structural Results

The following proposition contains basic facts about platypuses. These will mostly be used tacitly in the remainder of this chapter.

**Proposition 5.1** [145]

- (i) *Every platypus is 2-connected.*
- (ii) *A platypus containing a triangle with at least one cubic vertex is traceable.*
- (iii) *Let  $G$  be a platypus containing a  $k$ -ear  $P$ ,  $k \in \{3, 4\}$ . Then for every non-adjacent  $v, w \in V(P)$  the graph  $G + vw$  is a platypus.*
- (iv) *Every bipartite platypus must be balanced.*
- (v) *A platypus does not contain naughty vertices. In particular: no platypus contains a  $k$ -ear,  $k \geq 5$ .*
- (vi) *If a platypus contains a vertex of degree 2, then it is traceable.*
- (vii) *If a platypus contains a 4-ear  $H$ , then  $H$  can be replaced with a 3-ear, and the resulting graph is a platypus, too.*

*Proof.* (i) Let  $v$  be a cut-vertex of a platypus  $G$ . Then  $G - v$  cannot be traceable.

(ii) Let  $T$  be a triangle with  $V(T) = \{v_1, v_2, v_3\}$  in a platypus  $G$ , and let  $v_3$  be cubic.  $G - v_1$  contains a hamiltonian path  $\mathbf{p}$ . If  $v_2v_3 \in E(\mathbf{p})$ , then  $(\mathbf{p} - v_2v_3) \cup v_2v_1v_3$  is a hamiltonian path in  $G$ . If  $v_2v_3 \notin E(\mathbf{p})$ , then  $v_3$  is an end-vertex of  $\mathbf{p}$  and  $\mathbf{p} + v_3v_1$  shows that  $G$  is traceable.

(iii) Certainly, all paths in  $G$  remain intact after adding  $vw$ . Now assume  $G + vw$  is hamiltonian, and let  $x, y$  be the end-vertices of  $P$ . Then there exists a hamiltonian path in  $G - (V(P) \setminus \{x, y\})$ , the end-vertices of which are  $x$  and  $y$ . But then  $G$  is hamiltonian, a contradiction.

(iv) If  $(A, B)$  is a bipartition of  $G$  with  $|A| < |B|$ , then deleting a vertex from  $A$  cannot yield a traceable graph.

(v) Let  $G$  be a platypus containing a naughty vertex  $v$ . Let  $v', v'' \in N(v)$  have degree 2. Since  $G$  is a platypus,  $G - v$  contains a hamiltonian path  $\mathbf{p}$ . The end-vertices of  $\mathbf{p}$  must be  $v'$  and  $v''$ . But then  $\mathbf{p} \cup v'vv''$  is a hamiltonian cycle in  $G$ , a contradiction.

(vi) Let the platypus  $G$  have a vertex  $v$  of degree 2. If  $w \in N(v)$ , then  $G - w$  contains a hamiltonian path  $\mathbf{p}$  ending in  $v$ .  $\mathbf{p}$  can now be extended to a hamiltonian path in  $G$ .

(vii) Let  $G$  be the original and  $G'$  the resulting graph. Clearly,  $G'$  is non-hamiltonian. Denote the vertices of  $H$  which are not its end-vertices by  $x$  and  $y$  and let  $w$  be the vertex which replaces  $x, y$ . Any hamiltonian path in  $G$  using  $xy$  may now use  $w$ . It remains to see that  $G' - w$  is traceable. Let  $\mathbf{p}$  be a hamiltonian path in  $G - x$ . Now  $\mathbf{p} - y$  is a hamiltonian path in  $G' - w$ .  $\square$

In fact, the idea behind (vii) can be extended significantly, as we shall explore in Section 5.4. Applying (iii) to the graph shown in Fig. 42 (a) we obtain a planar platypus with no cubic vertices. This contrasts Thomassen's theorem [128] stating that every planar hypohamiltonian graph contains a cubic vertex—we will come back to this intriguing fact in Section 5.3. Since the graph from Fig. 42 (a) will appear frequently in future arguments, we will call it  $\Delta$  in the remainder of this chapter.

As mentioned in Section 2.4, Collier and Schmeichel [29, p. 196] were the first to publish that the vertices of a triangle in a hypohamiltonian graph have degree at least 4. (This is true for hypotraceable graphs, as well.) Thus, hypohamiltonian and hypotraceable graphs do not fulfil the condition from Prop. 5.2, but  $\Delta$ , the graph shown in Fig. 42 (a), does satisfy it. Yet again we see that there exist platypuses which are neither hypohamiltonian nor hypotraceable.

**Proposition 5.2** [145] *Let  $G$  be a platypus containing a triangle  $v_1v_2v_3$  with  $v_1, v_2 \in V(G)$  cubic. Consider vertices  $w, v'_1, v'_2, v'_3 \notin V(G)$ . Then*

$$R(G) = (V(G) \cup \{w\}, E(G) \cup \{wv_i\}_{i=1}^3),$$

$$S(G) = (V(G) \cup \{v'_1, v'_2\}, E(G) \cup \{v_1v'_1, v'_1v'_2, v_2v'_1, v_2v'_2, v_1v'_2, v_3v'_1\}),$$

and

$$T(G) = (V(G) \cup \{v'_i\}_{i=1}^3, E(G) \cup \{v'_1v'_2, v'_2v'_3, v'_3v'_1\} \cup \{v_iv'_i\}_{i=1}^3),$$

are platypuses, as well.  $R$ ,  $S$ , and  $T$  preserve planarity and 3-connectedness.

*Proof.* In the remainder of this proof,  $G$  will be considered as a subgraph of  $R(G)$ ,  $S(G)$  or  $T(G)$ , depending on which operation we are studying. We first show that  $R(G)$  is a platypus. Assume  $R(G)$  contains a hamiltonian cycle  $\mathfrak{h}$ . Ignoring symmetric cases, since  $w$  is cubic we have either  $v_1wv_2 \subset \mathfrak{h}$  or  $v_1wv_3 \subset \mathfrak{h}$ . In both situations we can modify  $\mathfrak{h}$  to a hamiltonian cycle in  $G = R(G) - w$  by replacing  $v_1wv_2$  with  $v_1v_2$  or  $v_1wv_3$  with  $v_1v_3$ , yielding a contradiction. Hence  $R(G)$  is non-hamiltonian.

Let  $v \in V(G)$  and let  $\mathfrak{r}$  be a hamiltonian path in  $G - v$ . If  $v_1v_2 \in E(\mathfrak{r})$ ,  $v_2v_3 \in E(\mathfrak{r})$  or  $v_1v_3 \in E(\mathfrak{r})$ , then  $\mathfrak{r}$  can be transformed into a hamiltonian path in  $R(G) - v$ . If neither of these three edges occurs in  $\mathfrak{r}$ , then necessarily  $v_1$  or  $v_2$  must be an end-vertex of  $\mathfrak{r}$ . Then  $\mathfrak{r} + v_1w$  or  $\mathfrak{r} + v_2w$  is a hamiltonian path in  $R(G) - v$ . Since  $G$  is a platypus containing a triangle which has a cubic vertex, Prop. 5.1 (ii) implies that  $G = R(G) - w$  is traceable.

Now we prove that  $S(G)$  is a platypus. Assume  $S(G)$  contains a hamiltonian cycle. Then this cycle contains a subpath from  $v_1$  to  $v_2$  hamiltonian in  $S(G)[\{v_1, v_2, v'_1, v'_2\}]$  or in  $S(G)[\{v_1, v_2, v_3, v'_1, v'_2\}]$ . In each case, substitute  $v_1v_2$  or  $v_1v_3v_2$  for the subpath, respectively.

Let  $\mathfrak{s}$  be a hamiltonian path in  $G - v$ , where  $v \in V(G) \setminus \{v_1, v_2\}$ . Assume  $v_1v_2 \in E(\mathfrak{s})$ . Then  $(\mathfrak{s} - v_1v_2) \cup v_1v'_1v'_2v_2$  is a hamiltonian path in  $S(G) - v$ . Assume now that  $v_1v_2 \notin E(\mathfrak{s})$ . If  $v_2$  is an end-vertex of  $\mathfrak{s}$ , then  $\mathfrak{s} \cup v_2v'_1v'_2$  is a hamiltonian path in  $S(G) - v$ . If  $v_2$  is not an end-vertex of  $\mathfrak{s}$ , then  $v_3v_2 \in E(\mathfrak{s})$ . In this case  $(\mathfrak{s} - v_3v_2) \cup v_2v'_2v'_1v_3$  is a hamiltonian path in  $S(G) - v$ . The discussion

for  $v \in \{v_1, v_2\}$  is very similar. Consider a hamiltonian path  $\mathfrak{s}'$  in  $G - v_2$ . If  $v_1v_3 \in E(\mathfrak{s}')$ , then  $(\mathfrak{s}' - v_1v_3) \cup v_3v_2v'_2v_1$  is a hamiltonian path in  $S(G) - v'_1$ . If  $v_1v_3 \notin E(\mathfrak{s}')$ , then  $v_1$  is an end-vertex of  $\mathfrak{s}'$ . Then  $\mathfrak{s}' \cup v_1v'_2v_2$  is a hamiltonian path in  $S(G) - v'_1$ . The argument for  $S(G) - v'_2$  is exactly the same.

We now show that  $T(G)$  is a platypus. We have shown that  $R(G)$  is non-hamiltonian, and  $T(G)$  is non-hamiltonian if and only if  $R(G)$  is non-hamiltonian.

Consider  $v \in V(G) \setminus \{v_1\}$  and a hamiltonian path  $\mathfrak{t}$  in  $G - v$ . If  $v_1$  is an end-vertex of  $\mathfrak{t}$ , then  $\mathfrak{t} \cup v_1v'_1v'_2v'_3$  is a hamiltonian path in  $T(G) - v$ . If  $v_1$  is not an end-vertex of  $\mathfrak{t}$ , then  $v_1v_2 \in E(\mathfrak{t})$  or  $v_1v_3 \in E(\mathfrak{t})$ . In the former case, substitute in  $\mathfrak{t}$  the path  $v_1v'_1v'_3v'_2v_2$  for the edge  $v_1v_2$ , and in the latter case substitute in  $\mathfrak{t}$  the path  $v_1v'_1v'_2v'_3v_3$  for the edge  $v_1v_3$ , and we obtain a hamiltonian path in  $T(G) - v$ . Since we have dealt with the case  $v = v_2$ , the case  $v = v_1$  follows directly.

A hamiltonian path in  $T(G) - v'_1$  can be obtained by considering the hamiltonian path  $\mathfrak{t}$  in  $G - v_3$ . We have  $v_1v_2 \in E(\mathfrak{t})$ . Substitute  $v_1v_3v'_3v'_2v_2$  for  $v_1v_2$  in  $\mathfrak{t}$ . Similarly for  $T(G) - v'_2$ . Consider a hamiltonian path  $\mathfrak{t}'$  in  $G - v_2$ . If  $v_1$  is an end-vertex of  $\mathfrak{t}'$ , then  $\mathfrak{t}' \cup v_1v'_1v'_2v_2$  is a hamiltonian path in  $T(G) - v'_3$ . If  $v_1$  is not an end-vertex of  $\mathfrak{t}'$ , then  $v_1v_3 \in E(\mathfrak{t}')$ . Replacing  $v_1v_3$  with  $v_1v'_1v'_2v_2v_3$  yields a hamiltonian path in  $T(G) - v'_3$ .  $\square$

**Theorem 5.3** [145] *Let  $G$  be a platypus of connectivity  $k$ . If  $H$  is a subgraph of  $G$  of order  $k$ , then  $G - H$  has at most  $k$  components.*

*Proof.* By Prop. 5.1 (i),  $G$  is 2-connected, so  $k \geq 2$ . Let  $v \in V(H)$  and consider a hamiltonian path  $\mathfrak{p}$  of  $G - v$ . The set of  $k - 1$  vertices  $M$  of  $H - v$  lies on  $\mathfrak{p}$  and determines at most  $k$  subpaths of  $\mathfrak{p}$ , where each subpath either (i) contains exactly one vertex from  $M$  and an end-vertex of  $\mathfrak{p}$  or (ii) exactly two vertices from  $M$ . Each such subpath visits exactly one component of  $G - H$  if it has more than two vertices, and none if it has just two vertices. Hence  $G - H$  has at most  $k$  components.  $\square$

Let  $G$  be a graph, consider its Cartesian product with  $P_2$ ,  $G \square P_2$ , and replace each copy of  $P_2$  with a copy of  $P_3$ . We will call the resulting graph the *dotted prism over  $G$*  and denote it by  $\dot{G}$ .

**Theorem 5.4** [145]

- (i) *The dotted prism over a hamiltonian graph  $G$  of odd order  $n \geq 3$  is a platypus.*
- (ii) *Let  $G$  be a hamiltonian graph of even order containing an edge  $e$  which lies on all hamiltonian cycles occurring in  $G$ . Let  $v'w' = e'$  and  $v''w'' = e''$  be the two copies of  $e$  in  $\dot{G}$ . Then  $H = \dot{G} - e' - e'' + v'w'' + v''w'$  is a platypus.*

*Proof.* (i) Denote the two copies of  $G$  in  $\dot{G}$  by  $G'$  and  $G''$ . Assume  $\dot{G}$  is hamiltonian. Every copy of  $P_3$  must be traversed. Since there is an odd number of copies of  $P_3$ , either we begin in  $G'$  and end up in  $G''$  or vice-versa. In both cases we obtain a contradiction, so  $\dot{G}$  is non-hamiltonian.

Denote the end-vertices of the  $n$  copies of  $P_3$  by  $v'_i$  and  $v''_i$ , where  $v'_i \in V(G')$  and  $v''_i \in V(G'')$ ,  $i \in \{1, \dots, n\}$ . Call  $w_i$  the vertex with neighbourhood  $\{v'_i, v''_i\}$  and put  $W = \{w_i\}_{i=1}^n$ . We now show that there exists a hamiltonian path in  $H = \dot{G} - w_i$  with  $i$  arbitrary, but fixed. It is easy to see that in  $H$  there exists a cycle  $\mathfrak{c}$  which visits all vertices with the exception of  $v'_i$ . Let  $x \in N(v'_i) \setminus \{w_i\}$  and  $\{y\} = N(x) \cap W$ . Then  $(\mathfrak{c} - xy) + xv'_i$  is a hamiltonian path in  $H$ .

Let  $v \in V(\dot{G}) \setminus W$ . W.l.o.g.  $v \in V(G')$ , so there exists a  $j$  such that  $v = v'_j$ . As before,  $\dot{G} - v'_j - w_j$  contains a hamiltonian cycle  $\mathfrak{h}$ . Let  $z \in N(v''_j) \cap V(\mathfrak{h})$ . Then  $(\mathfrak{h} - zv''_j) + v''_jw$  is a hamiltonian path in  $\dot{G} - v'_j$ .

(ii) Assume  $H$  has a hamiltonian cycle  $\mathfrak{h}$ . Due to the condition that  $G$  contains an edge  $e$  belonging to all hamiltonian cycles in  $G$ ,  $v'w''$  and  $v''w'$  are contained in  $\mathfrak{h}$ . (Since if there was a hamiltonian cycle in  $H$  not using the edges  $v'w''$  and  $v''w'$ , we would immediately obtain a hamiltonian cycle in  $G$  which would not contain  $e$ .) All copies of  $P_3$  must be traversed by  $\mathfrak{h}$ . Hence, due to the fact that there is an even number of copies of  $P_3$  and  $v'w'', v''w' \in E(\mathfrak{h})$ , we obtain a contradiction.

Showing that  $H - v$  is traceable for every  $v \in V(H)$  is very similar to the proof given in (i). □

Graphs obtained from  $G$  in the same manner as  $H$  in the statement of Theorem 5.4 will be called *modified dotted prisms of  $G$*  and will be denoted by  $G^\times$ .

We present two corollaries of Theorem 5.4, the first of which follows directly from Theorem 5.4 (ii), while the second one requires further preparations.

**Corollary 5.5** [145] *There exist infinitely many balanced bipartite platypuses.*

Using a computer, Van Cleemput [134] determined that there exist no platypuses on fewer than 9 vertices. His program also determined that there are exactly four such graphs on 9 vertices (Van Cleemput verified these results using two independent implementations); they are  $\Delta$ , i.e. the graph from Fig. 42 (a), and the three graphs obtained as follows. Apply Prop. 5.1 (iii) to one, two, or three of the 3-ears in  $\Delta$ . For order 10, the situation changes dramatically and there are many platypuses: at least forty, as communicated by Van Cleemput [134].

**Lemma 5.6** *Consider  $G \in \{\dot{C}_k\}_{k \geq 3, k \text{ odd}} \cup \{C_k^\times\}_{k \geq 4, k \text{ even}}$ . If we replace in  $G$  exactly  $\ell$  3-ears with 4-ears,  $0 \leq \ell \leq k$ , then the graph we obtain is a platypus.*

*Proof.* We prove the statement by induction.  $G$  is a platypus due to Theorem 5.4. Now assume we have replaced in  $G$  exactly  $p < k$  3-ears with 4-ears. We denote the resulting platypus by  $G'$ . In  $G'$ , replace a 3-ear  $uvw$  with a 4-ear  $uvv'w$  and denote the graph we obtain by  $G''$ . Any hamiltonian cycle in  $G''$  would use the path  $uvv'w$ , which if replaced with  $uvw$  would imply the hamiltonicity of  $G'$ , a contradiction.

Since  $G' - u$  and  $G' - w$  are traceable, so are  $G'' - u$  and  $G'' - w$ . It remains to show that  $G'' - v$  is traceable. Arguing as in the proof of Theorem 5.2 (since it makes no difference whether we must traverse 3-ears or 4-ears), there exists a hamiltonian path in  $G - v$  which has  $w$  as one of its end-vertices. Now consider this path in  $G'$  and add the edge  $wv'$ . The argument for  $G' - v'$  is exactly the same.  $\square$

**Corollary 5.7** (Van Cleemput and Zamfirescu [145]) *There exists a platypus of order  $n$  for every  $n \geq 9$ , there are no platypuses on fewer vertices, and there exist exactly four platypuses on 9 vertices.*

*Proof.* Adding to Theorem 5.4 and the arguments preceding the statement, we consider the dotted prism  $\dot{C}_k$  over  $C_k$  for  $k$  odd and the modified dotted prism

$C_k^\times$  over  $C_k$  for  $k$  even. (Fig. 42 shows (a)  $\dot{C}_3$ , (b)  $C_4^\times$ , and (c)  $\dot{C}_5$ .) Since we can replace in each of the graphs one or two 3-ears with 4-ears (see Lemma 5.6), we have covered all orders.  $\square$

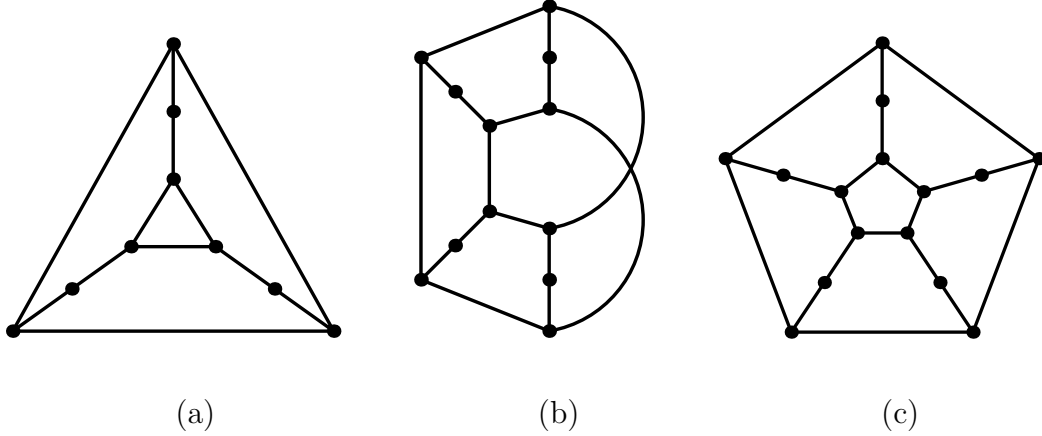


Fig. 42: (a) shows a platypus on 9 vertices, in this Chapter called  $\Delta$ , while (b) and (c) present platypuses of order 12 and 15, resp.

We briefly comment upon the graphs shown in Fig. 42. Graph (b) was already given by T. Zamfirescu, see [114, Fig. 19 (a)], who asked for the largest integer  $c$  such that any  $c$  longest cycles of any 2-connected graph have a common vertex. (This is in the same spirit as Gallai's famous question from 1966 whether in every graph the intersection of all longest paths is non-empty [43]. Walther [137] showed that this is not the case. See Sections 3.5 and 3.6, as well as Chapter 4 for more details.) Graph (b) proves that  $c \leq 7$ . Jendrol' and Skupień [76] showed  $c \leq 6$ , and this is the best bound we have. Graph (c) was already given by Thomassen, see [114, Fig. 16], to improve the bound for the smallest order of a planar 2-connected graph in which every vertex is missed by a longest cycle. Brinkmann and Van Cleemput [17] showed that Thomassen's example is the graph of smallest order with this property.

Herz defines in [66] the *cyclability* of a graph  $G$  as the greatest integer  $k$  such that for every set  $S \subset V(G)$  of cardinality  $k$  there exists a cycle in  $G$  containing  $S$ . The family of all graphs  $G$  of cyclability  $|V(G)|$  coincides with the family of all

hamiltonian graphs, and cyclability  $|V(G)| - 1$  simply means hypohamiltonicity. Cyclability has been studied extensively, for instance by Chvátal [26], Halin [62], and Holton, McKay, Plummer, and Thomassen [69].

**Theorem 5.8** [145] *For every integer  $t \geq 2$  there exists a platypus of cyclability  $t$ .*

*Proof.* Consider  $\dot{C}_k$  and  $C_k^\times$  as above. Denote by  $W_k$  the set of vertices added to the copies of  $P_2$  when constructing the dotted or modified dotted prism. We have  $|W_k| = k$ . Since  $\dot{C}_k$  and  $C_k^\times$  are non-hamiltonian, they have no cycle containing  $W_k$ , so their cyclability is at most  $k - 1$ . Consider  $G \in \{\dot{C}_k\}_{k \geq 5, k \text{ odd}} \cup \{C_k^\times\}_{k \geq 4, k \text{ even}}$ . Arguing as in the proof of Theorem 5.4, we obtain the following.

Claim. *The circumference of  $G$  is  $|V(G)| - 2 = 3k - 2$ . Let  $P = uvw$  be a 3-ear in  $G$ . Then there exists in  $G$  a longest cycle avoiding  $u$  and  $v$ .*

It remains to show that for any set  $S$  of  $k - 1$  vertices in  $G$ , there is a cycle containing  $S$ .  $G$  contains at least one 3-ear  $P$  for which  $V(P) \cap S = \emptyset$ , since  $|S| = k - 1$ . Using the Claim, we are done.

Finally, the circumference of  $\dot{C}_3$  is  $|V(\dot{C}_3)| - 1 = 8$ , see Fig. 42 (a). In the case of  $\dot{C}_3$  we are finished as well, as in a 2-connected graph any two vertices lie in a cycle.  $\square$

**Theorem 5.9** [145] *Let  $G$  be a platypus of order  $n$  and size  $m$ . We have  $m \geq \lceil 5n/4 \rceil$ , and this bound is sharp.*

*Proof.* We claim that at most half of the vertices of  $G$  have degree 2. Assume the contrary, i.e. that  $G$  has  $p < n/2$  vertices of degree greater than 2. By Prop. 5.1 (v),  $G$  contains no naughty vertices, which implies that every vertex of degree  $\neq 2$  has either no neighbours of degree 2 or one neighbour of degree 2, and every vertex of degree 2 has at least one neighbour of degree greater than 2. Therefore, the number of vertices of degree 2 is at most  $p$ . But then  $G$  has in total at most  $2p < n$  vertices, a contradiction. This immediately yields that  $2m \geq 2n/2 + 3n/2 = 5n/2$ , which gives the advertised bound.

Consider  $G \in \{\dot{C}_k\}_{k \geq 3, k \text{ odd}} \cup \{C_k^\times\}_{k \geq 4, k \text{ even}}$ . By replacing in  $G$  exactly  $\ell$  3-ears with 4-ears,  $0 \leq \ell \leq k$ , we have constructed graphs which prove the sharpness of the bound. That these graphs are indeed platypuses follows from Lemma 5.6.  $\square$



An immediate consequence of Corollary 5.7 and Theorem 5.9 is that  $\Delta$  is the smallest platypus both in terms of order and size.

### 5.3 Links to Other Families of Graphs

The first important distinction to be made is between traceable and non-traceable platypuses. While the latter coincides with the family of all hypotraceable graphs, the former contains (but does not coincide with) the family of all hypohamiltonian graphs.

We already mentioned in the Introduction that Fiorini showed that there exist infinitely many hypohamiltonian snarks [39]. Since every hypohamiltonian graph is a platypus, on one hand we have that there are infinitely many snarky platypuses (i.e. graphs which are both a snark and a platypus). On the other hand, Tait [124] proved that the Four Colour Theorem is equivalent to the statement that no snark is planar. As presented in Section 2.1, there exist infinitely many planar hypohamiltonian graphs [127], whence, there exist infinitely many non-snarky platypuses.

Consider the almost hypohamiltonian graph  $G$  constructed by the author in [144], see Fig. 29, and denote the exceptional vertex of  $G$  by  $w$ . It is easy to verify that  $G - w$  is traceable. (Note that a priori, in an almost hypohamiltonian graph  $G$  with exceptional vertex  $w$ , the graph  $G - w$  may be non-traceable; see the next paragraph.) As  $G$  is almost hypohamiltonian, for every  $v \in V(G) \setminus \{w\}$  the graph  $G - v$  is hamiltonian, so certainly traceable. It can be shown that this property is not lost if an infinite family is constructed by applying to  $G$  the operation  $\text{Th}$ , defined in Section 2.1.3. Thus, there exist infinitely many almost hypohamiltonian platypuses. (Here we have sketched a proof of the fact that there exist infinitely many polyhedral platypuses. A rigorous treatment is given in Section 5.5.)

Let  $H$  be a hypotraceable graph and  $w \notin V(H)$  a vertex. Construct a graph  $G$  by joining  $w$  with all vertices of  $H$ .  $G$  is almost hypohamiltonian with exceptional vertex  $w$ , yet not a platypus, as  $G - w = H$  is non-traceable. As there exist infinitely many hypotraceable graphs [125], we have shown that there are infinitely many almost hypohamiltonian graphs which are not platypuses. This holds vice-versa as well, since every hypotraceable graph is a platypus, and no graph can be both hypotraceable and almost hypohamiltonian. Furthermore, Wiener [140] showed recently that if  $G$  is a hypotraceable graph with a cut  $\{a, b\}$ , then  $G$  together with the edge  $ab$  is 2-leaf-stable, and thus a traceable platypus. Related to this, please see Prop. 5.10 (ii).

A further motivation for introducing platypuses follows. Chvátal [25] conjectured that if  $G$  is hypohamiltonian, and  $e \in E(G)$  an edge between vertices each of degree at least 4, then  $G - e$  is hypohamiltonian, too. Although Chvátal's conjecture is not true as shown by Thomassen [126] (and even has planar counterexamples as shown in Section 2.1.4), the following lemma does hold.

**Proposition 5.10** [145]

- (i) *Let  $G$  be a hypohamiltonian graph. For any  $e \in E(G)$ , the graph  $G - e$  is a platypus.*
- (ii) *Let  $H$  be hypotraceable graph, and  $v, w \in V(H)$  non-adjacent. Then  $H + vw$  is a platypus.*

*Proof.* (i) Put  $G' = G - e$ .  $G$  is non-hamiltonian, so  $G'$  is, too. Consider  $v \in V(G')$ . Since  $G$  is hypohamiltonian, there exists a hamiltonian cycle in  $G - v$ , so there exists a hamiltonian path in  $G' - v$ .

(ii) Put  $H' = H + vw$ . Since for every  $u \in V(H)$  there is a hamiltonian path in  $H - u$ , this evidently also holds for  $H'$ . Assume  $H'$  were hamiltonian. Then  $H' - vw = H$  would be traceable, a contradiction.  $\square$

Neither (i) nor (ii) can be inverted: (ii) is obvious, and concerning (i), consider a traceable platypus. Adding an edge between the end-vertices of a hamiltonian path would yield a hamiltonian graph. Indeed, for a platypus  $P$  and a pair of

non-adjacent vertices  $v, w$ , the graph  $P + vw$  is a platypus if and only if there exists no hamiltonian path between  $v$  and  $w$ . It may prove interesting to consider in future work “full” platypuses, i.e. platypuses to which no further edge may be added without losing the property of being a platypus.

The first part of the following proposition was mentioned, but not proven, in Chapter 3, while the second part is a special case of Prop. 5.10 (ii) (re-proven here for convenience), since a platypus is non-traceable if and only if it is hypo-traceable.

**Proposition 5.11** [145]

- (i) *An almost hypohamiltonian graph  $G$  minus its exceptional vertex  $w$  is a platypus.*
- (ii) *The join  $H'$  of a non-traceable platypus  $H$  and  $K_1 = (\{w\}, \emptyset)$  is almost hypohamiltonian with exceptional vertex  $w$ .*

*Proof.* (i) Put  $G' = G - w$ . By definition,  $G'$  is non-hamiltonian, too. Consider  $v \in V(G')$ . As  $G$  is almost hypohamiltonian, there exists a hamiltonian cycle in  $G - v$ , so there exists a hamiltonian path in  $G' - v$ .

(ii) Since  $H' - w = H$  is non-traceable,  $H' - w$  and  $H'$  are non-hamiltonian. Let  $v \in V(H') \setminus \{w\}$ . As  $H$  is a platypus,  $H - v$  contains a hamiltonian path  $\mathbf{p}$  with end-vertices  $v'$  and  $v''$ . Now  $\mathbf{p} \cup v'wv''$  is a hamiltonian cycle in  $H' - v$ .  $\square$

A non-hamiltonian graph  $G$  is *maximally non-hamiltonian* if for every pair of non-adjacent vertices  $v, w$  the graph  $G + vw$  is hamiltonian.

**Proposition 5.12** [145] *A maximally non-hamiltonian graph  $G$  is a platypus if and only if  $\Delta(G) < |V(G)| - 1$ .*

*Proof.* Consider a maximally non-hamiltonian graph  $G$  of order  $n$ . Assume  $G$  contains a vertex  $v$  of degree  $n - 1$ . If  $G$  is a platypus, then  $G - v$  contains a hamiltonian path  $\mathbf{p}$ . Let the end-vertices of  $\mathbf{p}$  be  $x$  and  $y$ . But then  $\mathbf{p} \cup xvy$  is a hamiltonian cycle in  $G$ , a contradiction. ( $xv, vy \in E(G)$  since the degree of  $v$  is  $n - 1$ .)

Now let  $G$  satisfy  $\Delta(G) < n - 1$ . Consider  $v \in V(G)$ . We know that for  $w \notin N[v]$ ,  $G + vw$  is hamiltonian, ergo  $G$  contains a hamiltonian path  $\mathbf{p}$  with end-vertices  $v$  and  $w$ . Then  $\mathbf{p} - v$  is a hamiltonian path in  $G - v$ .  $\square$

Lichiardopol and the author showed [92] that for every  $k \geq 1$  there exist infinitely many  $k$ -connected maximally non-hamiltonian graphs, but every graph  $G$  in that construction has maximum degree  $|V(G)| - 1$ , so the natural question whether 4-connected platypuses exist remains open at this point. For hypohamiltonian graphs it is a long-standing open problem whether 4-connected such graphs exist, see Thomassen's paper [128]. (The existence of 4-connected hypotractable graphs is undecided as well.) We shall see in Section 5.6 that, in stark contrast to hypohamiltonian graphs, a  $k$ -connected platypus exists *for every*  $k \geq 2$ .

We now repeat a useful definition already given in Sections 2.2 and 3.2. Consider graphs  $G, H$  containing cubic vertices  $x \in V(G)$  and  $y \in V(H)$ . Then  $G_x H_y$  is defined as one of the graphs obtained by taking  $G - x$  and  $H - y$ , and identifying, using a bijection,  $N(x)$  and  $N(y)$ . Thomassen [125] showed that if  $G$  and  $H$  are hypohamiltonian, then  $G_x H_y$  is hypohamiltonian, too (see Lemma 2.20). In Lemma 3.4, the author showed that if  $G$  is almost hypohamiltonian and  $H$  is hypohamiltonian, then  $G_x H_y$  is almost hypohamiltonian, too (under the condition that  $x$  is not the exceptional vertex of  $G$ ). The next theorem is inspired by Thomassen's result mentioned above. Note that if a graph is hypohamiltonian, then each vertex of a triangle contained in that graph has degree at least 4. Let  $G$  be a graph and  $e \in E(G)$ . Then  $G/e$  is the graph obtained by contracting  $e$ .

**Theorem 5.13** [145] *Let  $G$  be a hypohamiltonian graph and  $H$  a graph with cubic vertices  $x \in V(G)$  and  $y \in V(H)$ , and put  $N(x) = \{x_1, x_2, x_3\}$  and  $N(y) = \{y_1, y_2, y_3\}$ . Consider  $G - x$  and  $H - y$ , and denote the graph obtained by identifying  $x_1$  with  $y_1$  and  $x_2$  with  $y_2$  by  $\Gamma$ . If  $H$  is hypohamiltonian, then*

- (i)  $\Gamma$  is a 2-leaf-stable graph and
- (ii)  $\Gamma + x_3 y_3 = \Gamma'$  is a traceable platypus.
- (iii) If  $H$  is a platypus, then  $\Gamma' / x_3 y_3 = G_x H_y$  is a platypus.

*Proof.* In  $\Gamma$ , let  $z_1$  and  $z_2$  be the vertices obtained by identifying  $x_1$  with  $y_1$  and  $x_2$  with  $y_2$ , respectively. We denote by  $G_x$  and  $H_y$  the copy of  $G - x$  and  $H - y$  in  $\Gamma$ , respectively. (The same nomenclature holds in  $\Gamma'$  and  $G_x H_y$ .)

(i) We show in (ii) that  $\Gamma + x_3 y_3$  is non-hamiltonian, whence,  $\Gamma$  is non-hamiltonian and  $\text{ml}(\Gamma) \neq 1$ . We now prove that  $\Gamma - v$  is traceable, i.e.  $\text{ml}(\Gamma - v) \leq 2$ , for all  $v \in V(\Gamma)$ . W.l.o.g.  $v \in V(G_x)$ . First, we treat the case  $v \notin \{z_1, z_2\}$ . Let  $\mathbf{g}$  be a hamiltonian cycle in  $G$  avoiding the copy of  $v$  in  $G$ . Put  $\mathbf{p} = \mathbf{g} - x$ . W.l.o.g. one of the end-vertices of  $\mathbf{p}$  is  $z_1$ . Let  $\mathbf{h}$  be a hamiltonian cycle in  $H - y_2$ . Since we can treat  $\mathbf{p}$  and  $\mathbf{h} - y = \mathbf{q}$  as paths in  $G_x$  and  $H_y$ , respectively, and as  $\mathbf{q}$  has  $z_1$  as an end-vertex,  $\mathbf{p} \cup \mathbf{q}$  is a hamiltonian path in  $\Gamma - v$ .

Now assume that  $v \in \{z_1, z_2\}$ . W.l.o.g.  $v = z_1$ . Combining the path obtained by deleting from a hamiltonian cycle of  $G - x_1$  the vertex  $x$  and the path obtained by deleting from a hamiltonian cycle of  $H - y_1$  the vertex  $y$  yields the traceability of  $\Gamma - z_1$ .

Suppose there exists a vertex  $v$  such that  $\Gamma - v$  contains a hamiltonian cycle  $\mathbf{h}'$ . Obviously,  $v \notin \{z_1, z_2\}$ . W.l.o.g.  $v \in V(G_x)$ . Then  $(\mathbf{h}' \cap H_y) \cup z_1 y z_2$  corresponds to a hamiltonian cycle in  $H$ , which is absurd. We have shown that  $\Gamma - v$  is non-hamiltonian, i.e. that  $\text{ml}(\Gamma - v) \neq 1$  for every  $v \in V(\Gamma)$ . For  $\Gamma$  to be 2-leaf-stable, it remains to show that  $\Gamma$  is traceable—we do so in (ii).

(ii) Let  $\Gamma'$  contain a hamiltonian cycle  $\mathbf{h}$ . Thomassen [125] showed that  $\Gamma'/x_3 y_3 = G_x H_y$  is hypohamiltonian, ergo non-hamiltonian, so  $\mathbf{h}$  certainly does not contain  $x_3 y_3$ . Hence, the path  $\mathbf{p} = \mathbf{h} \cap G_x$  has end-vertices  $z_1$  and  $z_2$  and visits all vertices in  $G_x$ . Now consider  $\mathbf{p}$  to lie in  $G$ . But then  $\mathbf{p} \cup x_1 x x_2$  is a hamiltonian cycle in  $G$ , a contradiction. So  $\Gamma'$  is non-hamiltonian. That  $\Gamma' - v$  is traceable for every  $v \in V(\Gamma')$  follows directly from (i).

Combining the path obtained by deleting from a hamiltonian cycle of  $G - x_1$  the vertex  $x$  with the path obtained by deleting from a hamiltonian cycle in  $H - y$  the edge  $y_2 w$ , where  $w \in N(y_2) \setminus \{y\}$ , we obtain the traceability of  $\Gamma'$ . As  $x_3 y_3$  is not an edge of this path, we have also shown that  $\Gamma$  is traceable.

(iii) In  $\Gamma'/x_3 y_3 = G_x H_y$ , let  $z_3$  be the vertex obtained by identifying  $x_3$  with  $y_3$ . Assume that  $G_x H_y$  contains a hamiltonian cycle  $\mathbf{h}$ . Consider  $\mathbf{h} \cap G_x$ . If  $\mathbf{h} \cap G_x$  consists of two components, one of them is a single vertex. We denote

the other component, which must contain at least two vertices, by  $\mathfrak{p}$ . If  $\mathfrak{h} \cap G_x$  consists of one component, we call it  $\mathfrak{p}$ , as well. W.l.o.g.  $\mathfrak{p}$  has end-vertices  $z_2$  and  $z_3$ . If  $z_1 \in V(\mathfrak{p})$ , and we consider for a moment  $\mathfrak{p}$  to lie in  $G - x$ , then  $\mathfrak{p} \cup z_2xz_3$  corresponds to a hamiltonian cycle in  $G$  and we have a contradiction. If  $z_1 \notin V(\mathfrak{p})$ , consider the path  $\mathfrak{q} = \mathfrak{h} \cap H_y$  to lie in  $H$ . Now  $\mathfrak{q} \cup y_2yy_3$  is a hamiltonian cycle in  $H$ , once again a contradiction.

Consider  $v \in V(H) \setminus N[y]$ . Denote by  $\mathfrak{p}'$  a hamiltonian path in  $H - v$ . W.l.o.g.  $y_1y \in E(\mathfrak{p}')$ . Two cases arise: (a)  $y$  is an end-vertex of  $\mathfrak{p}'$ , or (b) it is not. We first treat case (a). Now clearly  $y_1$  cannot be an end-vertex of  $\mathfrak{p}'$ . Consider a hamiltonian cycle in  $G - x_2$  and delete from it  $x$ . We obtain a path  $\mathfrak{q}$  which visits all vertices in  $G$  excluding  $x$  and  $x_2$ , and has  $x_1$  and  $x_3$  as its end-vertices. Combining  $\mathfrak{q} - x_3$  with  $\mathfrak{p}' - y$ , we have shown that  $G_xH_y - v$  is traceable. Concerning case (b), w.l.o.g.  $y_3y \in E(\mathfrak{p}')$ . Construct  $\mathfrak{q}$  as in (a). Then  $\mathfrak{q} \cup (\mathfrak{p}' - y)$  corresponds to a hamiltonian path in  $G_xH_y - v$ .

Let  $v \in V(G) \setminus N[x]$ . Consider a hamiltonian cycle  $\mathfrak{g}$  in  $G - v$ . W.l.o.g.  $x_1xx_2 \subset \mathfrak{g}$ . Then the path  $\mathfrak{g} - x$  has end-vertices  $x_1$  and  $x_2$ . Let  $\mathfrak{q}''$  be a hamiltonian path in  $H - y_3$ . Then combining  $\mathfrak{q}'' - y$  (which consists of one or two components) with  $\mathfrak{g} - x$  yields a hamiltonian path (or hamiltonian cycle) in  $G_xH_y - v$ . (The treatment is very similar if  $y_2$  or  $y$  are an end-vertex of  $\mathfrak{q}''$ .)

Consider a hamiltonian cycle  $\mathfrak{g}'$  in  $G - x_1$ , and let  $\mathfrak{p}''$  be a hamiltonian path in  $H - y_1$ . Putting  $\mathfrak{g}' - x$  and  $\mathfrak{p}'' - y$  together gives a hamiltonian path (or hamiltonian cycle) in  $G_xH_y - z_1$ . (Note that although  $x_1x_2, x_2x_3, x_3x_1 \notin E(G)$ , some of the edges  $z_1z_2, z_2z_3, z_3z_1$  might be present in  $G_xH_y$ , but since we did not make use of such an edge, the argument for  $z_2$  and  $z_3$  is analogous.)  $\square$

By saying that “we add vertices  $x_1, x_2, \dots, x_n$  on an edge  $xy$ ” we mean that we replace the path isomorphic to  $K_2$  having vertices  $x, y$  by the path  $xx_1x_2\dots x_ny$  isomorphic to  $P_{n+2}$ .

**Proposition 5.14** [145] *Let  $G$  be a hypohamiltonian graph and  $e = v_1v_2 \in E(G)$ . By adding vertices  $v$  and  $v'$  on  $e$ , we obtain a platypus  $G'$ .*

*Proof.* W.l.o.g. assume  $v \in N(v_1)$  in  $G'$ . As  $G$  is non-hamiltonian,  $G'$  is non-hamiltonian, too. For every vertex  $u \in V(G) \setminus \{v, v', v_1, v_2\}$  we have that  $G - u$  contains a hamiltonian cycle  $\mathfrak{h}$ .  $\mathfrak{h}$  must contain some edge  $e'$  incident to  $v_1$ . If  $e' = e$ , we are done. If not, then  $(\mathfrak{h} - e') \cup v_1vv'$  is a hamiltonian path in  $G' - u$ .

Let  $\mathfrak{h}'$  be a hamiltonian cycle in  $G - v_1$  and  $w \in N(v_2)$  such that  $wv_2 \in E(\mathfrak{h}')$ . Then  $(\mathfrak{h}' - ww_2) \cup v_2v'v$  is a hamiltonian path in  $G' - v_1$ . In the same way it can be shown that  $G' - v_2$  is traceable. Consider  $w'w'' \in E(\mathfrak{h}')$ , where  $w' \in N(v_1)$ . Then  $(\mathfrak{h}' - w'w'') \cup w'v_1v$  is a hamiltonian path in  $G' - v'$ . Similarly,  $G' - v$  is traceable. Thus, the statement is shown.  $\square$

By Prop. 5.1 (vii), Prop. 5.14 holds as well if only a single vertex is added on the edge  $e$ .

A graph  $G$  is called *homogeneously traceable* if every vertex of the graph is an end-vertex of a hamiltonian path. It is easy to see that the following holds.

**Proposition 5.15** [145] *Every non-hamiltonian homogeneously traceable graph is a platypus.*

We now settle the open problem of Wiener [140] whether planar leaf-stable graphs without cubic vertices exist. Consider the dotted prism  $\dot{C}_k$  over  $C_k$  for  $k$  odd. In each graph, take the end-vertices of each 3-ear and join them by an edge. It is clear that these graphs are planar and have no cubic vertices. They are platypuses due to Prop. 5.1 (iii) and they are traceable due to Prop. 5.1 (vi). We have obtained the following.

**Theorem 5.16** [145] *There exist infinitely many planar 2-leaf-stable graphs which have no cubic vertices.*

Wiener [140] also poses a meta-question: are there leaf-stable or leaf-critical graphs *not* based on hypohamiltonian graphs, in the sense that their construction does not use hypohamiltonian graphs as building blocks? With Theorem 5.16, we have answered this question positively.

In his paper [140], Wiener goes on to ask for the smallest  $\ell$ -leaf-stable graphs, especially under the added condition of planarity. Consider  $\Delta$ , the graph from Fig. 42 (a), and replace each 3-ear with a 4-ear. We call this newly obtained graph  $\Delta'$ . For the proof of the following statement, which settles Wiener's question for  $\ell = 2$ , Van Cleemput and the author used a computer. (We exclude the graph  $K_2$  from the discussion.)

**Theorem 5.17** (Van Cleemput and Zamfirescu [145])  *$\Delta'$ , a planar graph, is the smallest 2-leaf-stable graph, both in terms of order and size.*

## 5.4 Connectivity 2

Consider  $G \in \mathcal{P}_2$ , and let  $X = \{x, y\}$  be a cut in  $G$ . We denote the pair  $(G, X)$  by  $G_X$ . By Theorem 5.3,  $G - X$  consists of exactly two connected components, say  $A$  and  $B$ . Put  $G'_X = G[V(A) \cup \{x, y\}]$  and  $G''_X = G[V(B) \cup \{x, y\}]$ . We call  $G'_X$  and  $G''_X$  *halves of  $G$  w.r.t.  $X$* .  $x, y$  will be called the *ends* of  $G'_X$  and  $G''_X$ . (In Thomassen's language [127],  $G'_X$  and  $G''_X$  are *2-fragments* of  $G$ , and  $x, y$  *vertices of attachment* of  $A$  and  $B$ .) When we simply speak of a “half” we are referring to a half of an arbitrary  $G \in \mathcal{P}_2$  w.r.t. to an arbitrary (but fixed) 2-cut in  $G$ . A half  $H$  is *traversable* if there exists a path between its ends which visits all vertices in  $H$ . Not both halves of  $G$  w.r.t.  $X$  can be traversable, so there are two cases:

- (i)  $G_X$  is *semi-traversable* if  $G'_X$  is non-traversable and  $G''_X$  is traversable.
- (ii)  $G_X$  is *non-traversable* if neither  $G'_X$  nor  $G''_X$  are traversable.

If, for all 2-cuts  $X$ ,  $G_X$  is semi-traversable (non-traversable), then we call  $G$  *semi-traversable* (*non-traversable*). If a platypus of connectivity 2 is neither semi-traversable nor non-traversable, we call it *mixed-traversable*.  $\Delta$  is an example of a semi-traversable platypus, and applying Theorem 5.13 (i) to two copies of the Petersen graph yields a mixed-traversable platypus. Every hypotraceable graph constructed using Thomassen's method introduced in [125] is a non-traversable



platypus—an example of such a graph is given in [125, Fig. 3] and reproduced in Fig. 18.

We now construct new platypuses from old ones by replacing halves. In order to do so, we need the following lemma.

**Lemma 5.18** [145] *For a half  $H$  with ends  $x$  and  $y$ , there exists a path which spans  $H - x$  and has  $y$  as an end-vertex.*

*Proof.* Since  $H$  is a half of a platypus (of connectivity 2)  $G$ ,  $G - x$  contains a hamiltonian path  $\mathbf{p}$ . As  $\{x, y\}$  is a cut in  $G$ ,  $\mathbf{p}$  must visit all vertices in  $H - \{x, y\}$  before reaching  $y$ , and in consequence  $|\{v \in V(H) : vy \in E(\mathbf{p})\}| = 1$ . Thus  $H \cap \mathbf{p}$  is the path we are looking for.  $\square$

**Theorem 5.19** [145] *Let  $G$  and  $H$  be platypuses such that there are 2-cuts  $X$  and  $Y$  in  $G$  and  $H$ , respectively. If*

- (i)  $G_X$  is non-traversable,  $H'_Y \notin \{P_3, K_3\}$ ,  $H'_Y$  contains a hamiltonian path with  $y \in Y$  as end-vertex, and  $H''_Y$  is non-traversable, or
- (ii)  $G_X$  is semi-traversable with  $G'_X$  containing a hamiltonian path with  $x \in X$  as end-vertex,  $G''_X \notin \{P_3, K_3\}$ ,  $G''_X$  is traversable, and  $H_Y$  is semi-traversable with  $H'_Y \notin \{P_3, K_3\}$  and  $H'_Y$  traversable,

*then by identifying (using a bijection) the ends of  $G'_X$  and  $H'_Y$  we obtain a platypus.*

*Proof.* We denote by  $\Gamma$  the resulting graph. Put  $X = \{x_1, x_2\}$  and  $Y = \{y_1, y_2\}$ . For the following, we denote by  $J_G$  and  $J_H$  the copy of  $G'_X$  and  $H'_Y$  in  $\Gamma$ , respectively, and put  $V(J_G) \cap V(J_H) = Z = \{z_1, z_2\}$ , where the identification of  $x_i$  and  $y_i$  yields  $z_i$ ,  $i \in \{1, 2\}$ . Assume that  $\Gamma$  contains a hamiltonian cycle  $\mathbf{h}$ . Then  $\mathbf{h} \cap J_G$  is a path in  $J_G$  between  $z_1$  and  $z_2$  visiting all vertices of  $J_G$ . This implies that  $G'_X$  is traversable, a contradiction. So  $\Gamma$  is non-hamiltonian.

For  $v \in Z$ , the traceability of  $\Gamma - v$  follows directly from Lemma 5.18.

(i) Let  $v \in V(G'_X) \setminus X$ . Consider the intersection of a hamiltonian path in  $G - v$  with  $G'_X$ , and denote it by  $\mathbf{p}$ . Clearly,  $\mathbf{p}$  has either one or two components. If  $\mathbf{p}$  has two components, both cannot consist of only one vertex simultaneously, since  $G'_X$  is non-traversable, so  $G'_X \notin \{P_3, K_3\}$ . There are two cases. (a) If both components contain more than one vertex, then  $G''_X$  must be traversable, a contradiction. (b) If one of the two components consists of one vertex, w.l.o.g.  $x_1$ , then we obtain the desired hamiltonian path in  $\Gamma - v$  by using the fact that  $H'_Y$  contains a hamiltonian path with  $y_2$  as end-vertex. Now consider  $\mathbf{p}$  to lie in  $J_G$  and let  $\mathbf{p}$  have one component. Since  $G''_X$  is non-traversable, it neither is  $P_3$  nor  $K_3$ , so  $z_1$  or  $z_2$  (possibly both) is (are) an end-vertex of  $\mathbf{p}$ —denote this fact by  $(\diamond)$ . Since  $\mathbf{p}$  visits all vertices in  $J_G$  with the exception of  $v$ , Lemma 5.18 yields a hamiltonian path in  $\Gamma - v$ .

Let  $v \in V(H'_Y) \setminus Y$ . Since  $H'_Y \notin \{P_3, K_3\}$  (this is needed in order to use  $(\diamond)$ ) and  $H''_Y$  is non-traversable, the arguments are the same as above.

(ii) Let  $v \in V(G'_X) \setminus X$ . Consider the intersection of a hamiltonian path in  $G - v$  with  $G'_X$ , and denote it by  $\mathbf{p}$ . Consider  $\mathbf{p}$  to lie in  $J_G$ . Let  $\mathbf{p}$  have two components. Then a hamiltonian path in  $\Gamma - v$  is obtained by using a path between  $z_1$  and  $z_2$  visiting all vertices in  $J_H$ . Let  $\mathbf{p}$  have one component. Since  $G''_X \notin \{P_3, K_3\}$ ,  $(\diamond)$  holds. Now Lemma 5.18 yields a hamiltonian path in  $\Gamma - v$ .

Let  $v \in V(H'_Y) \setminus Y$ . Consider the intersection of a hamiltonian path in  $H - v$  with  $H'_Y$ , and denote it by  $\mathbf{q}$ . If  $\mathbf{q}$  consists of two components, since  $H''_Y$  is non-traversable, there are two possibilities. (a) Each component is a single vertex. This is the case if and only if  $H'_Y \in \{P_3, K_3\}$ , but this was excluded in the theorem's statement. (b) One component is a single vertex, w.l.o.g.  $z_1$ , and one component contains more than one vertex. Since  $J_G$  contains a hamiltonian path with  $z_2$  as end-vertex, we obtain a hamiltonian path in  $\Gamma - v$ . Now let  $\mathbf{q}$  be connected.  $(\diamond)$  holds, so we can use Lemma 5.18.  $\square$

Thomassen asked in [128] whether hypohamiltonian graphs with minimum degree at least 4 exist. At the Cycles and Colourings conference in 2015, the author was asked by Tomáš Madaras the corresponding question for platypuses. After a solution to Madaras' problem was given by the author, Gunnar Brinkmann asked

the same question, but where 4 is replaced with 5. We can answer these questions positively.

**Theorem 5.20** [145]

- (i) *There exist infinitely many planar platypuses with minimum degree  $\ell$  for every  $\ell \in \{2, 3, 4, 5\}$ .*
- (ii) *For every  $d \geq 2$  there exists a platypus with minimum degree  $d$ .*

*Proof.* Consider the dotted prism  $\dot{C}_k$  over  $C_k$  for  $k$  odd. Replace in each such graph each 3-ear with a 4-ear. This yields platypuses due to Lemma 5.6. We denote the family we obtain by  $\mathcal{C}$ .

(i)  $\{\dot{C}_k\}_{k \text{ odd}}$  is an infinite family of planar platypuses with minimum degree 2. It was shown by Thomassen that there exist infinitely many planar hypohamiltonian graphs [127], and that planar hypohamiltonian graphs have minimum degree 3 [128], so there exist infinitely many planar platypuses with minimum degree 3. Now replace in each member of  $\mathcal{C}$  each 4-ear having end-vertices  $v, w$  with  $(G, v, w)$ , where  $G$  is the graph depicted in Fig. 43 (a). (Replacement as defined in Section 5.1.)

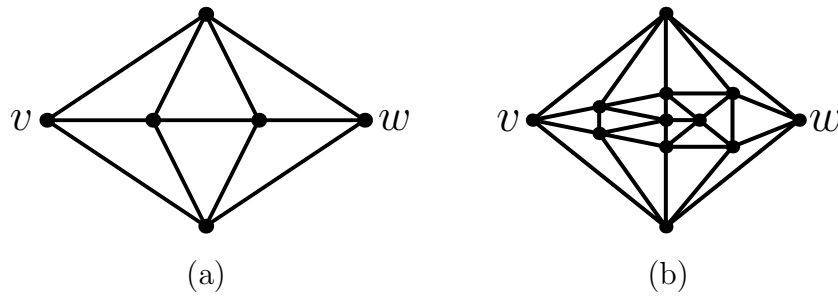


Fig. 43

The resulting graphs are platypuses due to Theorem 5.19 (ii). We have obtained an infinite family of planar platypuses with minimum degree 4. Finally, in each member of  $\mathcal{C}$ , replace each 4-ear having end-vertices  $v, w$  with  $(G', v, w)$ , where  $G'$  is defined in Fig. 43 (b). The resulting graphs are platypuses due to Theorem 5.19 (ii), and it is easily seen that each member has minimum degree 5.

(ii) Consider  $\Delta'$ , the platypus from Theorem 5.17. In  $\Delta'$ , replace each 4-ear with the complete graph  $(K_p, v, w)$ ,  $p \geq 4$ , where any choice of  $v, w \in V(K_p)$  will do. We obtain a platypus due to Theorem 5.19 (ii). It has minimum degree  $p - 1$ . For minimum degree 2, consider  $\Delta$ .  $\square$

Theorem 5.20 (i) is complete in the sense that for no other values of  $\ell$  the statement would be true. We shall see in Section 5.6 that, with more tools, Theorem 5.20 (ii) can be improved dramatically.

## 5.5 Connectivity 3

By Corollary 5.7, there are no 3-connected platypuses on fewer than 10 vertices. Of course, such a graph must have minimum degree 3. We have the following.

**Theorem 5.21** [145] *The Petersen graph is among 3-connected platypuses of both minimum order and minimum size.*

**Theorem 5.22** [145] *There exists a polyhedral platypus of order 40 and of order  $n$  for every  $n \geq 42$ .*

*Proof.* Since every hypohamiltonian graph is a platypus, the statement follows directly from Theorems 2.4 and 2.12.  $\square$

With Grinberg's Criterion and Prop. 5.2 we are in the position to present a result substantially stronger than Theorem 5.22.

**Theorem 5.23** [145] *There exists a polyhedral platypus of order  $n$  for every  $n \geq 25$ .*

*Proof.* Let  $\Lambda$  be the graph shown in Fig. 44. By Grinberg's Criterion,  $\Lambda$  is non-hamiltonian:  $(\dagger)$  becomes  $\pm 1 + 3(f_5 - f'_5) = 0$ , which is impossible. Now we prove that for every  $v \in V(\Lambda)$ , the graph  $\Lambda - v$  is indeed traceable. (Using “...” between two letters means a sequence of letters in their alphabetical order (possibly backwards).)

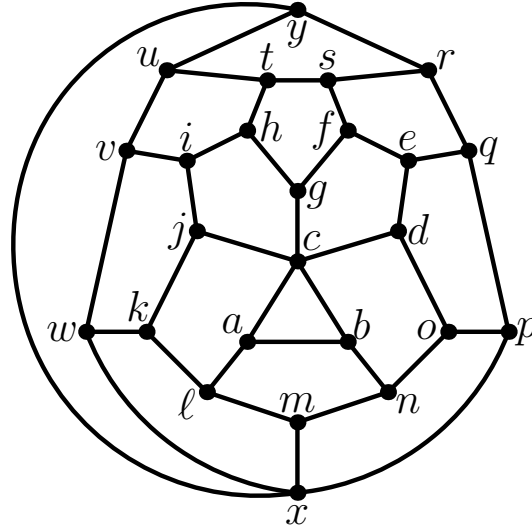


Fig. 44:  $\Lambda$ , a polyhedral platypus on 25 vertices.  
It is the smallest known polyhedral platypus.

**b:**  $ac...y$ . **c:**  $abn...do...y$ . **d:**  $abcgfeq...htsry...u$ . **e:**  $i...nbacdo...sfghtuyxwv$ .  
**f:**  $abnopqedcghtsryuvijkwxm\ell$ . **g:**  $a...fsth...ry...u$ . **m:**  $nbc\ell...do...y$ .  
**n:**  $yxm\ell abcdopqe...kw...r$ . **o:**  $w...sf...nbacdeqp\pi yr$ . **p:**  $abcdo...eq...y$ .  
**q:**  $a...px...ry$ . **r:**  $abnopqedcgfsth...mxwvuy$ . **s:**  $a...ry...t$ .  
**x:**  $abnm\ell kwvijkdopqefghtuys$ . **y:**  $a...x$ .

Put  $T^0(\Lambda) = \Lambda$ . The infinite family

$$\{T^k(\Lambda)\}_{k \geq 0} \cup \{R(T^k(\Lambda))\}_{k \geq 0} \cup \{S(T^k(\Lambda))\}_{k \geq 0}$$

yields the statement, where  $T$ ,  $R$ , and  $S$  are defined in Prop. 5.2. □

## 5.6 Higher Connectivity

One of the central problems concerning the theory of hypohamiltonian graphs is Thomassen's question from 1978 whether 4-connected such graphs exist [128]. (Such a graph cannot be planar due to a famous theorem of Tutte [133].) It is even unknown whether hypohamiltonian graphs with minimum degree at least 4 exist, a question also posed by Thomassen [128]. In Section 5.4, we settled the corresponding question for platypuses. In Section 3.1, we showed that there exist infinitely many *almost* hypohamiltonian graphs which are 4-connected. (But whether 5-connected such graphs exist is unknown, see Problem 8 in Chapter 6.) If we relax Thomassen's question concerning 4-connected hypohamiltonian graphs and ask for 4-connected platypuses, we realise that in fact much more can be shown.

**Theorem 5.24** [145] *There exists a  $k$ -connected platypus for every  $k \geq 2$ , and a  $k'$ -regular  $k'$ -connected platypus for every  $k' \geq 3$ .*

*Proof.* In Section 5.4 we discussed platypuses of connectivity 2. For the remaining cases, our main tool will be a method of Meredith [100]. We briefly repeat his construction. Label Petersen's graph  $P$  as in Fig. 45 and let  $H_k$  be the multigraph obtained from it as follows; replace each edge  $A_iB_i$  with  $b$  edges,  $1 \leq i \leq 5$ , and each of the other edges with  $a$  edges, where if  $k = 3\ell + \alpha$ ,  $\alpha \in \{-1, 0, 1\}$ , then  $a = \ell$  and  $b = \ell + \alpha$ .

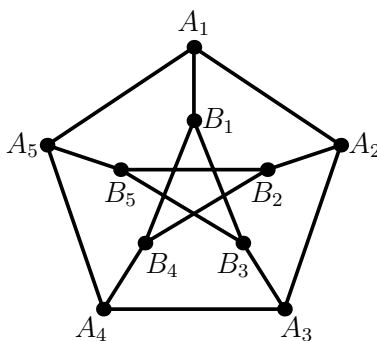


Fig. 45

$G_k$  is the (necessarily unique) graph obtained by expanding in  $H_k$  each vertex to a  $K_{k,k-1}$ . (For details, please see [100].) Meredith shows that for  $k \geq 3$ ,  $G_k$  is  $k$ -regular,  $k$ -connected, and non-hamiltonian.

It remains to prove that every vertex-deleted subgraph of  $G_k$  is traceable. Consider  $v \in V(P)$ . There exists a hamiltonian cycle in  $P - v$ . This yields a cycle  $\mathbf{c}$  of length  $n - (2k - 1)$  in  $G_k$ , where  $n$  is the order of  $G_k$ , since exactly one  $K_{k,k-1}$  is avoided by  $\mathbf{c}$ . (The one corresponding to the vertex  $v$ .) We denote the vertices of this avoided complete bipartite graph by  $a_1, \dots, a_k, b_1, \dots, b_{k-1}$ , where  $a_i a_j \notin E(G_k)$  and  $b_i b_j \notin E(G_k)$  for all  $i, j$ . In  $G_k$ , let  $x \in N(a_1) \setminus \{b_i\}_{i=1}^{k-1}$ . Consider  $xy \in E(\mathbf{c})$  and put  $\mathbf{c}' = \mathbf{c} - xy$ . Now  $\mathbf{c}'' = \mathbf{c}' \cup xa_1b_1a_2b_2\dots a_{k-1}b_{k-1}$  is a hamiltonian path in  $G_k - a_k$  with end-vertices  $y$  and  $b_{k-1}$ , and  $(\mathbf{c}'' \cup b_{k-1}a_k) - y$  is a hamiltonian path in  $G_k - y$ . All other vertices in  $G_k$  behave similarly.  $\square$

By a theorem of Petersen [106], every cubic bridgeless graph contains a perfect matching. There exist infinitely many cubic hypohamiltonian graphs; take for instance the so-called generalised Petersen graphs  $G(n, k)$ —which we have already seen in Section 2.2—originally defined by Coxeter [30] but baptised by Watkins [139]. Now  $G(n, 2)$  with  $n = 5$  (modulo 6) is hypohamiltonian. (Robertson showed their non-hamiltonicity [110].) In this situation, the approach above can be adapted to prove that for every  $k \geq 3$  there exist infinitely many  $k$ -connected platypuses.





# Chapter 6

## Outlook

Many problems of varying difficulty concerning hypohamiltonian and almost hypohamiltonian graphs remain unanswered. We here provide a (subjective) selection of questions which we consider to be especially interesting.

However, let us first provide a table with the best upper bounds we have for the orders of the smallest hypohamiltonian and almost hypohamiltonian graphs under certain additional criteria. The first and third value of the first row stem from Petersen's graph, see Fig. 1 or 45, the second value is due to the graph constructed by Jooyandeh, McKay, Östergård, Pettersson, and the author discussed in Theorem 2.8 (for an example of such a graph, see Fig. 8), and the graph discovered by Araya and Wiener [7] shown in Fig. 15 gives the fourth value. In the second row, the first and second value are due to the author (see Fig. 32 and Fig. 29, respectively) and consequences of Lemmas 3.9 and 3.3, while the third and fourth are due to McKay, see Theorem 3.13. We note here that no dedicated effort has been undertaken to improve the third value. Only the values provided by Petersen's graph are known to be optimal and therefore underlined in the table.

		planar	cubic	planar and cubic
hypohamiltonian	<u>10</u>	40	<u>10</u>	70
almost hypohamiltonian	17	39	68	68

Similar overviews have been provided if we additionally restrict the crossing number of the graph, see Section 2.2, or its girth, see Section 2.4.

We now discuss in more detail open questions concerning these values and other unsolved problems under the following three headings: hypohamiltonian graphs (6.1), almost hypohamiltonian graphs (6.2), and platypuses (6.3).

## 6.1 Hypohamiltonian Graphs

1. Thomassen's question from 1978 whether 4-connected hypohamiltonian graphs exist remains open [128]. (Even his weaker problem, whether hypohamiltonian graphs of minimum degree (at least) 4 exist, remains unanswered [128].) This is one of the central open problems in the study of hypohamiltonicity. We have seen that for almost hypohamiltonian graphs (Chapter 3) as well as platypuses (Chapter 5) these questions admit positive answers. In addition to Thomassen's questions, we ask here: do 4-connected hypotraceable graphs exist? (Horton [72] showed that 3-connected hypotraceable graphs exist.) Note that, by Lemma 3.1, a positive answer to this question would imply a positive answer to Problem 8, but not necessarily vice-versa.

Related to this, as mentioned in Section 4.2, nothing is known concerning  $C_4^1$ , i.e. we do not know whether there exists a 4-connected graph in which every vertex is missed by a longest cycle. This question was first raised by T. Zamfirescu [150, p. 232] in 1976.

2. Related to girth restrictions, we mention two open questions. Máčajová and Škoviera [95] ask whether there exist infinitely many hypohamiltonian cubic graphs with both cyclic connectivity and girth 7. (Máčajová and Škoviera use Coxeter's graph, see Fig. 24, to construct an infinite family of cubic hypohamiltonian graphs of girth 7 and cyclic connectivity 6. No hypohamiltonian graph of girth greater than 7 is known.)

McKay asks in his recent paper [96] whether infinitely many planar cubic hypohamiltonian graphs of girth 5 exist. In [96], he provides three such graphs,

each of order 76, and each having only identity as automorphism. One of them is shown in Fig. 26. Subsequently, Goedgebeur showed that there exists such a graph of order 78 which has non-trivial automorphism group, see Fig. 27.

**3.** Häggkvist [61] conjectures that every cubic hypohamiltonian graph has six perfect matchings which together cover every edge exactly twice. (Note that in [61], there is a minor yet confusing error in the definition of hypohamiltonicity: it lacks the demand that a hypohamiltonian graph must be non-hamiltonian! What Häggkvist defines coincides with the disjoint union of the family of all hypohamiltonian graphs with the family of all 1-hamiltonian graphs.)

For this paragraph, we follow recent work of Steffen [122]. He writes that Jaeger and Swart [75] conjectured that (i) the girth and (ii) the cyclic connectivity of a snark is at most 6. (i) was disproved by Kochol [88], while (ii) is still open. Steffen believes that both statements of Jaeger and Swart are true for hypohamiltonian snarks, and conjectures that for a hypohamiltonian snark  $G$ ,  $\mu_3(G) = 3$  holds, where  $\mu_i$  is defined as follows. Let  $G$  be a cubic graph,  $k \geq 1$ , and  $S_k$  be a list of  $k$  1-factors of  $G$ . By a list we mean a collection with possible repetition. For  $i \in \{0, \dots, k\}$  let  $E_i(S_k)$  be the set of edges that are in precisely  $i$  elements of  $S_k$ . We define

$$\mu_k(G) = \min\{|E_0(S_k)| : S_k \text{ is a list of } k \text{ 1-factors of } G\}.$$

**4.** Holton and Plummer [70] define a graph  $G$  of order at least  $m+n$  to be  $C(m, n)$  if for any set  $S = \{u_1, \dots, u_m, v_1, \dots, v_n\} \subset V(G)$ ,  $G$  has a cycle which visits  $u_1, \dots, u_m$  but avoids  $v_1, \dots, v_n$ . They conjecture that if the implication  $C(k, 1) \rightarrow C(k+1, 0)$  fails, the only exceptions are hypohamiltonian.

**5.** We explicitly formulate the obvious question: what is the order of the smallest planar hypohamiltonian graph? The upper bound is 40, see Theorem 2.4, and the lower bound is 18, see [5]. So despite significant progress on finding smaller and smaller planar hypohamiltonian graphs, there is still a wide gap. One explanation for this gap is the fact that no extensive computer search has been carried out to increase the lower bound. This is currently being done by

the author in collaboration with Jan Goedgebeur. Looking at the automorphism group, it would be somewhat surprising if no extremal graphs would have non-trivial automorphisms—the graphs of order 40 presented in Chapter 2 have no nontrivial automorphisms. An exhaustive study of graphs with prescribed automorphisms might lead to the discovery of new, smaller graphs. Whether or not a planar hypohamiltonian graph of order 41 exists is also unknown.

Concerning the cubic case, we ask the same natural question: what is the order of the smallest planar cubic hypohamiltonian graph? Currently the best lower bound is 44, see [7], and the upper bound is 70, given by Araya and Wiener in 2011, see [141]. In Theorem 2.17, we showed that there exist planar cubic hypohamiltonian graphs on 70 vertices and on  $n$  vertices for every even  $n \geq 74$ . Is there a planar cubic hypohamiltonian graph of order 72?

Finally, the smallest known hypotraceable has 34 vertices and was constructed by Thomassen [125] in 1974. It is depicted in Fig. 18. Since then, no smaller hypotraceable graph has been published. Are there any such graphs of order less than 34? (Also, no non-trivial lower bound seems to have been published.)

## 6.2 Almost Hypohamiltonian Graphs

**6.** What is the smallest order of a (planar) almost hypohamiltonian graph? The smallest known almost hypohamiltonian (planar almost hypohamiltonian) graph has 17 (39) vertices, see Fig. 32 (Fig. 29). We would also like to fill the gaps and know whether almost hypohamiltonian graphs of order  $n \in \{18, 19, 21, 24\}$  exist.

**7.** Is there an almost hypohamiltonian graph  $G$  with (i) cubic exceptional vertex  $w$  and (ii) all vertices in  $V(G) \setminus N[w]$  of degree at least 4? (The degrees of the vertices in  $N(w)$  do not matter.) Solving this would answer, by using Theorem 3.11, Thomassen’s question whether hypohamiltonian graphs with minimum degree (at least) 4 exist [128].

**8.** Are there 4-connected almost hypohamiltonian graphs on fewer than 41 vertices? It would be very interesting to construct 4-connected almost hypohamiltonian graphs *without* using hypotraceable graphs as building blocks. Going beyond connectivity 4, we ask: do 5-connected almost hypohamiltonian graphs exist?

**9.** Thomassen [128] showed that every planar hypohamiltonian graph contains a cubic vertex. Taking a 4-cycle  $v_1v_2v_3v_4$ , adding the vertex  $v_5$ , and the edges  $v_1v_3$ ,  $v_1v_5$  and  $v_3v_5$ , we obtain a planar 2-hypohamiltonian graph with no cubic vertex. Does Thomassen's result hold for planar almost hypohamiltonian graphs, as well?

**10.** The smallest known cubic almost hypohamiltonian graph was found by McKay (see Theorem 3.13) and has order 68, and is in fact planar. No explicit search has been undertaken to find a small cubic almost hypohamiltonian graph, so it seems likely that the upper bound on the order of the smallest such graph can be lowered dramatically.

## 6.3 Platypuses

**11.** Prop. 5.1 (vii) states that if a platypus  $G$  contains a 4-ear  $H$ , then  $H$  can be replaced with a 3-ear, and the resulting graph is a platypus, as well. (And Lemma 5.20 discusses this more generally.) A priori there does not seem to be an argument that vice-versa this must hold—but all examples encountered above allowed this; is there a platypus which contains a 3-ear  $D$ , which has the property that replacing  $D$  with a 4-ear, the resulting graph is not a platypus?

**12.** In light of Theorem 5.9, we ask for an *upper* bound on the size of a platypus. We can show that for every  $n \geq 9$  there exists a platypus of order  $n$  and size  $\binom{n-6}{2} + 12$ . Following [19], the size of a 2-connected non-hamiltonian graph on at least ten vertices is at most  $\binom{n-2}{2} + 4$ .

**13.** In Section 5.3, we proposed the study of “full” platypuses, i.e. platypuses  $G$  for which for every pair of non-adjacent vertices  $v, w \in V(G)$ , the graph  $G + vw$  is *not* a platypus. An example of such a well-fed platypus is the graph we obtain if we add in  $\Delta$  to each 3-ear an edge between its end-vertices. That this is indeed a full platypus follows from Corollary 5.7.

**14.** Although  $\Lambda$ , the graph from Fig. 44, is the smallest known polyhedral platypus, it does not address the question of the existence of a small polyhedral  $\ell$ -leaf-critical or  $\ell$ -leaf-stable graph, since  $\text{ml}(\Lambda - a) \neq 1$  (as for a hamiltonian cycle  $\mathfrak{h}$  in  $\Lambda - a$  we would have  $bc \in E(\mathfrak{h})$ , so then  $(\mathfrak{h} - bc) \cup bac$  would be a hamiltonian cycle in  $\Lambda$ , a contradiction), so  $\text{ml}(\Lambda - a) = 2$ , but  $\text{ml}(\Lambda - y) = 1$ . Since among leaf-stable and leaf-critical graphs, only the families of 2-leaf-stable and  $\{2, 3\}$ -leaf-critical graphs are contained in the class of platypuses, we focus thereon. The smallest known polyhedral 2-leaf-critical (i.e. hypohamiltonian) graphs have 40 vertices—see Theorem 2.4. Fig. 8 depicts such a graph. Currently, the smallest known polyhedral 3-leaf-critical (i.e. hypotraceable) graph has order 190, see Section 2.3. The smallest known polyhedral 2-leaf-stable graph is due to Wiener and has order 152, see [140], and is also based on the aforementioned smallest known planar hypohamiltonian graph.

**15.** Let  $p_k$  ( $\bar{p}_k$ ) be the order of the smallest platypus (smallest planar platypus) of connectivity  $k$ . In Chapter 5, we have shown that

$$p_2 = \bar{p}_2 = 9, \quad p_3 = 10, \quad \bar{p}_3 \leq 25, \quad \text{and} \quad p_k \leq 20k - 10.$$

We ask here for improving these bounds or showing their optimality, especially in the cases of  $\bar{p}_3$  and  $p_4$ .

# Chapter 7

## Appendix

4-Face Count	Face Sequence	Degree Sequence	Count
5	$5 \times 4, 22 \times 5$	$30 \times 3, 10 \times 4$	4
		$31 \times 3, 8 \times 4, 1 \times 5$	10
		$32 \times 3, 6 \times 4, 2 \times 5$	9
		$33 \times 3, 4 \times 4, 3 \times 5$	2
		All	25

Table 1: Properties of  $\mathcal{H}_5^4(40)$ .

4-Face Count	Face Sequence	Degree Sequence	Count
1	$1 \times 4, 26 \times 5$	$34 \times 3, 8 \times 4$	5
		$35 \times 3, 6 \times 4, 1 \times 5$	2
7	$7 \times 4, 22 \times 5$	$30 \times 3, 12 \times 4$	4
		$31 \times 3, 10 \times 4, 1 \times 5$	28
		$32 \times 3, 8 \times 4, 2 \times 5$	57
		$33 \times 3, 6 \times 4, 3 \times 5$	49
		$33 \times 3, 7 \times 4, 1 \times 5, \times 6$	11
		$34 \times 3, 4 \times 4, 4 \times 5$	10
		$34 \times 3, 5 \times 4, 2 \times 5, 1 \times 6$	5
		$34 \times 3, 6 \times 4, 2 \times 6$	6
		$35 \times 3, 4 \times 4, 1 \times 5, 2 \times 6$	2
All	All	All	179

Table 2: Properties of  $\mathcal{H}_1^4(42)$  and  $\mathcal{H}_7^4(42)$ .



4-Face Count	Face Sequence	Degree Sequence	Count
4	$4 \times 4, 23 \times 5, 1 \times 7$	$36 \times 3, 6 \times 4, 1 \times 6$	1
		$37 \times 3, 4 \times 4, 1 \times 5, 1 \times 6$	1
5	$5 \times 4, 22 \times 5, 1 \times 8$	$34 \times 3, 9 \times 4$	8
		$35 \times 3, 7 \times 4, 1 \times 5$	20
		$36 \times 3, 5 \times 4, 2 \times 5$	19
		$37 \times 3, 3 \times 4, 3 \times 5$	1
		$37 \times 3, 4 \times 4, 1 \times 5, 1 \times 6$	1
	$5 \times 4, 24 \times 5$	$32 \times 3, 11 \times 4$	52
		$33 \times 3, 9 \times 4, 1 \times 5$	148
		$34 \times 3, 7 \times 4, 2 \times 5$	175
		$34 \times 3, 8 \times 4, 1 \times 6$	2
		$35 \times 3, 5 \times 4, 3 \times 5$	56
		$35 \times 3, 6 \times 4, 1 \times 5, 1 \times 6$	6
		$36 \times 3, 3 \times 4, 4 \times 5$	1
		$36 \times 3, 4 \times 4, 2 \times 5, 1 \times 6$	4
		$37 \times 3, 2 \times 4, 3 \times 5, 1 \times 6$	1
		$37 \times 3, 3 \times 4, 1 \times 5, 2 \times 6$	1
All	All	All	497

Table 3: Properties of  $\mathcal{H}_4^4(43)$  and  $\mathcal{H}_5^4(43)$ .

$j, k$	1,1	1,2	1,3	2,1	2,2	2,3
$C_k^j$	<u>6</u>	<u>10</u>	<u>10</u>	<u>9</u>	75	75
$P_k^j$	<u>12</u>	26	36	93	270	270
$\overline{C}_k^j$	<u>6</u>	<u>15</u>	40	<u>9</u>	135	2625
$\overline{P}_k^j$	17	32	156	308	914	10350

Table 4: Upper bounds for  $C_k^j$ ,  $P_k^j$ ,  $\overline{C}_k^j$ ,  $\overline{P}_k^j$ , where  $k \in \{1, 2, 3\}$  and  $j \in \{1, 2\}$ , as defined in Chapter 4. Underlined entries are optimal.

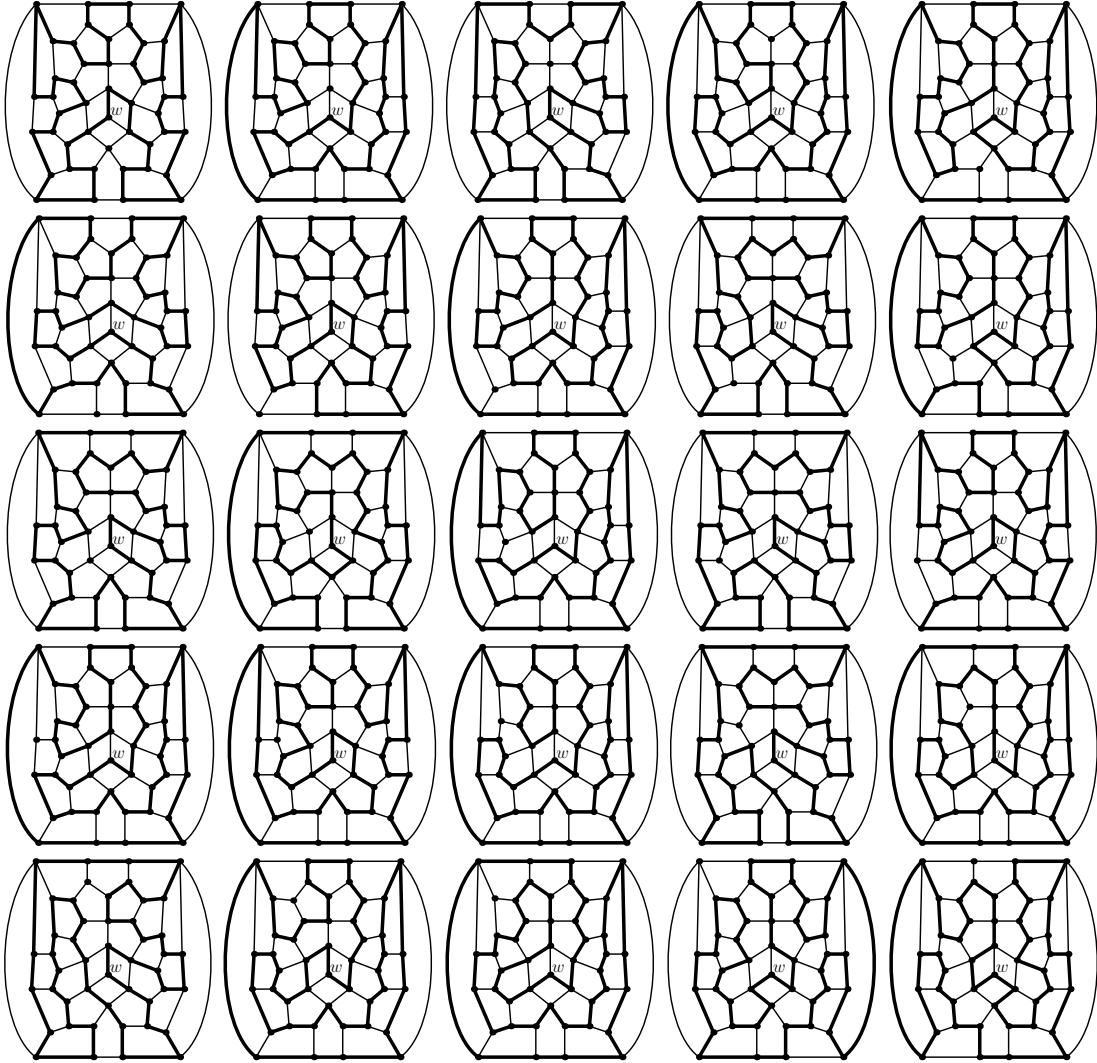


Fig. A1: A planar almost hypohamiltonian graph of order 47.

Its exceptional vertex  $w$  is cubic.

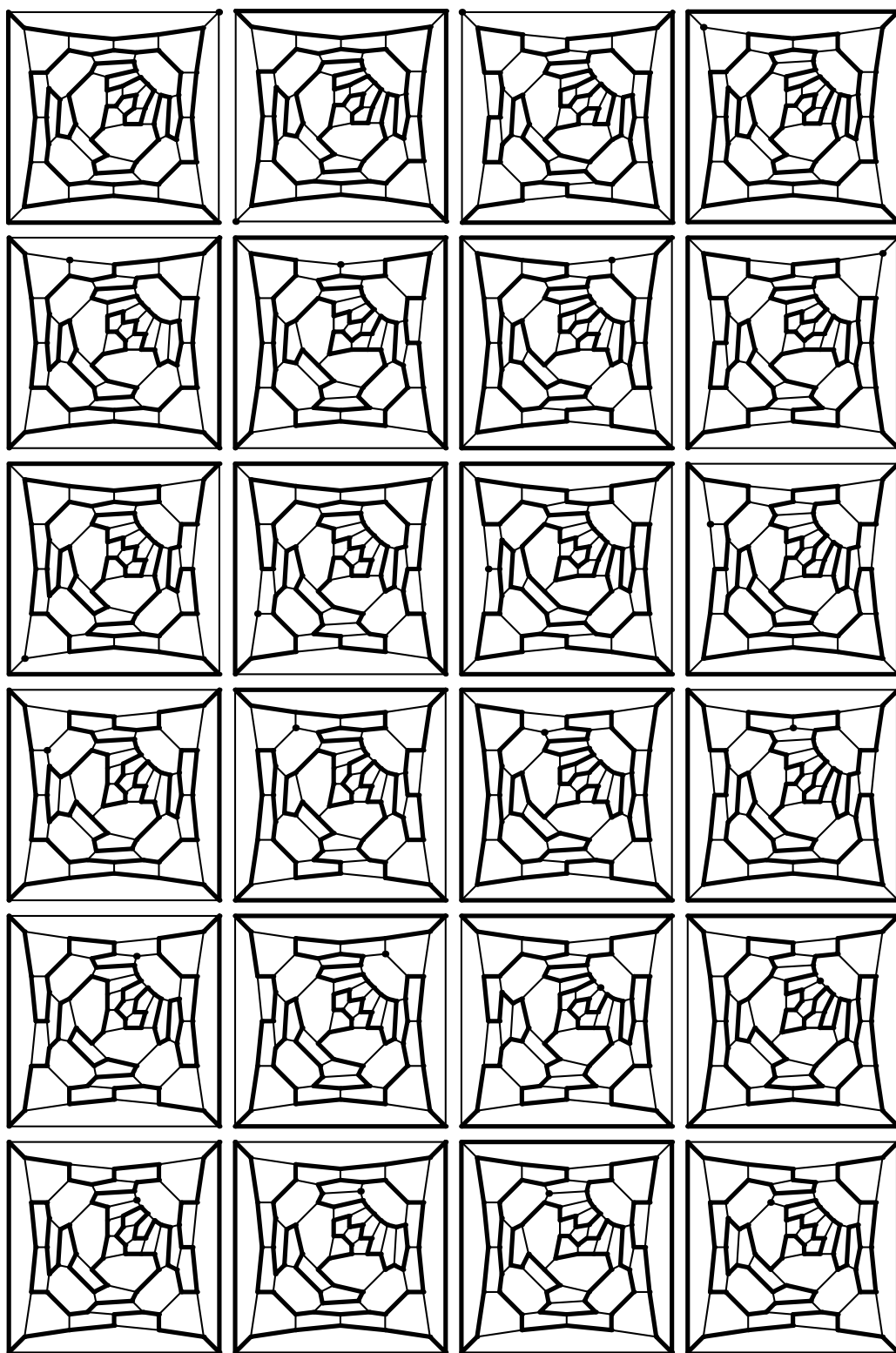


Fig. A2. A planar cubic hypohamiltonian graph of order 76 (part 1 of 2).

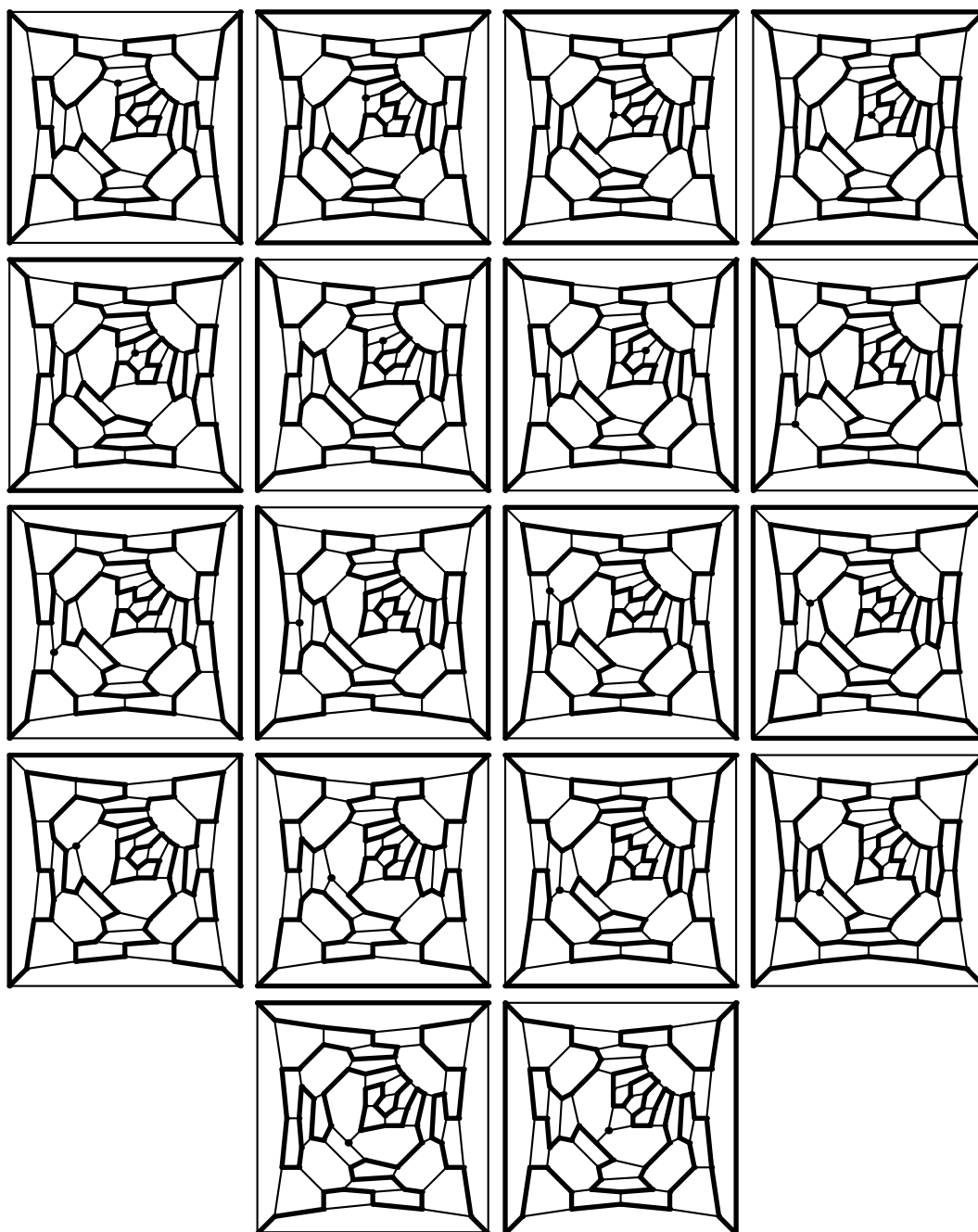


Fig. A3. A planar cubic hypohamiltonian graph of order 76 (part 2 of 2).

# Chapter 8

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# Nederlandse Samenvatting

Deze thesis is op de volgende manier gestructureerd. In Hoofdstuk 1 geven wij een kort historisch overzicht en definiëren fundamentele concepten. Het tweede hoofdstuk bevat een duidelijk verhaal over de vooruitgang die werd gemaakt om de kleinste planaire hypohamiltoniaanse graaf te kunnen vinden, met alle noodzakelijke theoretische werktuigen—vooral het Criterium van Grinberg. Gevolgen van deze vooruitgang zijn verdeeld over alle secties en vormen het hoofdmotief van deze dissertatie. In Hoofdstuk 2 bediscussiëren wij ook de taille en het kruisingsgetal van hypohamiltoniaanse grafen. Het derde hoofdstuk is een grondige bespreking van de onlangs geïntroduceerde bijna hypohamiltoniaanse grafen en hun verbinding met hypohamiltoniaanse grafen. Alweer speelt het planaire geval een bijzondere rol. Op het einde van het hoofdstuk bestuderen wij bijna hypotraceerbare grafen en het probleem van Gallai over langste paden. Het laatstgenoemde leidt naar Hoofdstuk 4, waarin het verwantschap tussen hypohamiltoniaanse grafen en diverse problemen over langste paden en langste cykels gepresenteerd wordt. Hoofdstuk 5 introduceert en bestudeert niet-hamiltoniaanse grafen met de eigenschap dat het verwijderen van elke top een traceerbare graaf geeft—een klasse van grafen die hypohamiltoniaanse en hypotraceerbare grafen omvat. Wij eindigen met een vooruitzicht in Hoofdstuk 6, waar wij een selectie van open problemen samen met commentaren en deeloplossingen voorstellen.



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