On the geometry of the laws of physics

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Abstract

In 2010, a conference was organized at the Perimeter Institute where mathematicians, philosophers and physicist debated on *Laws of Nature: Their Nature and Knowability*. Fundamental questions as *What is a law of nature? How many laws are there? What governs the laws of nature?* were discussed. We elaborate further on this topic, but narrow the discussion to *laws of physics* which express relations between physical quantities. We study the *geometry* of the relations between the physical quantities that result in these laws of physics. The mathematical structure S classifying the physical quantities is presently unknown. We prove that classes of physical quantities can be represented by integer lattice points and that ternary laws of physics are geometrically represented by parallelograms in the integer lattice $\mathbb{Z}^7$. The classifier that reveals the unknown mathematical structure S is the *perimeter of the parallelogram*. The distribution of perimeters displays frequencies with a value of *one* that indicate the existence of *unique representations* of laws of physics. The most famous law of physics $E = mc^2$ is an element of the set of laws of physics that are *unique*. The perimeter of a parallelogram is *invariant* under a signed permutation of the coordinates of the lattice points of the parallelogram. We prove that the isoperimeter property generates an equivalence relation in each hypercubic shell that has a Chebyshev norm equal to s. We demonstrate that each equivalence class(orbit) can be associated to a monomial. Each class(orbit) contains a finite number of physical quantities represented by vertices forming a centrally symmetric 7-polytope. The physical quantities of a class(orbit) are *mathematically* equivalent. We calculate the cardinality of each hypercubic shell for $s \leq 10$. The appendices contain a preliminary classification of common physical quantities based on the hypercubic shells of the mathematical structure $S = \bigcup_{s=0}^{\infty} hcs_7^s$, that is the *infinite* union of hypercubic shells of the integer lattice $\mathbb{Z}^7$.

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1. Introduction

Physicists, mathematicians and philosophers [17] search to understand why the laws of nature are the way we observe them. Wigner [23] stated:

> The regularities in the phenomena which physical science endeavors to uncover are called the laws of nature. ... We have ceased to expect from physics an explanation of all events, even of the gross structure of the universe, and we aim only at the discovery of the laws of nature, that is the regularities, of the events. The preceding section gives reason for the hope that the regularities form a sharply defined set, and are clearly separable from what we call initial conditions, in which there is a strong element of randomness. However, we are far from having found that set. In fact, if it is true that there are precise regularities, we have reason to believe that we know only an infinitesimal fraction of these.

Tegmark [20, 21] proposes the mathematical universe hypothesis (MUH) as framework for the laws of the universe. Lange [12] formulates the question *Must the fundamental laws of physics be complete?*. Rickles [18] elaborates also on certain properties of the mathematical structure S that governs the laws of physics. We elaborate about research on the mathematical structure S containing the physical quantities and on the relations between these physical quantities that are expressed as laws of physics. The research question which has already been posited by Wigner and Feynman becomes *What are the laws governing the laws of physics?*. We follow a bottom-up approach starting from the structure's building blocks, that are the physical quantities. The intrinsic properties of the mathematical structure S are the relations between the physical quantities. Each physical quantity is represented by a symbol or label. Physical quantities are found in the form of scalars, vectors, matrices and/or tensors. All the physical quantities are ultimately *measured* through their respective components and thus we restrict our analysis to the components of physical quantities. We use as mathematical framework a 7-dimensional integer lattice $\mathbb{Z}^7$. The basis of the integer lattice represents the 7 base units of the SI. On the contrary of dimensional exploration, see Roche [19, Chap.11], we strongly rely on geometric properties related to regular systems of points, see Hilbert [10] to study the geometric properties of the components of physical quantities. We know that Maxwell [15] in 1874 addressed partly the research question in his presentation "On the mathematical classification of physical quantities" to the London Mathematical Society. Maxwell stated:

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> ... the physical quantity called Energy or Work can be conceived as the product of two factors in many different ways. The dimensions of this quantity are $\frac{ML^2}{T^2}$,where L, M and T represent the concrete units of length, time, and mass. If we divide the energy into two factors, one of which contains L^2, both factors will be scalars. If, on the other hand, both factors contain L, they will be both vectors. The energy itself is always a scalar quantity. ...Thus, instead of dividing kinetic energy into the factors "mass" and "square of velocity", the latter of which has no meaning, we may divide it into "momentum" and "velocity",...

We study the factorization of energy in detail and discover a discrete value distribution of the geometric representation of "energy" that by inference results in the classification of the physical quantities. We demonstrate that the intuition of Maxwell about the physical meaning of $E = \boldsymbol{v} \cdot \boldsymbol{p}$ and $E = \frac{1}{2}mv^2$ can be related to the *principle of minimum perimeter of non-degenerated parallelograms* of a general ternary law of physics of the form $[z] = [\kappa][x][y]$.

1.1. Outline of the paper

Section 1 comprises the definitions and preliminaries that are needed to allow a mathematical elaboration of the geometry of the laws of physics. In section 2, we discuss the representation of classes of physical quantities as integer lattice points of $\mathbb{Z}^7$. We demonstrate in section 3 that ternary operations of the type $[z] = [\kappa][x][y]$, where $[\kappa], [x], [y], [z]$ represent classes of physical quantities, have a geometric representation in the integer lattice $\mathbb{Z}^7$. In section 4, we discuss the cardinality of the isoperimeter distribution. In this study we select for $[z]$ the class(orbit) of physical quantities representing *energy*. We propose in section 5 that the classification of classes of physical quantities is based on an equivalence relation applied to a hypercubic shell. Section 6 contains the future work and conclusion of the present research. Section 7 contains the appendices.

1.2. Preliminaries

In the Convocation of the General Conference on Weights and Measures - 24th meeting (17-21 October 2011) it is proposed that new definitions for the SI units be adopted, see [2]. In this article we follow these recommendations. A component of physical quantity is a quantity that is used in the description of physical processes. The universal set of components of physical quantities is called $\mathbb{U}_p$. This set can be partitioned in equivalence classes with notation $[a]$ where a is the representative of the equivalence class(orbit). In the class(orbit) energy $[E]$ we find physical quantities like potential energy, kinetic energy, work, heat, internal energy, ... which are all represented by $[E]$. A set of base quantities is a finite number of classes of physical quantities, which by *convention* are regarded as *dimensionally independent* in a system of physical quantities and equations defining the relationships between them. The "International System of Units (SI)" base quantities are length, mass, time, electric current, thermodynamic temperature, amount of substance and luminous intensity. The set of classes of base physical quantities is called $\mathcal{B} \doteq \{[l],[m],[t],[i],[T],[n],[L]\}$. The base units are the set $\mathcal{U} \doteq \{u_i \mid u_1 = \mathrm{m},\ u_2 = \mathrm{kg},\ u_3 = \mathrm{s},\ u_4 = \mathrm{A},\ u_5 = \mathrm{K},\ u_6 = \mathrm{mol},\ u_7 = \mathrm{cd}\}$. The dimensional product is the expression of a class(orbit) of a physical quantity as a product of powers of base quantities. Each class(orbit) of a physical quantity has parameters X^i, called dimensional exponents. We can write $[a]$ in terms of the SI base units with $X^i \in \mathbb{Z}$ and $u_i \in \mathcal{U}$,

$$[a] \doteq \{a\} \cdot \prod_{i=1}^{7} u_i^{X^i} . \tag{1.1}$$

It is known that some physical quantities (rms of a quantity, noise spectral density, specific detectivity, thermal inertia, thermal effusivity, ...) are defined as the square root of some product or fraction of other physical quantities. These physical quantities will have fractional exponents, where $X^i \in \mathbb{Q}$ and so will not comply with the above definition. Each of these physical quantities can be, by a proper exponentiation, transformed to a physical quantity having integer exponents.

2. Representation of a class(orbit) of physical quantities

The representation of a class(orbit) of physical quantities in the integer lattice $\mathbb{Z}^7$ requires the following definitions. The representation of a class(orbit) of physical quantities $[a]$ has the notation $\breve{a}$ which clearly indicates the distinction with physical quantities represented by scalars, vectors, matrices and/or tensors. We will see further that there is a mathematical justification for this notation. The representation of the class(orbit) energy $[E]$ is $\breve{E}$. The set of integer septuples $\mathbb{Z}^7 \doteq \{(X^1,\ldots,X^7) \mid X^i \in \mathbb{Z}\}$ is also called the 7-dimensional integer lattice.

Definition 1. The function 'dex' is defined from $\mathbb{U}_p$ into $\mathbb{Z}^7$ and formally as dex : $\mathbb{U}_{\mathrm{p}} \to \mathbb{Z}^7 \mid \mathrm{dex}([a]) \doteq \breve{a} = (A^1,\ldots,A^7)$ where $A^i \in \mathbb{Z}$.

The A^is are the contravariant components of the lattice point $\breve{a}$. This means that the exponents of the units of a class(orbit) of physical quantities, taken in the *correct order*, form the coordinates of a point in the integer lattice $\mathbb{Z}^7$. Every possible integer lattice point is the image of one class(orbit) of physical quantities and so the mapping 'dex' is bijective from $\mathbb{U}_p$ on $\mathbb{Z}^7$ and expresses 'dex' as an isomorphism between $\mathbb{U}_p$ and $\mathbb{Z}^7$. It is known, see Lipschutz [14], that $\mathbb{Z}^7, +$ is an Abelian group and so all the properties of an Abelian group will be used without proof. Remark: the scalar multiplication on $\breve{a}$ is only closed in $\mathbb{Z}^7$ when the scalar is an integer. So, the algebraic structure $\mathbb{Z}^7, +, \cdot$ is a ring and not a field $\mathbb{F}$. The prerequisite for the creation of a vector space is the existence of a field $\mathbb{F}$ for the scalars. The elements of the vector space are vectors. This justifies why the notation $\breve{a}$ is used instead of $\vec{a}$. We can select 7 linearly independent lattice points $\breve{e}_1, \ldots, \breve{e}_7$ of $\mathbb{Z}^7$. The $\breve{e}_i$s form a covariant basis, see Coxeter (section 10.4) [6], for the integer lattice in $\mathbb{Z}^7$. Every lattice point can be expressed in a unique way as the linear combination: $\breve{x} = X^1\breve{e}_1 + \ldots + X^7\breve{e}_7$ where the coefficients X^i are called the contravariant components of $\breve{x}$. The inner product is defined as the expression: $\breve{x} \cdot \breve{y} = \sum_{i=1}^{7}\sum_{j=1}^{7} a_{ij}X^iY^j$ where $a_{ij} = a_{ji}$. Consider seven lattice points $\breve{e}^i$ satisfying the expression $\breve{e}^i = \sum_{k=1}^{7} a^{ik}\breve{e}_k$. This contravariant basis spans the space $\mathbb{Z}^7$ resulting in the equations $\sum_{i=1}^{7} a_{ij}\breve{e}^i = \sum_{i=1}^{7}\sum_{k=1}^{7} a_{ij}a^{ik}\breve{e}_k = \sum_{k=1}^{7} \delta_j^k\breve{e}_k = \breve{e}_j$. A lattice point $\breve{x}$ has covariant components X_i, such that $\breve{x} = \sum_{i=1}^{7} X_i\breve{e}^i$. These components are related to the contravariant components by the expressions: $X^j = \sum_{i=1}^{7} a^{ij}X_i$ and $X_i = \sum_{i=1}^{7} a_{ij}X^j$. With this notation the inner product can be represented as $\breve{x} \cdot \breve{y} = \sum_{i=1}^{7} X^iY_i = \sum_{k=1}^{7} X_kY^k$. Observe that, since $\breve{e}^i \cdot \breve{e}_j = \sum_{i=k}^{7} a^{ik}\breve{e}_k \cdot \breve{e}_j = \sum_{i=k}^{7} a^{ik}a_{jk} = \delta_j^i$, each $\breve{e}^i$ is orthogonal to every $\breve{e}_j$ except $\breve{e}_i$. This means that $\breve{e}^i$ is orthogonal to the 6-dimensional space spanned by 6 of the $\breve{e}_j$ and $\breve{e}_i$ is related similarly to 6 of the $\breve{e}^j$. We obtain that $\breve{e}^i \cdot \breve{e}_j = 1$. The covariant components of $\breve{x}$ could have been defined as the inner product $\breve{x} \cdot \breve{e}_j = \sum_{i=1}^{7} X_i\breve{e}^i \cdot \breve{e}_j = \sum_{i=1}^{7} X_i\delta_j^i = X_j$ and similarly $\breve{x} \cdot \breve{e}^k = X^k$. For $\breve{x} = \breve{e}^i$ we have $\breve{e}^i \cdot \breve{e}^k = a^{ik}$, and thus the reciprocity between "covariant" and "contravariant" is complete. It is known, see Coxeter [6], that the points $\breve{x}$, whose covariant coordinates are integers, form a lattice. The lattice is formed by a set of transforms of a point by a group of translations. The generating translations are given by the contravariant basis $\breve{e}^i$. The points, whose contravariant coordinates are integers, form another lattice. Both lattices are called *reciprocal* . We are free to select seven basis lattice points. These points will receive the agreed, see BIPM [3], symbol for the dimension. We define: $\breve{l} \doteq \breve{e}_1 = L = (1,0,0,0,0,0,0)$, $\breve{m} \doteq \breve{e}_2 = M = (0,1,0,0,0,0,0)$, $\breve{t} \doteq \breve{e}_3 = T = (0,0,1,0,0,0,0)$, $\breve{i} \doteq \breve{e}_4 = I = (0,0,0,1,0,0,0)$, $\breve{T} \doteq \breve{e}_5 = \Theta = (0,0,0,0,1,0,0)$, $\breve{n} \doteq \breve{e}_6 = N = (0,0,0,0,0,1,0)$, $\breve{L} \doteq \breve{e}_7 = J = (0,0,0,0,0,0,1)$, with $\breve{e}_i \in \mathbb{Z}^7$. This basis generates a "hypercubic lattice", see Chapter 4 of Conway and Sloane [7]. This hypercubic lattice is its own reciprocal lattice. This basis has also the property of being *orthonormal*. We claim without giving proofs of the following 'dex' identities:

$$\forall [a],[b] \in \mathbb{U}_p \mid \mathrm{dex}\,([a][b]) = \mathrm{dex}\,(a) + \mathrm{dex}\,(b) \ , \tag{2.1a}$$

$$\forall [a],[b] \in \mathbb{U}_p \mid \mathrm{dex}\left(\frac{[a]}{[b]}\right) = \mathrm{dex}\,(a) - \mathrm{dex}\,(b) \ , \tag{2.1b}$$

$$\forall [a],[b],[c] \in \mathbb{U}_p \mid \mathrm{dex}\,([a][b][c]) = \mathrm{dex}\,([a]([b][c])) \ , \tag{2.1c}$$

$$= \mathrm{dex}\,(([a][b])[c]) \ , \tag{2.1d}$$

$$\forall p \in \mathbb{Z} \mid \mathrm{dex}\,([a]^p) = p\,\mathrm{dex}\,(a) \ . \tag{2.1e}$$

Definition 2. The inverse of the 'dex' function is a function of $\mathbb{Z}^7$ into $\mathbb{U}_p$, and defined as $\mathrm{dex}^{-1} : \forall \breve{\mathrm{a}} \in \mathbb{Z}^7, \exists [\mathrm{a}] \in \mathbb{U}_\mathrm{p} \mid \mathrm{dex}^{-1}(\breve{\mathrm{a}}) = [\mathrm{a}]$.

We claim without giving proofs of the following dex^{-1} identities:

$$\forall \breve{a},\breve{b} \in \mathbb{Z}^7 \mid [a][b] = \mathrm{dex}^{-1}\left(\breve{a} + \breve{b}\right) \ , \tag{2.2a}$$

$$\forall \breve{a},\breve{b} \in \mathbb{Z}^7 \mid \frac{[a]}{[b]} = \mathrm{dex}^{-1}\left(\breve{a} - \breve{b}\right) \ , \tag{2.2b}$$

$$\forall \breve{a},\breve{b},\breve{c} \in \mathbb{Z}^7 \mid \mathrm{dex}^{-1}\left(\breve{a} + \breve{b} + \breve{c}\right) = \mathrm{dex}^{-1}\left(\breve{a} + (\breve{b} + \breve{c})\right) \ , \tag{2.2c}$$

$$= \mathrm{dex}^{-1}\left((\breve{a} + \breve{b}) + \breve{c}\right) \ , \tag{2.2d}$$

$$\forall p \in \mathbb{Z} \mid [a]^p = \mathrm{dex}^{-1}\,(p\,\breve{a}) \ . \tag{2.2e}$$

We call the expression $N(\breve{x}) \doteq \|\breve{x}\|_1 = \sum\limits_{i=1}^{7}\sum\limits_{k=1}^{7} a_{ik}X^iX^k$, the ℓ_1-norm of $\breve{x}$ in $\mathbb{Z}^7$. We call the expression $\|\breve{x}\|_2 \doteq \sqrt{\sum\limits_{i=1}^{7}\sum\limits_{k=1}^{7} a_{ik}X^iX^k}$ the ℓ_2-norm or Euclidean norm of $\breve{x}$ in

$\mathbb{Z}^7$. We call the expression $\|\breve{x}\|_\infty = max\{|X^1|,\ldots,|X^7|\}$ the Chebyshev norm or infinity norm of $\breve{x}$ in $\mathbb{Z}^7$. Let $\breve{x},\breve{y}$ be lattice points of $\mathbb{Z}^7$. The distance between the points $\breve{x},\breve{y}$ is defined by: $d(\breve{x},\breve{y}) = \|\breve{x}-\breve{y}\|_2$, where $\breve{x}-\breve{y} = (X^1-Y^1,\ldots,X^7-Y^7)$ if $\breve{x} = (X^1,\ldots,X^7)$ and $\breve{y} = (Y^1,\ldots,Y^7)$. The Euclidean distance between $\breve{x}$ and $\breve{y}$ is $d(\breve{x},\breve{y}) = \sqrt{\sum_{i=1}^{7}(X_i-Y_i)(X^i-Y^i)}$. To each lattice point $\breve{x}$ of $\mathbb{Z}^7$ one can associate a hyperplane $H_{\breve{x}}$. A set $H_{\breve{x}}$ in $\mathbb{Z}^7$ is a hyperplane if and only if there exist scalars $C_0,C_1,\ldots,C_7$, where not all $C_1,\ldots,C_7$ are zero, such that $H_{\breve{x}} = \{(X^1,\ldots,X^7) \mid C_0 + C_1X^1 + \ldots + C_7X^7 = 0\}$, see Webster [22]. Consider now the lattice point $\breve{y} = (Y^1,\ldots,Y^7)$ and select its associated hyperplane $H_{\breve{y}}$ that contains the lattice point $\breve{o}$. The lattice point $\breve{x}$ will lie in the hyperplane $H_{\breve{y}}$ when it satisfies the equation $\sum_{i=1}^{7} Y^iX_i = 0$. The distance between the lattice point $\breve{x}$ and the hyperplane $H_{\breve{y}}$, measured along the perpendicular, is the projection of $\breve{o}\breve{x}$ in the direction of $\breve{o}\breve{y}$ that is given by the equation $\dfrac{\breve{x}\cdot\breve{y}}{\|\breve{y}\|_2} = \dfrac{\sum_{i=1}^{7} X_iY^i}{\sqrt{\sum_{i=1}^{7} Y_iY^i}}$. Let the lattice point $\breve{x}'$ be the image of $\breve{x}$ by reflection in the hyperplane $H_{\breve{y}}$. Consider the lattice point $\breve{z}$ satisfying $\breve{z} = \breve{x} - \breve{x}'$, then the line $\breve{o}\breve{z}$ is parallel to the line $\breve{o}\breve{y}$. We define now, see Coxeter [6], a general reflection in the hyperplane $H_{\breve{y}}$ as $\breve{x} - \breve{x}' = 2\dfrac{\breve{x}\cdot\breve{y}}{\breve{y}\cdot\breve{y}}\breve{y}$. We call, see chapter 4 of Conway and Sloane [7], the lattice point $\breve{y}$ the *root* of the reflecting hyperplane $H_{\breve{y}}$. The root system for the Lie algebra B_7 has the basis $\breve{\alpha}_1,\ldots,\breve{\alpha}_7$ defined by $\breve{\alpha}_1 = \breve{e}_1-\breve{e}_2,\ \breve{\alpha}_2 = \breve{e}_2-\breve{e}_3,\ \ldots,\breve{\alpha}_6 = \breve{e}_6-\breve{e}_7,\ \breve{\alpha}_7 = \breve{e}_7$. It is known, see chapter 4 of Conway and Sloane [7], that this root system generates the $\mathbb{Z}^7$ integer lattice as root lattice by reflections in the hyperplanes associated with the roots. The reflections are characterized by signed permutation matrices. As we will connect points in the integer lattice forming parallelograms, we use the term *k-cycle* from graph theory, see Diestel [9], where the k-cycle is a simple graph of length k, i.e., consisting of k vertices and k edges and represented by a sequence of consecutive vertices $\breve{x}_0\ldots\breve{x}_{k-1}\breve{x}_0$. Ternary laws of physics are represented by 4-cycles. Let the function psc, represent the *parity of the sum of coordinates* of a lattice point of $\mathbb{Z}^7$. We define the parity of the sum of coordinates:

Definition 3. $$\mathrm{psc} : \mathbb{Z}^7 \to \{0,1\} \mid \mathrm{psc}\,(\breve{x}) = |\sum_{i=1}^{7} X^i| \pmod{2}\,, X^i \in \mathbb{Z}\,. \tag{2.3}$$

Applying the 'psc' function to all lattice points creates a 2-coloring of the integer lattice. We have an *evensum* lattice point when $\mathrm{psc}\,(\breve{x}) = 0$ and an *oddsum* lattice point when $\mathrm{psc}(\breve{x}) = 1$ where $\breve{x} \in \mathbb{Z}^7$. Observe that the lattice points $\breve{x}$ for which $\mathrm{psc}(\breve{x}) = 0$ are elements of D_7 that is an indecomposable root lattice, see Coppel [4], defined as $D_7 = \{(X^1, \ldots, X^7) \in \mathbb{Z}^7 \mid \sum_{i=1}^{7} X^i \textit{ is even}\}$. The lattice D_7 has 84 minimal points, that are $\pm\breve{e}_j \pm \breve{e}_k$ where $(1 \leq j \leq k \leq 7)$. These 84 points form a simple basis derived from the canonical basis $\breve{e}_1, \ldots, \breve{e}_7$ of $\mathbb{Z}^7$. Consider a lattice point $\breve{x}_0$ and points $\breve{x}$, which have the property $\breve{x}_0 + \breve{x} \in S \Leftrightarrow \breve{x}_0 - \breve{x} \in S$ then we call S a centrally symmetric set. In the remainder of the article we will assume that $\breve{x}_0 = \breve{o}$ is the origin of $\mathbb{Z}^7$. As we are operating in a hypercubic lattice, we introduce the concept of "hypercubic shell" in analogy with spherical shells. A hypercubic shell hcs^s_n of edge-length $2s$ is a subset of $\mathbb{Z}^n$ with the following property $hcs^s = \{\breve{x} \in \mathbb{Z}^7 \mid \|\breve{x}\|_\infty = s\}$, where $X^i \in \mathbb{Z}$.

3. Geometric representation of ternary laws of physics

A relationship between n components of physical quantities which may be used to describe a phenomenon, without exception, is a n-ary law of physics. The present study focuses on the case where $n = 3$ and so it investigates ternary laws of physics. The ternary laws under study are of the type $[z] = [\kappa][x][y]$ and the ternary operator is the multiplication operator.

Theorem 1. *If $[\kappa], [x], [y], [z]$ are distinct classes of physical quantities for which*

$$\mathrm{dex}^{-1}(\mathrm{dex}([z])) = [z], \qquad \mathrm{dex}^{-1}(\mathrm{dex}([\kappa])) = [\kappa],$$
$$\mathrm{dex}^{-1}(\mathrm{dex}([x])) = [x], \qquad \mathrm{dex}^{-1}(\mathrm{dex}([y])) = [y].$$

then the ternary operation $[z] = [\kappa][x][y]$ is a law of physics with $[\kappa]$ a dimensionless quantity, if and only if, the 4-cycle $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$ is a parallelogram in the integer lattice $\mathbb{Z}^7$ and $\mathrm{dex}([x]) = \breve{x}$, $\mathrm{dex}([y]) = \breve{y}$, $\mathrm{dex}([z]) = \breve{z}$, $\mathrm{dex}([\kappa]) = \breve{o}$ *are distinct integer lattice points with $\breve{o}$ being the origin of the integer lattice $\mathbb{Z}^7$.*

PROOF. *The proof is of the 'if and only if'–type where it is split in a necessary and sufficient condition. We aim to prove that a ternary law of physics is equivalent with a* 4*-cycle, being a parallelogram and vice-versa.*

Condition 1 (Necessary)**.** Let $[\kappa], [x], [y], [z] \in \mathbb{U}_p$ be distinct classes of physical quantities and $\mathrm{dex}([\kappa]) = \breve{o}$ be a dimensionless quantity. Suppose that the ternary operation $[z] = [\kappa][x][y]$ is a law of physics. By the 'dex' identity (2.1a) we obtain $\mathrm{dex}([z]) = \mathrm{dex}([\kappa]) + \mathrm{dex}([x][y]) = \mathrm{dex}([\kappa]) + \mathrm{dex}([x]) + \mathrm{dex}([y])$. By the definition of 'dex', see 1, one writes

$$\breve{z} = \breve{o} + \breve{x} + \breve{y}\,, \tag{3.1}$$

where the addition is performed component-wise. The integer coordinates $(X^1, \ldots, X^7)$ of $\breve{x}$, $(Y^1, \ldots, Y^7)$ of $\breve{y}$ and the origin $\breve{o}$ determine uniquely the coordinates of a lattice point $\breve{z}$ according to the above equation (3.1) . As no degree of freedom is left over for the coordinates of $\breve{z}$, one can claim that a parallelogram (Fig. 3.1) represented by the 4-cycle $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$ has been constructed in $\mathbb{Z}^7$.

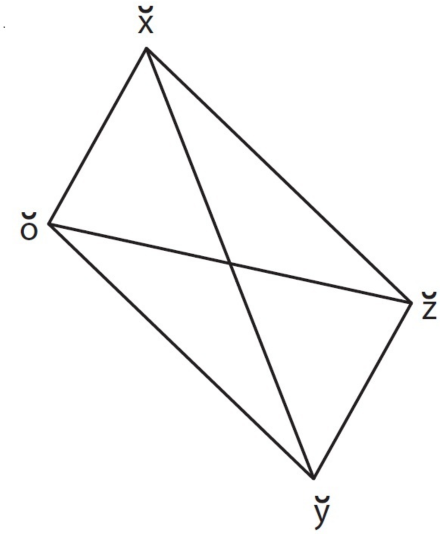

Figure 3.1: Parallelogram $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$ representing the ternary operation $[z] = [\kappa][x][y]$ in $\mathbb{Z}^7$.

Condition 2 (Sufficient)**.** Let the 4-cycle $\breve{o}\breve{y}\breve{z}\breve{x}\breve{o}$ be a parallelogram (Fig. 3.1) with as diagonals the lines $\breve{o}\breve{z}$ and $\breve{x}\breve{y}$. Let $\breve{o}$ be, without loss of generality, the origin of the integer lattice $\mathbb{Z}^7$. By the definition of a 4-cycle one writes $\breve{o} = \breve{z} - \breve{x} - \breve{y}$. This equation can be rewritten as $\breve{z} = \breve{o} + \breve{x} + \breve{y}$. We apply on both sides of the equation the function dex^{-1} , see 2, and obtain the equation $\mathrm{dex}^{-1}(\breve{z}) = \mathrm{dex}^{-1}(\breve{o} + \breve{x} + \breve{y})$. By the definition, of dex^{-1} identity (2.2a) we obtain

$$\mathrm{dex}^{-1}(\mathrm{dex}(z)) = \mathrm{dex}^{-1}(\mathrm{dex}(\kappa)) \cdot \mathrm{dex}^{-1}(\mathrm{dex}(x)) \cdot \mathrm{dex}^{-1}(\mathrm{dex}(y)) \ . \tag{3.2}$$

As the product function $(\mathrm{dex}^{-1} \circ \mathrm{dex})$ results in the identity function we claim that there exists a set $\{[\kappa], [x], [y], [z]\} \subset \mathbb{U}_p$ for which

$$\begin{aligned} \mathrm{dex}^{-1}(\mathrm{dex}([z])) &= [z], & \mathrm{dex}^{-1}(\mathrm{dex}([\kappa])) &= [\kappa], \\ \mathrm{dex}^{-1}(\mathrm{dex}([x])) &= [x], & \mathrm{dex}^{-1}(\mathrm{dex}([y])) &= [y] \, . \end{aligned}$$

So, one obtains from the equation (3.2) the ternary operation $[z] = [\kappa][x][y]$ that is to be considered as a law of physics.

$\square$

Conjecture 3.1. *If* $[\kappa], [x_1], [x_2], \ldots, [x_n]$ *are distinct classes of physical quantities then the n-ary operation* $[x_n] = [\kappa][x_1] \ldots [x_{n-1}]$ *is a law of physics with* $[\kappa]$ *a dimensionless quantity, if and only if, the* $(n+1)$*-cycle contains the origin of the integer lattice* $\mathbb{Z}^7$ *.*

4. Cardinality of isoperimeter parallelograms

Based on theorem 1 we could explore the integer lattice and search for "new laws of physics" by selecting at random 2 points $\breve{z}$ and $\breve{x}$ and create a parallelogram by deriving the coordinates of $\breve{y}$ and connect it to the origin. This prescription at first was *astonishing* for the author and I assume it should be for most physicists. One starts then asking a lot of research questions. Some that popped up were:

- Is the perimeter of the parallelogram the major characteristic of the law of physics or is it perhaps the area of the parallelogram?
- Are connections between parallelograms indications of relations between the laws of physics?
- Can we retrieve characteristics of the laws of physics by calculating the possible combinations of parallelograms for a fixed $\breve{z}$?
- Is the distance between $\breve{z}$ and the origin a classifier for laws of physics?
- Is a parallelogram for a fixed $\breve{z}$ an element of a set that itself is a partition of a finite set?

"Distance from $\breve{z}$ to the origin" was first studied but without success. Inspired by concepts of random walk, the path length through the lattice was considered an interesting parameter to be studied. The followed approach was to select a fixed point $\breve{z}$ and to vary the point $\breve{x}$. For ease of calculation perimeters of triangles p_t instead of parallelograms p_p were recorded and then converted. The fixed point to start the survey through the integer lattice was selected to be $\breve{z} = \breve{E}$, representing "energy" . The question became now more specific: Which lattice points are generating triangles resulting in parallelograms representing an "energy" law of physics and how many of these triangles have the same perimeter? A program in MATLAB®

was first created, but rapidly computational/memory problems occurred due to the large amount of data to be processed. The program was adapted and written in the programming language $C\#$. The algorithm is given in appendix A. The absolute frequency of occurrence of these parallelogram perimeters p_p are tabulated as a sequence of positive integers and represented graphically for $\breve{z} = \breve{E}$, as a *discrete value distribution* in accordance with the recommendations of Barford [1].

4.1. Case study for the physical quantity energy

The lattice point $\breve{z} = (2, 1, -2, 0, 0, 0, 0) = \breve{E}$ represents the physical quantity "energy". The graphical representation (Fig. 4.1) of the discrete value distribution of parallelogram perimeters p_p for parallelograms representing ternary laws of physics in $\mathbb{Z}^7$ resulting in the physical quantity "energy" shows a rich structure. It reveals the "distribution of energy laws". The enumeration as class(orbit) 6 (Table E.6) of the first 50 frequencies is not found in the OEIS database [16]. Finding the generating function for this integer sequence could be interesting. Observe that the lowest frequency f_{min} in Fig. 4.1 for the *non-degenerated* parallelograms is $f_{min} = 1$ with exception of the point with perimeter $p_p = 6$, that is a degenerated parallelogram. This isoperimeter distribution shows that *unique* non-degenerated parallelograms exist, that form *unique representations* of laws of physics! At perimeter $p_p = 7,657$ we find the well-known equation $E = \gamma m_0 c^2$ represented in its generic form as $E = \kappa_2 m_0 v^2$ (Table 1). Observe (Table 1) that the parity of the sum of the coordinates of the lattice points $\breve{x}$ are *odd* while those of the lattice points $\breve{y}$ are *even*. The components of physical quantities which are unknown to the author are marked u_n in the ternary relations of components of physical quantities resulting in the physical quantity energy. The first row represents a degenerated parallelogram. The dimensionless quantity κ_0 can be associated to the dimensionless quantity γ. The second row is recognized as the product of the linear momentum and the velocity. This law was considered "more important" by Maxwell than the one from the third row. The "more important" can be translated in a *principle of minimum parallelogram perimeter* for ranking the laws of physics. The third row is recognized as the kinetic energy and if $v = c$, as the famous equation $E = \gamma m_0 c^2$. Observe that the lattice points $\breve{x}$ and $\breve{y}$ are *orthogonal* for $E = \gamma m_0 c^2$. The fourth row is a well-known form appearing as a term in a Hamiltonian. The other rows express forms of laws of physics that are unknown to the author. According to Maxwell [15] we can state that a physical quantity is a vector if it contains the dimensional symbol $L^{\pm 1}$. It means that physical quantities which are *vectors* are elements of the hyperplanes $H_{\breve{a}} = \{(A^1, \ldots, A^7) \mid 1 + A^1 = 0\}$ and $H_{\breve{b}} = \{(B^1, \ldots, B^7) \mid -1 + B^1 = 0\}$. Thus, we expect u_1 and u_2 to be vectors. Observe that u_2 is the reciprocal of the velocity.

We expect u_3 and u_4 to be vectors. Observe that u_3 is the reciprocal of the linear momentum. We expect u_5 and u_6 to be scalars. The distribution in Fig. 4.1 is truncated at $p_p = 25$ due to edge effects at the hypercube surface. The edge effects are related to the memory capacity of the author's personal computer. The computation of the distribution was performed for a Chebyshev norm $\|\breve{x}\|_\infty = 5$.

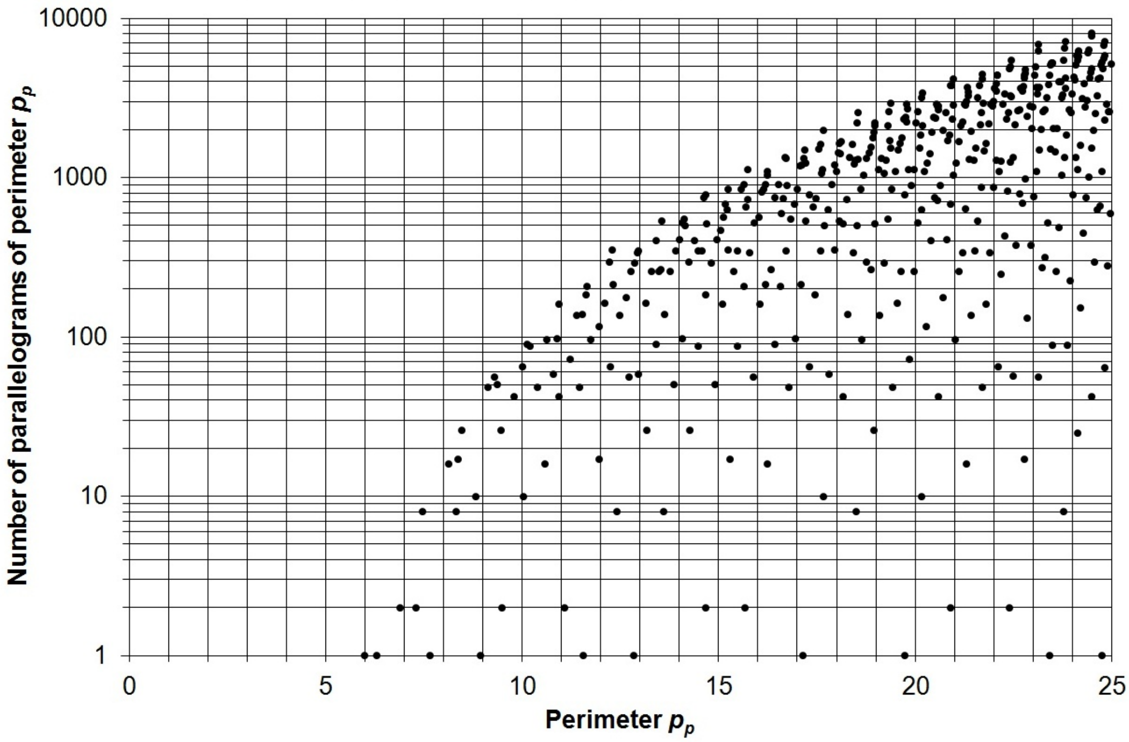

Figure 4.1: Discrete value distribution of parallelogram perimeters p_p for parallelograms representing ternary laws of physics in $\mathbb{Z}^7$ resulting in the physical quantity "Energy".

Table 1: Unique parallelograms in $\mathbb{Z}^7$ for the physical quantity "Energy".

p_p	$\breve{x}$	$\breve{y}$	Ternary operation
6,000	(2,1,-2,0,0,0,0)	(0,0,0,0,0,0,0)	$E = \kappa_0 E_0$
...	...	...	...

p_p	$\breve{x}$	$\breve{y}$	Ternary operation
6,293	(1,1,-1,0,0,0,0)	(1,0,-1,0,0,0,0)	$E = \kappa_1 pv$
7,657	(0,1,0,0,0,0,0)	(2,0,-2,0,0,0,0)	$E = \kappa_2 m_0 v^2$
8,928	(0,-1,0,0,0,0,0)	(2,2,-2,0,0,0,0)	$E = \kappa_3 \dfrac{p^2}{m_0}$
11,546	(3,1,-3,0,0,0,0)	(-1,0,1,0,0,0,0)	$E = \kappa_4 u_1 u_2$
12,845	(-1,-1,1,0,0,0,0)	(3,2,-3,0,0,0,0)	$E = \kappa_5 u_3 u_4$
17,146	(4,1,-4,0,0,0,0)	(-2,0,2,0,0,0,0)	$E = \kappa_6 \dfrac{u_5}{v^2}$
19,734	(4,3,-4,0,0,0,0)	(-2,-2,2,0,0,0,0)	$E = \kappa_7 \dfrac{u_6}{p^2}$
23,415	(-3,-1,3,0,0,0,0)	(5,2,-5,0,0,0,0)	$E = \kappa_8 u_7 u_8$
24,743	(5,3,-5,0,0,0,0)	(-3,-2,3,0,0,0,0)	$E = \kappa_9 u_9 u_{10}$

The list of unique parallelograms generates the folowing research questions:

- Can the next unique parallelogram be found without calculating the complete distribution for a Chebyshev norm $\|\breve{x}\|_\infty = s$?
- Is the number of unique parallelograms finite?
- Are there distributions which have no unique parallelograms?
- Are unique parallelograms only possible in the sublattice of $\mathbb{Z}^7$ for which $\breve{z}$ is a representative?

4.2. Invariance of the isoperimeter distribution

Theorem 2. *The isoperimeter distribution, for parallelograms containing the integer lattice points $\breve{o}$ and $\breve{z}$, is invariant when the coordinates of the integer lattice point $\breve{z}$ are subjected to a signed permutation.*

Proof. *The invariance can be explained based on the isometric property of the above mapping and mapping combinations. The perimeter of the parallelogram is based on the Euclidean distance between the lattice points and so neither a permutation of the* *coordinates nor a change in the sign of the coordinates will modify the value of the distance between the lattice points, see Conway and Sloane [7].* □

Example 1. The components of the physical quantity "force", represented by (1, 1, -2, 0, 0, 0, 0), and the components of the physical quantity "angular momentum", represented by (2, 1, -1, 0, 0, 0, 0), have the same isoperimeter distribution. The components of the physical quantity "mass", represented by (0, 1, 0, 0, 0, 0, 0), and the components of the physical quantity "frequency", represented by (0, 0, -1, 0, 0, 0, 0), have the same isoperimeter distribution.

The fact that some physical quantities are related through a signed permutation implies that they are qualitatively indistinguishable, see Rickles [18, Chap. 9.3].

5. Classification of components of physical quantities

To classify the *infinite* set of components of physical quantities, we observe that our present knowledge of physical quantities mainly describe lattice points close to the origin of the integer lattice $\mathbb{Z}^7$, probably due the *principle of minimum parallelogram perimeter*. We also observed that the Euclidean distance of a lattice point to the origin is not the classifier that generates the isoperimeter distribution of components of a physical quantity. We claim that lattice points which have the same isoperimeter distribution form a finite set. By ordering these sets one can observe that hypercubic shells are constructed. We therefore postulate that a classification of the components of physical quantities can be generated by applying the relation "the component of a physical quantity z_1 has the same isoperimeter distribution as the component of a physical quantity z_2" on the finite set of hypercubic shells of edge-length $2s$. This relation is reflexive, symmetric and transitive and complies with the definition of an equivalence relation. Applying this equivalence relation on a hypercubic shell generates equivalence classes which in group theory are defined as *orbits*.

5.1. Hypercubic shell properties

Theorem 3. *Let hcs_n^s be a centrally symmetric n-dimensional hypercube of edge-length 2s then the cardinality of hcs_n^s is $(2s+1)^n$.*

Proof. *For $n = 0$ the result is trivial.*
For $n = 1$ we have the set $hcs_1^s = \{-s, \dots, 0, \dots, s\}$ with edge-length $2s$. Let us denote the cardinality of the set S by $\#(S)$ then $\#(hcs_1^s) = 2s + 1$.
For $n = 2$ we have to increase the dimension n by 1, which corresponds to calculate

the Cartesian product of the sets $hcs_1^s \times hcs_1^s = hcs_2^s$ *.*
It is a property, see Lipschutz [13], of cardinal numbers that:

$$\#\left(hcs_2^s\right) = \#\left(hcs_1^s\right) \times \#\left(hcs_1^s\right) = \#\left(hcs_1^s\right) \cdot \#\left(hcs_1^s\right) = (2s+1)^2 \ . \tag{5.1}$$

Assume that $\#\left(hcs_{n-1}^s\right) = (2s+1)^{n-1}$ *. Then* $\#\left(hcs_n^s\right) = \#\left(hcs_{n-1}^s\right) \cdot \#\left(hcs_1^s\right) = (2s+1)^{n-1} \cdot (2s+1) = (2s+1)^n$ *.* □

We distinguish the hypercubic shells hcs_n^s by the parameters n and s, where n represents the dimension of the integer lattice and s represents the shell number. We define the class(orbit) of a hypercubic shell as:

Definition 4. A class(orbit) of a hypercubic shell is the set of lattice points $\mathbb{Z}^7$ that have the same isoperimeter distribution.

The class(orbit) of a hypercubic shell of $\mathbb{Z}^7$ is noted as $[(X^1,\ldots,X^7)]$ where $(X^1,\ldots,X^7)$ are the coordinates of the representative lattice point. The cardinality of a class(orbit) of a hypercubic shell is calculated using elementary combinatorics. Let $A = \{0,1,2,\ldots,k\}$ be the alphabet of hypercubic shell k. The representative of a class(orbit) of a hypercubic shell can be considered as a word w constructed from the alphabet A. The words w have a length n that corresponds to the dimension of $\mathbb{Z}^7$. Let n_i be the number of characters i of the alphabet A. Suppose that the characters can be subjected to permutation and change of sign, then the cardinality is given by the equation

$$\#\left(w\right) = 2^{n-n_0}\frac{n!}{n_0!n_1!n_2!\ldots n_k!} \ . \tag{5.2}$$

Observe that each class(orbit) of the hypercubic shells in $\mathbb{Z}^7$ represents a centrally symmetric 7-polytope. The theory of polytopes is well-known and references are Coxeter [6], Grünbaum [11] and Ziegler [26]. The number of vertices of the 7-polytopes is equal to the cardinality of w. Several of these polytopes are enumerated in [24]. Observe also that the representative lattice point has only coordinates that are *non-negative integers*. We define the total degree of a monomial as:

Definition 5. A monomial m in $u_1, u_2, \ldots, u_7$ is a product of the form:

$$m = \prod_{i=1}^{7} u_i^{X^i} \ , \tag{5.3}$$

where all the exponents $X^i \in \mathbb{Z}_+$ and $u_i \in \mathcal{U}$ see section 1. The total degree deg of this monomial is the sum $X^1 + \ldots + X^7$.

It is shown in Cox [5] that from the 7-tuple of non-negative integer exponents $(X^1, \ldots, X^7) \in \mathbb{Z}^7_+$ a monomial can be constructed one-to-one of the form $m = \prod_{i=1}^7 u_i^{X^i}$ that can be compared with equation (1.1). It means that a lot of results known from the commutative ring of monomials can be applied in the classification of the components of physical quantities. The number of classes of monomials (Table 2) with Chebyshev norm $\|\breve{x}\|_\infty \leq s$ in $\mathbb{Z}^7$ is the result from application of lemma 4 of chapter 9 in Cox [5].

Table 2: Properties of the structure S representing physical quantities in $\mathbb{Z}^7$ for $s \leq 10$.

s	sum(# ([a]))	cumul(sum(# ([a])))	# (hcs_7^s)	cumul(# (hcs_7^s))
0	1	1	1	1
1	2186	2187	7	8
2	75938	78125	28	36
3	745418	823543	84	120
4	3959426	4782969	210	330
5	14704202	19487171	462	792
6	43261346	62748517	924	1716
7	108110858	170859375	1716	3432
8	239479298	410338673	3003	6435
9	483533066	893871739	5005	11440
10	907216802	1801088541	8008	19448

Proposition 1. *The symmetries of the mathematical structure S are found in the automorphism group Aut($\mathbb{Z}^7$), that is the automorphism of the seven-dimensional integer lattice and is of order $2^7 \cdot 7!$.*

As a matrix group it is given by the set of all $n \times n$ signed permutation matrices. This group is isomorphic to the semidirect product $(\mathbb{Z}_2)^7 \rtimes S_7$ where the symmetric group S_7 acts on $(\mathbb{Z}_2)^7$ by permutation [25].

5.2. Classification tables of hypercubic shells

The classification table of each hypercubic shell hcs_7^s consists of 7 columns. The first column is the row identifier. The second column gives the representative of the equivalence class(orbit). The third column contains the sum of the absolute value of the coordinates of the lattice points being elements of the equivalence class(orbit) that is exclusively the total degree of the monomial associated with the equivalence class(orbit). The fourth column gives the parity of the representative of the equivalence class(orbit). The fifth column gives the ℓ_1-norm of the representative. The sixth column gives the cardinality of the equivalence class(orbit). The seventh column contains the name, if known, of the 7-polytope (or polyexon). The ordering of the classes is based on *graded reverse lex order* that is explained in definition 6 of chapter 2 in Cox [5]. We derive from Table 2 that hypercubic shells hcs_7^s are partitioned in $\binom{7+s-1}{s}$ equivalence classes. The appendices contain the complete list of classes in the following hypercubic shells hcs_7^1 (Table B.3), hcs_7^2 (Table D.5), hcs_7^3 (Table F.7). The common physical quantities (Table H.9) which belong to these hypercubic shells, where the variable s taking values from 0 to 10, are enumerated. Table H.9 is far from exhaustive, but it highlights the sparse distribution of the common physical quantities when taking in consideration the cardinalities (Table 2) of classes and vertices.

6. Future work and conclusion

We construct the mathematical foundation for *the geometry of the laws of physics*. We prove that ternary operations between components of physical quantities are equivalent to a parallelogram in the integer lattice $\mathbb{Z}^7$. This equivalence opens the path to search for laws of physics based on geometric properties between the integer lattice points of $\mathbb{Z}^7$, which are the representatives of components of physical quantities. We develop an algorithm that creates a listing of the ternary operations of the type $z = \kappa xy$ where κ, x, y, z represents components of physical quantities. Application of the algorithm for the case where z is representing the physical quantity "energy" results in a discrete value distribution that is characteristic for the equivalence class(orbit) $[(2, 2, 1, 0, 0, 0, 0)]$. The analysis of the discrete value distribution for the physical quantity "energy" indicates the existence of *unique representations* of laws of physics. One could define these *unique* laws as *fundamental* and form a mathematical criterion for what are *fundamental laws of physics.* This algorithm can now be applied to any other component of a physical quantity. The compilation of the listings generated by the algorithm, will result in a catalogue of components of physical ternary operations of the type $z = \kappa xy$. The equivalence relation z_1 *has the same isoperimeter distribution as* z_2 applied on a finite set, representing a hypercubic shell of $\mathbb{Z}^7$, results in the classification of physical quantities by revealing the mathematical structure S. The symmetries of the mathematical structure S are found in the automorphism group $\mathrm{Aut}(\mathbb{Z}^7)$, that is the automorphism of the seven-dimensional integer lattice. We show that a relation exists between these equivalence classes(orbits), monomials and 7-polytopes. The geometry of the laws of physics provides inherently a predictive property for finding the *form* of laws of physics that are yet to be discovered. This research shows that our knowledge about the components of physical quantities and about their relations is far from being understood and that large “volumes” of $\mathbb{Z}^7$, are still to be explored. We conclude that the number of laws of physics are *infinite* as their cardinality is equal to the cardinality of the integer lattice $\mathbb{Z}^7$. The mathematical structure S in which the laws of physics are embedded is consistent and complies with Gödel’s incompleteness theorem. We will never know *all* the laws of physics.

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Appendix A. 3-cycle isoperimeter distribution algorithm

Algorithm. *Calculate for each integer lattice point $\breve{x}$ of a centrally symmetric 7-dimensional hypercubic lattice the following:*

(i) d_1*, the Euclidean distance to the lattice point* $\breve{z}$*, representing “a component of a physical quantity” with coordinates* $(Z^1, \ldots, Z^7)$,

(ii) d_2*, the Euclidean distance to the origin* $\breve{o}$,

(iii) the cosine of the angle between $\breve{z}$ *and* $\breve{z}$,

(iv) $2a = d_1 + d_2$ *that is a characteristic of an ellipse,*

(v) the perimeter of the 3-cycle $p_t = d_1 + d_2 + d(\breve{x}, \breve{z})$,

(vi) store these results in a data structure allowing sorting by perimeter,

(vii) query the data structure to obtain the number of lattice points $\breve{x}$ *generating the same perimeter,*

(viii) find for each triangle perimeter p_t *the number of points corresponding to this triangle perimeter and record the discrete value distribution,*

(ix) select the set of vertices having the same perimeter starting with the shortest 3-cycle perimeter,

(x) calculate for each of these vertices the complementary vertices and write them in adjacent rows creating a listing of increasing perimeter.

Appendix B. Hypercubic shell 1

Column 2, Table B.3 contains seven classes. Observe that the representative lattice points of the classes generate the *successive minima* R_i of the lattice $\mathbb{Z}^7$ as defined by Davenport [8] . The successive minima R_i are given in the column 5 and correspond to the values of $N(\breve{z})$. The representative lattice points of the classes form a set of minimal points of the lattice $\mathbb{Z}^7$, see Davenport [8] .

Table B.3: Hypercubic shell 1.

Id	Class(orbit)	deg	psc $(\breve{z})$	$N(\breve{z})$	# ([a])	Polytope name
1	[(1,0,0,0,0,0,0)]	1	1	1	14	7-orthoplex
2	[(1,1,0,0,0,0,0)]	2	0	2	84	rectified 7-orthoplex
3	[(1,1,1,0,0,0,0)]	3	1	3	280	birectified 7-orthoplex
4	[(1,1,1,1,0,0,0)]	4	0	4	560	trirectified 7-cube
5	[(1,1,1,1,1,0,0)]	5	1	5	672	birectified 7-cube
6	[(1,1,1,1,1,1,0)]	6	0	6	448	rectified 7-cube
7	[(1,1,1,1,1,1,1)]	7	1	7	128	7-cube

Appendix C. Isoperimeter distributions of classes of hypercubic shell 1

Table C.4 consists of 8 columns. The first column is the index of the integer sequence. The other columns are numbered from 1 to 7 and represent the first 50 integers of the isoperimeter distribution corresponding to the classes with id from 1 to 7 from the hypercubic shell hcs_7^1 . Study of the minimum frequencies f_{min} in the 7 distributions and the corresponding vertices results in finding the classes that have *unique* ternary laws. The results for the hypercubic shell hcs_7^1 are that *only* class(orbit) 2 contains *unique* parallelograms. The ternary law for class(orbit) 2 is represented by a physical quantity that is expressed as $length \times mass$. Observe that the frequencies in the sequence of class(orbit) 1 also appear in the OEIS [16] sequence A000141 given by $r_6(m) = 1, 12, 60, 160, 252, 312, 544, 960, \ldots$. The sequence represents the number of ways of writing a positive integer m as a sum of *six* integral squares. It is known that the OEIS sequence A000141 is related to the theta function, as described in Conway and Sloane [7].

Table C.4: Truncated ($n \leq 50$) integer sequences of the isoperimeter distributions of classes of the hypercubic shell hcs^1_7.

n	$cl1$	$cl2$	$cl3$	$cl4$	$cl5$	$cl6$	$cl7$
1	1	1	1	1	1	1	1
2	12	1	3	4	5	6	7
3	1	10	8	3	10	15	21
4	60	10	24	6	4	10	35
5	12	2	3	24	20	2	7
6	160	42	30	18	40	12	42
7	60	40	75	4	5	30	105
8	252	20	24	24	24	26	147
9	160	100	80	60	50	30	147
10	312	80	120	40	65	60	21
11	1	1	3	24	20	66	105
12	252	80	75	80	80	30	210
13	544	170	168	104	120	12	252
14	12	91	150	48	100	60	315
15	312	10	24	6	10	120	441
16	960	160	120	60	50	15	35
17	60	272	240	156	114	132	147
18	544	122	288	180	170	60	252
19	1020	42	1	78	200	60	350
20	160	182	75	36	40	92	595
21	960	420	150	104	120	102	735
22	876	280	246	156	128	165	574
23	252	100	504	264	160	110	35
24	1020	244	8	176	10	30	147
25	1560	544	120	4	320	120	315
26	312	400	288	80	65	180	595
27	876	2	400	180	170	20	882
28	2400	170	528	192	260	180	840
29	1	560	30	328	320	270	854
30	544	682	150	240	375	180	1260
31	1560	290	504	24	40	66	21
32	2080	20	750	96	100	102	147
33	12	272	510	264	160	200	441
34	960	800	80	480	400	360	735
...	...	...	...	...	...	...	...

n	$cl1$	$cl2$	$cl3$	$cl4$	$cl5$	$cl6$	$cl7$
35	2400	910	288	480	5	342	840
36	2040	362	528	193	560	166	1050
37	60	80	728	60	340	132	1575
38	1020	420	840	156	65	180	1785
39	2080	580	3	328	200	15	1470
40	3264	1040	168	636	320	280	7
41	160	800	504	624	424	480	147
42	876	160	510	219	520	420	441
43	2040	544	576	6	20	132	574
44	4160	724	1227	104	530	60	854
45	252	1220	24	352	100	165	1575
46	1560	880	240	480	320	360	1750
47	3264	1	528	438	560	450	1533
48	4092	182	840	680	1	30	1932
49	312	682	1200	468	484	390	2387
50	2400	1600	1200	24	500	570	1

Appendix D. Hypercubic shell 2

Table D.5 contains in the second column 28 classes. Each of these classes are representing a 7-polytope. In the seventh column we find the name of the 7-polytope for those polytopes known to the author. Observe that the class(orbit) 6 contains 840 integer lattice points with the same geometrical properties as the physical quantity "energy". So, there exists 840 laws with the same importance as $E = \gamma m_0 c^2$.

Table D.5: Hypercubic shell 2.

Id	Class(orbit)	deg	psc ($\breve{z}$)	$N(\breve{z})$	# ([a])	Polytope name
1	[(2,0,0,0,0,0,0)]	2	0	4	14	7-orthoplex (size 2)
2	[(2,1,0,0,0,0,0)]	3	1	5	168	truncated 7-orthoplex
3	[(2,1,1,0,0,0,0)]	4	0	6	840	cantellated 7-orthoplex
4	[(2,2,0,0,0,0,0)]	4	0	8	84	rectified 7-orthoplex (size 2)
5	[(2,1,1,1,0,0,0)]	5	1	7	2240	runcinated 7-orthoplex
6	[(2,2,1,0,0,0,0)]	5	1	9	840	bitrunctated 7-orthoplex
7	[(2,1,1,1,1,0,0)]	6	0	8	3360	stericated 7-orthoplex
8	[(2,2,1,1,0,0,0)]	6	0	10	3360	bicantellated 7-orthoplex
9	[(2,2,2,0,0,0,0)]	6	0	12	280	birectified 7-orthoplex (size 2)
...	...	...	...	...	...	...

Id	Class(orbit)	deg	psc ($\breve{z}$)	$N(\breve{z})$	# ([a])	Polytope name
10	[(2,1,1,1,1,1,0)]	7	1	9	2688	pentellated 7-orthoplex
11	[(2,2,1,1,1,0,0)]	7	1	11	6720	biruncinated 7-orthoplex
12	[(2,2,2,1,0,0,0)]	7	1	13	2240	tritruncated 7-orthoplex
13	[(2,1,1,1,1,1,1)]	8	0	10	896	hexicated 7-cube
14	[(2,2,1,1,1,1,0)]	8	0	12	6720	bistericated 7-cube
15	[(2,2,2,1,1,0,0)]	8	0	14	6720	tricantellated 7-cube
16	[(2,2,2,2,0,0,0)]	8	0	16	560	trirectified 7-cube (size 2)
17	[(2,2,1,1,1,1,1)]	9	1	13	2688	pentellated 7-cube
18	[(2,2,2,1,1,1,0)]	9	1	15	8960	biruncinated 7-cube
19	[(2,2,2,2,1,0,0)]	9	1	17	3360	tritruncated 7-cube
20	[(2,2,2,1,1,1,1)]	10	0	16	4480	stericated 7-cube
21	[(2,2,2,2,1,1,0)]	10	0	18	6720	bicantellated 7-cube
22	[(2,2,2,2,2,0,0)]	10	0	20	672	birectified 7-cube (size 2)
23	[(2,2,2,2,1,1,1)]	11	1	19	4480	runcinated 7-cube
24	[(2,2,2,2,2,1,0)]	11	1	21	2688	bitruncated 7-cube
25	[(2,2,2,2,2,1,1)]	12	0	22	2688	cantellated 7-cube
26	[(2,2,2,2,2,2,0)]	12	0	24	448	rectified 7-cube (size 2)
27	[(2,2,2,2,2,2,1)]	13	1	25	896	truncated 7-cube
28	[(2,2,2,2,2,2,2)]	14	0	28	128	7-cube (size 2)

Appendix E. Isoperimeter distributions of classes of hypercubic shell 2

Table E.6 consists of 11 columns. The first column is the index of the integer sequence. The other columns represent the first 50 integers of the isoperimeter distribution corresponding to the classes containing known physical quantities from the hypercubic shell hcs_7^2. Observe that minimum frequencies $f_{min} = 1$ are present in the distributions. Listing the vertices that correspond to those frequency minima results in finding the classes that have *unique* ternary laws. The class(orbit) 6 (see 4.1) has been studied in detail.

Table E.6: Truncated ($n \leq 50$) integer sequences of the isoperimeter distributions of classes of the hypercubic shell hcs_7^2.

n	$cl1$	$cl2$	$cl3$	$cl4$	$cl5$	$cl6$	$cl7$	$cl8$	$cl11$	$cl12$
1	1	1	1	2	1	1	1	1	1	1
2	6	1	1	2	1	1	1	1	1	1
...	...	...	...	...	...	...	...	...	...	...

n	$cl1$	$cl2$	$cl3$	$cl4$	$cl5$	$cl6$	$cl7$	$cl8$	$cl11$	$cl12$
3	12	1	1	5	3	2	4	1	3	3
4	30	10	2	20	3	2	3	2	2	3
5	60	10	9	31	9	8	4	4	6	3
6	81	11	8	80	19	1	10	8	3	6
7	160	40	8	50	6	16	20	8	10	3
8	126	1	18	42	21	8	17	13	14	1
9	12	40	34	2	3	17	20	6	11	18
10	252	1	26	160	36	26	4	26	4	19
11	156	50	26	85	45	10	40	28	28	18
12	60	81	1	100	18	1	44	14	36	18
13	312	11	64	20	1	48	20	16	29	6
14	272	80	74	182	57	56	16	2	18	21
15	160	120	34	136	83	50	1	34	3	40
16	544	100	18	170	63	26	44	60	32	9
17	480	10	50	80	21	2	80	2	48	45
18	252	50	112	244	50	42	20	16	12	47
19	960	90	9	211	82	65	80	60	62	39
20	511	1	120	272	9	10	32	24	62	3
21	312	170	41	560	120	90	60	52	45	18
22	1020	152	64	432	122	88	10	16	18	57
23	438	40	2	10	57	48	80	57	72	45
24	12	120	88	420	3	16	91	62	57	60
25	544	114	114	800	114	96	140	98	75	36
26	876	202	185	341	108	58	88	55	44	96
27	780	10	104	182	135	98	44	36	132	9
28	60	320	34	42	36	42	4	13	11	43
29	960	81	112	544	249	160	106	88	68	81
30	1560	170	164	580	82	2	140	100	106	44
31	1200	260	16	455	150	72	40	52	45	78
32	160	352	164	244	19	136	122	84	134	18
33	1020	411	264	100	210	48	184	144	6	104
34	2400	40	184	682	219	139	130	98	140	111
35	1040	100	74	724	276	1	80	82	160	83
36	252	202	114	520	83	184	96	94	96	36
37	876	400	1	560	3	208	20	34	32	66
38	2080	1	240	910	108	96	184	1	93	3
39	1020	560	368	170	150	17	280	166	1	102
40	312	322	330	1600	339	116	244	234	105	172
...	...	...	...	...	...	...	...	...	...	...

n	$cl1$	$cl2$	$cl3$	$cl4$	$cl5$	$cl6$	$cl7$	$cl8$	$cl11$	$cl12$
41	1560	81	194	610	45	162	176	201	228	78
42	2040	152	120	2	399	296	6	170	68	210
43	1632	352	164	800	246	65	140	26	251	108
44	544	360	9	1040	120	352	160	136	147	39
45	2400	520	304	272	210	212	44	128	28	3
46	3264	11	480	272	19	8	244	57	116	120
47	2081	530	427	1760	300	136	400	212	162	153
48	960	100	160	850	366	176	364	324	72	83
49	2080	320	68	20	435	56	128	8	194	192
50	4160	560	185	580	63	256	91	262	10	21

Appendix F. Hypercubic shell 3

Table F.7 contains in the second column 84 classes. Each of these classes are representing a 7-polytope. In the seventh column we find the name of the 7-polytope for those polytopes known to the author.

Table F.7: Hypercubic shell 3.

Id	Class(orbit)	deg	psc ($\breve{z}$)	$N(\breve{z})$	# ([a])	Polytope name
1	[(3,0,0,0,0,0,0)]	3	1	9	14	7-orthoplex (size 3)
2	[(3,1,0,0,0,0,0)]	4	0	10	168	()
3	[(3,1,1,0,0,0,0)]	5	1	11	840	()
4	[(3,2,0,0,0,0,0)]	5	1	13	168	()
5	[(3,1,1,1,0,0,0)]	6	0	12	2240	()
6	[(3,2,1,0,0,0,0)]	6	0	14	1680	()
7	[(3,3,0,0,0,0,0)]	6	0	18	84	rectified 7-orthoplex (size 3)
8	[(3,1,1,1,1,0,0)]	7	1	13	3360	()
9	[(3,2,1,1,0,0,0)]	7	1	15	6720	()
10	[(3,2,2,0,0,0,0)]	7	1	17	840	()
11	[(3,3,1,0,0,0,0)]	7	1	19	840	()
12	[(3,1,1,1,1,1,0)]	8	0	14	2688	()
13	[(3,2,1,1,1,0,0)]	8	0	16	13440	()
14	[(3,2,2,1,0,0,0)]	8	0	18	6720	()
15	[(3,3,1,1,0,0,0)]	8	0	20	3360	()
16	[(3,3,2,0,0,0,0)]	8	0	22	840	()
17	[(3,1,1,1,1,1,1)]	9	1	15	896	()
...	...	...	...	...	...	...

Id	Class(orbit)	deg	psc ($\check{z}$)	$N(\check{z})$	# ([a])	Polytope name
18	[(3,2,1,1,1,1,0)]	9	1	17	13440	()
19	[(3,2,2,1,1,0,0)]	9	1	19	20160	()
20	[(3,2,2,2,0,0,0)]	9	1	21	2240	()
21	[(3,3,1,1,1,0,0)]	9	1	21	6720	()
22	[(3,3,2,1,0,0,0)]	9	1	23	6720	()
23	[(3,3,3,0,0,0,0)]	9	1	27	280	birectified 7-orthoplex (size 3)
24	[(3,2,1,1,1,1,1)]	10	0	18	5376	()
25	[(3,2,2,1,1,1,0)]	10	0	20	26880	()
26	[(3,2,2,2,1,0,0)]	10	0	22	13440	()
27	[(3,3,1,1,1,1,0)]	10	0	22	6720	()
28	[(3,3,2,1,1,0,0)]	10	0	24	20160	()
29	[(3,3,2,2,0,0,0)]	10	0	26	3360	()
30	[(3,3,3,1,0,0,0)]	10	0	28	2240	()
31	[(3,2,2,1,1,1,1)]	11	1	21	13440	()
32	[(3,2,2,2,1,1,0)]	11	1	23	26880	()
33	[(3,2,2,2,2,0,0)]	11	1	25	3360	()
34	[(3,3,1,1,1,1,1)]	11	1	23	2688	()
35	[(3,3,2,1,1,1,0)]	11	1	25	26880	()
36	[(3,3,2,2,1,0,0)]	11	1	27	20160	()
37	[(3,3,3,1,1,0,0)]	11	1	29	6720	()
38	[(3,3,3,2,0,0,0)]	11	1	31	2240	()
39	[(3,2,2,2,1,1,1)]	12	0	24	17920	()
40	[(3,2,2,2,2,1,0)]	12	0	26	13440	()
41	[(3,3,2,1,1,1,1)]	12	0	26	13440	()
42	[(3,3,2,2,1,1,0)]	12	0	28	40320	()
43	[(3,3,2,2,2,0,0)]	12	0	30	6720	()
44	[(3,3,3,1,1,1,0)]	12	0	30	8960	()
45	[(3,3,3,2,1,0,0)]	12	0	32	13440	()
46	[(3,3,3,3,0,0,0)]	12	0	36	560	trirectified 7-cube (size 3)
47	[(3,2,2,2,2,1,1)]	13	1	27	13440	()
48	[(3,2,2,2,2,2,0)]	13	1	29	2688	()
49	[(3,3,2,2,1,1,1)]	13	1	29	26880	()
50	[(3,3,2,2,2,1,0)]	13	1	31	26880	()
51	[(3,3,3,1,1,1,1)]	13	1	31	4480	()
52	[(3,3,3,2,1,1,0)]	13	1	33	26880	()
53	[(3,3,3,2,2,0,0)]	13	1	35	6720	()
54	[(3,3,3,3,1,0,0)]	13	1	37	3360	()
55	[(3,2,2,2,2,2,1)]	14	0	30	5376	()
...	...	...	...	...	...	...

Id	Class(orbit)	deg	psc $(\breve{z})$	$N(\breve{z})$	$\#([a])$	Polytope name
56	[(3,3,2,2,2,1,1)]	14	0	32	26880	()
57	[(3,3,2,2,2,2,0)]	14	0	34	6720	()
58	[(3,3,3,2,1,1,1)]	14	0	34	17920	()
59	[(3,3,3,2,2,1,0)]	14	0	36	26880	()
60	[(3,3,3,3,1,1,0)]	14	0	38	6720	()
61	[(3,3,3,3,2,0,0)]	14	0	40	3360	()
62	[(3,2,2,2,2,2,2)]	15	1	33	896	()
63	[(3,3,2,2,2,2,1)]	15	1	35	13440	()
64	[(3,3,3,2,2,1,1)]	15	1	37	26880	()
65	[(3,3,3,2,2,2,0)]	15	1	39	8960	()
66	[(3,3,3,3,1,1,1)]	15	1	39	4480	()
67	[(3,3,3,3,2,1,0)]	15	1	41	13440	()
68	[(3,3,3,3,3,0,0)]	15	1	45	672	birectified 7-cube (size 3)
69	[(3,3,2,2,2,2,2)]	16	0	38	2688	()
70	[(3,3,3,2,2,2,1)]	16	0	40	17920	()
71	[(3,3,3,3,2,1,1)]	16	0	42	13440	()
72	[(3,3,3,3,2,2,0)]	16	0	44	6720	()
73	[(3,3,3,3,3,1,0)]	16	0	46	2688	()
74	[(3,3,3,2,2,2,2)]	17	1	43	4480	()
75	[(3,3,3,3,2,2,1)]	17	1	45	13440	()
76	[(3,3,3,3,3,1,1)]	17	1	47	2688	()
77	[(3,3,3,3,3,2,0)]	17	1	49	2688	()
78	[(3,3,3,3,2,2,2)]	18	0	48	4480	()
79	[(3,3,3,3,3,2,1)]	18	0	50	5376	()
80	[(3,3,3,3,3,3,0)]	18	0	54	448	rectified 7-cube (size 3)
81	[(3,3,3,3,3,2,2)]	19	1	53	2688	()
82	[(3,3,3,3,3,3,1)]	19	1	55	896	()
83	[(3,3,3,3,3,3,2)]	20	0	58	896	()
84	[(3,3,3,3,3,3,3)]	21	1	63	128	7-cube (size 3)

Appendix G. Isoperimeter distributions of classes of hypercubic shell 3

Table G.8 consists of 11 columns. The first column is the index of the integer sequence. The other columns represent the first 50 integers of the isoperimeter distribution corresponding to the classes containing known physical quantities from the hypercubic shell hcs_7^3. Observe that minimum frequencies $f_{min} = 1$ are present in the distributions. Listing the vertices that correspond to those frequency minima results in finding the classes that have *unique* ternary laws.

Table G.8: Truncated ($n \leq 50$) integer sequences of the isoperimeter distributions of classes of the hypercubic shell hcs_7^3.

n	$cl1$	$cl2$	$cl3$	$cl4$	$cl5$	$cl6$	$cl9$	$cl14$	$cl15$	$cl22$
1	2	1	1	1	1	1	1	1	1	1
2	12	1	1	1	1	1	1	1	1	1
3	12	1	2	1	3	1	1	1	2	1
4	60	10	1	1	3	1	2	2	1	1
5	160	10	8	10	1	1	1	1	2	2
6	60	1	2	1	6	1	2	2	2	1
7	1	1	16	10	3	8	1	6	2	2
8	252	10	8	40	18	9	7	2	6	2
9	160	40	3	10	18	1	13	2	4	6
10	312	40	8	10	6	9	9	8	12	8
11	12	1	26	10	6	1	8	2	6	3
12	252	10	48	40	6	9	2	13	12	1
13	544	10	28	1	19	24	15	1	12	8
14	60	10	16	1	39	8	26	8	4	13
15	312	80	2	80	3	9	9	14	2	13
16	960	40	24	40	18	32	30	15	12	2
17	544	80	48	40	42	9	34	26	16	7
18	160	10	26	1	18	32	26	13	28	1
19	1020	40	64	40	36	1	2	14	6	14
20	960	1	64	80	18	10	15	6	24	13
21	252	10	49	11	50	33	43	13	20	26
22	876	41	1	90	42	35	38	30	30	13
23	1020	90	16	1	60	57	35	38	29	21
24	1	90	74	10	44	32	1	1	24	30
25	312	40	74	80	42	33	34	27	2	26
26	1560	80	51	1	1	1	70	32	32	6
27	876	1	48	80	18	24	14	46	28	15
28	12	80	120	90	78	56	46	40	12	22
29	544	90	3	80	96	66	1	40	40	8
30	2400	40	72	50	44	1	61	2	56	25
31	1560	112	112	10	66	40	43	32	52	1
32	2080	112	49	112	99	25	78	32	65	45
33	960	90	128	90	84	25	15	14	30	31
34	60	90	8	40	60	64	66	57	16	56
...	...	...	...	...	...	...	...	...	...	...

n	$cl1$	$cl2$	$cl3$	$cl4$	$cl5$	$cl6$	$cl7$	$cl8$	$cl11$	$cl12$
35	2400	91	120	10	84	66	90	80	56	33
36	2040	10	176	90	42	65	70	60	62	30
37	1020	1	72	10	6	57	26	82	2	9
38	160	130	24	112	116	34	62	39	40	44
39	2080	240	76	120	168	9	9	10	64	1
40	3264	241	2	90	174	96	71	68	106	50
41	876	170	122	240	152	128	143	50	12	43
42	252	40	192	113	36	97	61	44	30	62
43	2040	112	72	40	3	136	164	84	90	14
44	4160	122	267	81	99	40	103	132	17	28
45	1560	41	194	112	120	9	43	13	38	52
46	312	192	26	40	60	88	8	24	80	75
47	3264	320	112	240	145	83	90	92	64	2
48	4092	10	160	1	240	40	108	40	5	39
49	2400	330	74	40	19	152	66	60	104	53
50	544	112	224	170	225	216	146	100	32	48

Appendix H. Classification of common physical quantities

Table H.9 contains 5 columns. The first column represents the name of a common physical quantity. The second column indicates to which shell that the physical quantity belongs. The third column gives the "id" of the class(orbit) within the shell for the physical quantity. The fourth column lists the class(orbit) that contains the physical quantity. The fifth column identifies the physical quantity by its integer lattice point in $\mathbb{Z}^7$.

Table H.9: Classification of common physical quantities.

physical quantity	s	id	class(orbit)	vertex
plane angle	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
solid angle	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
linear strain	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
shear strain	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
bulk strain	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
relative elongation	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
refractive index	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
...	...	...	...	...

physical quantity	s	id	class(orbit)	vertex
electric susceptibility	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
mass ratio	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
fine-structure constant	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
redshift	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
Poisson's ratio	0	1	[(0,0,0,0,0,0,0)]	(0,0,0,0,0,0,0)
length	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
height	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
breadth	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
thickness	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
distance	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
radius	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
diameter	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
path length	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
persistence length	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
length of arc	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
Planck length	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
wavelength	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
Compton wavelength	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
relaxation length	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
luminosity distance	1	1	[(1,0,0,0,0,0,0)]	(1,0,0,0,0,0,0)
mass	1	1	[(1,0,0,0,0,0,0)]	(0,1,0,0,0,0,0)
reduced mass	1	1	[(1,0,0,0,0,0,0)]	(0,1,0,0,0,0,0)
Planck mass	1	1	[(1,0,0,0,0,0,0)]	(0,1,0,0,0,0,0)
time	1	1	[(1,0,0,0,0,0,0)]	(0,0,1,0,0,0,0)
period	1	1	[(1,0,0,0,0,0,0)]	(0,0,1,0,0,0,0)
relaxation time	1	1	[(1,0,0,0,0,0,0)]	(0,0,1,0,0,0,0)
time constant	1	1	[(1,0,0,0,0,0,0)]	(0,0,1,0,0,0,0)
time interval	1	1	[(1,0,0,0,0,0,0)]	(0,0,1,0,0,0,0)
proper time	1	1	[(1,0,0,0,0,0,0)]	(0,0,1,0,0,0,0)
Planck time	1	1	[(1,0,0,0,0,0,0)]	(0,0,1,0,0,0,0)
half-life time	1	1	[(1,0,0,0,0,0,0)]	(0,0,1,0,0,0,0)
specific impulse	1	1	[(1,0,0,0,0,0,0)]	(0,0,1,0,0,0,0)
electric current	1	1	[(1,0,0,0,0,0,0)]	(0,0,0,1,0,0,0)
thermodynamic temperature	1	1	[(1,0,0,0,0,0,0)]	(0,0,0,0,1,0,0)
Planck temperature	1	1	[(1,0,0,0,0,0,0)]	(0,0,0,0,1,0,0)
thermal expansion coefficient	1	1	[(1,0,0,0,0,0,0)]	(0,0,0,0,-1,0,0)
amount of substance	1	1	[(1,0,0,0,0,0,0)]	(0,0,0,0,0,1,0)
luminous intensity	1	1	[(1,0,0,0,0,0,0)]	(0,0,0,0,0,0,1)
...	...	...	...	...

physical quantity	s	id	class(orbit)	vertex
luminous flux	1	1	[(1,0,0,0,0,0,0)]	(0,0,0,0,0,0,1)
wave number	1	1	[(1,0,0,0,0,0,0)]	(-1,0,0,0,0,0,0)
optical power	1	1	[(1,0,0,0,0,0,0)]	(-1,0,0,0,0,0,0)
spatial frequency	1	1	[(1,0,0,0,0,0,0)]	(-1,0,0,0,0,0,0)
absorption coefficient	1	1	[(1,0,0,0,0,0,0)]	(-1,0,0,0,0,0,0)
laser gain	1	1	[(1,0,0,0,0,0,0)]	(-1,0,0,0,0,0,0)
rotational constant	1	1	[(1,0,0,0,0,0,0)]	(-1,0,0,0,0,0,0)
Rydberg constant	1	1	[(1,0,0,0,0,0,0)]	(-1,0,0,0,0,0,0)
frequency	1	1	[(1,0,0,0,0,0,0)]	(0,0,-1,0,0,0,0)
angular frequency	1	1	[(1,0,0,0,0,0,0)]	(0,0,-1,0,0,0,0)
circular frequency	1	1	[(1,0,0,0,0,0,0)]	(0,0,-1,0,0,0,0)
activity	1	1	[(1,0,0,0,0,0,0)]	(0,0,-1,0,0,0,0)
specific material permeability	1	1	[(1,0,0,0,0,0,0)]	(0,0,-1,0,0,0,0)
angular velocity	1	1	[(1,0,0,0,0,0,0)]	(0,0,-1,0,0,0,0)
decay constant	1	1	[(1,0,0,0,0,0,0)]	(0,0,-1,0,0,0,0)
Avogadro constant	1	1	[(1,0,0,0,0,0,0)]	(0,0,0,0,0,-1,0)
velocity	1	2	[(1,1,0,0,0,0,0)]	(1,0,-1,0,0,0,0)
group velocity	1	2	[(1,1,0,0,0,0,0)]	(1,0,-1,0,0,0,0)
volumetric flux	1	2	[(1,1,0,0,0,0,0)]	(1,0,-1,0,0,0,0)
speed	1	2	[(1,1,0,0,0,0,0)]	(1,0,-1,0,0,0,0)
speed of light in vacuum	1	2	[(1,1,0,0,0,0,0)]	(1,0,-1,0,0,0,0)
magnetic field strength	1	2	[(1,1,0,0,0,0,0)]	(-1,0,0,1,0,0,0)
magnetisation	1	2	[(1,1,0,0,0,0,0)]	(-1,0,0,1,0,0,0)
temperature gradient	1	2	[(1,1,0,0,0,0,0)]	(-1,0,0,0,1,0,0)
electric charge	1	2	[(1,1,0,0,0,0,0)]	(0,0,1,1,0,0,0)
electric flux	1	2	[(1,1,0,0,0,0,0)]	(0,0,1,1,0,0,0)
catalytic activity	1	2	[(1,1,0,0,0,0,0)]	(0,0,-1,0,0,1,0)
molar mass	1	2	[(1,1,0,0,0,0,0)]	(0,1,0,0,0,-1,0)
second radiation constant	1	2	[(1,1,0,0,0,0,0)]	(1,0,0,0,1,0,0)
luminous energy	1	2	[(1,1,0,0,0,0,0)]	(0,0,1,0,0,0,1)
linear density	1	2	[(1,1,0,0,0,0,0)]	(-1,1,0,0,0,0,0)
mass flow rate	1	2	[(1,1,0,0,0,0,0)]	(0,1,-1,0,0,0,0)
electric dipole moment	1	3	[(1,1,1,0,0,0,0)]	(1,0,1,1,0,0,0)
linear momentum	1	3	[(1,1,1,0,0,0,0)]	(1,1,-1,0,0,0,0)
Faraday constant	1	3	[(1,1,1,0,0,0,0)]	(0,0,1,1,0,-1,0)
dynamic viscosity	1	3	[(1,1,1,0,0,0,0)]	(-1,1,-1,0,0,0,0)
fluidity	1	3	[(1,1,1,0,0,0,0)]	(1,-1,1,0,0,0,0)
magnetogyric ratio	1	3	[(1,1,1,0,0,0,0)]	(0,-1,1,1,0,0,0)
...	...	...	...	...

physical quantity	s	id	class(orbit)	vertex
area	2	1	[(2,0,0,0,0,0,0)]	(2,0,0,0,0,0,0)
elastic modulus	2	1	[(2,0,0,0,0,0,0)]	(2,0,0,0,0,0,0)
Thomson cross section	2	1	[(2,0,0,0,0,0,0)]	(2,0,0,0,0,0,0)
space-time curvature	2	1	[(2,0,0,0,0,0,0)]	(-2,0,0,0,0,0,0)
angular acceleration	2	1	[(2,0,0,0,0,0,0)]	(0,0,-2,0,0,0,0)
acceleration	2	1	[(2,1,0,0,0,0,0)]	(1,0,-2,0,0,0,0)
areal velocity	2	2	[(2,1,0,0,0,0,0)]	(2,0,-1,0,0,0,0)
mass attenuation coefficient	2	2	[(2,1,0,0,0,0,0)]	(2,-1,0,0,0,0,0)
radiant exposure	2	2	[(2,1,0,0,0,0,0)]	(0,1,-2,0,0,0,0)
diffusion constant	2	2	[(2,1,0,0,0,0,0)]	(2,0,-1,0,0,0,0)
thermal diffusivity	2	2	[(2,1,0,0,0,0,0)]	(2,0,-1,0,0,0,0)
kinematic viscosity	2	2	[(2,1,0,0,0,0,0)]	(2,0,-1,0,0,0,0)
quantum of circulation	2	2	[(2,1,0,0,0,0,0)]	(2,0,-1,0,0,0,0)
electric current density	2	2	[(2,1,0,0,0,0,0)]	(-2,0,0,1,0,0,0)
luminance	2	2	[(2,1,0,0,0,0,0)]	(-2,0,0,0,0,0,1)
illuminance	2	2	[(2,1,0,0,0,0,0)]	(-2,0,0,0,0,0,1)
luminous emittance	2	2	[(2,1,0,0,0,0,0)]	(-2,0,0,0,0,0,1)
irradiance	2	2	[(2,1,0,0,0,0,0)]	(-2,0,0,0,0,0,1)
magnetic dipole moment	2	2	[(2,1,0,0,0,0,0)]	(2,0,0,1,0,0,0)
Bohr magneton	2	2	[(2,1,0,0,0,0,0)]	(2,0,0,1,0,0,0)
surface density	2	2	[(2,1,0,0,0,0,0)]	(-2,1,0,0,0,0,0)
surface tension	2	2	[(2,1,0,0,0,0,0)]	(0,1,-2,0,0,0,0)
stiffness	2	2	[(2,1,0,0,0,0,0)]	(0,1,-2,0,0,0,0)
compliance	2	2	[(2,1,0,0,0,0,0)]	(0,-1,2,0,0,0,0)
moment of inertia	2	2	[(2,1,0,0,0,0,0)]	(2,1,0,0,0,0,0)
accelerator luminosity	2	2	[(2,1,0,0,0,0,0)]	(-2,0,-1,0,0,0,0)
force	2	3	[(2,1,1,0,0,0,0)]	(1,1,-2,0,0,0,0)
energy density	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
radiant energy density	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
sound energy density	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
toughness	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
pressure	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
modulus of elasticity	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
Young's modulus	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
shear modulus	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
compression modulus	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
normal stress	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
shear stress	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
...	...	...	...	...

physical quantity	s	id	class(orbit)	vertex
energy momentum tensor	2	3	[(2,1,1,0,0,0,0)]	(-1,1,-2,0,0,0,0)
Planck constant	2	3	[(2,1,1,0,0,0,0)]	(2,1,-1,0,0,0,0)
angular momentum	2	3	[(2,1,1,0,0,0,0)]	(2,1,-1,0,0,0,0)
action	2	3	[(2,1,1,0,0,0,0)]	(2,1,-1,0,0,0,0)
spin	2	3	[(2,1,1,0,0,0,0)]	(2,1,-1,0,0,0,0)
acoustic impedance	2	3	[(2,1,1,0,0,0,0)]	(-2,1,-1,0,0,0,0)
mass flux	2	3	[(2,1,1,0,0,0,0)]	(-2,1,-1,0,0,0,0)
magnetic flux density	2	3	[(2,1,1,0,0,0,0)]	(0,1,-2,-1,0,0,0)
magnetic induction	2	3	[(2,1,1,0,0,0,0)]	(0,1,-2,-1,0,0,0)
surface charge density	2	3	[(2,1,1,0,0,0,0)]	(-2,0,1,1,0,0,0)
dielectric polarisation	2	3	[(2,1,1,0,0,0,0)]	(-2,0,1,1,0,0,0)
electrical displacement	2	3	[(2,1,1,0,0,0,0)]	(-2,0,1,1,0,0,0)
electrical quadrupole moment	2	3	[(2,1,1,0,0,0,0)]	(2,0,1,1,0,0,0)
luminous exposure	2	3	[(2,1,1,0,0,0,0)]	(-2,0,1,0,0,0,1)
absorbed dose	2	4	[(2,2,0,0,0,0,0)]	(2,0,-2,0,0,0,0)
dose equivalent	2	4	[(2,2,0,0,0,0,0)]	(2,0,-2,0,0,0,0)
specific energy	2	4	[(2,2,0,0,0,0,0)]	(2,0,-2,0,0,0,0)
gravitational potential	2	4	[(2,2,0,0,0,0,0)]	(2,0,-2,0,0,0,0)
molar Planck constant	2	5	[(2,1,1,1,0,0,0)]	(2,1,-1,0,0,-1,0)
magnetic vector potential	2	5	[(2,1,1,1,0,0,0)]	(1,1,-2,-1,0,0,0)
thermal conductivity	2	5	[(2,1,1,1,0,0,0)]	(1,1,-2,0,-1,0,0)
thermal resistivity	2	5	[(2,1,1,1,0,0,0)]	(-1,-1,2,0,1,0,0)
torque	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
moment of a force	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
specific heat capacity	2	6	[(2,2,1,0,0,0,0)]	(2,0,-2,0,-1,0,0)
energy	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
potential energy	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
kinetic energy	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
work	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
Lagrange function	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
Hamilton function	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
Hartree energy	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
ionization energy	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
electron affinity	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
electronegativity	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
dissociation energy	2	6	[(2,2,1,0,0,0,0)]	(2,1,-2,0,0,0,0)
magnetic constant	2	8	[(2,2,1,1,0,0,0)]	(1,1,-2,-2,0,0,0)
permeability	2	8	[(2,2,1,1,0,0,0)]	(1,1,-2,-2,0,0,0)
...	...	...	...	...

physical quantity	s	id	class(orbit)	vertex
magnetic flux	2	8	[(2,2,1,1,0,0,0)]	(2,1,-2,-1,0,0,0)
magnetic moment	2	8	[(2,2,1,1,0,0,0)]	(2,1,-2,-1,0,0,0)
entropy	2	8	[(2,2,1,1,0,0,0)]	(2,1,-2,0,-1,0,0)
specific heat	2	8	[(2,2,1,1,0,0,0)]	(2,1,-2,0,-1,0,0)
Boltzmann constant	2	8	[(2,2,1,1,0,0,0)]	(2,1,-2,0,-1,0,0)
Josephson constant	2	8	[(2,2,1,1,0,0,0)]	(-2,-1,2,1,0,0,0)
magnetic flux quantum	2	8	[(2,2,1,1,0,0,0)]	(2,1,-2,-1,0,0,0)
chemical potential	2	8	[(2,2,1,1,0,0,0)]	(2,1,-2,0,0,-1,0)
molar energy	2	8	[(2,2,1,1,0,0,0)]	(2,1,-2,0,0,-1,0)
molar heat capacity	2	8	[(2,2,1,1,0,0,0)]	(2,1,-2,0,-1,-1,0)
molar gas constant	2	11	[(2,2,1,1,1,0,0)]	(2,1,-2,0,-1,-1,0)
molar entropy	2	11	[(2,2,1,1,1,0,0)]	(2,1,-2,0,-1,-1,0)
inductance	2	12	[(2,2,2,1,0,0,0)]	(2,1,-2,-2,0,0,0)
self-inductance	2	12	[(2,2,2,1,0,0,0)]	(2,1,-2,-2,0,0,0)
mutual inductance	2	12	[(2,2,2,1,0,0,0)]	(2,1,-2,-2,0,0,0)
magnetisability	2	12	[(2,2,2,1,0,0,0)]	(2,-1,2,2,0,0,0)
volume	3	1	[(3,0,0,0,0,0,0)]	(3,0,0,0,0,0,0)
Loschmidt constant	3	1	[(3,0,0,0,0,0,0)]	(-3,0,0,0,0,0,0)
number density	3	1	[(3,0,0,0,0,0,0)]	(-3,0,0,0,0,0,0)
mass density	3	2	[(3,1,0,0,0,0,0)]	(-3,1,0,0,0,0,0)
specific volume	3	2	[(3,1,0,0,0,0,0)]	(3,-1,0,0,0,0,0)
amount of substance concentration	3	2	[(3,1,0,0,0,0,0)]	(-3,0,0,0,0,1,0)
molar volume	3	2	[(3,1,0,0,0,0,0)]	(3,0,0,0,0,-1,0)
heat flux density	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
Poynting vector	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
radiative flux	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
thermal emittance	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
sound intensity	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
radiance	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
irradiance	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
radiant exitance	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
radiant emittance	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
radiosity	3	2	[(3,1,0,0,0,0,0)]	(0,1,-3,0,0,0,0)
volume rate of flow	3	2	[(3,1,0,0,0,0,0)]	(3,0,-1,0,0,0,0)
jerk	3	2	[(3,1,0,0,0,0,0)]	(1,0,-3,0,0,0,0)
electric field gradient	3	3	[(3,1,1,0,0,0,0)]	(0,1,-3,-1,0,0,0)
electric charge density	3	3	[(3,1,1,0,0,0,0)]	(-3,0,1,1,0,0,0)
heat transfer coefficient	3	3	[(3,1,1,0,0,0,0)]	(0,1,-3,0,-1,0,0)
...	...	...	...	...

physical quantity	s	id	class(orbit)	vertex
thermal insulance	3	3	[(3,1,1,0,0,0,0)]	(0,-1,3,0,1,0,0)
spectral exitance	3	3	[(3,1,1,0,0,0,0)]	(-1,1,-3,0,0,0,0)
spectral radiance	3	3	[(3,1,1,0,0,0,0)]	(-1,1,-3,0,0,0,0)
spectral irradiance	3	3	[(3,1,1,0,0,0,0)]	(-1,1,-3,0,0,0,0)
spectral power	3	3	[(3,1,1,0,0,0,0)]	(1,1,-3,0,0,0,0)
spectral intensity	3	3	[(3,1,1,0,0,0,0)]	(1,1,-3,0,0,0,0)
luminous energy density	3	3	[(3,1,1,0,0,0,0)]	(-3,0,1,0,0,0,1)
catalytic activity concentration	3	3	[(3,1,1,0,0,0,0)]	(-3,0,-1,0,0,1,0)
reaction rate	3	3	[(3,1,1,0,0,0,0)]	(-3,0,-1,0,0,1,0)
absorbed dose rate	3	4	[(3,2,0,0,0,0,0)]	(2,0,-3,0,0,0,0)
thermal conductivity	3	5	[(3,1,1,1,0,0,0)]	(1,1,-3,0,-1,0,0)
first hyper-susceptibility	3	5	[(3,1,1,1,0,0,0)]	(-1,-1,3,1,0,0,0)
electric field	3	5	[(3,1,1,1,0,0,0)]	(1,1,-3,-1,0,0,0)
radiant intensity	3	6	[(3,2,1,0,0,0,0)]	(2,1,-3,0,0,0,0)
radiant flux	3	6	[(3,2,1,0,0,0,0)]	(2,1,-3,0,0,0,0)
Newton constant of gravitation	3	6	[(3,2,1,0,0,0,0)]	(3,-1,-2,0,0,0,0)
power	3	6	[(3,2,1,0,0,0,0)]	(2,1,-3,0,0,0,0)
sound energy flux	3	6	[(3,2,1,0,0,0,0)]	(2,1,-3,0,0,0,0)
bolometric luminosity	3	6	[(3,2,1,0,0,0,0)]	(2,1,-3,0,0,0,0)
responsivity	3	6	[(3,2,1,1,0,0,0)]	(-2,-1,3,1,0,0,0)
electric potential difference	3	9	[(3,2,1,1,0,0,0)]	(2,1,-3,-1,0,0,0)
electric potential	3	9	[(3,2,1,1,0,0,0)]	(2,1,-3,-1,0,0,0)
thermal conductance	3	9	[(3,2,1,1,0,0,0)]	(2,1,-3,0,-1,0,0)
thermal resistance	3	9	[(3,2,1,1,0,0,0)]	(-2,-1,3,0,1,0,0)
electromotive force	3	9	[(3,2,1,1,0,0,0)]	(2,1,-3,-1,0,0,0)
luminous efficacy	3	9	[(3,2,1,1,0,0,0)]	(-2,1,3,0,0,0,1)
electrical resistance	3	14	[(3,2,2,1,0,0,0)]	(2,1,-3,-2,0,0,0)
reactance	3	14	[(3,2,2,1,0,0,0)]	(2,1,-3,-2,0,0,0)
impedance	3	14	[(3,2,2,1,0,0,0)]	(2,1,-3,-2,0,0,0)
conductance	3	14	[(3,2,2,1,0,0,0)]	(-2,-1,3,2,0,0,0)
admittance	3	14	[(3,2,2,1,0,0,0)]	(-2,-1,3,2,0,0,0)
susceptance	3	14	[(3,2,2,1,0,0,0)]	(-2,-1,3,2,0,0,0)
characteristic impedance of vacuum	3	14	[(3,2,2,1,0,0,0)]	(2,1,-3,-2,0,0,0)
von Klitzing constant	3	14	[(3,2,2,1,0,0,0)]	(2,1,-3,-2,0,0,0)
specific resistance	3	15	[(3,3,1,1,0,0,0)]	(3,1,-3,-1,0,0,0)
electrical resistivity	3	22	[(3,3,2,1,0,0,0)]	(3,1,-3,-2,0,0,0)
electrical conductivity	3	22	[(3,3,2,1,0,0,0)]	(-3,-1,3,2,0,0,0)
second moment of area	4	1	[(4,0,0,0,0,0,0)]	(4,0,0,0,0,0,0)
...	...	...	...	...

physical quantity	s	id	class(orbit)	vertex
jounce	4	2	[(4,1,0,0,0,0,0)]	(1,0,-4,0,0,0,0)
electric polarisability	4		[(4,2,1,0,0,0,0)]	(0,-1,4,2,0,0,0)
Stefan-Boltzmann constant	4		[(4,3,1,0,0,0,0)]	(0,1,-3,0,-4,0,0)
first radiation constant	4		[(4,3,1,0,0,0,0)]	(4,1,-3,0,0,0,0)
electrical mobility	4		[(4,3,1,1,0,0,0)]	(3,1,-4,-1,0,0,0)
electric capacitance	4		[(4,2,2,1,0,0,0)]	(-2,-1,4,2,0,0,0)
electric constant	4		[(4,3,2,1,0,0,0)]	(-3,-1,4,2,0,0,0)
permittivity	4		[(4,3,2,1,0,0,0)]	(-3,-1,4,2,0,0,0)
second hyper-susceptibility	6		[(6,2,2,2,0,0,0)]	(-2,-2,6,2,0,0,0)
first hyper-polarisability	7		[(7,3,2,1,0,0,0)]	(-1,-2,7,3,0,0,0)
second hyper-polarisability	10		[(10,4,3,2,0,0,0)]	(-2,-3,10,4,0,0,0)

[1] N.C. Barford, Experimental measurements: precision, error and truth, 2nd edition, John Wiley & Sons, Chichester New York Brisbane Toronto Singapore, 1990.

[2] BIPM. http://www.bipm.org. accessed 14 Jun 2011.

[3] BIPM, International vocabulary of metrology - Basic and general concepts and associated terms(VIM), JCGM 200:2008

[4] W.A. Coppel, Number Theory, An introduction to Mathematics, Second Edition, Springer Science+Business Media, New York, 2009.

[5] D. Cox, J. Little, D. O'Shea, Ideal, Varieties, and Algorithms, An Introduction to Computational Algebraic Geometry and Commutative Algebra, Third Edition, Springer Science+Business Media, New York, 2007.

[6] H.S.M. Coxeter, Regular Polytopes, Third Edition, Dover Publications, New York, 1973.

[7] ,J.H. Conway, N.J.A. Sloane, Sphere packings, lattices and groups, Third Edition, Springer-Verlag, Berlin Heidelberg New York, 1999.

[8] H. Davenport, Analytic Methods for Diophantine Equations and Diophantine Inequalities, Second edition, Cambridge University Press, Cambridge, 2005.

[9] R. Diestel, Graph Theory, Second Edition, Springer-Verlag, New York, 2000.

[10] D. Hilbert, S. Cohn-Vossen, Geometry and the imagination, Translated by P. Nemenyi, AMS Chelsea Publishing, Providence, 1999.

[11] B. Grünbaum, Convex Polytopes, Second Edition, Springer-Verlag, New York, 2003.

[12] ,M. Lange, Must the Fundamental Laws of Physics be Complete?, Philosophy and Phenomenological Research, Vol. LXXVIII No. 2, (2009), 312–345

[13] S. Lipschutz, Theory and problems of Set Theory and Related Problems, McGraw-Hill book company, 1964.

[14] S. Lipschutz, Theory and problems of Linear Algebra, McGraw-Hill book company, New York, 1968.

[15] J.C. Maxwell, On the mathematical classification of physical quantities, in Proceedings of the London Mathematical Society (1874) 258–266.

[16] OEIS. http://oeis.org. accessed 14 Jun 2011.

[17] PIRSA:C10001-Laws of Nature: Their Nature and Knowability. http://pirsa.org/C10001. accessed 28 Oct 2011.

[18] D. Rickles, Symmetry, Structure and Spacetime, Elsevier, Amsterdam, 2008.

[19] J.J. Roche, The mathematics of measurement: a critical history, The Athlone Press, London, 1998.

[20] M. Tegmark, Is âĂIJthe theory of everythingâĂİ merely the ultimate ensemble theory?, Annals of Physics 270 (1998), 1-51.

[21] M. Tegmark, The Mathematical Universe, Foundations of Physics 36 (2006) 765-794.

[22] R.J. Webster, Convexity, Oxford University Press, Oxford, 2002.

[23] E.P. Wigner, Events, laws of nature, and invariance principles, Nobel Lecture, December 12, 1963.

[24] Wikipedia.http://en.wikipedia.org/wiki/Uniform_7-polytope. accessed 14 Jun 2011.

[25] Wikipedia. http://en.wikipedia.org/wiki/Integer_lattice. accessed 09 Oct 2011.

[26] G.M. Ziegler, Lectures on Polytopes, Updated Seventh Printing of the First Edition, Springer Science+Business Media, New York, 2006.