**Analytical Mechanics.** *By Antonio Fasano and Stefano Marmi*. Oxford Graduate Texts, Oxford University Press, Oxford, 2006. \$. xiii+772 pp., hardcover. ISBN 0-19-850802-6.

This is the English translation of "Meccanica Analytica" which was published in Italian in 2002 (Bollati Boringhieri editore, Torino).

It is quite an achievement to write a book of more than 700 pages, even if it comes out of lecture notes and is primarily an account of the way you have been teaching a subject at various levels. The book by Fasano and Marmi is sufficiently original in its perception and composition and rich in contents to deserve a place on the shelf of every mechanics teacher, despite the abundance of books on analytical mechanics which are available already. Whether it can be used as a syllabus for students is a different matter. It certainly covers too much material for one course. An additional complication in that respect is a rather unfortunate (at least in my opinion) ambiguity in the way the authors approach the very basics of the theory. I mean, they do want to talk about the fundamental laws of mechanics, for example, but assume at the same time that there has been a first course on mechanics preceding the one they offer. As a result, one will wonder occasionally why we get a clear definition of concept $A$, while concept $B$ is taken for granted as something everybody knows. Anyhow, the writing style of the book makes it fluent and pleasant to read.

Let us go through a survey of the contents first, and I will do my best to restrict myself to little remarks here and there, leaving the discussion of more fundamental issues for later.

One of the original elements of the book is its refreshing, quite extensive account of differential geometry of curves and surfaces in the first chapter. The rather unexpected step to offer then already a basic idea of differentiable (Riemannian) manifolds is digestible and serves the introduction of 'Lagrangian coordinates' (more commonly called 'generalized coordinates') for systems of particles with holonomic constraints. It is in that context that we are warned for the first time that the fundamental notions of the mechanics of a single point are assumed to be known. I do not quite see a reason for this as Chapter 2 is about the general laws and dynamics of a point particle and could, with a little extra effort, just as well be presented as starting from scratch, even though an elementary first course in mechanics is indeed a useful prerequisite. For example, why not include in this book a proof of the 'well known property' that $\operatorname{curl}\mathbf{F} = \mathbf{0}$ is the condition for having $\mathbf{F} = \nabla U$? Incidentally, all chapters end with interesting additional bibliographical notes and additional solved problems. One of those in chapter 2 involves phase space analysis and thus seems to anticipate the next chapter, which is about the qualitative analysis of one-dimensional motion. It is a bit strange that central force problems then only come in chapter 5, while in chapter 4 we immediately jump to systems of particles. That starts by the equations for the total linear and angular momentum of the system, which are given the typical Italian name of cardinal equations. In deriving those equations, the authors make the observation that "internal forces are in equilibrium"; I suppose that this means that the forces add up to zero, but the authors fail to observe that this is not enough to conclude that internal forces do not appear in the angular momentum equation. For holonomic systems with 'smooth constraints' (often called 'ideal constraints' in the literature), we immediately pass to Lagrange's equations, which is good! I less appreciate that $L$ is written as $T + U$ for gradient forces and that $U$ is called the potential energy of the system. Noether's theorem is presented in a restricted form. On the positive side certainly is that attention is paid again to qualitative aspects in the study of dynamical systems, by introducing Lyapunov functions and presenting some instability results. Again, the additional solved problems are very interesting but exhibit the ambiguity of this unidentified course we are supposed to have undergone already: there is a problem which involves moments of inertia and the concept of rolling without slipping, but the theory about rigid bodies yet has to come! As already indicated, in chapter 5 we go back to one particle with a case which certainly deserves a separate chapter, namely central force problems. We get an excellent account of properties one can prove in all generality and of course the integration of Kepler's problem. But there is more: Bertrand's theorem is proved; a series solution of Kepler's equation is discussed, plus its relation to a formula derived by Lagrange; also a few interesting general features of the $n$-body problem are presented. The one thing I miss in this chapter is the derivation of the Laplace-Runge-Lenz vector and particularly the way this extra conserved vector allows to find the orbits without needing to integrate the second-order equation which is usually set up for this purpose. Also, the effect that the centre of mass reduction in the 2-body problem has on Kepler's third law, would have been worth an extra line.

Chapters 6 and 7 are about the kinematics and dynamics of rigid bodies. I must admit that I am less enthusiastic about a number of issues in this part. It starts by defining rigid bodies as a set of points with rigid constraints. Towards the end of chapter 6 we are put at ease by the statement that the results can be easily extended to continua by replacing sums by integrals everywhere, but I disagree with such a point of view. The fundamental formula for the instantaneous angular velocity in rigid motions (here called Poisson's formula?) can be done better. When it comes to introducing the important concept of inertia tensor, we first learn about the computation of a matrix and then a rather awkward reasoning is set up to argue that there is something intrinsic behind it which justifies the term 'tensor of inertia'. Further on, the 'cardinal equations' referred to above in the context of systems of particles, are claimed to hold automatically also for bodies with mass distribution. Even stronger: it is almost mentioned in passing that this is equivalent to Lagrange's equations as well. And indeed, to study the motion of a spinning top a little further, the authors simply start by writing down the Lagrangian $L = T - V$ without any further justification. I am afraid I don't buy this! Leaving such questions aside, we do get a good, complete discussion of Poisson motion (while the treatment of the spinning top is much less complete). As before, there are a number of interesting additional solved problems.

Chapters 8 to 10 develop the basics of the Hamiltonian formalism. There is a good discussion of the Legendre transformation in chapter 8, and also some attention for qualitative features again via the statement and proof of Poincaré's recurrence theorem (which has a link with ergodic theory in chapter 13). Chapter 9 explains variational principles; it includes a discussion of Maupertuis' 'principle of stationary action' and the Jacobi metric. Chapter 10 has the bulk of classical Hamiltonian mechanics on $\mathbb{R}^{2n}$. What distinguishes the book from others here is that the authors first present a thorough discussion of linear Hamiltonian systems, with the introduction of symplectic matrices and symplectic vector spaces, for example. In fact another distinguishing feature, to some extent, is a proof of the symplectic rectification theorem further on. In between, we get a good account of canonical transformations, integral invariants, the canonicity of the flow of a Hamiltonian system, generating functions and Poisson brackets. The authors show an interest in canonical perturbation theory also and of course in the relationship between symmetries and first integrals. The chapter ends with a brief excursion to Hamiltonian systems on general symplectic manifolds.

The very appreciable story of chapter 10 goes crescendo, in my opinion, in the next couple of chapters. I mean, chapter 11 is for me a masterly exposé of Hamilton-Jacobi theory with a perfect mix of classical and modern aspects. First, there is the classical discussion of separation of variables, with examples in spherical, parabolic and elliptic coordinates; an interesting discussion of action-angle variables for one degree of freedom as a warm up; a proof of the classical Liouville theorem about completely integrable systems. This is followed by an extensive account of the modern version, i.e. Arnol'd's theorem on global aspects of complete integrability and related issues (part of the credit for this section is given to unpublished lecture notes of Giorgilli). The additional bibliographical notes, by the way, contain many more hints to the rich Italian history in this field and to the important contributions of Russian mathematicians. Also the section on action-angle variables for several degrees of freedom is excellent, and Kepler's problem comes back with interesting features from celestial mechanics. In chapter 12, a brief discussion is presented of a number of aspects of canonical perturbation theory for nearly-integrable Hamiltonian systems, that is Hamiltonians which are completely integrable to zeroth order in some small parameter $\epsilon$. Through interesting examples, we see a gradual progress to more complicated modern achievements in this field (which still owe credit, however, to the pioneering work of Poincaré). The "simple" problem is to design algorithmic procedures to eliminate dependence on the angle variables in the perturbed system, order by order. The hard questions are about convergence of the methods and it is shown that, in a way, the odds are very much against it by proving Poincaré's result on non-existence of an analytic first integral (different from the Hamiltonian) under some conditions of genericness and non-degeneracy. In addition, dropping the last requirement and using 'Birkhoff series' to construct a formal solution, the resulting series will generally diverge. But some related more positive statements can be made when the frequencies of the system satisfy a certain diophantine condition. Proceeding in this way, the authors prepare the stage for the famous KAM-theorem about preservation of “most” of the invariant tori of the unperturbed Hamiltonian. Including a full scale proof of this remarkable achievement would indeed go beyond the scope of the book, but the reader will perhaps hope at this point to find a sketchy idea of the different steps of a proof (in the style of the preceding sections, where the authors also skipped some of the mathematical details).

Chapter 13 on aspects of ergodic theory starts with a good introduction to the basic concepts, definitions and results in the theory of measurable dynamical systems (with reference to the book by Mañe (1987)). It is certainly interesting to get an idea of things like Lyapunov's characteristic exponents, which measure the exponential rate of divergence of orbits which are initially close, but this is a bit disconnected from the rest in a book on ‘Analytical Mechanics’. The last section, for example, which gives useful info about the important issue of the stability (or chaotic nature?) of the solar system, could have been told also at the end of the preceding chapter. Chapters 14 and 15 on statistical mechanics lead us still further away. Chapter 14 essentially models the study of diluted gases and tells us about things like the distribution function (on a 6-dimensional space), the Boltzmann equation and Boltzmann's so-called H-theorem, based on, for example, the assumption that the particles are elastic spheres. Chapter 15 is about the continuation of Boltzmann's ideas in the work of Gibbs. It discusses topics like the ergodic hypothesis, the microcanonical set and its relation to e.g. the Maxwell-Boltzmann distribution, the ‘grand canonical set’ and thermodynamical limit, with a brief link to the theory of phase transitions. Obviously, some knowledge of thermodynamics is assumed here, so it is my feeling that chapters 13 to 15 would better be part of a separate book on statistical mechanics and thermodynamics. A lot of credit is given in this part to work of Gallavotti.

The final chapter 16 gives a brief sketch of the Lagrangian formalism in field theory, with reference, in particular, to the equations of continuum mechanics. Of course, this is again a different subject, which could in fact have been incorporated as appendix to chapter 9, for example.

In conclusion, I can only repeat what I said at the beginning: this is a very interesting book for mechanics teachers, who will have to select what material is best adapted to the aims and scopes of the course(s) they are in charge of. As for critical remarks on the way the fundamentals of classical mechanics are presented, I am afraid that, despite my good intentions, I have already given most of them away in discussing the kinematics and dynamics of rigid bodies. Allow me to formulate one more element of criticism in this respect, by going back to the very first fundamental issue, namely the formulation of ‘Newton's laws’, a slippery point in probably every book! I think that the four axioms presented at the start of chapter 2 (which incorporate the existence of inertial frames and mass ratio, and anticipate so to speak on the principle of action and reaction) are quite acceptable. But these axioms say nothing about ‘force’. In fact, the authors go on to say: “If we now *define* force by ... $m\mathbf{a} = \mathbf{F} \ldots$”. And two lines further they simply state: “When $\mathbf{F}$ is *specified* as a function of position, velocity and time, the above relation is the well-known fundamental equation of the dynamics of a point particle”. In my opinion, these two phrases are incompatible: how can we specify such an $\mathbf{F}$ if we have just defined it to be $m\mathbf{a}$? I will not elaborate on this point further, however, as I have done that on a previous occasion.

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