UNIVERSITEIT GENT

Vakgroep Zuivere Wiskunde
en Computeralgebra

A study of (α, β)-geometries fully embedded in projective spaces

Sara Cauchie

Promotoren:
Prof. Dr. F. De Clerck
Dr. N. Hamilton

Mei 2003

Proefschrift voorgelegd
aan de Faculteit Wetenschappen
tot het behalen van de graad van
Doctor in de Wetenschappen
richting Wiskunde

Preface

As is already suggested by the title, the main subject of this thesis is the study of (α,β)-geometries. The idea for defining (α,β)-geometries comes from the study of partial geometries, semipartial geometries, polar spaces and generalized quadrangles. These structures were studied extensively in the past by mathematicians from all over the world. F. De Clerck and H. Van Maldeghem defined the concept of (α,β)-geometries as a generalization of all these previously mentioned structures. One of the reasons for this generalization was that the class of, for example, partial geometries turned out to be a bit too restrictive, or in other words, not too many examples exist. The class of (α,β)-geometries is much larger and therefore promises to overcome this problem. Note that all the previously mentioned structures are subclasses of the class of (α,β)-geometries and hence their examples are examples of (α,β)-geometries too. The (α,β)-geometries in this thesis are always fully embedded in projective spaces, as this gives extra conditions on the (α,β)-geometries and hence provides examples with a nice geometric structure. The main goals of my research, which resulted in writing this thesis, were to construct new (α,β)-geometries, to characterize the known examples of (α,β)-geometries, and to classify fully embedded (α,β)-geometries in projective spaces. Therefore the most important results of my thesis are classifications and characterizations of (α,β)-geometries fully embedded in projective spaces.

In the first chapter some preliminary results are mentioned. It is our aim to give an overview of some basic definitions and results of mathematical objects and concepts that will be needed later in the thesis. We have tried to give all of the definitions needed later in the first chapter. However, the results in the first chapter are only stated in function of the rest of this thesis, so they are not meant to be a reference on their own about the current state of research in the distinct mathematical research fields. We first define graphs and more specifically, strongly regular graphs. The reason for this is that with each (α,β)-geometry we can associate a graph, namely its point graph. So there is a link between (α, β)-geometries and graph theory. Next, some definitions from incidence geometry are given, and also (α, β)-geometries and their special cases, being partial geometries, semipartial geometries, polar spaces, copolar spaces, generalized quadrangles and partial quadrangles, are defined. As fully embedded (α, β)-geometries in projective spaces are the subject of this thesis, we describe what is known about full embeddings of the special cases of (α, β)-geometries. In the case of generalized quadrangles and partial geometries, a complete classification is known, for $(0, \alpha)$-geometries there exists a partial classification. In the next section, we mention some results about strongly regular (α, β)-geometries. These results were obtained by N. Hamilton and R. Mathon; strongly regular (α, β)-geometries are interesting because of their connection with strongly regular graphs. The last section contains definitions of various mathematical objects, that are not always clearly linked with each other, but that appear later in the thesis and therefore it is useful to define them in this first chapter.

In the second chapter a classification is obtained of fully embedded (α, β)-geometries in projective spaces, for q odd and $\alpha > 1$, under some additional assumptions. This classification is only valid in the case that q is odd. In the q even case most of the theorems also hold and give examples of fully embedded (α, β)-geometries, but they do not form a complete classification. The reason for this is that we use the important result of S. Ball, A. Blokhuis and F. Mazzocca that says that for q odd there do not exist non-trivial maximal arcs in Desarguesian projective planes. We exclude in our classification also the case that $\alpha = 1$, as this is somehow a special case. In the first section of this chapter, we study $(1, \beta)$-geometries fully embedded in $\mathrm{PG}(3, q)$ and prove a non-existence result. In projective spaces of dimension greater than three, it turned out not to be possible to get a classification of full projective embeddings of $(1, \beta)$-geometries with the techniques that we have used in three dimensional spaces.

In the third chapter we have completely classified fully embedded $(1, q)$-geometries in $\mathrm{PG}(n, q)$, for $q \neq 2$. When the values of α and β are fixed from the start, this gives us strong conditions, which makes it easier to get a complete classification. Since $(1, q+1)$-geometries are a special class of polar spaces, their full embeddings in projective spaces were already classified by F. Buekenhout and C. Lefèvre, and for the $(0, q)$-geometries a classification has been proved by J. I. Hall. Moreover $(0, 1)$-geometries have been studied by P. J. Cameron and proper $(0, q + 1)$-geometries turn out to be a disjoint union of partial geometries for which $\alpha = q + 1$. Hence the case of $(1, q)$-geometries was the next case to be studied in this sense. In this chapter we therefore also give an overview of the known results for proper (α, β)-

geometries with $\alpha, \beta \in \{0, 1, q, q+1\}$.

In the fourth chapter, we obtain characterization theorems for some of the (α, β)-geometries discovered in the second chapter. In the first section we give a characterization of two $(q, q+1)$-geometries fully embedded in an n-dimensional projective space, assuming the axiom of Pasch, also called axiom of Veblen or axiom of Veblen-Young. The idea for this characterization comes from the existing characterization of a certain partial geometry, by F. De Clerck and J. A. Thas. This partial geometry is indeed very similar, from a geometrical point of view, to the $(q, q+1)$-geometries that we have characterized. In the second section, a characterization is obtained for the $((q-1)/2, (q+1)/2)$-geometry fully embedded in a three dimensional projective space, that was constructed by J. A. Thas.

Acknowledgements

I am very much indebted to the GOA-project "Incidence Geometries: Constructions, Characterizations and Applications" of Ghent University, designed by Prof. Dr. F. De Clerck, Prof. Dr. L. Storme, Prof. Dr. J. A. Thas and Prof. Dr. H. Van Maldeghem, which gave me the financial support that made it possible for me to do research during the last three and a half years. Also I am grateful to my promoter Prof. Dr. F. De Clerck, for stimulating me to start my mathematical research and giving me the opportunity to do so, for introducing me to the world of research and for reading my abstracts, preprints and a draft of this thesis. Next, I want to say thanks to Dr. N. Hamilton, who always believed in my value as a researcher. Without his encouragement, I would probably not have written this thesis. Dr. N. Hamilton and I have spent many hours discussing mathematics, which was very stimulating for me and which was a great help for my research. Also he carefully read my proofs and taught me how to formulate them in a better way, which is surely a useful thing. Finally, I want to say thanks to my colleagues, my family and my friends for being there for me, for listening to my worries and problems and for the nice moments that we shared in the past few years.

Contents

Chapter 1

Introduction

In this chapter we will briefly summarize some basic concepts on graphs and incidence structures. We are not aiming to give a full overview, as this would lead us too far. We will give references to more detailed literature for the interested reader. One should keep in mind that the main reason of this introduction is to give some definitions and examples that will be important for the rest of the thesis and to introduce the notation that we will use further on.

1.1 Graphs

1.1.1 Some general definitions from graph theory

A finite *graph* $\Gamma = (V, E)$ consists of a non-empty set V and a set E of unordered pairs of elements of V. The elements of V are called the *vertices* of the graph Γ, while the elements of E are called the *edges*. Two vertices are called *adjacent* if they are contained in an edge of Γ. If x and y are distinct adjacent vertices of Γ, then we write $x \sim y$. If x and y are distinct non-adjacent vertices of Γ, then we write $x \not\sim y$. Furthermore we will always assume that $x \not\sim x$, for every vertex x of Γ. If E is the set of all unordered pairs of V, then Γ is called the *complete graph on v vertices* and is denoted by K_v. The *complement* of a graph Γ is the graph Γ^C, that has the same set V of vertices as Γ, but in which two distinct vertices x and y of V are adjacent if and only if x and y are non-adjacent vertices of Γ.

Let x and y be two vertices of Γ. A *path* of length m from x to y, is a sequence of vertices $x = x_0, x_1, x_2, \ldots, x_m = y$ in the graph, such that $x_i \neq x_{i+2}$, $0 \leq i \leq m-2$ and $x_i \sim x_{i+1}$, $0 \leq i \leq m-1$. If $x = y$ then any such path of length at least three will be called a *circuit*. Two distinct vertices x and y of a graph Γ are said to be at *distance* $\delta(x, y)$, if there exists a path of length $\delta(x, y)$ between x and y, but not a shorter path. By definition a vertex has distance 0 from itself, and a vertex has distance 1 from all the vertices adjacent to it. We will denote by $\Gamma_i(x)$ the set of all vertices of Γ at distance i from the vertex x of Γ. The set $\Gamma_1(x)$ will also be denoted by $\Gamma(x)$, for reasons of convenience. If for all vertices x of Γ we have that $|\Gamma(x)| = k$, then Γ is a *regular* graph of *valency* or *degree* k. A graph Γ is called *connected* if and only if for any two distinct vertices x and y of Γ, there is at least one path from x to y. The *diameter* d of a connected graph Γ is the maximum value of the distance function $\delta(x, y)$. The *girth* of a graph Γ having at least one circuit, is the length of its shortest circuit.

1.1.2 Strongly regular graphs

A *strongly regular graph* $\mathrm{srg}(v, k, \lambda, \mu)$ is a graph Γ, that has v vertices, that is regular of degree k and that satisfies the following two conditions.

1. For every two adjacent vertices x and y of Γ, there are exactly λ vertices of Γ that are adjacent with both x and y.

2. For every two distinct non-adjacent vertices x and y or Γ, there are exactly μ vertices of Γ that are adjacent with both x and y.

As we do not want to have to consider disconnected graphs and their complements, we assume moreover that $0 < \mu < k < v - 1$. It is easy to prove that the complement of a strongly regular graph $\mathrm{srg}(v, k, \lambda, \mu)$ is a $\mathrm{srg}(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$.

Let Γ be a graph with v vertices, numbered as $1, \ldots, v$. The *adjacency matrix* $A = (a_{ij})$ of Γ is the $(v \times v)$-matrix defined as follows:

$$\begin{cases} a_{ij} = a_{ji} = 1 \iff i \sim j & i, j = 1, \ldots, v, \\ a_{ij} = a_{ji} = 0 \iff i \not\sim j & i, j = 1, \ldots, v. \end{cases}$$

Clearly the matrix A is a symmetric matrix and $a_{ii} = 0$ for $i = 1, \ldots, v$. The *Bose-Mesner algebra* of a strongly regular graph Γ is the three dimensional algebra generated by I, J and A, where J is the matrix with all of its entries equal to 1.

In the following theorem, some properties of a $\mathrm{srg}(v, k, \lambda, \mu)$ are summarized. These properties imply necessary conditions for the existence of strongly regular graphs. For the proofs and more information on strongly regular graphs we refer to [5, 57].

Theorem 1.1.1 *If* Γ *is a* $\mathrm{srg}(v,k,\lambda,\mu)$, *then the following holds.*

1. $v-2k+\mu-2\geq 0$.

2. $k(k-\lambda-1)=\mu(v-k-1)$.

3. *If* A *is the adjacency matrix of* Γ, *then* $AJ=kJ$,

$$A^2+(\mu-\lambda)A+(\mu-k)I=\mu J,$$

and A *has three eigenvalues* k, r *and* l *such that*

$$r,l=\frac{\lambda-\mu\pm\sqrt{(\lambda-\mu)^2+4(k-\mu)}}{2} \qquad r>l,$$

with multiplicities respectively

$$1, \quad f=\frac{-k(l+1)(k-l)}{(k+rl)(r-l)}, \quad g=\frac{k(r+1)(k-r)}{(k+rl)(r-l)},$$

and clearly f *and* g *must be integers.*

4. *The eigenvalues* $r>0$ *and* $l<0$ *are both integers, except for one family of graphs, the so-called conference graphs. A conference graph is a* $\mathrm{srg}(2k+1,k,\frac{k}{2}-1,\frac{k}{2})$, *for which number* $2k+1$ *of vertices can be written as a sum of two squares, and the eigenvalues are* $\frac{-1+\sqrt{v}}{2}$ *and* $\frac{-1-\sqrt{v}}{2}$.

Besides the properties stated in the previous theorem, there are some other important necessary conditions on the existence of strongly regular graphs. In the following theorem we mention four such conditions, namely the Krein conditions, the two absolute bounds , the claw bound and the Hoffman bound.

Theorem 1.1.2 ([5]) *If* Γ *is a* $\mathrm{srg}(v,k,\lambda,\mu)$, *then the following holds.*

1. *The two Krein conditions:*

$$\begin{aligned}(r+1)(k+r+2rl) &\leq (k+r)(l+1)^2,\\ (l+1)(k+l+2rl) &\leq (k+l)(r+1)^2.\end{aligned}$$

2. *The two absolute bounds:*

- *$v \leq \frac{1}{2}f(f+3)$, and if there is no equality in the first Krein condition then $v \leq \frac{1}{2}f(f+1)$,*
- *$v \leq \frac{1}{2}g(g+3)$, and if there is no equality in the second Krein condition then $v \leq \frac{1}{2}g(g+1)$.*

3. *The claw bound: If $\mu \neq l^2$ and $\mu \neq l(l+1)$, then the following inequality holds: $2(r+1) \leq l(l+1)(\mu+1)$.*

4. *The Hoffman bound:*

   - *If C is a clique of Γ, then $|C| \leq 1 - \frac{k}{l}$, with equality if and only if every vertex $x \notin C$ has the same number of neighbours in C (this number is then $-\frac{\mu}{l}$),*
   - *If C is a coclique of Γ, then $|C| \leq v(1-\frac{k}{l})^{-1}$, with equality if and only if every vertex $x \notin C$ has the same number of neighbours in C (this number is then $-l$).*

1.2 (α, β)-geometries

1.2.1 Definitions

A *partial linear space* of order (s,t), for some s and t, is a connected incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with $\mathcal{P}$ a finite non-empty set of elements called *points*, $\mathcal{L}$ a family of subsets of $\mathcal{P}$ called *lines* and a symmetric incidence relation $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ satisfying the following axioms.

1. Any two distinct points are incident with at most one line.
2. Each line is incident with exactly $s+1$ points, $s \geq 1$.
3. Each point is incident with exactly $t+1$ lines, $t \geq 1$.

If a point $p \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$, then we also say that the point p *lies on* or *belongs to* the line L, or that the line L *contains* or *goes through* the point p. Two points p_1, $p_2 \in \mathcal{P}$, are *collinear* if there is a line $L \in \mathcal{L}$ such that $p_1 \mathrm{I} L$ and $p_2 \mathrm{I} L$. Two lines L_1, $L_2 \in \mathcal{L}$, are *concurrent* if there is a point $p \in \mathcal{P}$ such that $p \mathrm{I} L_1$ and $p \mathrm{I} L_2$. The elements of I will also be called the *flags* of $\mathcal{S}$, while the elements that belong to $((\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})) \setminus \mathrm{I}$ will be called the *antiflags* of $\mathcal{S}$.

The *incidence number* $i(x,L)$ of an antiflag (x,L) of $\mathcal{S}$ is the number, of points collinear with the point $x \in \mathcal{P}$ and incident with the line $L \in \mathcal{L}$. The *point graph* of a partial linear space $\mathcal{S}$ is the graph $\Gamma(\mathcal{S})$ with vertices the points of $\mathcal{S}$ and such that two vertices of $\Gamma(\mathcal{S})$ are adjacent if and only if the corresponding points of $\mathcal{S}$ are collinear in $\mathcal{S}$.

An (α, β)-*geometry* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ of order (s, t), for some s and t, such that for any $x \in \mathcal{P}$ and any $L \in \mathcal{L}$, x not incident with L, we have that $i(x, L) = \alpha$ or $i(x, L) = \beta$, and both cases occur. Although the concept of an (α, β)-geometry was commonly known for special values of α and β, the general definition appeared for the first time in [22].

An (α, β)-geometry is said to be *proper* if $\alpha > 0$, $\beta > 0$ and $\alpha \neq \beta$. We will always assume that $\alpha < \beta$, unless in the case $\alpha = 0$, where we will sometimes speak of $(0, \alpha)$-geometries instead of $(0, \beta)$-geometries. An (α, β)-geometry is *strongly regular* if its point graph is a strongly regular graph (see [32]). We will give some more information on strongly regular (α, β)-geometries in section 1.3.

An (α, β)-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is said to be *fully embedded* in a projective space $\mathrm{PG}(n, q)$ if $\mathcal{P}$ is a subset of the point set of $\mathrm{PG}(n, q)$, $\mathcal{L}$ is a subset of the line set of $\mathrm{PG}(n, q)$, I is the incidence inherited from $\mathrm{PG}(n, q)$ and $s = q$. We require that the points of $\mathcal{S}$ span $\mathrm{PG}(n, q)$. The subject of this thesis is to study fully embedded (α, β)-geometries in $\mathrm{PG}(n, q)$.

1.2.2 Some special (α, β)-geometries

Although the terminology of (α, β)-geometries has not been used until recently, several particular (α, β)-geometries have been the subject of lots of research in the past.

The first class of (α, β)-geometries being studied extensively was the one of the polar spaces [9]. It follows from [9] that non-degenerate polar spaces are in fact nothing else then $(1, s+1)$-geometries. Inspired by this work, J. I. Hall started studying $(0, s)$-geometries, which he called copolar spaces [29]. He managed to get a complete classification of the so called *reduced* $(0, s)$-geometries; these are $(0, s)$-geometries $\mathcal{S}$ such that for every two distinct points x and y of $\mathcal{S}$, $\Gamma(x) \neq \Gamma(y)$, where $\Gamma(\mathcal{S})$ is the point graph of $\mathcal{S}$.

Another special class of (α, β)-geometries that were studied before are the generalized quadrangles. Generalized quadrangles are (α, β)-geometries that have $\alpha = \beta = 1$. On generalized quadrangles there is a lot of literature. We refer here to [42], which is a standard work on this topic and in which other useful references can be found.

A generalization of the class of the generalized quadrangles is the class of the partial quadrangles. Partial quadrangles are $(0, 1)$-geometries, with the extra condition that their point graph has to be a strongly regular graph (see [10]).

Next, there is the class of the partial geometries. Partial geometries are (α, β)-geometries that have $\alpha = \beta$. They are denoted as $\mathrm{pg}(s, t, \alpha)$. Partial geometries were introduced by R. C. Bose in [4]. Later on lots of papers on partial geometries appeared (see for instance [47, 48, 20, 49, 57, 19, 51, 16, 17, 14]). A partial geometry is called proper if $1 < \alpha < \min(s, t)$.

A last special class of (α, β)-geometries is the one of the semipartial geometries. Semipartial geometries are $(0, \alpha)$-geometries that have a strongly regular point graph. Semipartial geometries have been studied for example by I. Debroey and J. A. Thas ([25, 24, 23]), J. A. Thas ([52, 53]) and by M. Delanote [26]. The paper [15] is a nice summary on the present status of knowledge on proper partial and semipartial geometries.

1.2.3 Full embeddings of generalized quadrangles in $\mathrm{PG}(n, q)$

Generalized quadrangles fully embeddable in a projective space have been completely classified.

Theorem 1.2.1 ([7]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a generalized quadrangle fully embedded in $\mathrm{PG}(n, q)$. Then either*

1. *$n = 3$ and $\mathcal{S}$ is formed by the points and lines of a non-singular hyperbolic quadric of $\mathrm{PG}(3, q)$, $\mathcal{S}$ is formed by the points and lines of a non-singular Hermitian variety of $\mathrm{PG}(3, q)$ (in this case q is a square), or the points of $\mathcal{S}$ are all the points of $\mathrm{PG}(3, q)$, while the lines of $\mathcal{S}$ are the lines of $\mathrm{PG}(3, q)$ that are totally isotropic with respect to a symplectic polarity in $\mathrm{PG}(3, q)$;*

2. *$n = 4$ and $\mathcal{S}$ is formed by the points and lines of a non-singular parabolic quadric of $\mathrm{PG}(4, q)$, or $\mathcal{S}$ is formed by the points and lines of a non-singular Hermitian variety of $\mathrm{PG}(4, q)$ (in this case q is a square);*

3. *$n = 5$ and $\mathcal{S}$ is formed by the points and lines of a non-singular elliptic quadric of $\mathrm{PG}(5, q)$.*

The generalized quadrangles that appear in the classification are called *classical* generalized quadrangles, because they all are associated with classical groups.

1.2.4 Full embeddings of partial geometries in $\mathrm{PG}(n, q)$

Partial geometries fully embedded in a projective space $\mathrm{PG}(n, q)$ have been studied by F. De Clerck and J. A. Thas [20]. They managed to prove that the only partial geometry fully embedded in $\mathrm{PG}(n,q)$, for which $1 < \alpha < q+1$ and $\alpha < t+1$, is the partial geometry that we will define now, and which we will denote by H_q^n. The set $\mathcal{P}$ of points of H_q^n is the set of points of the projective space $\mathrm{PG}(n,q)$, that are not contained in an $(n-2)$-dimensional subspace H of $\mathrm{PG}(n,q)$. The set $\mathcal{L}$ of lines of H_q^n is the set of lines of $\mathrm{PG}(n,q)$ that have no point in common with H. The incidence relation I of H_q^n is the incidence of $\mathrm{PG}(n,q)$ restricted to $(\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$. It is easy to prove that H_q^n is indeed a partial geometry and that it has the following parameters: $s = q$, $t = q^{n-1} - 1$ and $\alpha = q$.

The following theorem classifies all partial geometries fully embeddable in a projective space.

Theorem 1.2.2 ([20]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial geometry with parameters $s = q$, t and α. Assume that $\mathcal{S}$ is fully embeddable in a projective space $\mathrm{PG}(n,q)$. Then the following cases may occur.*

1. *$\mathcal{S}$ is the design of points and lines of $\mathrm{PG}(n,q)$, and $\alpha = q+1$.*
2. *$\mathcal{S}$ is a classical generalized quadrangle (classified in theorem 1.2.1), and $\alpha = 1$.*
3. *$\mathcal{S}$ is a dual design in $\mathrm{PG}(2,q)$, and $\alpha = t+1$.*
4. *$\mathcal{S} = \mathrm{H}_q^n$ and $\alpha = q$ $(n \geq 3)$.*

1.2.5 Full embeddings of $(0,\alpha)$-geometries in $\mathrm{PG}(n,q)$

Full embeddings of $(0,\alpha)$-geometries in $\mathrm{PG}(n,q)$, $\alpha > 1$, have been studied in [21] and [55]. In these papers, a complete classification is obtained, except for the case of $(0,\alpha)$-geometries in $\mathrm{PG}(3,q)$ and the case of $(0,2)$-geometries in $\mathrm{PG}(n,2)$, for $n > 4$.

We will first give an example of a fully embeddable semipartial geometry. Therefore let $Q^-(3,2)$ be a three dimensional elliptic quadric in $\mathrm{PG}(3,2)$. Let $\mathcal{P}$ be the set of points of $\mathrm{PG}(3,2) \setminus Q^-(3,2)$, let $\mathcal{L}$ be the set of lines of $\mathrm{PG}(3,2)$ that are skew to $Q^-(3,2)$, and let I be the incidence of $\mathrm{PG}(3,2)$ restricted to $(\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$. Then $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a semipartial geometry fully embedded in $\mathrm{PG}(3,2)$. This $(0,2)$-geometry is usually denoted by $\mathrm{NQ}^-(3,2)$. The same construction can be used in $\mathrm{PG}(3,2^h)$, but then with a hyperbolic quadric $Q^+(3,2^h)$ replacing $Q^-(3,2)$, $h \geq 2$. We denote the semipartial geometry coming from this construction by $\mathrm{NQ}^+(3,2^h)$. Also in $\mathrm{PG}(4,2)$, with a parabolic quadric $Q(4,2)$, this construction can be used and gives a semipartial geometry, that we denote by $\mathrm{NQ}(4,2)$.

It is clear that a $(0,\alpha)$-geometry fully embedded in a plane $\mathrm{PG}(2,q)$ has to be a partial geometry. The following theorem partly classifies the $(0,\alpha)$-geometries, with $\alpha > 1$ and fully embedded in $\mathrm{PG}(3,q)$, that are not contained in a plane. The theorem uses the following terminology. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in a $\mathrm{PG}(3,q)$, but not contained in a plane. A point p is said to be *isolated* in a plane π if there are no lines of $\mathcal{S}$ in π that contain the point p.

Theorem 1.2.3 ([21]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in a $\mathrm{PG}(3,q)$. If there is a plane of $\mathrm{PG}(3,q)$ that contains an antiflag of $\mathcal{S}$ and no isolated points, then $\mathcal{S}$ is one of the following.*

1. *$\alpha = q + 1$ and $\mathcal{S}$ is the design of points and lines of $\mathrm{PG}(3,q)$;*
2. *$\alpha = q$ and $\mathcal{S} = \mathrm{H}_q^3$;*
3. *$\alpha = q = 2$ and $\mathcal{S} = \mathrm{NQ}^-(3,2)$.*

The cases not covered by the previous theorem (that is, if every plane of $\mathrm{PG}(3,q)$ that contains an antiflag of $\mathcal{S}$, contains at least one isolated point) remain still open.

We consider the following example of a semipartial geometry, which we will denote by $W(n,2k,q)$. Let W be a symplectic polarity of rank $2k$. Then $W(n,2k,q) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ has points the set of points of $\mathrm{PG}(n,q) \setminus \mathrm{rad}\, W$, lines the set of lines of $\mathrm{PG}(n,q)$ that are not totally isotropic with respect to the polarity W, and incidence that of $\mathrm{PG}(n,q)$ restricted to the points and lines of $W(n,2k,q)$. In [55] it has been conjectured that each $(0,\alpha)$-geometry, $\alpha > 1$, fully embedded in $\mathrm{PG}(3,q)$, either contains a plane in which there is an antiflag of $\mathcal{S}$, but no isolated points (and hence it is classified by theorem 1.2.3), or it is one of the following:

1. each plane of $\mathrm{PG}(3,q)$ contains an antiflag of $\mathcal{S}$, and $\mathcal{S} = W(3,4,q)$;
2. there is a plane containing points of $\mathcal{S}$ but no lines of $\mathcal{S}$, and $\mathcal{S} = \mathrm{NQ}^+(3,2^h)$.

In the case $n \geq 4$ and $q \neq 2$, $(0,\alpha)$-geometries fully embedded in $\mathrm{PG}(n,q)$ are also completely classified. A complete classification of $(0,\alpha)$-geometries fully embedded in $\mathrm{PG}(n,q)$, $n > 3$ and $q > 2$, is given in the following theorem.

Theorem 1.2.4 ([55]) *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(0,\alpha)$-geometry, $\alpha \neq 1$, fully embedded in $\mathrm{PG}(n,q)$, $n > 3$ and $q > 2$. Then $\mathcal{S}$ is the design of points and lines of $\mathrm{PG}(n,q)$, $\mathcal{S} = \mathrm{H}_q^n$ or $\mathcal{S} = W(n,2k,q)$.*

A last theorem completely classifies $(0, 2)$-geometries fully embedded in $\mathrm{PG}(4, 2)$. Note that to classify $(0, 2)$-geometries, it suffices to classify reduced $(0, 2)$-geometries, as defined in section 1.2.2. In the next theorem the following notation is used: $U_{2,3}(m)$ is the semipartial geometry with points the set of all non-ordered pairs of elements of the set $\{1, \ldots, m\}$, with lines the set of all non-ordered triples of elements of the set $\{1, \ldots, m\}$, and with incidence being inclusion.

Theorem 1.2.5 ([55]) *If $\mathcal{S}$ is a reduced $(0, 2)$-geometry fully embedded in $\mathrm{PG}(4, 2)$, then $\mathcal{S}$ is one of the following.*

1. $\mathcal{S} = \mathrm{NQ}(4, 2)$ *(then $v = 15$, $t = 3$);*
2. *$\mathcal{S}$ is a representation of $U_{2,3}(7)$ which is unique up to a projectivity of $\mathrm{PG}(4, 2)$ (then $v = 21$, $t = 4$).*

The general problem of classifying all $(0, 2)$-geometries fully embeddable in $\mathrm{PG}(n, 2)$, $n > 4$, turns out to be quite complicated. Note that these geometries are in fact copolar spaces, as defined is section 1.2.2, and the copolar spaces are classified (see [29]). An example of a copolar space is $U_{2,3}(m)$, as described above. Some full embeddings of $U_{2,3}(m)$ in $\mathrm{PG}(n, q)$ are known (see therefore [40]), but not all possible projective embeddings of $U_{2,3}(m)$ are classified. Moreover the problem of determining all full embeddings of $U_{2,3}(m)$ in $\mathrm{PG}(n, q)$ is equivalent to determining (up to equivalence) all binary codes of length n with all weights even and minimum weight greater than 4 (for more information see [30]). Hence there is not much hope that a complete classification will be found in the near future.

1.3 Strongly regular (α, β)-geometries

In this section we will discuss strongly regular (α, β)-geometries. The results mentioned here are taken from a paper of N. Hamilton and R. Mathon [32].

1.3.1 Necessary conditions for existence

By definition an (α, β)-geometry is called strongly regular if its point graph is a strongly regular graph. Hence from the conditions of existence of a strongly regular graph, one can deduce similar conditions for the existence of strongly regular (α, β)-geometries (see theorems 1.1.1 and 1.1.2). Some other conditions for the existence of (α, β)-geometries are summarized in the next theorem.

Theorem 1.3.1 ([32]) *Let $\mathcal{S}$ be a proper strongly regular (α,β)-geometry with parameters (s,t). Then the following conditions hold.*

1. $\beta-\alpha \mid \beta(v-s-1)-st(s+1)$.
2. $v \leq b$ *and* $s \leq t$.
3. *If* $\beta = s+1$, *then* $(s+1-\alpha) \mid (t+1)(s-\alpha)(t-c)$, *where* c *is a constant equal to* $(st+s-\lambda-1)/(s+1-\alpha)$.

1.3.2 Strongly regular (α,β)-reguli

In [53], J. A. Thas defined SPG reguli in order to construct semipartial geometries. This definition can be generalized to the concept of (α,β)-reguli, which allowed N. Hamilton and R. Mathon to construct strongly regular (α,β)-geometries.

A *strongly regular* (α,β)*-regulus* is a collection $\mathcal{R}$ of m-dimensional subspaces of $\mathrm{PG}(n,q)$, $|\mathcal{R}|>1$, satisfying the following conditions.

1. $\pi_i \cap \pi_j = \emptyset$ for every $\pi_i,\pi_j \in \mathcal{R}$, $\pi_i \neq \pi_j$.
2. If an $(m+1)$-dimensional subspace of $\mathrm{PG}(n,q)$ contains some $\pi_i \in \mathcal{R}$, then it has a point in common with either α or β subspaces of $\mathcal{R}\setminus\{\pi_i\}$. Such an $(m+1)$-dimensional subspace that meets α (respectively β) elements of $\mathcal{R}\setminus\{\pi_i\}$ is said to be an α-secant (respectively β-secant) to $\mathcal{R}$ at π_i.
3. If a point of $\mathrm{PG}(n,q)$ is contained in an element π_i of $\mathcal{R}$, then it is contained in a constant number p of α-secant $(m+1)$-dimensional spaces on elements of $\mathcal{R}\setminus\{\pi_i\}$.
4. If a point of $\mathrm{PG}(n,q)$ is contained in no element of $\mathcal{R}$ then it is contained in a constant number r of α-secant $(m+1)$-dimensional spaces of $\mathcal{R}$.

Note that if $\alpha=0$, then a strongly regular (α,β)-regulus is an SPG regulus as defined in [53]. Now with this definition of a strongly regular (α,β)-regulus, the following theorem can be proved.

Theorem 1.3.2 ([32]) *Let $\mathcal{R}$ be a strongly regular (α,β)-regulus, such that the elements of $\mathcal{R}$ are contained in $\mathrm{PG}(n,q)$ and of dimension m. Embed $\mathrm{PG}(n,q)$ as a hyperplane Π in $\mathrm{PG}(n+1,q)$, and define an incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ of points and lines as follows.*

1. *The point set* $\mathcal{P}$ *is the set of points of* $\mathrm{PG}(n+1,q)\setminus\Pi$.

2. *The line set* $\mathcal{L}$ *is the set of all the* $(m+1)$*-dimensional subspaces of* $\mathrm{PG}(n+1,q)$ *that meet* Π *in an element of* $\mathcal{R}$.

3. *Incidence* I *is containment.*

Then $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ *is a strongly regular* (α,β)*-geometry with parameters* $s=q^{m+1}-1$ *and* $t=|\mathcal{R}|-1$.

An interesting example of a strongly regular (α,β)-geometry constructed from an (α,β)-regulus is the following. Let $Q^{\pm}(2n+1,q)$ be a non-degenerate hyperbolic or elliptic quadric contained in $\mathrm{PG}(2n+1,q)$. Assume that there exists a partition Σ of the points of $\mathrm{PG}(2n+1,q)\setminus Q^{\pm}(2n+1,q)$ into lines. Every plane through an element L of Σ intersects $Q^{\pm}(2n+1,q)$ in either one point or in a conic. Hence every such plane contains q^2-q-1 or q^2-1 points of $\mathrm{PG}(n,q)\setminus(Q^{\pm}(2n+1,q)\cup L)$. It follows that the partition Σ is a strongly regular (α,β)-regulus with $\alpha=q^2-q-1$ and $\beta=q^2-1$. Indeed, the first two conditions of the definition of an (α,β)-regulus follow immediately, and the other conditions follow since $Q^{\pm}(2n+1,q)$ has two intersection sizes with respect to hyperplanes.

Note that the partition Σ does not exist for every choice of n and q. It has been proved by J. A. Thas that such a partition exists for an elliptic quadric $Q^{-}(2n+1,q)$ if and only if n is even, and for a hyperbolic quadric $Q^{+}(2n+1,q)$ if and only if n is odd. Hence we get in this way two classes of (q^2-q-1,q^2-1)-geometries, namely the ones coming from a partition of the points of $\mathrm{PG}(4n+1,q)\setminus Q^{-}(4n+1,q)$ into lines and the ones coming from a partition of the points of $\mathrm{PG}(4n+3,q)\setminus Q^{+}(4n+3,q)$ into lines.

1.4 Some other useful definitions

In this section, we will give the definitions of some mathematical objects that will appear in later chapters of this thesis. It is not necessary to give very detailed information here, we will only mention some basic properties, and give references for whom wants to know more.

A *maximal arc* is a set of points $\mathcal{K}$ in a projective plane π of order q, such that on each line of π there lie either 0 or d points of $\mathcal{K}$. The number d is called the *order* of the maximal arc. It follows immediately that $|\mathcal{K}|=qd+d-q$. Hence through each point of π not in $\mathcal{K}$ there are $q+1-q/d$ lines containing d points of $\mathcal{K}$. This implies that, if not all points of π are points of $\mathcal{K}$, then $d|q$. A maximal arc $\mathcal{K}$ is called *trivial* if $\mathcal{K}=\emptyset$

($d = 0$), if $\mathcal{K}$ consists of exactly one point ($d = 1$), if $\mathcal{K}$ is the set of all points of an affine plane ($d = q$) or if $\mathcal{K}$ is the set of all points of the plane π ($d = q + 1$). A *hyperoval* is a maximal arc of order 2. Lots of interesting results on maximal arcs have been obtained. It has been proved by R. H. F. Denniston [28] that for every $d|q$ and q even, there exists a maximal arc of order d in $\mathrm{PG}(2, q)$. In the case q odd, it has been proved by S. Ball, A. Blokhuis and F. Mazzocca that there exists no non-trivial maximal arc in a Desarguesian projective plane $\mathrm{PG}(2, q)$ [1]. For more information and constructions of maximal arcs that are different from the one given by R. H. F. Denniston, we refer to the paper of J. A. Thas [50], the paper of R. Mathon [41] and the one of N. Hamilton and R. Mathon [31].

A *hyperplane* of a projective space $\mathrm{PG}(n, q)$ is an $(n-1)$-dimensional subspace of $\mathrm{PG}(n, q)$.

Let π be a projective plane of square order q. A *unital* is a set of $q\sqrt{q} + 1$ points in π, such that every line of π contains either 1 or $\sqrt{q} + 1$ points of this set. A *Baer subplane* of π is a subplane of order $\sqrt{q}$ of π. A Baer subplane of π has the property that every line of π intersects it in either 1 or $\sqrt{q}+1$ points. More detailed information about unitals and Baer subspaces in Desarguesian projective planes can be found in [34] and the references that are given there.

A *Baer subspace* of the projective space $\mathrm{PG}(n, q)$ is an m-dimensional subspace of order $\sqrt{q}$, with $3 \leq m \leq n$. In other words, it is a subspace of the form $\mathrm{PG}(m, \sqrt{q})$.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial geometry $\mathrm{pg}(s, t, \alpha)$. A *spread* of the partial geometry $\mathcal{S}$ is a partition of the points of $\mathcal{S}$ into lines. If $\mathcal{S}$ has a spread, then the number $s + 1$ of points of $\mathcal{S}$ on a line of $\mathcal{S}$ has to divide the total number of points of $\mathcal{S}$. So a partial geometry $\mathcal{S}$ can only have a spread if $\alpha \mid st$. An *ovoid* of the partial geometry $\mathcal{S}$ is a set of points $\mathcal{O}$ of $\mathcal{S}$ such that every line of $\mathcal{S}$ contains exactly one point of $\mathcal{O}$. Counting in two different ways the pairs (p, L), where p is an element of $\mathcal{O}$ and L is a line of $\mathcal{S}$, we get that $(t + 1)(st + \alpha)/\alpha = (t + 1)|\mathcal{O}|$. Hence an ovoid $\mathcal{O}$ of the partial geometry $\mathcal{S}$ contains exactly $1 + st/\alpha$ points of $\mathcal{S}$.

A *set of type* $(r_1, r_2, \ldots, r_s)$ in a projective space $\mathrm{PG}(n, q)$ is a set of points of $\mathrm{PG}(n, q)$ such that for each line L of $\mathrm{PG}(n, q)$ we have that $|L \cap \mathcal{K}| \in \{r_1, r_2, \ldots, r_s\}$. A set of type $(r_1, r_2, \ldots, r_s)$ is also sometimes called a *set of type* $(r_1, r_2, \ldots, r_s)$ *with respect to lines* or an $(r_1, r_2, \ldots, r_s)$*-set*. A subset $\mathcal{K}$ of points of $\mathrm{PG}(n, q)$ is called a *quadratic set* if

1. $\mathcal{K}$ is a set of type $\{0, 1, 2, q + 1\}$,

2. for each element p of $\mathcal{K}$, the union of the lines through p that intersect

$\mathcal{K}$ in either 1 element (namely p) or in $q+1$ elements, form the *tangent space* at the point p, which is either a hyperplane or the whole projective space $\mathrm{PG}(n,q)$.

A point p of the quadratic set $\mathcal{K}$ is called *singular* if the tangent space at the point p is the projective space $\mathrm{PG}(n,q)$. If $\mathcal{K}$ has a singular point, then $\mathcal{K}$ is called *singular*. If a quadratic set $\mathcal{K}$ does not contain a line, then it is an *ovoid*.

1.5 Sets of type $(0,1,r,q+1)$ in $\mathrm{PG}(n,q)$

Sets of type $(0,1,r,q+1)$ in a projective space $\mathrm{PG}(n,q)$ have been studied extensively in earlier years and many nice results were obtained. As such $(0,1,r,q+1)$-sets appear at several places in this thesis, we give here a summary of the results that are known. We however restrict ourselves to $(0,1,r,q+1)$-sets $\mathcal{K}$ in $\mathrm{PG}(n,q)$ with the property that there is a line of $\mathrm{PG}(n,q)$ that contains 0 points of $\mathcal{K}$, as $(1,r,q+1)$-sets do not appear further in this thesis. If no line of $\mathrm{PG}(n,q)$ contains r points of $\mathcal{K}$, then $\mathcal{K}$ is the set of points of a subspace of $\mathrm{PG}(n,q)$. Hence from now on we always assume that there is a line of $\mathrm{PG}(n,q)$ that contains exactly r points of $\mathcal{K}$.

1.5.1 Sets of type $(0,1,2,q+1)$ in $\mathrm{PG}(n,q)$

In the previous section we have defined what a quadratic set is. From this definition it follows that a quadratic set is a $(0,1,2,q+1)$-set satisfying an additional condition. The following theorem classifies non-singular quadratic sets in $\mathrm{PG}(n,q)$.

Theorem 1.5.1 ([35], theorem 22.10.23) *A non-singular quadratic set $\mathcal{K}$ in $\mathrm{PG}(n,q)$ is a quadric or an ovoid. If $\mathcal{K}$ is an ovoid, then it is one of the following:*

1. *a $(q+1)$-arc in $\mathrm{PG}(2,q)$;*
2. *an ovaloid of $\mathrm{PG}(3,q)$, $q>2$;*
3. *an elliptic quadric in $\mathrm{PG}(3,2)$.*

More in general, $(0,1,2,q+1)$-sets that are not necessarily quadratic sets, have been studied by G. Tallini. He obtained the following results.

Theorem 1.5.2 ([45]) *A set $\mathcal{K}$ of type $(0,1,2,q+1)$ in $\mathrm{PG}(n,q)$, $n \geq 3$, $q > 2$, such that $\frac{q^{n+1}-1}{q-1} > |\mathcal{K}| \geq \frac{q^n-1}{q-1}$, is one of the following:*

1. *the points of a hyperplane together with the points of an r-dimensional subspace, for $r \in \{-1,0,1,\ldots,n-1\}$;*
2. *the points of a (possibly degenerate) parabolic quadric;*
3. *the points of a (possibly degenerate) hyperbolic quadric;*
4. *if q is even: the points of a (possibly degenerate) parabolic quadric $\Pi[r]Q(n-r-1,q)$, for $r \in \{-1,0,1,\ldots,n-3\}$, together with the points of a d-dimensional subspace that is contained in $\langle \Pi[r], x\rangle$, where x is the nucleus of a base $Q(n-r-1,q)$.*
5. *if q is even: the points of a cone with vertex an $(n-3)$-dimensional subspace $\Pi[n-3]$ and base a $(q+1)$-arc $\mathcal{K}'$ contained in a plane skew to $\Pi[n-3]$, together with the points of an r-dimensional subspace contained in $\langle \Pi[n-3], x\rangle$, where x is the nucleus of $\mathcal{K}'$.*

Theorem 1.5.3 ([46]) *A $(0,1,2,q+1)$-set $\mathcal{K}$ in $\mathrm{PG}(n,q)$, with $q > 3$ and $n \geq 4$, such that $|\mathcal{K}| = \frac{q^n-1}{q-1} - q^{g+1}$, where g is the largest dimension of a subspace in $\mathcal{K}$, is one of the following:*

1. *the points of a (possibly degenerate) elliptic quadric;*
2. *the points of an $(n-2)$-dimensional subspace;*
3. *if q is even: the points of a cone with vertex an $(n-4)$-dimensional subspace $\Pi[n-4]$ and base a (q^2+1)-cap in a three dimensional space skew to $\Pi[n-4]$.*

The last theorem was improved by C. Lefèvre-Percsy as follows.

Theorem 1.5.4 ([37]) *In $\mathrm{PG}(n,q)$, with $q > 3$, with $n \geq 3$, and q odd, a set $\mathcal{K}$ of type $(0,1,2,q+1)$ with $\frac{q^{n+1}-1}{q-1} > |\mathcal{K}| > \frac{q^n-1}{q-1} - q^{n-2} + q^{n-3}$ is either a quadric or the union of two subspaces of dimension respectively $n-1$ and r, for some $r \in \{-1,0,1,\ldots,n-1\}$.*

The following result is a corollary of the previous three theorems, restricted to the non-singular case.

Theorem 1.5.5 ([35], theorem 22.11.6) *In $\mathrm{PG}(n,q)$ with $n \geq 4$ and $q > 2$, let $\mathcal{K}$ be a non-singular set of type $(0,1,2,q+1)$.*

1. *If* $\frac{q^{n+1}-1}{q-1} > |\mathcal{K}| \geq \frac{q^n-1}{q-1}$*, then one of the following holds:*

   (a) *n is even and $\mathcal{K}$ is an n-dimensional non-degenerate parabolic quadric;*

   (b) *n is odd and $\mathcal{K}$ is an n-dimensional non-degenerate hyperbolic quadric;*

   (c) *if q is even: $\mathcal{K}$ is either a (possibly degenerate) parabolic quadric $\Omega[r]Q(n-r-1)$ together with the nucleus of a base $Q(n-r-1)$ or $\mathcal{K}$ is a cone with vertex an $(n-3)$-dimensional space $\Pi[n-3]$ and base a $(q+1)$-arc $\mathcal{K}'$ in a plane skew to $\Pi[n-3]$, together with the nucleus of $\mathcal{K}'$.*

2. *If* $|\mathcal{K}| = \frac{q^n-1}{q-1} - q^{g+1}$*, where g is the largest dimension of a subspace in $\mathcal{K}$, then $\mathcal{K}$ is a non-degenerate elliptic quadric.*

3. *If q is odd, $q > 3$ and* $\frac{q^{n+1}-1}{q-1} > |\mathcal{K}| > \frac{q^n-1}{q-1} - q^{n-2} + q^{n-3}$*, then*

   (a) *n is even and $\mathcal{K}$ is a non-degenerate parabolic quadric;*

   (b) *n is odd and $\mathcal{K}$ is a non-degenerate hyperbolic or elliptic quadric.*

1.5.2 Sets of type $(0,1,q,q+1)$ in $\mathrm{PG}(n,q)$

C. Lefèvre-Percsy completely classified sets of type $(0,1,q,q+1)$ in $\mathrm{PG}(n,q)$. The case $q=2$ is trivial as any set of points in $\mathrm{PG}(n,2)$ is a set of type $(0,1,2,3)$. For $q>2$, C. Lefèvre-Percsy proved the following theorem.

Theorem 1.5.6 ([38]) *Let $\mathcal{K}$ be a set of type $(0,1,q,q+1)$ in $\mathrm{PG}(n,q)$, for $q>2$. Then $\mathcal{K}$ consists of the set of points not contained in a subspace $\Pi[m]$ of $\mathrm{PG}(n,q)$, $0 \leq m < n$, together with the points of a $(0,1,q,q+1)$-set $\mathcal{K}'$ in $\Pi[m]$.*

1.5.3 Sets of type $(0,1,r)$ in $\mathrm{PG}(n,q)$

Sets of type $(0,1,r)$ in projective spaces have been studied by J. Ueberberg in [56]. He obtained a classification of such sets in the case that $r \geq \sqrt{q}+1$. In the next theorem, the classification of $(0,1,r)$-sets, $r \geq \sqrt{q}+1$, in a projective plane is given.

Theorem 1.5.7 ([56]) *Let $\mathcal{K}$ be a set of type $(0,1,r)$ in a projective plane $\mathrm{PG}(2,q)$, such that $\mathcal{K}$ is not a maximal arc. If $r \geq \sqrt{q}+1$, then one of the following possibilities occurs.*

- *$\mathcal{K}$ consists of n collinear points.*
- *q is a square and $\mathcal{K}$ is a Baer subplane of* $\mathrm{PG}(2,q)$. *In particular $r = \sqrt{q}+1$.*
- *q is a square and $\mathcal{K}$ is a unital of* $\mathrm{PG}(2,q)$. *In particular $r = \sqrt{q}+1$.*
- *$\mathcal{K}$ is a maximal arc.*

If $r = \sqrt{q}+1$, then this theorem has the following corollary.

Corollary 1.5.8 ([56]) *Let $\mathcal{K}$ be a set of type $(0,1,\sqrt{q}+1)$ in* $\mathrm{PG}(2,q)$. *Then $\mathcal{K}$ is a set of $\sqrt{q}+1$ collinear points, a Baer subplane or a unital.*

In the next theorem, the n-dimensional case is treated.

Theorem 1.5.9 ([56]) *Let $\mathcal{K}$ be a set of type $(0,1,r)$ in a projective space* $\mathrm{PG}(n,q)$. *Suppose that the points $\mathcal{K}$ span a subspace Π of* $\mathrm{PG}(n,q)$ *of dimension at least three. If $r \geq \sqrt{q}+1$, then $\mathcal{K}$ is a Baer subspace of Π, an affine subspace of Π, or $\mathcal{K}$ is the set of all points of Π.*

1.5.4 Sets of type $(0,1,r,q+1)$ in $\mathrm{PG}(n,q)$, for $3 \leq r \leq q-1$

C. Lefèvre-Percsy studied $(0,1,r,q+1)$-sets in $\mathrm{PG}(n,q)$, for $3 \leq r \leq q-1$. She defines a degenerate $(0,1,r,q+1)$-set $\mathcal{K}$, for $3 \leq r \leq q-1$, as a $(0,1,r,q+1)$-set that contains a point x such that all points of $\mathcal{K}\setminus\{x\}$ lie on a line through x that contains $q+1$ points of $\mathcal{K}$. She then proves that each degenerate $(0,1,r,q+1)$-set $\mathcal{K}$, for $3 \leq r \leq q-1$, is a cone $\Pi[m]\mathcal{K}'$, where $\Pi[m]$ is an m-dimensional subspace of $\mathrm{PG}(n,q)$ and $\mathcal{K}'$ is a non-degenerate $(0,1,r,q+1)$-set in a $(n-m-1)$-dimensional subspace skew to $\Pi[m]$, for $3 \leq r \leq q-1$. Hence it suffices to classify non-degenerate $(0,1,r,q+1)$-sets, for $3 \leq r \leq q-1$. In the next theorem, a classification of non-degenerate $(0,1,r,q+1)$-sets in $\mathrm{PG}(n,q)$ is given, for $3 \leq r \leq q-1$.

Theorem 1.5.10 ([39]) *Let $\mathcal{K}$ be a non-degenerate set of type $(0,1,r,q+1)$ in* $\mathrm{PG}(n,q)$, *with $n \geq 3$ and $3 \leq r \leq q-1$, that contains a line and such that no plane of* $\mathrm{PG}(n,q)$ *intersects $\mathcal{K}$ in the set of points not contained in a maximal arc and no plane of* $\mathrm{PG}(n,q)$ *intersects $\mathcal{K}$ in the set of points of a maximal arc together with the points of a line exterior to this maximal arc. Then $\mathcal{K}$ is the set of points of a Hermitian variety in* $\mathrm{PG}(n,q)$.

Chapter 2

On the classification of fully embedded (α, β)-geometries in $\mathrm{PG}(n, q)$, q odd

In this chapter we will give a classification of (α, β)-geometries fully embedded in $\mathrm{PG}(n, q)$, under some assumptions that will be explained later. The classification only works in case q is odd. For q even, most of the results will remain valid, and give examples of (α, β)-geometries fully embedded in $\mathrm{PG}(n, q)$. However it is possible that there exist much more classes of (α, β)-geometries fully embedded in $\mathrm{PG}(n, q)$, q even, under our assumptions. All the results of this chapter are taken from [12, 13].

Let $\mathcal{S}$ be an (α, β)-geometry fully embedded in $\mathrm{PG}(n, q)$. If $\alpha = 0$, then $\mathcal{S}$ is a $(0, \beta)$-geometry. Full embeddings of $(0, \beta)$-geometries have been studied in [21, 55]. Therefore we will exclude here the case $\alpha = 0$. The case $\alpha = 1$ turns out to be a bit special. The reason for this will become more clear later in this chapter. In the first section, we will study proper $(1, \beta)$-geometries in $\mathrm{PG}(3, q)$, q even or odd. With the methods we are using, it appeared not to be possible to classify $(1, \beta)$-geometries fully embedded in a projective space $\mathrm{PG}(n, q)$ for every $n \in \mathbb{N}$. That is why from the second section on the case $\alpha = 1$ is excluded. The case $\alpha = 1$ will be examined further in the next chapter, where we will study $(1, q)$-geometries fully embedded in $\mathrm{PG}(n, q)$.

Note that a proper (α, β)-geometry $\mathcal{S}$ can not be fully embedded in a projective plane $\mathrm{PG}(2, q)$. Indeed, in a plane every two lines intersect. Since $\mathcal{S}$ is proper, there is an antiflag (p_1, L_1) of $\mathcal{S}$ for which $i(p_1, L_1) = \alpha$ and an antiflag (p_2, L_2) of $\mathcal{S}$ for which $i(p_2, L_2) = \beta$. Hence through p_1 there are α lines of $\mathcal{S}$, while through p_2 there are β lines of $\mathcal{S}$. It follows that $t + 1 = \alpha$

and $t+1=\beta$, where $t+1$ is the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$. This is a contradiction, since $t+1$ is a constant and $\alpha \neq \beta$.

In the first section of this chapter, we will study fully embeddable $(1, \beta)$-geometries in $\mathrm{PG}(3, q)$. In the second section an overview will be given of the results that are proved in the later sections. More precisely a classification of fully embeddable (α, β)-geometries in $\mathrm{PG}(n, q)$, q odd and $\alpha > 1$ will be given there, under certain assumptions that will be explained later. In the third, fourth and fifth sections the proofs of the classification results of the second section are given.

2.1 Proper $(1, \beta)$-geometries that are fully embeddable in $\mathrm{PG}(3, q)$

If $\mathcal{S}$ is a $(1, \beta)$-geometry fully embedded in a projective space, then the restriction of $\mathcal{S}$ to a plane that contains an antiflag of $\mathcal{S}$, is a partial linear space, but has not necessarily an order. In case it has an order, it follows immediately that it is a partial geometry $\mathrm{pg}(s, \beta-1, \beta)$. The results of [2] show that the points and lines of a partial geometry fully embedded in a projective plane are either all points and lines of the plane, or the points not contained in a maximal arc $\mathcal{K}$ of the plane, and the lines exterior to $\mathcal{K}$. A plane in which the restriction of $\mathcal{S}$ is a partial geometry $\mathrm{pg}(s, \beta-1, \beta)$, we call a *$\beta$-plane*. A plane, that contains an antiflag of $\mathcal{S}$, in which the set of lines of $\mathcal{S}$ is a set of lines mutually intersecting in one point x, we call a *degenerate plane*. The point x will be called the *center* of the degenerate plane. A plane that contains an antiflag of $\mathcal{S}$ and that is not a degenerate plane or a β-plane, we call a *mixed* plane. In such a mixed plane, every point of $\mathcal{S}$ is incident with either 1 line or with β lines of $\mathcal{S}$ contained in the plane, and both of the cases occur. It is immediately clear that if a plane contains an antiflag of $\mathcal{S}$ then it must be either a β-plane, a mixed plane or a degenerate plane.

In this section we will prove that there are no proper $(1, \beta)$-geometries fully embedded in $\mathrm{PG}(3, q)$, under the assumption that $\mathrm{PG}(3, q)$ contains at least one degenerate plane and at least one β-plane.

Lemma 2.1.1 *There exists no proper $(1, t+1)$-geometry of order (q, t) fully embedded in $\mathrm{PG}(3, q)$.*

Proof. Let $\mathcal{S}$ be a proper $(1, t+1)$-geometry fully embedded in $\mathrm{PG}(3, q)$. Since $\mathcal{S}$ is proper, there exists a point $x \in \mathcal{P}$ and a line $L \in \mathcal{L}$ for which

$i(x,L) = t+1$. So all lines of $\mathcal{S}$ through x are contained in the plane $\langle x, L\rangle$. Let p_1 and p_2 be distinct points of L collinear with x. The lines $\langle x, p_1\rangle$ and L are both lines of $\mathcal{S}$ through p_1 in the plane $\langle x, L\rangle$. It follows that $i(p_1, \langle x, p_2\rangle) \neq 1$, and so $i(p_1, \langle x, p_2\rangle) = t+1$, i.e. all lines of $\mathcal{S}$ through p_1 are contained in the plane $\langle x, L\rangle$. In the same way, one can see that all the lines of $\mathcal{S}$ through p_2 are contained in $\langle x, L\rangle$. Assume that there is a line $M \in \mathcal{L}$ not contained in $\langle x, L\rangle$. Let $M \cap \langle x, L\rangle = \{y\}$. If $\langle x, y\rangle \notin \mathcal{L}$, then M contains 0 points collinear with x, a contradiction. Hence $\langle x, y\rangle \in \mathcal{L}$, and necessarily $\langle y, p_1\rangle \notin \mathcal{L}$ or $\langle y, p_2\rangle \notin \mathcal{L}$. Indeed, if $\langle y, p_1\rangle \in \mathcal{L}$ and $\langle y, p_2\rangle \in \mathcal{L}$, arguing as before implies that all the lines of $\mathcal{S}$ on y are in $\langle x, L\rangle$, which is a contradiction. So either p_1 or p_2 is collinear with no points of M, a contradiction. This proves that all the lines of $\mathcal{S}$ meeting $\langle x, L\rangle$ are contained in $\langle x, L\rangle$ and since $\mathcal{S}$ is connected, all the points of $\mathcal{S}$ are contained in $\langle x, L\rangle$. So $\mathcal{S}$ is contained in a plane of $\mathrm{PG}(3,q)$, a contradiction with the assumption that the points of $\mathcal{S}$ span $\mathrm{PG}(3,q)$. This proves that a proper $(1,t+1)$-geometry fully embedded in $\mathrm{PG}(3,q)$, does not exist. □

Remark. From the proof of lemma 2.1.1, it follows that the result of lemma 2.1.1 also holds in $\mathrm{PG}(n,q)$, for $n \geq 3$. However, we need it only in $\mathrm{PG}(3,q)$ and hence we restricted the statement of the lemma to this case. Note that the lemma also holds for a $\mathrm{pg}(q,t,t+1)$ fully embeddable in $\mathrm{PG}(n,q)$, for $n \geq 3$.

Lemma 2.1.2 *There exists no proper $(1,q+1)$-geometry fully embedded in* $\mathrm{PG}(3,q)$.

Proof. Let $\mathcal{S}$ be a proper $(1,q+1)$-geometry fully embedded in $\mathrm{PG}(3,q)$. Then from [9] it follows that $\mathcal{S}$ is a polar space. The non-degenerate polar spaces in $\mathrm{PG}(3,q)$ are all generalized quadrangles (for which $\alpha = \beta = 1$). In a degenerate polar space the number of lines through a point is never a constant. This proves that there does not exist a proper $(1,q+1)$-geometry fully embedded in $\mathrm{PG}(3,q)$. □

Lemma 2.1.3 *If $\mathcal{S}$ is a proper $(1,\beta)$-geometry fully embedded in* $\mathrm{PG}(3,q)$, *such that there exists at least one degenerate plane and at least one β-plane, then $t+1 = q\beta + \beta - q$.*

Proof. Let $\mathcal{S}$ be a proper $(1,\beta)$-geometry fully embedded in $\mathrm{PG}(3,q)$, such that there is at least one degenerate plane π, call its center x, and at least one β-plane ρ. From the definition of a β-plane, it follows that the points and lines of $\mathcal{S}$ in ρ are the points not on a maximal arc of degree q/β and the lines exterior to this maximal arc. Note that from lemma 2.1.2 it follows that $\beta \neq q+1$, and hence that $\beta | q$.

Assume that the center x of the degenerate plane π is incident with $r+1$ lines of $\mathcal{S}$ contained in π. Then $|\mathcal{L}| = b = t+1+(r+1)qt$, hence in each degenerate plane the center is incident with $r+1$ lines of $\mathcal{S}$.

Assume that the β-plane ρ does not contain the center x of the degenerate plane π. As the line $\pi \cap \rho$ is not passing through x, it is not a line of $\mathcal{S}$ and hence it contains $q+1-q/\beta$ points of $\mathcal{S}$. This implies that $r+1 = q+1-q/\beta$. Counting the lines of $\mathcal{S}$ intersecting π, we get that $b = t+1+(q+1-q/\beta)qt$. Counting the lines of $\mathcal{S}$ intersecting ρ, we obtain that $b = (q+1-q/\beta)(qt+q+t+1-q\beta)$. From both expressions it follows that $t+1 = q\beta+\beta-q$. Hence, if $t \neq (q+1)(\beta-1)$, then every β-plane contains all centers of the degenerate planes.

Let us assume for the rest of the proof that $t \neq (q+1)(\beta-1)$. We will first prove that $r \neq q$, i.e. that through the center x of π there is a line in π that contains exactly one point of $\mathcal{S}$. Assume that $r = q$. Let M be a line in π that does not contain x. Then M contains $q+1$ points of $\mathcal{S}$, but $M \notin \mathcal{L}$. It is clear that M cannot be contained in a β-plane, since we have proved that every β-plane contains x. Hence a plane through M is a degenerate plane, a mixed plane or a plane that contains no antiflag of $\mathcal{S}$.

Assume that there is a mixed plane σ through M. We count the number of lines of $\mathcal{S}$ in σ in two different ways. Let L be a line of σ that is an element of $\mathcal{L}$. Let a be the number of points on M through which there are β lines of $\mathcal{S}$ in σ. Let a' be the number of points on L through which there are β lines of $\mathcal{S}$ in σ. Counting the lines of $\mathcal{S}$ in σ intersecting M, we get that there are $a\beta+q+1-a$ lines of $\mathcal{S}$ in σ. Counting the lines of $\mathcal{S}$ in σ intersecting L, we get that there are $a'(\beta-1)+1$ lines of $\mathcal{S}$ in σ. Comparing these two results, it follows that $a = a' - q/(\beta-1)$. Since a and a' are both integers, this implies that $(\beta-1)|q$. However, we noted above that $\beta|q$. Hence $\beta = 2$. Since M is a line not belonging to $\mathcal{S}$ but containing $q+1$ points of $\mathcal{S}$, there are at least $q+1$ lines of $\mathcal{S}$ contained in σ. Since $\beta = 2$, no three lines of $\mathcal{S}$ in σ are concurrent. This forces the lines of $\mathcal{S}$ in σ to be $q+1$ lines of a dual oval, M being the nucleus line of this dual oval. So in a mixed plane there is exactly one line of $\mathcal{S}$ through each point of M. In a degenerate plane through M there is also one line of $\mathcal{S}$ through each point of M. In a plane containing no antiflag of $\mathcal{S}$ there are no lines of $\mathcal{S}$. Hence we may conclude that if there is a mixed plane σ through M, then $t+1 \leq q+1$, but as $r = q$, it follows that $t+1 = q+1$.

Assume that there is not a mixed plane through M. The line M contains $q+1$ points of $\mathcal{S}$, but $M \notin \mathcal{L}$. So in this case every plane through M is either a degenerate plane or a plane that contains no antiflag of $\mathcal{S}$. In a degenerate plane there is one line through every point of M. In a plane through M that contains no antiflag of $\mathcal{S}$ there are no lines of $\mathcal{S}$. Counting all the lines of $\mathcal{S}$ through a point of M, we get that $t+1 \leq q+1$, and hence again it follows that $t+1 = q+1$.

Now from the first paragraph of the proof, we know that every β-plane contains x. Since $t = q = r$, all of the lines of $\mathcal{S}$ through x are contained in π and there can not be a β-plane through x. This proves that there cannot be a β-plane contained in $\mathrm{PG}(3,q)$, a contradiction with our assumption. It follows that the degenerate plane π has to contain a tangent N at $\mathcal{S}$.

Now we look at the planes through the tangent line N through x in the degenerate plane π with center x. Every plane through N has to be either a degenerate plane or a plane containing no antiflag of $\mathcal{S}$. A plane spanned by N and a line through x in the β-plane ρ that does not belong to $\mathcal{S}$ clearly cannot be a degenerate plane. So such a plane contains no antiflag of $\mathcal{S}$. Let L' be a line of $\mathcal{S}$ in ρ, with $x \in L'$. Assume that $\langle N, L' \rangle$ contains no antiflag of $\mathcal{S}$. Since $\langle N, L' \rangle$ contains L', all lines of $\langle N, L' \rangle$ different from L' are tangent lines to $\mathcal{S}$. Let y be a point of $\mathcal{S}$, $y \notin L'$. The plane $\langle y, L' \rangle$ contains an antiflag of $\mathcal{S}$. Let L_y be a line of $\mathcal{S}$ through y intersecting L'. Then all the planes through L_y different from $\langle L_y, L' \rangle$ intersect $\langle N, L' \rangle$ in a line that is tangent to $\mathcal{S}$. This implies that in all these planes there is exactly one line of $\mathcal{S}$ through y, namely L_y. Hence, counting the lines of $\mathcal{S}$ through y, we obtain that $t+1 \leq \beta$, so $t+1 = \beta$ a contradiction with lemma 2.1.1. We conclude that $\langle N, L' \rangle$ is a degenerate plane. Since there are β lines of $\mathcal{S}$ in ρ through x, there are β degenerate planes through N. Counting the lines of $\mathcal{S}$ through $x \in N$, we obtain that $t+1 = (r+1)\beta$. Hence

$$b = (r+1)q((r+1)\beta-1)+(r+1)\beta = q\beta-q+\beta+(q+1)(q-\frac{q}{\beta}+1)((r+1)\beta-\beta).$$

Solving this quadratic equation in $r+1$ yields $r+1 = q+1-\frac{q}{\beta}$ which implies $t = (q+1)(\beta-1)$ which is against the assumption, or $r+1 = 1$ which implies that $t+1 = \beta$, which is also a contradiction. Hence, if $\mathcal{S}$ is a proper $(1,\beta)$-geometry fully embedded in $\mathrm{PG}(3,q)$, such that there exists at least one degenerate plane and at least one β-plane, then $t+1 = q\beta+\beta-q$. $\square$

This leaves us with the case $t = (q+1)(\beta-1)$. It will be proved in lemma 2.1.4 that there is no proper $(1,\beta)$-geometry fully embedded in $\mathrm{PG}(3,q)$, such that every plane that contains an antiflag of $\mathcal{S}$ is either a degenerate plane or a β-plane. In lemma 2.1.5 we will prove however that under the assumption that there is at least a degenerate plane and a β-plane, mixed planes cannot occur if $t = (q + 1)(\beta - 1)$.

Lemma 2.1.4 *Let $\mathcal{S}$ be a proper $(1, \beta)$-geometry fully embedded in $\mathrm{PG}(3, q)$, such that $t = (q + 1)(\beta - 1)$, then there is at least one mixed plane.*

Proof. Let $\mathcal{S}$ be a proper $(1, \beta)$-geometry fully embedded in $\mathrm{PG}(3, q)$, such that $t + 1 = q\beta + \beta - q$. Assume that there are no mixed planes. We will get a contradiction. Since $\mathcal{S}$ is proper, there is at least one degenerate plane π and at least one β-plane ρ.

Assume that the center x of π is incident with $q + 1$ lines of $\mathcal{S}$ in π. Let M be a line in π not through x. Then there cannot be a β-plane through M. Indeed, in a β-plane the points and lines of $\mathcal{S}$ are the points and lines exterior to a maximal arc. So if M is contained in a β-plane, then since M contains $q+1$ points of $\mathcal{S}$, it follows that $M \in \mathcal{L}$, which is a contradiction. So every plane through M is either a degenerate plane or a plane that contains no antiflag of $\mathcal{S}$. Counting the lines of $\mathcal{S}$ through a point $w \in M$, we get that $t + 1 \leq q + 1$, but since $t + 1 = q\beta + \beta - q$, this yields a contradiction. Hence π contains a tangent line N to $\mathcal{P}$ at x.

We will prove now that there are at least two distinct degenerate planes through N. All the planes through N are degenerate planes or planes that do not contain an antiflag of $\mathcal{S}$. By assumption $t + 1 = q\beta + \beta - q$. In particular, as $\beta \geq 2$, $t + 1 \geq q + \beta$. The β-plane ρ intersects N in a point. If $x \notin \rho$, then every plane through N in $\mathrm{PG}(3, q)$ contains $q + 1 - q/\beta$ points of $\mathcal{S}$ on its intersection line with ρ. In π there are at most q lines of $\mathcal{S}$ through x, hence since $t + 1 \geq q + \beta$, there is a plane through N in $\mathrm{PG}(3, q)$, different from π, containing a line of $\mathcal{S}$ through x. This plane intersects ρ in a line containing $q + 1 - q/\beta$ points of $\mathcal{S}$, so it contains an antiflag of $\mathcal{S}$, and hence it is a degenerate plane. This proves that if $x \notin \rho$, there are at least two distinct degenerate planes through N. If $x \in \rho$, then β planes through N that intersect ρ in a line of $\mathcal{S}$, while the $q + 1 - \beta$ other planes through N in $\mathrm{PG}(3, q)$ intersect ρ in a line containing $q + 1 - q/\beta$ points of $\mathcal{S}$. In π there are at most q lines of $\mathcal{S}$ through x. Since $x \in \rho$ by assumption and ρ cannot contain a tangent line, it follows that the intersection line of π and ρ is a line of $\mathcal{S}$. So π and ρ together contain at most $q + \beta - 1$ distinct lines of $\mathcal{S}$ through x. From $t + 1 \geq q + \beta$ it follows that there is a plane through N in $\mathrm{PG}(3, q)$, different from π, containing a line of $\mathcal{S}$ through x and intersecting ρ in a line through x containing $q + 1 - q/\beta$ points of $\mathcal{S}$. So this plane contains an antiflag of $\mathcal{S}$ and the tangent line N, which implies that it is a degenerate plane. Hence we may assume that there are at least two distinct degenerate planes π and π' through N.

Let L be a line of $\mathcal{S}$ not through x. We will prove that all planes through L are β-planes. We first consider the plane $\langle L, x\rangle$. It intersects both π and π' in a line of $\mathcal{S}$. Hence it contains a triangle of lines of $\mathcal{S}$. This proves that $\langle x, L\rangle$ is a β-plane.

All planes through L different from $\langle L, x\rangle$ contain an antiflag of $\mathcal{S}$. Indeed, they contain the line L of $\mathcal{S}$ and a point of $\mathcal{S}$ not on L on their intersection line with π. Hence they are degenerate planes or β-planes. Assume that $c+1$ planes through L are β-planes. We denote the point $L \cap \pi$ by z. We count the lines of $\mathcal{S}$ through z in the planes through L. The $(c+1)$ β-planes through L contain β lines of $\mathcal{S}$ through z. In each of the $(q-c)$ degenerate planes through L there is exactly one line of $\mathcal{S}$ through z, namely L, since the line of that plane through z in π does not belong to $\mathcal{S}$ and it is not a tangent. It follows that $t = (q+1)(\beta-1) = (c+1)(\beta-1)$. Hence $c = q$, which means that all the planes through L in $\mathrm{PG}(3, q)$ are β-planes.

We will now prove that there exists a plane through N that does not contain an antiflag of $\mathcal{S}$. By assumption there is a β-plane ρ contained in $\mathrm{PG}(3, q)$. Let L' be a line of $\mathcal{S}$ in ρ, such that $x \notin L'$. From the previous paragraph it follows that the plane $\langle x, L'\rangle$ is a β-plane. Let η be a plane through the tangent line N at x and a line through x in $\langle x, L'\rangle$ that does not belong to $\mathcal{S}$. Then η contains a tangent line to $\mathcal{S}$ at x and a line through x containing $q+1-q/\beta$ points of $\mathcal{S}$. Since $q+1-q/\beta \neq 1$ (otherwise it follows that $\beta = 1$, which is a contradiction), this proves that η cannot be a degenerate plane. It follows that η is a plane that does not contain an antiflag of $\mathcal{S}$.

We will count the points of $\mathcal{S}$ in η in two different ways. Let L_w be a line of $\mathcal{S}$ that intersects η in a point w different from x. By the above argument we know that all planes through L_w are β-planes. They intersect η in a line that does not belong to $\mathcal{S}$. Hence all lines through w in η contain $q+1-q/\beta$ points of $\mathcal{S}$. It follows that η contains $(q+1)(q-q/\beta)+1$ points of $\mathcal{S}$. Now we look at the lines through x in η. If a line through x in η contains a point u of $\mathcal{S}$ different from x, then this line contains $q+1-q/\beta$ points of $\mathcal{S}$. Indeed, we can take a line of $\mathcal{S}$ through u. By the above argument , all planes through this line are β-planes. It follows that $\langle x, u\rangle$ is contained in a β-plane. Since $\langle x, u\rangle \notin \mathcal{L}$, it contains $q+1-q/\beta$ points of $\mathcal{S}$. Now assume that there are c' lines through x in η that contain a point of $\mathcal{S}$ different from x. Then the number of points of $\mathcal{S}$ in η is equal to $c'(q-q/\beta)+1$, which should be equal to $(q+1)(q-q/\beta)+1$; hence $c' = q+1$ and all the lines through x in η contain $q+1-q/\beta$ points of $\mathcal{S}$. This is a contradiction, since η contains the tangent line N at x to $\mathcal{S}$.

This proves that under the assumption that $t = (q+1)(\beta-1)$, there should be at least one mixed plane. □

Lemma 2.1.5 *Let $\mathcal{S}$ be a proper $(1, \beta)$-geometry fully embedded in* $\mathrm{PG}(3, q)$*, such that* $\mathrm{PG}(3, q)$ *contains both a degenerate plane and a β-plane. Then there cannot be a mixed plane contained in* $\mathrm{PG}(3, q)$*.*

Proof. Let $\mathcal{S}$ be a proper $(1, \beta)$-geometry fully embedded in $\mathrm{PG}(3, q)$, such that $\mathrm{PG}(3, q)$ contains both a degenerate plane and a β-plane. By lemma 2.1.3 we know that $t + 1 = q\beta + \beta - q$. We will separate the cases $\beta = q$ and $\beta \neq q$.

1. Assume first of all that $\beta = q$, hence $t + 1 = q^2$ and the incidence structure of points and lines of $\mathcal{S}$ in a q-plane is a dual affine plane.

   We will prove that there are no mixed planes contained in $\mathrm{PG}(3, q)$. Assume therefore that σ is a mixed plane contained in $\mathrm{PG}(3, q)$. Hence σ contains an antiflag (w, L) such that $i(w, L) = q$, which means that there exists a unique point w' on L such that the projective line $\langle w, w' \rangle$ is not a line of $\mathcal{S}$. Let p be a point of L such that the projective line $\langle w, p \rangle$ is a line of $\mathcal{S}$. Then p is incident with at least two lines of $\mathcal{S}$ in σ, and so is incident with $\beta = q$ lines of $\mathcal{S}$ in σ. Counting the points of $\mathrm{PG}(3, q)$ that do not belong to $\mathcal{P}$ on the lines through p and w in σ it follows that either all points of σ belong to $\mathcal{S}$, or exactly one point y in σ does not belong to $\mathcal{S}$.

   Assume first that all the points of σ belong to $\mathcal{S}$. Then through all the points of σ there are 1 or q lines of $\mathcal{S}$. Now, if we dualize, we get a plane σ^D in which the lines of $\mathcal{S}$ form a set $\mathcal{K}$ of points such that every line of σ^D contains either 1 or q points of $\mathcal{K}$. Hence $\mathcal{K}$ is a $(1, q)$-set in the plane σ^D, a contradiction because such a set does not exist (see [34], theorem 12.3.6).

   Assume next that σ contains one point y that does not belong to $\mathcal{S}$. Then, dualizing, we get a plane σ^D in which the lines of $\mathcal{S}$ are a set of points $\mathcal{K}$ such that every line of σ^D different from y^D contains 1 or q points of $\mathcal{K}$, while y^D contains 0 points of $\mathcal{K}$. Hence $\mathcal{K}$ is a set of type $(1, q)$ in the affine plane $\sigma^D \setminus y^D$. If $q > 2$ such a set is an affine plane, but this implies that the incidence structure of points and lines of $\mathcal{S}$ in σ is a dual affine plane, which contradicts the assumption that σ is a mixed plane. The case $q = 2$ is also easily ruled out.

   So, if $\beta = q$ then there cannot be a mixed plane contained in $\mathrm{PG}(3, q)$.

2. Assume now that $\beta \neq q$. By lemma 2.1.2 we may assume that $\beta < q$. Also, q has to be even, since for q odd there exists no non-trivial maximal arc in a Desarguesian projective plane [1]. Since $\beta | q$, we conclude that $q = 2^h$ for $h \in \mathbb{N}$ and that $\beta = 2^r$ for an $r \in \mathbb{N}$, $0 < r < h$.

   We will prove again that $\mathrm{PG}(3,q)$ cannot contain a mixed plane. Assume therefore that $\mathrm{PG}(3,q)$ contains a mixed plane σ. Let b_σ be the number of lines of $\mathcal{S}$ in σ. The number of points of σ through which there are β lines (respectively 1 line) of $\mathcal{S}$ contained in σ, we denote by m_β (respectively m_1). It follows that

$$\begin{cases} m_1 + \beta m_\beta &= (q+1)b_\sigma \\ \beta(\beta-1)m_\beta &= b_\sigma(b_\sigma - 1). \end{cases}$$

   So

$$m_\beta = \frac{b_\sigma(b_\sigma-1)}{\beta(\beta-1)} \qquad \text{and} \qquad m_1 = (q+1)b_\sigma - \frac{b_\sigma(b_\sigma-1)}{\beta-1}.$$

   Now we count the lines of $\mathcal{S}$ through the points of $\mathcal{S}$ in σ. We get that $|\mathcal{L}| = b_\sigma + m_1 t + m_\beta(t+1-\beta)$. Counting the lines of $\mathcal{S}$ through the points of $\mathcal{S}$ in a β-plane, we get that $|\mathcal{L}| = (q+1)(q+1-q/\beta)(t+1-\beta) + q\beta + \beta - q$. By assumption $t+1 = q\beta + \beta - q$. Substituting these expressions for $|\mathcal{L}|$, $t+1$, m_1 and m_β in the equation $|\mathcal{L}| = b_\sigma + m_1 t + m_\beta(t+1-\beta)$, we get that

$$\begin{aligned}(q/\beta - q - 1)b_\sigma^2 + (q^2\beta - q^2 + 2q\beta + \beta - q + 1 - q/\beta)b_\sigma \\ -(q+1-q/\beta)(q^2\beta - q^2 + q\beta - q + \beta) = 0.\end{aligned}$$

   The solutions of this quadratic equation in b_σ are (with $\beta = 2^r$)

$$\frac{q^2(2^r-1) + q(2^r-1) + 2^r}{q(2^r-1)+2^r} \qquad \text{and} \qquad q(2^r-1)+2^r$$

   Assume first that $b_\sigma = \frac{q^2(2^r-1)+q(2^r-1)+2^r}{q(2^r-1)+2^r}$. Since $b_\sigma \in \mathbb{N}$, it follows that $((2^r-1)q + 2^r) \mid (-q + 2^r)$. In particular, $(2^r-1)q + 2^r \leq q - 2^r$. Substituting $q = 2^h$, we get that $2^{h-1} \leq 2^{h-r} - 1$, which is clearly a contradiction since $r \geq 1$.

   Assume next that $b_\sigma = q(2^r-1) + 2^r$. Then $m_1 = 0$ and hence σ is a β-plane, a contradiction with our assumption.

   This proves that also in the case $\beta \neq q$ there cannot be a mixed plane contained in $\mathrm{PG}(3,q)$.

Hence in both cases we have found a contradiction. We may conclude that if $\mathcal{S}$ is a proper $(1,\beta)$-geometry, fully embedded in $\mathrm{PG}(3,q)$, such that there exists both a degenerate plane and a β-plane, then $\mathrm{PG}(3,q)$ cannot contain a mixed plane. □

From all the lemmas above follows the main classification theorem.

Theorem 2.1.6 *There exist no proper* $(1,\beta)$*-geometry, fully embedded in* $\mathrm{PG}(3,q)$*, under the assumption that there is at least one degenerate plane and at least one* β*-plane.*

Remark. In the previous theorem, it is assumed that $\mathrm{PG}(3,q)$ contains at least one degenerate plane and at least one β-plane. The cases not included in the theorem can be put into 3 categories.

1. Planes containing an antiflag of $\mathcal{S}$ are either mixed planes or β-planes, and there is at least one β-plane. Since there is a β-plane, either $\beta = q$ or $\beta = 2^m$, for a $m \in \mathbb{N}$. It will be proved in chapter 3 (see lemma 3.4.2) that $(1,q)$-geometries fully embeddable in $\mathrm{PG}(3,q)$, for $q \neq 2$ can not contain mixed planes. Hence from the above theorem it follows that $(1,q)$-geometries fully embeddable in $\mathrm{PG}(3,q)$ do not exist. Hence $\beta = 2^m$ and $q = 2^h$, for $m, h \in \mathbb{N}$, $m < h$.

   Let σ be a mixed plane. Let v_1 (respectively v_β) be the number of points of σ through which there is 1 line of $\mathcal{S}$ in σ (resp. there are β lines of $\mathcal{S}$ in σ). Let b_σ be the number of lines of $\mathcal{S}$ in σ, and let b be the number of lines of $\mathcal{S}$ in $\mathrm{PG}(n,q)$. Then, by counting arguments, one obtains that

$$\left\{\begin{array}{ccc} v_1 + \beta v_\beta & = & (q+1)b_\sigma \\ \beta(\beta-1)v_\beta & = & b_\sigma(b_\sigma - 1) \\ v_1 t + v_\beta(t+1-\beta) + b_\sigma & = & b. \end{array}\right.$$

   Now let ρ be a β-plane. Counting the lines of $\mathcal{S}$ intersecting ρ and the lines of $\mathcal{S}$ in ρ, we get that $b = q\beta - q + \beta + (q+1)(q - q/\beta + 1)(t+1-\beta)$. From these equations, it follows that $b_\sigma = q + 1 - q\beta/(t+1)$. Hence $(t+1) \mid q\beta$. Let L be a line of $\mathcal{S}$ in σ and assume that there are a points of L through which there are β lines of $\mathcal{S}$ in σ. Then, counting the lines of $\mathcal{L}$ in σ intersecting L, we get that $a = q(t+1-\beta)/((\beta-1)(t+1))$. So $(\beta - 1) \mid q(t+1-\beta)$ and hence $(\beta-1) \mid qt$. Since we proved above that $\beta = 2^m$ and $q = 2^h$, it follows that the greatest common divisor of $(\beta - 1)$ and q equals 1. So we get that $(\beta-1) \mid t$. Now from $(t+1) \mid q\beta$, it follows that $t + 1 = 2^c$ for a $c \in \mathbb{N}$. So q, β and $t+1$ have to be even prime powers and $(\beta - 1) \mid t$.

2. Planes containing an antiflag of $\mathcal{S}$ are either mixed planes or degenerate planes, and there is at least one degenerate plane.

3. All planes containing an antiflag of $\mathcal{S}$ are mixed planes.

2.2 Classification results on the full embedding of (α, β)-geometries in $\mathrm{PG}(n, q)$, $\alpha > 1$ and q odd

Let $\mathcal{S}$ be an (α, β)-geometry fully embedded in $\mathrm{PG}(n, q)$, q odd and $\alpha > 1$. As in section 2.1 we examine the behavior of planes containing an antiflag of $\mathcal{S}$. Such planes can be divided into three types, namely *α-planes* (in which the restriction of $\mathcal{S}$ is a $\mathrm{pg}(s, \alpha - 1, \alpha)$), *$\beta$-planes* (in which the restriction of $\mathcal{S}$ is $\mathrm{pg}(s, \beta - 1, \beta)$), and *mixed planes*. Let π be an α-plane or a β-plane contained in $\mathrm{PG}(n, q)$. Then as before the points and lines of $\mathcal{S}$ in π are either all points and lines of π, or all points not contained in a maximal arc of π and all lines not intersecting this maximal arc. Now for q odd, there exist no non-trivial maximal arcs in a Desarguesian projective plane [1]. Hence the points and lines of $\mathcal{S}$ in π are either all points and lines of π, or all points of π except one point p and all lines of π not through p. In what follows we will give a classification of all fully embedded proper (α, β)-geometries in $\mathrm{PG}(n, q)$, q odd and $\alpha > 1$, under the assumption that there is at least one α-plane or one β-plane contained in $\mathrm{PG}(n, q)$. With this assumption there are 3 possibilities:

(i) $\alpha = q$ and $\beta = q + 1$,

(ii) $\alpha < q$, in which case there are no α-planes and $\beta = q + 1$,

(iii) $\alpha < q$, in which case there are no α-planes and $\beta = q$.

Our classification is not a complete classification of (α, β)-geometries fully embedded in $\mathrm{PG}(n, q)$, for q odd. The following example illustrates that there do exist (α, β)-geometries fully embedded in $\mathrm{PG}(n, q)$, q odd, that are not included in the classification.

Example 2.2.1 *This example of an (α, β)-geometry fully embeddable in $\mathrm{PG}(3, q)$, q odd, is due to J. A. Thas (personal communication). Define an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ as follows: $\mathcal{P}$ is the set of points not on a three dimensional non-degenerate hyperbolic quadric, $\mathcal{L}$ is the set of lines exterior to this quadric and incidence is the one inherited from $\mathrm{PG}(3, q)$. Then $\mathcal{S}$ is a $((q-1)/2, (q+1)/2)$-geometry fully embedded in $\mathrm{PG}(3, q)$.*

The example 2.2.1 is an (α, β)-geometry for which every plane containing an antiflag of $\mathcal{S}$ is a mixed plane. Indeed, every plane containing an antiflag of $\mathcal{S}$ intersects the hyperbolic quadric in a conic. So this is an example of an (α, β)-geometry fully embedded in $\mathrm{PG}(3, q)$, such that $\mathrm{PG}(3, q)$ contains no α-planes and no β-planes. This shows that there do exist (α, β)-geometries fully embedded in $\mathrm{PG}(n, q)$, q odd, that are not covered by our classification.

In the remainder of this section we give an overview of the classification results for (α, β)-geometries in $\mathrm{PG}(n, q)$, q odd, that will be proved later in the chapter. We also mention the theorems that remain valid in the case q is even.

We will use the notation $\mathrm{P}(n, q)$ for the point set of $\mathrm{PG}(n, q)$. Furthermore $\Pi[m]$, $\Omega[m]$ and $\Lambda[m]$ will denote m-dimensional subspaces of $\mathrm{PG}(n, q)$, for $-1 \leq m \leq n-1$.

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper (α, β)-geometry fully embedded in $\mathrm{PG}(n, q)$, q odd and $\alpha > 1$. Assume that $\mathrm{PG}(n, q)$ contains at least one α-plane or one β-plane. Then $\mathcal{S}$ is one of the following.

1. $\mathcal{S}$ is a $(q, q+1)$-geometry, with points the points of $\mathrm{PG}(n, q) \setminus \Pi[m]$, for some $0 \leq m < n-2$, and lines those lines of $\mathrm{PG}(n, q)$ that are disjoint from $\Pi[m]$.

2. $\mathcal{S}$ is a $(q, q+1)$-geometry, with points the points of $\mathrm{PG}(n, q) \setminus \Pi[m]$, for $0 \leq m < n-3$, and with lines defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ be a partition of the points of $\mathcal{S}$, where $l = (q^{n-m} - 1)/(q^{m'-m} - 1)$, such that for $i = 1, \ldots, l$ we have that $\sigma_i = \Omega_i[m'] \setminus \Pi[m]$, with $\Omega_i[m']$ an m'-dimensional subspace of $\mathrm{PG}(n, q)$ that contains $\Pi[m]$, and with $m + 2 \leq m' \leq n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. A necessary and sufficient condition for this partition and the $(q, q+1)$-geometry to exist is that $(m' - m)|(n-m)$.

3. $\mathcal{S}$ is a $(q-1, q)$-geometry, with points the points of $\mathrm{PG}(n, q) \setminus \Pi[n-2]$, and lines defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{n-r}\}$ be a partition of the points of $\mathcal{S}$, such that for $i = 1, \ldots n-r$ we have that $\sigma_i = \Omega_i[r] \setminus \Pi[n-2]$, with $\Omega_i[r]$ an r-dimensional subspace of $\mathrm{PG}(n, q)$ that intersects $\Pi[n-2]$ in an $(r-2)$-dimensional space, for $1 \leq r \leq n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. Further, such a partition exists for every $1 \leq r \leq n-2$, and gives a $(q-1, q)$-geometry.

4. $\mathcal{S}$ is a $(q-1, q)$-geometry with points the points of $\mathrm{PG}(n, q)$ not contained in one of the two subspaces $\Pi[n-2]$ and $\Omega[r]$ of $\mathrm{PG}(n, q)$, for

$1 \leq r \leq n-2$, for which $\Omega[r] \cap \Pi[n-2]$ is an $(r-2)$-dimensional space. The lines of $\mathcal{S}$ are either all lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathcal{S}$, or they are defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ be a partition of the points of $\mathcal{S}$, where $l = (q^{n-r}-1)/(q^{d-r}-1)$, and such that for $i = 1, \ldots, l$ we have that $\sigma_i = \Lambda_i[d] \setminus (\Pi[n-2] \cup \Omega[r])$, with $\Lambda_i[d]$ a d-dimensional subspace of $\mathrm{PG}(n,q)$ that contains $\Omega[r]$, and $r+2 \leq d \leq n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. A necessary and sufficient condition for such a partition to exist is that $(d-r)|(n-r)$. Further, if $(d-r)|(n-r)$ and $n-2 \geq d \geq r+2$, then this partition gives a $(q-1,q)$-geometry.

5. $\mathcal{S}$ is a $(q-\sqrt{q}, q)$-geometry with points the points of a cone $\Pi[m]\mathcal{S}'$ that are not contained in the vertex $\Pi[m]$, where $\Pi[m]$ is an m-dimensional subspace of $\mathrm{PG}(n,q)$, for $m = n-4$ or $m = n-5$, and where $\mathcal{S}'$ is a $(q-\sqrt{q}, q)$-geometry fully embedded in an $(n-m-1)$-dimensional space $\Omega[n-m-1]$ skew to $\Pi[m]$ described as follows. The points of $\mathcal{S}'$ are the points of $\Omega[n-m-1]$ that do not belong to an $(n-m-1)$-dimensional Baer subspace of $\Omega[n-m-1]$, the lines of $\mathcal{S}'$ are the lines of $\Omega[n-m-1]$ that have no point in common with this Baer subspace.

6. $\mathcal{S}$ is a $(q-\sqrt{q}, q)$-geometry in $\mathrm{PG}(3,q)$, with points all points of $\mathrm{PG}(3,q)$, such that if p is a point of $\mathcal{S}$ and π is a plane of $\mathrm{PG}(3,q)$ not containing p, the lines of $\mathcal{S}$ through p intersect π in the points not on a unital, and such that in every plane of $\mathrm{PG}(3,q)$ the lines of $\mathcal{S}$ are the lines that intersect some unital in this plane in $\sqrt{q}+1$ point. It is not known to us whether such a $(q-\sqrt{q}, q)$-geometry exists. If such a $(q-\sqrt{q}, q)$-geometry exists, then also a cone $\Pi[n-4]\mathcal{S}$, with $\Pi[n-4]$ an $(n-4)$-dimensional subspace of $\mathrm{PG}(n,q)$ and $\mathcal{S}$ as before, gives rise to a $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$.

7. $\mathcal{S}$ is a $(q-\sqrt{q}, q)$-geometry in $\mathrm{PG}(3,q)$, with points all points of $\mathrm{PG}(3,q)$, such that if p is a point of $\mathcal{S}$ and π is a plane of $\mathrm{PG}(3,q)$ not containing p, the lines of $\mathcal{S}$ through p intersect π in the points not contained in a Baer subplane of π, and such that in every plane of $\mathrm{PG}(3,q)$ the lines of $\mathcal{S}$ are the lines that are tangent to a Baer subplane in this plane. It is not known to us whether such a $(q-\sqrt{q}, q)$-geometry exists. If such a $(q-\sqrt{q}, q)$-geometry exists, then also a cone $\Pi[n-4]\mathcal{S}$, with $\Pi[n-4]$ an $(n-4)$-dimensional subspace of $\mathrm{PG}(n,q)$ and $\mathcal{S}$ as before, gives rise to a $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$.

For q even, then we obtain the following results in later sections.

1. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q, q+1)$-geometry fully embedded in $\mathrm{PG}(n, q)$, q even. Assume that every plane of $\mathrm{PG}(n, q)$ that contains an antiflag of $\mathcal{S}$ is a q-plane or a $(q+1)$-plane. Then $\mathcal{P}$ is the set of points of $\mathrm{PG}(n, q) \setminus \Pi[m]$, for some $0 \leq m < n-2$ and $\mathcal{L}$ is the set of the lines of $\mathrm{PG}(n, q)$ that are disjoint from $\Pi[m]$.

2. Let $\mathcal{S}$ be a $(q, q+1)$-geometry, fully embedded in $\mathrm{PG}(n, q)$, q even, such that there is at least one mixed plane. Then the points of $\mathcal{S}$ are the points of $\mathrm{PG}(n, q) \setminus \Pi[m]$, with $0 \leq m < n-3$, and the lines are defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ be a partition of the points of $\mathcal{S}$, where $l = (q^{n-m} - 1)/(q^{m'-m} - 1)$, such that for $i = 1, \ldots, l$ we have that $\sigma_i = \Omega_i[m'] \setminus \Pi[m]$, with $\Omega_i[m']$ an m'-dimensional subspace of $\mathrm{PG}(n, q)$ that contains $\Pi[m]$, and $m+2 \leq m' \leq n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. A necessary and sufficient condition for this partition and the $(q, q+1)$-geometry to exist is that $(m'-m)|(n-m)$.

3. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q-1, q)$-geometry fully embedded in $\mathrm{PG}(n, q)$, q even, $q \neq 2$. Assume that there is no plane that contains an antiflag of $\mathcal{S}$ and two points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. Then the points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ are the points of a subspace $\Pi[n-2]$, and the lines are defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_{n-r}\}$ be a partition of the points of $\mathcal{S}$, such that for $i = 1, \ldots, n-r$ we have that $\sigma_i = \Omega_i[r] \setminus \Pi[n-2]$, with $\Omega_i[r]$ an r-dimensional subspace of $\mathrm{PG}(n, q)$ that intersects $\Pi[n-2]$ in an $(r-2)$-dimensional space, for $1 \leq r \leq n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. Further, such a partition exists for every $1 \leq r \leq n-2$, and gives a $(q-1, q)$-geometry.

4. Let $\mathcal{S}$ be a $(q-1, q)$-geometry fully embedded in $\mathrm{PG}(n, q)$, q even, but $q \neq 2$. Assume that there is a plane that contains exactly two distinct points of $\mathrm{P}(n, q) \setminus \mathcal{P}$, while the lines of $\mathcal{S}$ are all the lines not containing one or both of these points. Then the points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ are the points of two subspaces $\Pi[n-2]$ and $\Omega[r]$ of $\mathrm{PG}(n, q)$, for $0 \leq r \leq n-2$, with $\Omega[r] \cap \Pi[n-2]$ an $(r-2)$-dimensional space. The lines of $\mathcal{S}$ are either all lines of $\mathrm{PG}(n, q)$ that contain $q+1$ points of $\mathcal{S}$, or they are defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ be a partition of the points of $\mathcal{S}$, where $l = (q^{n-r} - 1)/(q^{d-r} - 1)$, such that for $i = 1, \ldots, l$ we have that $\sigma_i = \Lambda_i[d] \setminus (\Pi[n-2] \cup \Omega[r])$, with $\Lambda_i[d]$ a d-dimensional subspace of $\mathrm{PG}(n, q)$ that contains $\Omega[r]$, and $r+2 \leq d \leq n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. A necessary and sufficient condition for such a partition to exist is that

$(d-r)|(n-r)$. Further, if $(d-r)|(n-r)$ and $n-2 \geq d \geq r+2$, then this partition gives a $(q-1,q)$-geometry.

Remark. The point graphs of the (α, β)-geometries described above are never strongly regular graphs.

In the next sections of this chapter, the proofs of the results mentioned above, will be given.

2.3 The case in which every plane of $\mathrm{PG}(n,q)$ that contains an antiflag of $\mathcal{S}$ is an α-plane or a β-plane

Let $\mathcal{S}$ be a proper (α, β)-geometry fully embedded in $\mathrm{PG}(n,q)$, $\alpha > 1$ and q odd. Assume that every plane containing an antiflag of $\mathcal{S}$ is an α-plane or a β-plane, i.e. there are no mixed planes contained in $\mathrm{PG}(n,q)$. Since α and β both have to occur, there has to be at least one α-plane and at least one β-plane contained in $\mathrm{PG}(n,q)$. As explained in the beginning of section 2.2, and assuming $\alpha < \beta$, it follows that the points and lines of $\mathcal{S}$ in a β-plane are all the points and all the lines of the plane, and that the points and lines of $\mathcal{S}$ in an α-plane are all the points of the plane except one point p and all the lines of the plane that do not contain p. Hence $\beta = q+1$ and $\alpha = q$.

The next lemma is valid for both q odd and q even. Again, the point set of $\mathrm{PG}(n,q)$ is denoted as $\mathrm{P}(n,q)$, and the notation $\Pi[m]$ is used for fixed m-dimensional subspace of $\mathrm{PG}(n,q)$, $-1 \leq m \leq n-1$.

Lemma 2.3.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q, q+1)$-geometry fully embedded in $\mathrm{PG}(n,q)$. Assume that every plane of $\mathrm{PG}(n,q)$ that contains an antiflag of $\mathcal{S}$ is either a q-plane or a $(q+1)$-plane. Then*

1. *a line of $\mathrm{PG}(n,q)$, that contains $q+1$ points of $\mathcal{S}$, belongs to $\mathcal{L}$;*
2. *every line in $\mathrm{PG}(n,q)$ contains 0, 1, q or $q+1$ points that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$.*

Proof.

1. Let M be a line of $\mathrm{PG}(n,q)$ that contains $q+1$ points of $\mathcal{S}$. Since $t+1 > 1$, we can take a line L of $\mathcal{S}$ that intersects M in a point. The plane $\langle L, M \rangle$ clearly contains an antiflag of $\mathcal{S}$. So it is a q-plane or a $(q+1)$-plane. Now, in both a q-plane and a $(q+1)$-plane, a line containing $q+1$ points of $\mathcal{S}$ is a line of $\mathcal{S}$. Hence M is a line of $\mathcal{S}$.

2. Assume that $\mathrm{PG}(n,q)$ contains a line M on which there are r points that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$, with $r \notin \{0,1,q,q+1\}$. Then M contains at least two points y_1 and y_2 that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$ and at least two points p_1 and p_2 of $\mathcal{S}$. Since $t+1>0$, there is a plane through M containing a line L of $\mathcal{S}$ through p_1. This plane contains the antiflag (p_2, L) of $\mathcal{S}$ and hence it is a q-plane or a $(q+1)$-plane. However, the plane $\langle M, L\rangle$ contains the two points y_1 and y_2, that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$. This is a contradiction. Hence every line in $\mathrm{PG}(n,q)$ contains 0, 1, q or $q+1$ points that are elements of $\mathrm{P}(n,q)\setminus\mathcal{P}$.

□

Theorem 2.3.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper (α,β)-geometry, $\alpha > 1$, fully embedded in $\mathrm{PG}(n,q)$, for q odd. Assume that every plane of $\mathrm{PG}(n,q)$ that contains an antiflag of $\mathcal{S}$ is an α-plane or a β-plane. Then $\mathcal{P}$ is the set of points of $\mathrm{PG}(n,q) \setminus \Pi[m]$, for some $0 \le m < n-2$ and $\mathcal{L}$ is the set of the lines of $\mathrm{PG}(n,q)$ that are disjoint from $\Pi[m]$.*

Proof. Let $\mathcal{S}$ be a proper (α,β)-geometry fully embedded in $\mathrm{PG}(n,q)$, for $\alpha > 1$ and q odd. Assume that $\mathrm{PG}(n,q)$ contains no mixed planes. It follows immediately that $\alpha = q$ and $\beta = q+1$.

We will first prove that the points that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of a subspace $\Pi[m]$ of $\mathrm{PG}(n,q)$. Suppose not, and let N be a line containing at least two points that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$ and at least one point $p \in \mathcal{P}$. Then from lemma 2.3.1 it follows that N contains q points that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Since $t+1 \ge 2$, we can take a plane π through two lines of $\mathcal{S}$ intersecting in p. The plane π has to be a q-plane or a $(q+1)$-plane. So there are q or $q+1$ lines of $\mathcal{S}$ through p in π.

Now we look at the three dimensional space $\langle N, \pi\rangle$. Every plane through N in $\langle N, \pi\rangle$ contains at least q points that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$, so such a plane cannot contain an antiflag of $\mathcal{S}$. There are at least q of these planes that intersect π in a line of $\mathcal{S}$. They contain exactly q^2 points that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$. If the remaining plane through N in $\langle N, \pi\rangle$ would contain a point w of $\mathcal{S}$, $w \notin \pi$, then a plane through w and a line of $\mathcal{S}$ in π through p contains an antiflag of $\mathcal{S}$ and more than two points that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$, a contradiction. We conclude that all points of $\langle N, \pi\rangle \setminus \pi$ are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$.

Since $\mathcal{S}$ is connected, there is a line L of $\mathcal{S}$ intersecting $\langle N, \pi\rangle$ in a point. Since all points of $\mathcal{S}$ in $\langle N, \pi\rangle$ are contained in π, L intersects π in a point. The planes through L in $\langle L, N, \pi\rangle$ partition the points of $\langle L, N, \pi\rangle$. Every plane through L intersecting $\langle N, \pi\rangle$ in a line not contained in π contains the

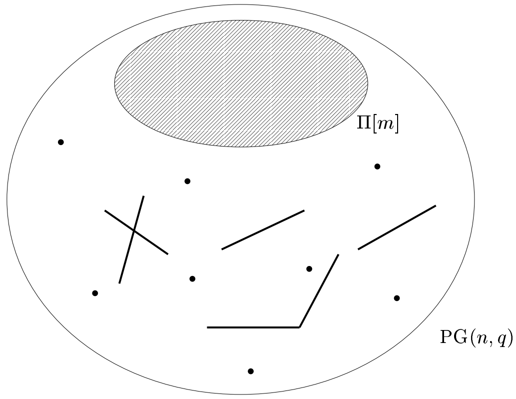

Figure 2.1: The $(q, q+1)$-geometry $\mathrm{H}_q^{n,m}$

line L of $\mathcal{S}$ and at least q points that are elements of $\mathrm{P}(n, q) \setminus \mathcal{P}$. So every such plane contains q^2 points that are elements of $\mathrm{P}(n, q) \setminus \mathcal{P}$, being all the points not on L. Hence all the points of $\mathcal{S}$ in the four dimensional space $\langle L, N, \pi\rangle$ are contained in the three dimensional space $\langle L, \pi\rangle$.

Continuing in this way, we see that all the points of $\mathcal{S}$ in $\mathrm{PG}(n, q)$ are contained in an $(n-1)$-dimensional subspace of $\mathrm{PG}(n, q)$. This is a contradiction, because by assumption the points of $\mathcal{S}$ span $\mathrm{PG}(n, q)$. Hence we have proved that the points that are elements of $\mathrm{P}(n, q) \setminus \mathcal{P}$ are the points of an m-dimensional subspace $\Pi[m]$ of $\mathrm{PG}(n, q)$. It is clear that $m \leq n-2$, as otherwise $\mathrm{PG}(n, q)$ cannot contain lines of $\mathcal{S}$.

From lemma 2.3.1 it follows that every line that contains $q+1$ points of $\mathcal{S}$ belongs to $\mathcal{S}$. Hence the lines of $\mathcal{S}$ in $\mathrm{PG}(n, q)$ are the lines not intersecting $\Pi[m]$. If $m = n-2$ or $m = -1$, then $\mathcal{S}$ would be a partial geometry, a contradiction since we assumed $\mathcal{S}$ to be proper. Hence $0 \leq m \leq n-3$. $\square$

For q even, we get the following result. It is clear that this theorem also holds for q odd, as the above theorem for q odd is a stronger result.

Theorem 2.3.3 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q, q+1)$-geometry fully embedded in $\mathrm{PG}(n, q)$. Assume that every plane of $\mathrm{PG}(n, q)$ that contains an antiflag of $\mathcal{S}$ is a q-plane or a $(q+1)$-plane. Then $\mathcal{P}$ is the set of points of $\mathrm{PG}(n, q) \setminus \Pi[m]$, for some $0 \leq m < n-2$ and $\mathcal{L}$ is the set of the lines of $\mathrm{PG}(n, q)$ that are disjoint from $\Pi[m]$.*

Since the above described $(q, q+1)$-geometry is very similar to the partial geometry H_q^n, we will denote it by $\mathrm{H}_q^{n,m}$. In chapter 4, a characterization of $\mathrm{H}_q^{n,m}$ will be given, based on the known characterization result for the partial geometry H_q^n that uses the axiom of Pasch (also sometimes called the axiom of Veblen-Young) [54].

2.4 The case in which there is a mixed plane and $\beta = q + 1$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper $(\alpha, q+1)$-geometry fully embedded in $\mathrm{PG}(n, q)$, for $\alpha > 1$ and q odd. Assume that there is a mixed plane σ.

We will determine what the restriction of $\mathcal{S}$ to σ can be. To do that, suppose first that σ contains only one point p through which there are $q+1$ lines of $\mathcal{S}$ in σ. Then all points of σ belong to $\mathcal{S}$, since through every point of σ there is a line of $\mathcal{S}$. Now in $\sigma \setminus \{p\}$ there are $\alpha - 1$ lines of $\mathcal{S}$ through every point. Hence the restriction of $\mathcal{S}$ to $\sigma \setminus \{p\}$ is a partial geometry $\mathrm{pg}(q, \alpha - 2, \alpha - 1)$. Now using the formula for the number of points of a partial geometry, we get that it contains $(q + 1)(q + 1 - q/(\alpha - 1))$ points, which should be equal to $(q + 1)q$, which is the number of points contained in $\sigma \setminus \{p\}$. So $\alpha = q + 1 = \beta$, a contradiction since we assumed $\mathcal{S}$ to be a proper $(\alpha, q + 1)$-geometry.

Now suppose that σ contains more than one point through which there are $q+1$ lines of $\mathcal{S}$. Let $\mathcal{K}$ be the set of points of σ through which there are $q + 1$ lines of $\mathcal{S}$. We will prove that $\mathcal{K}$ is a maximal arc. Note that, since σ is a mixed plane, $\mathcal{K} \neq \emptyset$ and hence all points of σ are points of $\mathcal{S}$. Let L be a line of $\mathcal{S}$ in σ that contains c points of $\mathcal{K}$. Counting the lines of $\mathcal{S}$ in σ intersecting L, it follows that there are $1+cq+(q+1-c)(\alpha-1)$ lines of $\mathcal{S}$ in σ. Let M be a line of $\mathcal{S}$ in σ that contains c' points of $\mathcal{K}$. Then counting the lines of $\mathcal{S}$ in σ intersecting M, we get that there are $1+c'q+(q+1-c')(\alpha-1)$ such lines. Comparing these two results tells us that $c = c'$. Hence on every line of $\mathcal{S}$ in σ there is a constant number c of points of $\mathcal{K}$. It is clear that on a line in σ that does not belong to $\mathcal{S}$, there are 0 points of $\mathcal{K}$. So $\mathcal{K}$ is a maximal arc. Now since q is odd, $\mathcal{K}$ has to be trivial (see [1]), or in other words $\mathcal{K} = \emptyset$, $|\mathcal{K}| = 1$ or $\mathcal{K}$ is the set of all points of an affine or a projective plane. Since it was assumed that there is more than one point with $q + 1$ lines of $\mathcal{S}$ through it, the points in σ with $q + 1$ lines through them are the points of an affine plane. This implies that $\alpha = q$. Hence all the points of σ are points of $\mathcal{S}$, while there is exactly one line of σ that does not belong to $\mathcal{S}$.

We conclude that there are three types of planes containing an antiflag of $\mathcal{S}$, namely q-planes, $(q + 1)$-planes and planes in which all points belong

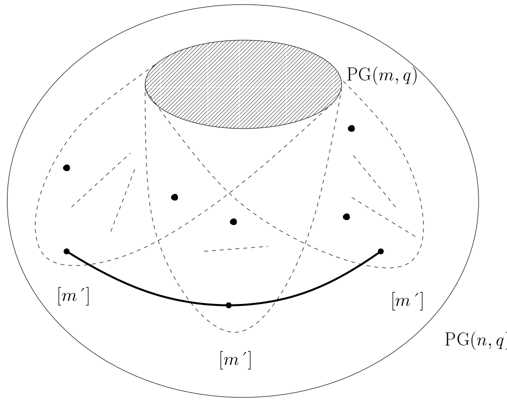

Figure 2.2: The $(q, q+1)$-geometry $\mathrm{SH}_q^{n,m}$

to $\mathcal{S}$ and all lines but one line belong to $\mathcal{S}$. In the next theorem, we use again the notation $\mathrm{P}(n,q) \setminus \mathcal{P}$ for the points set of $\mathrm{PG}(n,q)$, and $\Pi[m]$ and $\Lambda[m]$ for fixed m-dimensional subspaces of $\mathrm{PG}(m,q)$.

Theorem 2.4.1 *Let $\mathcal{S}$ be a proper $(\alpha, q+1)$-geometry, $\alpha > 1$ and q odd, fully embedded in $\mathrm{PG}(n,q)$, such that there is at least one mixed plane. Then the points of $\mathcal{S}$ are the points of $\mathrm{PG}(n,q) \setminus \Pi[m]$, with $-1 \leq m < n-3$, and the lines of $\mathcal{S}$ are defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ be a partition of the points of $\mathcal{S}$, where $l = (q^{n-m}-1)/(q^{m'-m}-1)$, such that for $i = 1, \ldots, l$ we have that $\sigma_i = \Omega_i[m'] \setminus \Pi[m]$, with $\Omega_i[m']$ an m'-dimensional subspace of $\mathrm{PG}(n,q)$ that contains $\Pi[m]$, and with $m+2 \leq m' \leq n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. A necessary and sufficient condition for this partition and the $(q, q+1)$-geometry to exist is that $(m'-m)|(n-m)$.*

Proof. Let $\mathcal{S}$ be a proper $(\alpha, q+1)$-geometry, $\alpha > 1$ and q odd, fully embedded in $\mathrm{PG}(n,q)$, such that there is at least one mixed plane.

Consider a new geometry $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$, with $\mathcal{P}^* = \mathcal{P}$ and with $\mathcal{L}^* = \mathcal{L} \cup \mathcal{B}$, where $\mathcal{B}$ is the set of all lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathcal{S}$ but that do not belong to $\mathcal{L}$. Then $\mathcal{S}^*$ satisfies the hypotheses of theorem 2.3.2. So, applying that theorem to $\mathcal{S}^*$, it follows that the points of $\mathrm{PG}(n,q)$ that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of a subspace $\Pi[m]$ of $\mathrm{PG}(n,q)$, for $m \leq n-2$. If $m = n-2$, then every plane would contain a point that is an element of $\mathrm{P}(n,q) \setminus \mathcal{P}$, a contradiction since we assumed that there is a mixed plane. Hence $m < n-2$. Note that, since in this case there does not necessarily exist an α-plane, it is possible that $m = -1$.

It remains to prove that the lines of $\mathcal{S}$ are the lines not belonging to a partition of the points of $\mathcal{S}$ in m'-dimensional spaces through $\Pi[m]$, intersecting pairwise in $\Pi[m]$, with $m+2 \leq m' \leq n-2$. By assumption there is at least one mixed plane in $\mathrm{PG}(n, q)$. Such a plane contains a line M of $\mathcal{B}$.

Let σ be a plane spanned by M and a point of $\Pi[m]$. Then σ cannot contain an antiflag of $\mathcal{S}$. So σ does not contain a line of $\mathcal{S}$. Since σ was a plane through M and an arbitrary point of $\Pi[m]$, this implies that all lines contained in the $(m+2)$-dimensional space $\langle M, \Pi[m]\rangle$ and intersecting M, do not belong to $\mathcal{S}$. Let N be a line in $\langle M, \Pi[m]\rangle$ that does not intersect M. Then a plane spanned by N and a point u of M contains $q+1$ lines through u that do not belong to $\mathcal{S}$. This implies that $N \notin \mathcal{L}$, as otherwise $i(u, N)$ would be 0. Hence $\langle M, \Pi[m]\rangle$ contains no lines of $\mathcal{S}$. Through each point of M, there are at least $(q^{m+2}-1)/(q-1)$ lines not belonging to $\mathcal{S}$, namely all the lines through such a point that are contained in the $(m+2)$-dimensional space $\langle M, \Pi[m]\rangle$.

Assume that not all lines of $\mathcal{B}$ through the points of M belong to $\langle M, \Pi[m]\rangle$. Then there is a line $M' \in \mathcal{B}$, that intersects M in a point u', such that M' does not belong to $\langle M, \Pi[m]\rangle$. In the same way as we did for M, we get that the $(m+2)$-dimensional space $\langle M', \Pi[m]\rangle$ contains no lines of $\mathcal{S}$. It follows immediately that the $(m+3)$-dimensional space spanned by $\langle M, \Pi[m]\rangle$ and $\langle M', \Pi[m]\rangle$ does not contain a line of $\mathcal{S}$. Indeed, if there would be a line L of $\mathcal{S}$ in $\langle M, M', \Pi[m]\rangle$, then L would intersect both of the $(m+2)$-dimensional spaces $\langle M, \Pi[m]\rangle$ and $\langle M', \Pi[m]\rangle$ in a point, as we have proved above that L can not be contained in $\langle M, \Pi[m]\rangle$ or in $\langle M', \Pi[m]\rangle$. Now if L is skew to $\langle M, \Pi[m]\rangle \cap \langle M', \Pi[m]\rangle$, then the plane $\langle L, u'\rangle$ contains no points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. Hence $\langle L, u'\rangle$ is a plane containing an antiflag of $\mathcal{S}$ and at least two lines of $\mathcal{B}$, being its intersection lines with $\langle M, \Pi[m]\rangle$ respectively $\langle M', \Pi[m]\rangle$. This is a contradiction, as such a plane can not exist. If the line L does intersect $\langle M, \Pi[m]\rangle \cap \langle M', \Pi[m]\rangle$ in a point z, then let N_z be a line of $\mathcal{B}$ through z in $\langle M, \Pi[m]\rangle$. The plane $\langle L, N_z\rangle$ is contained in the $(m+3)$-dimensional space $\langle M, M', \Pi[m]\rangle$ and hence it intersects the $(m+2)$-dimensional space $\langle M', \Pi[m]\rangle$ in a line N'_z. Clearly $N'_z \in \mathcal{B}$, since $\langle L, N_z\rangle$ is skew to $\Pi[m]$. Hence $\langle L, N_z\rangle$ is a plane that contains an antiflag of $\mathcal{S}$ and at least two distinct lines of $\mathcal{B}$, a contradiction. This proves that $\langle M, M', \Pi[m]\rangle$ does not contain lines of $\mathcal{S}$.

Continuing in this way, we obtain that all lines of $\mathcal{B}$ through the points of M are contained in a subspace $\mathrm{PG}(m', q)$, for $m+2 \leq m' \leq n-2$, that contains $\Pi[m]$, and that this subspace $\Lambda[m']$ contains no lines of $\mathcal{S}$. Clearly the number of lines through a point of $\mathcal{S}$, on which there lies a point that is an element of $\mathrm{P}(n,q) \setminus \mathcal{P}$, is a constant.

Since $t+1$ is a constant, through every point of $\mathcal{S}$ in $\mathrm{PG}(n,q)$ there are a constant number of lines that do not belong to $\mathcal{S}$. Hence through every point of $\mathcal{S}$ there is a constant number of lines of $\mathcal{B}$. It now follows from the above that the lines of $\mathcal{B}$ are the lines contained in a partition of the points of $\mathrm{P}(n,q) \setminus \Pi[m]$ into m'-dimensional spaces through $\Pi[m]$, that pairwise intersect in $\Pi[m]$.

In the above we proved that the points that are elements of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of a subspace $\Pi[m]$ of $\mathrm{PG}(n,q)$, $m < n-2$. The points of $\mathrm{PG}(n,q) \setminus \Pi[m]$ are partitioned into m'-dimensional subspaces of $\mathrm{PG}(n,q)$, $m+2 \leq m' \leq n-2$, that intersect pairwise in $\Pi[m]$. All lines of $\mathrm{PG}(n,q)$ contained in such an m'-dimensional subspace do not belong to $\mathcal{S}$, while all lines intersecting $q+1$ of these subspaces in a point are lines of $\mathcal{S}$.

A necessary and sufficient condition for this configuration to exist is that the points of $\mathrm{PG}(n,q) \setminus \Pi[m]$ can be partitioned into m'-dimensional spaces pairwise intersecting in $\Pi[m]$. Considering the factor space $\mathrm{PG}(n,q)/\Pi[m]$, this is equivalent to the existence of a partition of an $(n-m-1)$-dimensional projective space into $(m'-m-1)$-dimensional subspaces. Such a partition exists if and only if $(m'-m) \mid (n-m')$ ([34], theorem 4.1.1). □

Note that for q even, it does not necessarily follow that $\alpha = q$. Indeed, a mixed plane σ can be a plane containing q^2+q+1 points of $\mathcal{S}$, the lines of $\mathcal{S}$ in σ being all lines intersecting a (possibly non-trivial) maximal arc. In this case $\alpha = q+1-q/d$, where d is the order of the maximal arc. Hence if we do not assume that q is odd, then we have to assume that $\alpha = q$, in which case the above proof gives the following theorem.

Theorem 2.4.2 *Let $\mathcal{S}$ be a $(q,q+1)$-geometry, fully embedded in* $\mathrm{PG}(n,q)$*, such that there is at least one mixed plane. Then the points of $\mathcal{S}$ are the points of* $\mathrm{PG}(n,q) \setminus \Pi[m]$*, with $0 \leq m \leq n-3$, and the lines of $\mathcal{S}$ are defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ be a partition of the points of $\mathcal{S}$, where $l = (q^{n-m}-1)/(q^{m'-m}-1)$, such that for $i = 1, \ldots, l$ we have that $\sigma_i = \Omega_i[m'] \setminus \Pi[m]$, with $\Omega_i[m']$ an m'-dimensional subspace of* $\mathrm{PG}(n,q)$ *that contains $\Pi[m]$, and with $m+2 \leq m' \leq n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. A necessary and sufficient condition for this partition and the $(q,q+1)$-geometry to exist is that $(m'-m) | (n-m)$.*

In chapter 4, we will give a characterization of this $(q,q+1)$-geometry, that we will denote by $\mathrm{SH}_q^{n,m}$, because it is similar to the partial geometry

H_q^n and the $(q, q+1)$-geometry $\mathrm{H}_q^{n,m}$ defined in section 2.3.

2.5 The case in which $\beta = q$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper (α, q)-geometry fully embedded in $\mathrm{PG}(n, q)$, with $\alpha > 1$ and q odd. We use again the notation $\mathrm{P}(n, q)$ for the point set of $\mathrm{PG}(n, q)$. Assume that $\mathrm{PG}(n, q)$ contains a mixed plane σ. We determine what the restriction of $\mathcal{S}$ to σ can be. Let $\mathcal{K}$ be the set of points of σ through which there are q lines of $\mathcal{S}$ in σ.

Assume first that no pair of points of $\mathcal{K}$ lie on a line of $\mathcal{S}$. Let p be a point of $\mathcal{K}$. Then all the other points of $\mathcal{K}$ in σ lie on the unique line M_p through p that does not belong to $\mathcal{S}$. In the affine plane $\sigma \setminus M_p$, there are α lines of $\mathcal{S}$ through every point. Hence the restriction of $\mathcal{S}$ to the affine plane $\sigma \setminus M_p$ is a partial geometry. From [49] it follows that the points and lines of $\mathcal{S}$ in $\sigma \setminus M_p$ are the points of a net, or that the lines of $\mathcal{S}$ in σ, together with the line M_p, are the lines of a dual hyperoval of the projective closure of $\sigma \setminus M_p$. Since q is odd, the hyperoval case can not occur. Hence the points and lines of $\mathcal{S}$ in $\sigma \setminus M_p$ are the points of a net. The points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ in σ, which is the projective closure of $\sigma \setminus M_p$, are $q + 1 - \alpha$ points of M_p, while the lines of $\mathcal{S}$ in σ are all lines of σ intersecting M_p in a point of $\mathcal{S}$. We therefore call the intersection of $\mathcal{S}$ with σ the *closure of a net*.

Assume next that σ contains two points of $\mathcal{K}$ that lie on a line of $\mathcal{S}$. Then either all points of σ belong to $\mathcal{S}$, or there is exactly one point of σ that does not belong to $\mathcal{S}$.

Assume that σ contains one point y of $\mathrm{P}(n, q) \setminus \mathcal{P}$. There are α or q lines of $\mathcal{S}$ in σ through every point of σ different from y. Now we look at the dual plane σ^D. In this plane, every line different from y^D contains either α or q points of the form L^D, with $L \in \mathcal{L}$. The line y^D contains 0 such points. Hence the set of points $\{L^D | L \in \mathcal{L}, L \subset \sigma\}$ is a set of type (α, q) in the affine plane $\sigma^D \setminus y^D$. The complement of this set is a $(0, q-\alpha)$-set in $\sigma^D \setminus y^D$, which is a maximal arc. Since we assumed that q is odd, a non-trivial maximal arc cannot exist (see [1]). Hence the set $\{L^D | L \in \mathcal{L}, L \subset \sigma\}^C$ is either one point or all points of the affine plane $\sigma \setminus y^D$. The last case cannot occur, as it would imply that σ contains no lines of $\mathcal{S}$, a contradiction with the assumptions. Hence $\{L^D | L \in \mathcal{L}, L \subset \sigma\}^C$ is a point and this point is contained in $\sigma \setminus y^D$. Dualizing again we get that σ contains exactly one line, not through the point y, that does not belong to $\mathcal{S}$. It immediately follows that $\alpha = q - 1$.

Assume next that every point of σ belongs to $\mathcal{S}$. Then clearly there are either q or α lines of $\mathcal{S}$ in σ through every point of σ. This implies that in the dual plane σ^D every line contains either q or α points that are of the form L^D, with $L \in \mathcal{L}$. Hence the set of points $\{L^D | L \in \mathcal{L}, L \subset \sigma\}$ is an (α, q)-set in σ. The complement of this set is a $(1, q+1-\alpha)$-set in σ. From ([34], theorem 12.17) it follows that $\{L^D | L \in \mathcal{L}, L \subset \sigma)\}^C$ is a unital or a Baer subplane. This implies that q is a square, and that $q+1-\alpha = \sqrt{q}+1$, or thus $\alpha = q - \sqrt{q}$. Taking the complement and dualizing again, we get that the lines of $\mathcal{S}$ in σ are either the lines intersecting a unital in $\sqrt{q}+1$ points, or the lines tangent to a Baer subplane.

We conclude that there are three possibilities for the restriction of $\mathcal{S}$ to a mixed plane σ.

1. It is the closure of a net, as defined above.
2. The points of $\mathcal{S}$ in σ are the points of $\sigma \setminus \{x\}$, for a point $x \in \sigma$, while the lines of $\mathcal{S}$ in σ are all lines in σ not through x, except for one line M, such that $x \notin M$. In this case $\alpha = q - 1$.
3. All points of σ belong to $\mathcal{S}$. The lines of $\mathcal{S}$ in σ are either the lines intersecting a unital in $\sqrt{q}+1$ points or the lines tangent to a Baer subplane. In this case $\alpha = q - \sqrt{q}$.

So if $\alpha = q - 1$ or $\alpha = q - \sqrt{q}$, then there exist certain mixed planes, that do not occur for $q - \sqrt{q} \neq \alpha \neq q - 1$. We treat the cases $\alpha = q - 1$ and $\alpha = q - \sqrt{q}$ separately.

2.5.1 The case $\alpha = q - 1$

If $\alpha = q - 1$, then there are no α-planes contained in $\mathrm{PG}(n, q)$. This follows from the fact that for a maximal arc, the degree d of the maximal arc has to divide q. As $\alpha = q - 1$, it is clear that α does not divide q. So there are three different types of planes that contain an antiflag of $\mathcal{S}$.

- Type I are the q-planes.
- Type II are the planes in which the restriction of $\mathcal{S}$ is the closure of a net. Note that, as $\alpha = q - 1$, these planes contain exactly two points that do not belong to $\mathcal{S}$.
- Type III are the planes that contain one point x of $\mathrm{P}(n, q) \setminus \mathcal{P}$ and one line M that does not belong to $\mathcal{S}$, such that $x \notin M$.

Remark. Let $\mathcal{S}$ be a $(q-1, q)$-geometry fully embedded in $\mathrm{PG}(n, q)$, q even and $q > 2$. Then the planes containing an antiflag of $\mathcal{S}$ are precisely the planes of type I, II and III as defined above. This follows from the previous paragraphs, where we determined what the restriction of $\mathcal{S}$ to a mixed plane looks like. Note that also for q even, the case that the lines of $\mathcal{S}$ are $q+1$ lines of a dual hyperoval and the points of $\mathcal{S}$ are the points on these lines, does not occur. Indeed, in this kind of planes there are one or two lines of $\mathcal{S}$ through every point of $\mathcal{S}$. Hence $q-1=1$, a contradiction as we assumed that $q>2$. For this reason, the results in this section hold for any q, $q \neq 2$.

Lemma 2.5.1 *Let $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q-1, q)$-geometry, $q \neq 2$, fully embedded in* $\mathrm{PG}(n, q)$. *Then every line of* $\mathrm{PG}(n, q)$ *contains* 0, 1, 2 *or* $q+1$ *points of* $\mathrm{P}(n, q) \setminus \mathcal{P}$.

Proof. Let $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q-1, q)$-geometry fully embedded in $\mathrm{PG}(n, q)$, with $q \neq 2$. Assume that there exists a line M in $\mathrm{PG}(n, q)$ that contains r points of $\mathrm{P}(n, q) \setminus \mathcal{P}$, with $r \notin \{0,1,2, q+1\}$. Then M contains at least three points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ and at least one point w of $\mathcal{S}$.

Assume first that $r<q$. Then every plane spanned by M and a line of $\mathcal{S}$ through w contains an antiflag of $\mathcal{S}$ and hence it is of type I, II or III. Now it is clear that in a plane of type I, II or III every line contains 0, 1 or 2 points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. This is a contradiction, as we assumed that $r \notin \{0,1,2\}$.

Assume next that $r=q$. A plane through M cannot contain an antiflag of $\mathcal{S}$. In particular, every plane through M contains at most one line of $\mathcal{S}$ through w. Since $t+1>1$, there are at least two lines of $\mathcal{S}$ through w in $\mathrm{PG}(n, q)$. Hence there exists a plane π through w that contains an antiflag of $\mathcal{S}$. We denote the lines through w in π by $L_1, \ldots, L_{q+1}$. At least $\alpha=q-1$ of these lines belong to $\mathcal{S}$. So we may assume that $L_1, \ldots, L_{q-1}$ are lines of $\mathcal{S}$. The planes $\langle M, L_i\rangle$, for $i=1, \ldots, q-1$, contain a line of $\mathcal{S}$, but they can not contain an antiflag of $\mathcal{S}$. Hence each of them contains q^2 points of $\mathrm{P}(n, q) \setminus \mathcal{P}$, namely all the points not on the line L_i, for $i=1, \ldots, q-1$. Now assume that $\langle M, \pi\rangle$ contains a point u of $\mathcal{S}$, with $u \notin \pi$. Let L be a line of $\mathcal{S}$ in π not through w. Then clearly the plane $\langle u, L\rangle$ contains an antiflag of $\mathcal{S}$. It follows that $\langle u, L\rangle$ is of type I, II or III. This implies that $\langle u, L\rangle$ can contain at most two points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. However, $\langle u, L\rangle$ intersects each of the planes $\langle M, L_i\rangle$, for $i \in\{1, \ldots, q-1\}$, in a line that contains q points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. This is a contradiction. So we have proved that $\langle M, \pi\rangle \setminus \pi$ contains no points of $\mathcal{S}$.

If $n=3$, then the above gives a contradiction, as the points of $\mathcal{S}$ have to span $\mathrm{PG}(n, q)$. So we may assume that $n \geq 4$. Let z be a point of $\mathcal{S}$ that does not belong to $\langle M, \pi\rangle$. Since $\mathcal{S}$ is a $(q-1, q)$-geometry, there exists a line $N \in \mathcal{L}$ through z that intersects $\langle M, \pi\rangle$ in a point. Assume that there exists a point $u' \in \mathcal{P}$ in the four dimensional space $\langle N, M, \pi \rangle$ that is not contained in the three dimensional space $\langle N, \pi \rangle$. Then $\langle u', N \rangle$ contains q points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ on its intersection line with the three dimensional space $\langle M, \pi \rangle$. Hence $\langle u', N \rangle$ is a plane containing an antiflag of $\mathcal{S}$ and at least q points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. This is a contradiction, because every plane containing an antiflag of $\mathcal{S}$ has to be of type I, II or III. Hence all the points of $\mathcal{S}$ in the four dimensional space $\langle N, M, \pi \rangle$ are contained in the three dimensional space $\langle N, \pi \rangle$. If $n = 4$, then the above gives a contradiction, since the points of $\mathcal{S}$ have to span $\mathrm{PG}(n,q)$.

Now assume that $\Pi[m]$ is an m-dimensional subspace of $\mathrm{PG}(n,q)$, $m \geq 4$, such that $\langle M, N, \pi \rangle \subseteq \Pi[m]$ and such that all points of $\mathcal{S}$ in $\Pi[m]$ are contained in a hyperplane $\Upsilon[m-1]$ of $\Pi[m]$. If $n = m$ then we have found a contradiction, since the points of $\mathcal{S}$ have to span $\mathrm{PG}(n,q)$. So we may assume that $n > m$. Let p be a point of $\mathcal{S}$, $p \notin \Pi[m]$. Since $\mathcal{S}$ is connected, there exists a line N_p of $\mathcal{S}$ through p intersecting $\Pi[m]$, and hence $\Upsilon[m-1]$, in a point. Let $\Gamma[m+1]$ be the $(m+1)$-dimensional subspace spanned by $\Pi[m]$ and N_p. We will prove that all the points of $\mathcal{S}$ in $\Gamma[m+1]$ are contained in the m-dimensional space $\langle p, \Upsilon[m-1] \rangle$. Assume that there would be a point $p' \in \mathcal{P}$ that is contained in $\Gamma[m+1]$ but not in $\langle p, \Upsilon[m-1] \rangle$. Then the plane $\langle p', N_p \rangle$ intersects $\Pi[m]$ in a line containing q points that do not belong to $\mathcal{S}$. However $\langle p', N_p \rangle$ contains an antiflag of $\mathcal{S}$, so it is a plane of type I, II or III, and hence it contains at most two points that do not belong to $\mathcal{S}$. This is a contradiction. It follows that all points of $\mathcal{S}$ in $\Gamma[m+1]$ are contained in the m-dimensional subspace $\langle N_p, \Upsilon[m-1] \rangle$.

Continuing in this way, after a finite number of steps we get that all the points of $\mathcal{S}$ in $\mathrm{PG}(n,q)$ are contained in an $(n-1)$-dimensional subspace of $\mathrm{PG}(n,q)$. This is a contradiction, because we assumed that the points of $\mathcal{S}$ span $\mathrm{PG}(n,q)$. This proves that every line of $\mathrm{PG}(n,q)$ contains 0, 1, 2 or $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. □

Theorem 2.5.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q-1,q)$-geometry fully embedded in* $\mathrm{PG}(n,q)$*, with $q \neq 2$. Then every plane that contains a line of $\mathcal{S}$ is a plane of type* I, II *or* III.

Proof. From lemma 2.5.1 we know that $\mathrm{PG}(n,q)$ contains no lines on which there are q points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. It follows that every plane through a line of $\mathcal{S}$ contains an antiflag of $\mathcal{S}$. So every plane through a line of $\mathcal{S}$ is a plane of type I, II or III. □

Theorem 2.5.3 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q-1,q)$-geometry fully embedded in* $\mathrm{PG}(n,q)$*, with $q \neq 2$. Assume that all the planes containing an antiflag*

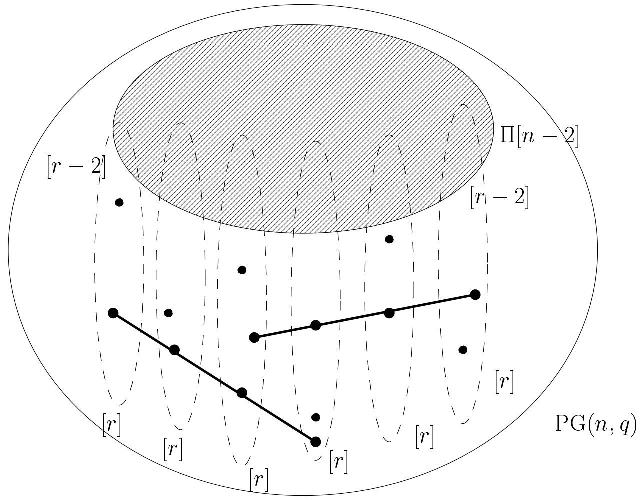

Figure 2.3: Fully embedded $(q-1,q)$-geometry in $\mathrm{PG}(n,q)$, with no planes of type II

of $\mathcal{S}$ are of type I *or of type* III*. Then the points of* $\mathrm{P}(n,q)\setminus\mathcal{P}$ *are the points of an $(n-2)$-dimensional subspace* $\Pi[n-2]$*. The lines of $\mathcal{S}$ are defined as follows. Let $\Sigma=\{\sigma_1,\ldots,\sigma_{n-r}\}$ be a partition of the points of $\mathcal{S}$, such that for $i=1,\ldots n-r$ we have that $\sigma_i=\Omega_i[r]\setminus\Pi[n-2]$, with $\Omega_i[r]$ an r-dimensional subspace of* $\mathrm{PG}(n,q)$ *that intersects $\Pi[n-2]$ in an $(r-2)$-dimensional space, for $1\le r\le n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. Further, such a partition exists for every $1\le r\le n-2$, and gives a $(q-1,q)$-geometry.*

Proof. Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with $q\neq 2$. Assume that all the planes containing an antiflag of $\mathcal{S}$ are of type I or of type III. From lemma 2.5.1 it follows that every line of $\mathrm{PG}(n,q)$ contains 0, 1, 2 or $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Since we assumed that there are no planes of type II, there cannot be lines in $\mathrm{PG}(n,q)$ that contain two points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. It follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ are the points of a subspace $\Pi[m]$ of $\mathrm{PG}(n,q)$ of dimension m, with $m\le n-2$. Now let L be a line of $\mathcal{S}$. We will prove that all points of $\mathcal{S}$ are contained in $\langle L,\Pi[m]\rangle$. Assume therefore that x is a point of $\mathcal{S}$ that is not contained in $\langle L,\Pi[m]\rangle$. The plane $\langle x,L\rangle$ contains an antiflag of $\mathcal{S}$ and hence it contains no point of $\mathrm{P}(n,q)\setminus\mathcal{P}$. This is a contradiction, as by assumption every plane containing an antiflag of $\mathcal{S}$ is of type I or III. Hence all the points of $\mathcal{S}$ are contained in the $(m+2)$-dimensional space $\langle L, \Pi[m]\rangle$. Since the points of $\mathcal{S}$ span $\mathrm{PG}(n,q)$, this proves that $m = n-2$.

Let $\mathcal{B}$ be the set of lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathcal{S}$ but that do not belong to $\mathcal{S}$. There are $t+1$ lines of $\mathcal{S}$ through every point of $\mathcal{S}$ and there are $(q^{n-1}-1)/(q-1)$ lines through a point of $\mathcal{S}$ that intersect the subspace $\Pi[n-2]$ in a point. So the number of elements of $\mathcal{B}$ through a point of $\mathcal{S}$ equals $q^{n-1}-t-1$ and hence it is constant. It is clear that $\mathcal{B} \neq \emptyset$, as otherwise it follows that $\mathcal{S}$ is a partial geometry H_q^n, a contradiction since we assumed that $\mathcal{S}$ is a $(q-1,q)$-geometry.

If through every point of $\mathcal{S}$ in $\mathrm{PG}(n,q)$ there is exactly one line of $\mathcal{B}$, then the elements of $\mathcal{B}$ are the lines of a line spread of $\mathrm{PG}(n,q) \setminus \Pi[n-2]$.

If through a point z of $\mathcal{S}$ there are two lines N_1 and N_2 of $\mathcal{B}$, then the plane spanned by N_1 and N_2 can not contain a line of $\mathcal{S}$, as otherwise this plane would contain an antiflag of $\mathcal{S}$, but a plane containing an antiflag of $\mathcal{S}$ can not contain two lines of $\mathcal{B}$. So all lines of $\langle N_1, N_2\rangle$ not through the point $\langle N_1, N_2\rangle \cap \Pi[n-2]$, belong to $\mathcal{B}$. If all elements of $\mathcal{B}$ through z are contained in $\langle N_1, N_2\rangle$, then through every point of $\mathcal{S}$ there has to be a plane containing all elements of $\mathcal{B}$ through this point and all these planes are disjoint or their intersection belongs to $\Pi[n-2]$. Hence the lines of $\mathcal{S}$ are the lines that are skew to $\Pi[n-2]$ and that are not contained in a partition Σ of the points of $\mathrm{PG}(n,q) \setminus \Pi[n-2]$ into planes intersecting $\Pi[n-2]$ in a point.

If there is a line N_3 of $\mathcal{B}$ through z, such that N_3 is not contained in the plane $\langle N_1, N_2\rangle$, then the three dimensional space spanned by N_1, N_2 and N_3 contains no lines of $\mathcal{S}$. Indeed, assume that there would be a line L_z of $\mathcal{S}$ through z in $\langle N_1, N_2, N_3\rangle$. The plane $\langle N_1, N_2\rangle$ contains no lines of $\mathcal{S}$. The plane $\langle L_z, N_3\rangle$ intersects $\langle N_1, N_2\rangle$ in a line of $\mathcal{B}$ through z or in a line that contains a point of $\Pi[n-2]$. In the first case, it follows that the plane $\langle L_z, N_3\rangle$ contains an antiflag of $\mathcal{S}$ and at least two lines of $\mathcal{B}$, a contradiction since every plane containing an antiflag of $\mathcal{S}$ is of type I or III. In the second case, $\langle L_z, N_3\rangle$ is a plane of type III. The plane $\langle L_z, N_1\rangle$ contains an antiflag of $\mathcal{S}$ and a line of $\mathcal{B}$, hence it is also a plane of type III. Let L'_z be a line of $\mathcal{S}$ through z in $\langle L_z, N_1\rangle$, $L'_z \neq L_z$ (L'_z exists since $q \neq 2$). The plane $\langle L'_z, N_3\rangle$ then intersects $\langle N_1, N_2\rangle$ in a line of $\mathcal{B}$. This is again a contradiction, since $\langle L'_z, N_3\rangle$ cannot contain an antiflag of $\mathcal{S}$ and two lines of $\mathcal{B}$. It follows that all lines through z in $\langle N_1, N_2, N_3\rangle$ are lines of $\mathcal{B}$. Now if there would be a line M of $\mathcal{S}$ in $\langle N_1, N_2, N_3\rangle$ not through z, then there would be 0 lines through z that intersect the line M, a contradiction since $\mathcal{S}$ is a $(q-1,q)$-geometry. So we have proved that $\langle N_1, N_2, N_3\rangle$ contains no lines of $\mathcal{S}$. If all elements of $\mathcal{B}$ through z are contained in $\langle N_1, N_2, N_3\rangle$ then through every point of $\mathcal{S}$ there has to be a three dimensional space that contains all the lines of

$\mathcal{B}$ through this point, and all these three dimensional spaces are disjoint or their intersections are contained in $\Pi[n-2]$. Hence the lines of $\mathcal{S}$ are the lines that are skew to $\Pi[n-2]$ and that are not contained in a partition Σ of the points of $\mathrm{PG}(n,q) \setminus \Pi[n-2]$ into three dimensional spaces intersecting $\Pi[n-2]$ in a line.

Now assume that $\Gamma[d]$ is a d-dimensional subspace of $\mathrm{PG}(n,q)$ through z, $d \geq 3$, that contains no lines of $\mathcal{S}$, such that $\Gamma[d]$ intersects $\mathrm{PG}(n,q)$ in a $(d-2)$-dimensional space. If all lines of $\mathcal{B}$ through z are contained in $\Gamma[d]$, then through every point p of $\mathcal{S}$ there has to be a d-dimensional space, intersecting $\Pi[n-2]$ in a $(d-2)$-dimensional space, and containing all elements of $\mathcal{B}$ through p. Moreover all such d-dimensional spaces are disjoint or their intersection belongs to $\Pi[n-2]$. If there is a line N_d of $\mathcal{B}$ through z, N_d not contained in $\Gamma[d]$, then $\langle \Gamma[d], N_d \rangle$ contains no lines of $\mathcal{S}$. Indeed, by assumption $\Gamma[d]$ contains no lines of $\mathcal{S}$. If there would be a line $\tilde{L}_z$ of $\mathcal{S}$ through z, $\tilde{L}_z \subset \langle \Gamma[d], N_d \rangle$, then the plane $\langle \tilde{L}_z, N_d \rangle$ would intersect $\Gamma[d]$ in a line of $\mathcal{B}$ or in a line that contains a point x of $\Pi[n-2]$.

In the first case, the plane $\langle \tilde{L}_z, N_d \rangle$ would contain a line of $\mathcal{S}$ and two lines of $\mathcal{B}$ through z. This is a contradiction, since every plane containing an antiflag of $\mathcal{S}$ is of type I or III.

In the second case, $\langle \tilde{L}_z, N_d \rangle$ is a plane of type III. The plane $\langle \tilde{L}_z, N_1 \rangle$ contains an antiflag of $\mathcal{S}$ and a line of $\mathcal{B}$, hence it is a plane of type III. Let $\tilde{L}'_z$ be a line of $\mathcal{S}$ through z in $\langle \tilde{L}_z, N_1 \rangle$, $\tilde{L}'_z \neq \tilde{L}_z$ ($\tilde{L}'_z$ exists since $q \neq 2$). Then $\langle \tilde{L}'_z, N_3 \rangle$ intersects $\Gamma[d]$ in a line of $\mathcal{B}$. Indeed, the three dimensional space $\langle \tilde{L}'_z, N_3, N_1 \rangle$ intersects $\Gamma[d]$ in the plane $\langle N_1, x \rangle$. Since the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of a subspace of $\mathrm{PG}(n,q)$, and N_1 does not contain a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$, x is the only point of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\langle N_1, x \rangle$. Clearly $x \notin \langle \tilde{L}'_z, N_3 \rangle$. Hence $\langle \tilde{L}'_z, N_3 \rangle$ intersects $\Gamma[d]$ in a line of $\mathcal{B}$. This is again a contradiction.

So we have proved that $\langle \Gamma[d], N_d \rangle$ contains no lines of $\mathcal{S}$ through z. If there would be a line M' of $\mathcal{S}$ in $\langle \Gamma[d], N_d \rangle$, $z \notin M'$, then it would follow from the above that $i(z, M') = 0$, a contradiction since $\mathcal{S}$ is a $(q-1, q)$-geometry. So $\langle \Gamma[d], N_d \rangle$ is a $(d+1)$-dimensional space through z that contain no lines of $\mathcal{S}$.

Continuing in this way, we get that all the elements of $\mathcal{B}$ through z are contained in an r-dimensional space through z intersecting $\Pi[n-2]$ in an $(r-2)$-dimensional space and that this space does not contain lines of $\mathcal{S}$, for $1 \leq r \leq n-1$. Since the number of elements of $\mathcal{B}$ through a point of $\mathcal{S}$ is a constant, it follows that the lines of $\mathcal{S}$ are the lines that are skew to $\Pi[n-2]$ and that do not belong to a partition Σ of the points of $\mathrm{PG}(n,q) \setminus \Pi[n-2]$ into r-dimensional spaces intersecting $\Pi[n-2]$ in $(r-2)$-spaces, for $1 \leq r \leq n-1$.

It remains to prove that such a partition Σ of r-dimensional spaces exists for each r, with $1 \leq r \leq n-2$.

It is clear that $r \neq n-1$, as otherwise Σ would be the set consisting of the $q+1$ $(n-1)$-dimensional spaces on $\Pi[n-2]$, as in $\mathrm{PG}(n,q)$ any two $(n-1)$-dimensional spaces intersect in a subspace of dimension at least $(n-2)$. However we have proved in the previous paragraph that the intersection of an element of Σ with $\Pi[n-2]$ has to be $(n-3)$-dimensional, a contradiction. Hence $r < n-1$.

If $r = 1$, then Σ is a partition of $\mathrm{PG}(n,q) \setminus \Pi[n-2]$ into lines. In other words, Σ is a partial spread of $\mathrm{PG}(n,q)$. In [3] a partial spread Λ of lines of $\mathrm{PG}(n,q)$ such that each line of Λ is skew to a given $(n-2)$-dimensional space is constructed as follows. Embed $\mathrm{PG}(n,q)$ in a $(2n-3)$-dimensional space $\mathrm{PG}(2n-3,q)$. In $\mathrm{PG}(2n-3,q)$ one can take a spread Λ' of $(n-2)$-dimensional spaces such that $\Pi[n-2] \in \Lambda'$. The elements of $\Lambda' \setminus \{\Pi[n-2]\}$ intersect $\mathrm{PG}(n,q)$ in a partial spread Λ of lines such that every point of $\mathrm{PG}(n,q) \setminus \Pi[n-2]$ is contained in a line of Λ and such that every element of Λ is skew to $\Pi[n-2]$. So for $r = 1$, the partition Σ is this partial spread Λ.

Assume that $2 \leq r \leq n-2$. We know that every element of Σ intersects $\Pi[n-2]$ in an $(r-2)$-dimensional space. Now let $\Upsilon[r-2]$ be an $(r-2)$-dimensional subspace of $\Pi[n-2]$. Let $\Omega[n-r+1]$ be an $(n-r+1)$-dimensional subspace of $\mathrm{PG}(n,q)$ skew to $\Upsilon[r-2]$. Then $\Omega[n-r+1] \cap \Pi[n-2]$ is an $(n-r-1)$-dimensional space. In the same way as in the previous paragraph, we can take a partial spread Λ of lines of $\Omega[n-r+1]$ such that every element of Λ is skew to $\Omega[n-r+1] \cap \Pi[n-2]$ and such that every point of $\Omega[n-r+1] \setminus \Pi[n-2]$ belongs to an element of Λ. Now the set $\Sigma := \{\langle \Upsilon[r-2], M\rangle | M \in \Lambda\}$ is a partition of the points of $\mathrm{PG}(n,q) \setminus \Pi[n-2]$ into r-dimensional spaces through $\Upsilon[r-2]$. Hence also for $2 \leq r \leq n-2$, the partition Σ exists.

That such a partition gives rise to a $(q-1,q)$-geometry is easy to show. □

Remark. If $r \neq 1$, then the elements of the partition Σ of the points of $\mathrm{PG}(n,q) \setminus \Pi[n-2]$ are not necessarily disjoint. Indeed, it is possible that two different elements of Σ both contain a point x of $\Pi[n-2]$. Note that in the example given in the proof of the theorem, all elements of Σ intersect $\Pi[n-2]$ in the same $(r-2)$-dimensional space. If $r = 1$, then the elements of Σ are lines that are skew to $\Pi[n-2]$. Hence in the case $r = 1$, the elements of Σ are pairwise disjoint.

Lemma 2.5.4 *Let $\mathcal{S}$ be a $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q \neq 2$. Assume that $\mathrm{PG}(n,q)$ contains a plane π of type II. Let u be a point of $\mathcal{S}$, $u \notin \pi$. Then the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the three dimensional space $\langle \pi, u \rangle$ are the points of two subspaces of $\mathrm{PG}(n,q)$, one of them being a line, while the other one is a point or a line.*

Proof. Let $\mathcal{S}$ be a $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q \neq 2$. Let π be a plane of type II. Let y_1 and y_2 be the two points of π that do not belong to $\mathcal{S}$. Since $t+1$ is a constant, we know that $n > 2$. The points of $\mathcal{S}$ span $\mathrm{PG}(n,q)$, so there is a point u of $\mathcal{S}$ not contained in π. Let ρ be a plane through u and a line of $\mathcal{S}$ in π. Then ρ contains an antiflag of $\mathcal{S}$. Hence ρ is of type I, II or III.

We denote the intersection point of the line $\langle y_1, y_2 \rangle$ with the plane ρ by w. Since ρ intersects π in a line of $\mathcal{S}$, w is a point of $\mathcal{S}$. Let L be a line of $\mathcal{S}$ in ρ through w, such that L is not contained in π. By theorem 2.5.2 we know that all planes through L contain an antiflag of $\mathcal{S}$. Hence in every plane through L there are one or two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Denote the set of points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ that are contained in $\langle \pi, \rho \rangle$ by X. Then it follows that $q+2 \leq |X| \leq 2q+2$.

Through the line $\langle y_1, y_2 \rangle$ there are at least $q-1$ planes that intersect ρ in a line of $\mathcal{S}$. Hence at least $q-1$ planes through $\langle y_1, y_2 \rangle$ are of type II. This implies that all the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\langle \pi, \rho \rangle$ are contained in at most two planes through $\langle y_1, y_2 \rangle$.

Assume first that all the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\langle \pi, \rho \rangle$ are contained in one plane τ_1. From lemma 2.5.1 it follows that every line of $\mathrm{PG}(n,q)$ contains 0, 1, 2 or $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. We will prove that τ_1 contains a line on which there are no points of $\mathcal{S}$. Assume therefore that every line of τ_1 contains a point of $\mathcal{S}$. Then every line of τ_1 contains 0, 1 or 2 points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. We proved above that there are at least $q+2$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\langle \pi, \rho \rangle$. It follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in τ_1 are the points of a hyperoval. Let L_{τ_1} be a line of τ_1 that contains no points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Then the q planes through L_{τ_1} in $\langle \pi, \rho \rangle$ different from τ_1, contain an antiflag of $\mathcal{S}$ and no point of $\mathrm{P}(n,q) \setminus \mathcal{P}$. This is a contradiction. So we have proved that τ_1 contains a line on which there are $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Now as on every line there are 0, 1, 2 or $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, and as $q+2 \leq |X| \leq 2q+2$, we can conclude that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in τ_1 are either the points on one line together with an extra point, or the points on two intersecting lines.

Assume next that all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\langle \pi, \rho \rangle$ are contained in two distinct planes τ_1 and τ_2, but not in one plane. Then both τ_1 and τ_2 cannot contain an antiflag of $\mathcal{S}$.

We will prove that either τ_1 or τ_2 contains a line on which there is no point of $\mathcal{S}$. Assume that τ_1 contains no such line. Then from lemma 2.5.1 it follows that every line in τ_1 contains 0, 1 or 2 points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, and hence the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in τ_1 are the points of a (subset of an) arc. This implies that τ_1 contains a line N on which there are $q+1$ points of $\mathcal{S}$. Since τ_1 does not contain an antiflag of $\mathcal{S}$, the line $N \notin \mathcal{L}$. All planes through N different from τ_1 intersect π in a line of $\mathcal{S}$. Hence these q planes contain an antiflag of $\mathcal{S}$. As the line N is contained in each of these planes, they all have to be of type III, which means that they contain exactly one point of $\mathrm{P}(n,q) \setminus \mathcal{P}$. As all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are contained in τ_1 and τ_2, it follows that τ_2 contains exactly $q+2$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. From lemma 2.5.1 it follows that every line in τ_2 contains 0, 1, 2 or $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, and hence the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in τ_2 are either the points of one line together with one extra point or it are the points of a hyperoval.

Now assume the latter case. Let $N \cap \tau_2$ be the point p. Then through p we can take a line N_p in τ_2 that contains no point of $\mathrm{P}(n,q) \setminus \mathcal{P}$. It follows that the plane $\langle N, N_p \rangle$ contains no point of $\mathrm{P}(n,q) \setminus \mathcal{P}$. However, in the three dimensional space $\langle \pi, \rho \rangle$ the plane $\langle N, N_p \rangle$ has to intersect π in a line. Since $\langle N, N_p \rangle$ does not contain points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, this line belongs to $\mathcal{S}$. Hence $\langle N, N_p \rangle$ contains an antiflag of $\mathcal{S}$ and no point of $\mathrm{P}(n,q) \setminus \mathcal{P}$, a contradiction as every plane containing an antiflag of $\mathcal{S}$ is of type I, II or III. This proves that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in τ_2 are the points on one line together with one extra point. This proves that either τ_1 or τ_2 contains a line on which there is no point of $\mathcal{S}$.

Hence we may assume that the line $\langle y_1, y_3 \rangle$ contains no points of $\mathcal{S}$, with $y_3 \in \tau_1$, $y_1 \neq y_3 \neq y_2$. By assumption not all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are contained in τ_1. Hence there is a point y_4 of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in τ_2, $y_1 \neq y_4 \neq y_2$. Every plane in $\langle \pi, \rho \rangle$ through the line $\langle y_2, y_4 \rangle$ contains at least 3 points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, hence every such plane contains no lines of $\mathcal{S}$. Now let σ be a plane through L that does not contain y_2 or y_4, with L again a line of $\mathcal{S}$ through w in ρ, $L \not\subset \pi$. Then σ intersects the line $\langle y_2, y_4 \rangle$ in a point z. All the lines through z in σ are contained in some plane through $\langle y_2, y_4 \rangle$ in $\langle \pi, \rho \rangle$, so there are no lines of $\mathcal{S}$ through z in σ. This implies that $z \notin \mathcal{P}$; since if $z \in \mathcal{P}$, then there would be 0 lines through z that intersect L, a contradiction since $\mathcal{S}$ is a $(q-1,q)$-geometry. Hence the line $\langle y_2, y_4 \rangle$ contains at least 3 points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. From lemma 2.5.1 it follows that $\langle y_2, y_4 \rangle$ contains $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Hence $\langle \pi, \rho \rangle$ contains at least $2q+2$ points of $\mathrm{P}(n,2) \setminus \mathcal{P}$, namely the points of the disjoint lines $\langle y_1, y_3 \rangle$ and $\langle y_2, y_4 \rangle$. Since we have proved above that $|X| \leq 2q+2$, these points are all points of $P(n,q) \setminus \mathcal{P}$ in $\langle \pi, \rho \rangle$. So the points of $P(n,q) \setminus \mathcal{P}$ are the points of two disjoint lines in $\langle \pi, \rho \rangle$.

We conclude that the points of $P(n,q) \setminus \mathcal{P}$ in the three dimensional space $\langle \pi, \rho \rangle$ are the points of two subspaces of $PG(n,q)$ of dimension less than or equal to 1 and that at least one of these subspaces is a line. □

Note that, since π was an arbitrarily chosen plane of type II and $\langle \pi, \rho \rangle$ an arbitrarily chosen three dimensional space through π and a point of $\mathcal{S}$ not in π, it follows from the above lemma that for every three dimensional subspace of $PG(n,q)$ through a plane of type II, the points of $P(n,q) \setminus \mathcal{P}$ in this three dimensional subspace are the points of two subspaces of dimension less than or equal to one, and that at least one of these subspaces has dimension one.

Lemma 2.5.5 *Let $\mathcal{S}$ be a $(q-1,q)$-geometry, for $q \neq 2$, fully embedded in* $PG(n,q)$. *Assume that* $PG(n,q)$ *contains a plane of type* II. *Then the points of* $P(n,q) \setminus \mathcal{P}$ *are the points of two distinct subspaces of* $PG(n,q)$. *One of these subspaces has dimension $n-2$, the other one has dimension less than or equal to $n-2$.*

Proof. Let $\mathcal{S}$ be a $(q-1,q)$-geometry fully embedded in $PG(n,q)$, for $q \neq 2$. Let π be a plane of type II. Let y_1 and y_2 be the two points of $P(n,q) \setminus \mathcal{P}$ that are contained in π. Let w be a point of $\mathcal{S}$, $w \notin \pi$. Then from lemma 2.5.4 it follows that the points of $P(n,q) \setminus \mathcal{P}$ in $\langle \pi, w \rangle$ are the points of two distinct subspaces of dimension at most one, one of them being a line.

If $n = 3$, then the lemma is proved. So we may assume that $n > 3$. Now let $\Pi[m]$ be an m-dimensional subspace of $PG(n,q)$, $m \geq 3$, such that the points of $P(n,q) \setminus \mathcal{P}$ in $\Pi[m]$ are the points of two subspaces $\Upsilon[m-2]$ and $\Omega[r]$ of dimension $m-2$ and r respectively, with $0 \leq r \leq m-2$, and such that $\langle \pi, w \rangle \subseteq \Pi[m]$. If $m = n$, then the lemma is proved. So we may assume that $m < n$. Let $\Pi'[m+1]$ be an $(m+1)$-dimensional subspace of $PG(n,q)$ through $\Pi[m]$ and a point u' of $\mathcal{S}$ in $PG(n,q) \setminus \Pi[m]$. Such a point u' exists because the points of $\mathcal{S}$ span $PG(n,q)$. We will prove that the points of $P(n,q) \setminus \mathcal{P}$ contained in $\Pi'[m+1]$ are the points of two subspaces of $\Pi'[m+1]$ of dimension $m-1$ and r' respectively, with $0 \leq r' \leq m-1$.

It immediately follows that $\Pi'[m+1] \setminus \Pi[m]$ contains a point y that does not belong to $\mathcal{S}$. Indeed, assume that all the points of $\Pi'[m+1] \setminus \Pi[m]$ would belong to $\mathcal{S}$. Then a plane in $\Pi'[m+1]$ that intersects $\Pi[m]$ in a line of $\mathcal{S}$ contains no point of $P(n,q) \setminus \mathcal{P}$ (Note that $\Pi[m]$ contains lines of $\mathcal{S}$, since $\langle \pi, w \rangle \subset \Pi[m]$). This is a contradiction because we know that every

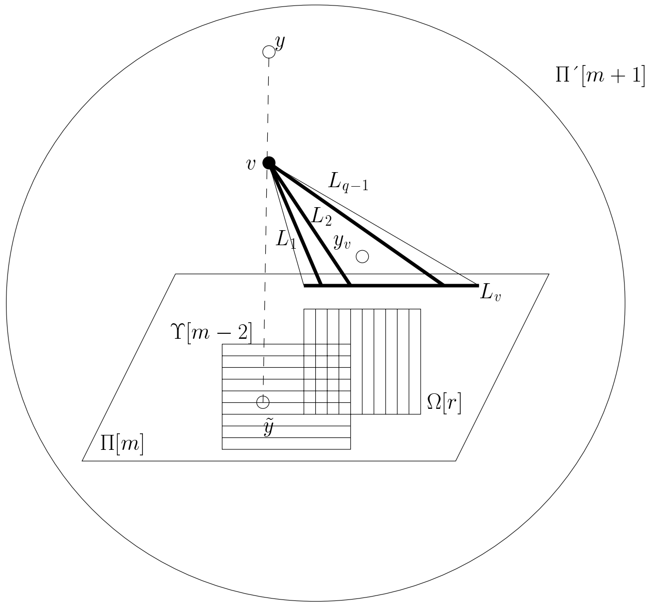

Figure 2.4: Step 1. Assume that $\langle y, \Upsilon[m-2]\rangle$ contains a point v of $\mathcal{S}$.

plane that contains an antiflag of $\mathcal{S}$ is of type I, II or III. So not all points of $\Pi'[m+1] \setminus \Pi[m]$ belong to $\mathcal{S}$.

Step 1. We will prove that there is an $(m-1)$-dimensional space through $\Upsilon[m-2]$ that contains no point of $\mathcal{S}$.

Assume therefore that the subspace $\langle y, \Upsilon[m-2]\rangle$ contains a point v of $\mathcal{S}$. We will construct an $(m-1)$-dimensional subspace $\Lambda[m-1]$, such that $\Lambda[m-1] \neq \langle y, \Upsilon[m-2]\rangle$ and such that $\Lambda[m-1]$ does not contain a point of $\mathcal{S}$. Let $\langle y, v\rangle \cap \Upsilon[m-2]$ be the point $\tilde{y}$. Let L_v be a line of $\mathcal{S}$ in $\Pi[m]$. The plane $\langle v, L_v\rangle$ is a plane containing an antiflag of $\mathcal{S}$. So in $\langle v, L_v\rangle$ there are at least $q-1$ lines of $\mathcal{S}$ through v that intersect $\Pi[m]$. We denote these lines by $L_1, \ldots, L_{q-1}$. By theorem 2.5.2, the plane $\langle L_1, y\rangle$ contains an antiflag of $\mathcal{S}$. It contains the points y and $\tilde{y}$ of $\mathrm{P}(n,q) \setminus \mathcal{P}$, so it is a plane of type II. Hence $\langle y, v, L_v\rangle$ is a three dimensional space that contains a plane of type II. From lemma 2.5.4, it follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle y, v, L_v\rangle$ are either the points of two lines or one point and one line. The plane $\langle v, L_v\rangle$ contains one or two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Let y_v be a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\langle v, L_v\rangle$. Then either $\langle y, y_v\rangle$ or $\langle \tilde{y}, y_v\rangle$ is a line containing $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Note that if $\langle v, L_v\rangle$ is of type II, then through both y and $\tilde{y}$ there is a line containing $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. The three dimensional space $\langle y, v, L_v \rangle$ intersects $\Pi[m]$ in a plane containing an antiflag of $\mathcal{S}$. Now there are two possibilities.

The first possibility is that the three dimensional space $\langle y, v, L_v \rangle$ *is disjoint from* $\Omega[r] \setminus \Upsilon[m-2]$. Then clearly the point $\{u\} = \langle y, y_v \rangle \cap \Pi[m]$ belongs to $\mathcal{S}$. Hence the line $\langle \tilde{y}, y_v \rangle$ is a line containing $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. There is a line L_u of $\mathcal{S}$ through u in $\Pi[m]$. Indeed, since $\langle w, \pi \rangle \subseteq \Pi[m]$, $\Pi[m]$ contains lines of $\mathcal{S}$. So if there are no lines of $\mathcal{S}$ through u in $\Pi[m]$, then $i(u, L) = 0$ for a line L of $\mathcal{S}$ in $\Pi[m]$, which is clearly a contradiction. The plane $\langle y, L_u \rangle$ contains two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, namely y and y_v. By theorem 2.5.2 every plane through L_u contains an antiflag of $\mathcal{S}$. So the plane $\langle y, L_u \rangle$ is a plane of type II. Hence by lemma 2.5.4, the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in every three dimensional space through $\langle y, L_u \rangle$ and a point of $\Upsilon[m-2]$ are either the points of two lines or it are the points of one line together with an extra point. Since $\langle y, y_v \rangle$ does not contain $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, this implies that for each point $\tilde{x}$ of $\Upsilon[m-2]$ either $\langle \tilde{x}, y \rangle$ or $\langle \tilde{x}, y_v \rangle$ is a line containing no points of $\mathcal{S}$.

Now we will prove that for every $\tilde{x} \in \Upsilon[m-2]$, the line $\langle y_v, \tilde{x} \rangle$ does not contain a point of $\mathcal{S}$. Let $N_{\tilde{y}}$ be a line through $\tilde{y}$ in $\Upsilon[m-2]$. Then $N_{\tilde{y}}$ contains $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. We look at the plane $\langle y, N_{\tilde{y}} \rangle$. Since $\langle y, \tilde{y} \rangle$ contains the point v of $\mathcal{S}$, lemma 2.5.1 implies that either y and the points of the line $N_{\tilde{y}}$ are all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\langle y, N_{\tilde{y}} \rangle$, or the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\langle y, N_{\tilde{y}} \rangle$ are the points of $N_{\tilde{y}}$ and the points of a line through y. Hence there are at least q points $z_1 = \tilde{y}, z_2, \ldots, z_q$ on $N_{\tilde{y}}$ for which the lines $\langle y, z_1 \rangle, \ldots, \langle y, z_q \rangle$ contain points of $\mathcal{S}$. By the previous paragraph, we know that for every $\tilde{z} \in \Upsilon[m-2]$ either $\langle y, \tilde{z} \rangle$ or $\langle y_v, \tilde{z} \rangle$ is a line that does not contain points of $\mathcal{S}$. Since $\langle y, z_1 \rangle, \ldots, \langle y, z_q \rangle$ are all lines that contain points of $\mathcal{S}$, we may conclude that $\langle y_v, z_1 \rangle, \ldots, \langle y_v, z_q \rangle$ are lines that contain no points of $\mathcal{S}$. Now we look at the plane $\langle y_v, N_{\tilde{y}} \rangle$. It contains the lines $\langle y_v, z_1 \rangle, \ldots, \langle y_v, z_q \rangle$. By lemma 2.5.1, this implies that $\langle y_v, N_{\tilde{y}} \rangle$ contains no points of $\mathcal{S}$. Since $N_{\tilde{y}}$ was an arbitrarily chosen line through $\tilde{y}$ in $\Upsilon[m-2]$, it follows that every point of $\langle y_v, \Upsilon[m-2] \rangle$ does not belong to $\mathcal{S}$. Hence $\langle y_v, \Upsilon[m-2] \rangle$ is an $(m-1)$-dimensional subspace of $\Pi'[m+1]$ that contains no points of $\mathcal{S}$.

The second possibility is that the three dimensional space $\langle y, v, L_v \rangle$ *contains a point* $\tilde{p}$ *of* $\Omega[r] \setminus \Upsilon[m-2]$. In this case $\tilde{y}$ is a point of $\Upsilon[m-2] \setminus \Omega[r]$. From lemma 2.5.4, it follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the three dimensional space $\langle y, v, L_v \rangle$ are either the points of two lines or one points and one line. Since $\langle y, \tilde{y} \rangle$ and $\langle \tilde{y}, \tilde{p} \rangle$ both contain points of $\mathcal{S}$, the line $\langle y, \tilde{p} \rangle$ has to be a line that contains $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$.

Assume first that $r < m-2$. Then the dimension of $\langle \tilde{y}, \Omega[r] \rangle$ is less than or equal to $m-2$. So there is a line $L \subset \Pi[m]$, $L \in \mathcal{L}$, that is skew to $\langle \tilde{y}, \Omega[r] \rangle$. The plane $\langle L, \tilde{y} \rangle$ is disjoint from $\Omega[r]$ and contains an antiflag of $\mathcal{S}$. Hence, by the previous part of the proof, the result follows, replacing the plane $\langle v, L_v \rangle$ by the plane $\langle v, L \rangle$.

Now assume that $r = m-2$. Take a line through $\tilde{y}$ and a point $\tilde{p}'$ of $\Omega[r] \setminus \Upsilon[m-2]$. Then $\langle \tilde{y}, \tilde{p}' \rangle$ contains a point w' of $\mathcal{S}$. Through w' we can take a line $L_{w'}$ of $\mathcal{S}$. The plane $\langle L_{w'}, \tilde{y} \rangle$ contains two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, hence it is of type II. From lemma 2.5.4, it follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the three dimensional space $\langle L_{w'}, \tilde{y}, y \rangle$ are either the points of two lines or it are the points of one line together with an extra point. Since $\langle \tilde{y}, y \rangle$ and $\langle \tilde{y}, \tilde{p}' \rangle$ both contain points of $\mathcal{S}$, this implies that $\langle y, \tilde{p}' \rangle$ is a line containing $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Since $\tilde{p}'$ was arbitrary chosen in $\Omega[r] \setminus \Upsilon[m-2]$, every line through y and a point of $\Omega[r] \setminus \Upsilon[m-2]$ is a line containing no point of $\mathcal{S}$. From lemma 2.5.1, it follows that no line of $\mathrm{PG}(n,q)$ can contain q points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Hence we get that the space $\langle y, \Omega[r] \rangle$ can not contain a point of $\mathcal{S}$. Since by assumption $r = m-2$, the space $\langle y, \Omega[r] \rangle$ is $(m-1)$-dimensional.

Hence we proved that $\Pi'[m+1]$ contains an $(m-1)$-dimensional subspace $\langle \overline{y}, \Upsilon[m-2] \rangle$ of points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, with $\overline{y} \in \Pi'[m+1] \setminus \Pi[m]$.

Step 2. It remains to prove that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Pi'[m+1]$ are the points of two subspaces of dimension less than or equal to $m-1$.

If all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are contained in $\Omega[r] \cup \langle \overline{y}, \Upsilon[m-2] \rangle$, then the lemma is proved. So assume that there is a point $\overline{z}$ of $\mathrm{P}(n,q) \setminus \mathcal{P}$ that is not contained in $\Omega[r] \cup \langle \overline{y}, \Upsilon[m-2] \rangle$. Let $\langle \overline{y}, \overline{z} \rangle \cap \Pi[m]$ be the point $\overline{x}$.

Assume first that $\overline{x}$ is a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Then the line $\langle \overline{y}, \overline{z} \rangle$ contains 3 points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. From theorem 2.5.2 it follows that $\langle \overline{y}, \overline{z} \rangle$ contains no points of $\mathcal{S}$. We will now prove that every line through $\overline{z}$ and a point of $\Upsilon[m-2] \setminus \Omega[r]$ contains $q-1$ points of $\mathcal{S}$. Indeed, let $\overline{y}_1$ be a point of $\Upsilon[m-2] \setminus \Omega[r]$. Since $\overline{x} \in \Omega[r] \setminus \Upsilon[m-2]$, the line $\langle \overline{y}_1, \overline{x} \rangle$ contains $q-1$ points of $\mathcal{S}$. Now we look at the plane $\langle \overline{y}, \overline{x}, \overline{y}_1 \rangle$. It contains the lines $\langle \overline{y}, \overline{x} \rangle$ and $\langle \overline{y}, \overline{y}_1 \rangle$, on which there are no points of $\mathcal{S}$. It contains also the line $\langle \overline{y}_1, \overline{x} \rangle$ on which there are $q-1$ points of $\mathcal{S}$. By lemma 2.5.1 we know that the plane $\langle \overline{y}, \overline{x}, \overline{y}_1 \rangle$ can not contain other points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Hence the line $\langle \overline{y}_1, \overline{z} \rangle$ contains $q-1$ points of $\mathcal{S}$. Since $\overline{y}_1$ was an arbitrary point of $\Upsilon[m-2] \setminus \Omega[r]$, all lines through $\overline{z}$ and a point of $\Upsilon[m-2] \setminus \Omega[r]$ contain $q-1$ points of $\mathcal{S}$. Now, let $\overline{p}$ be a point of $\Omega[r] \setminus \Upsilon[m-2]$, $\overline{p} \neq \overline{x}$. Let $N_{\overline{p}}$ be a line through $\overline{p}$ that intersects $\Upsilon[m-2] \setminus \Omega[r]$. Through a point of $\mathcal{S}$ on $N_{\overline{p}}$ we can take a line $L_{\overline{p}}$ of $\mathcal{S}$ in $\Pi[m]$. The plane $\langle L_{\overline{p}}, N_{\overline{p}} \rangle$ is of type II. By lemma 2.5.4, we know that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the three dimensional space

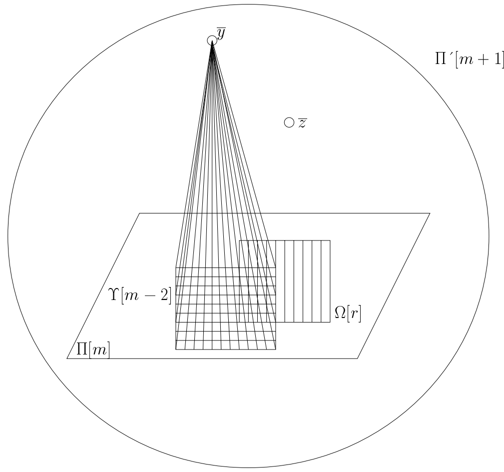

Figure 2.5: Assume that $\overline{z}$ is a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$, $\overline{z}$ not contained in $\langle \overline{y}, \Upsilon[m-2] \rangle \cup \Omega[r]$

$\langle L_{\overline{p}}, N_{\overline{p}}, \overline{z} \rangle$ have to be either the points of two lines or the points of one line together with an extra point. We have already shown that for every point $x \in \Upsilon[m-2] \setminus \Omega[r]$ the line $\langle \overline{z}, x \rangle$ contains points of $\mathcal{S}$. Hence $\langle \overline{z}, \overline{p} \rangle$ is a line containing no point of $\mathcal{S}$. Since $\overline{p}$ was an arbitrary point of $\Omega[r] \setminus \Upsilon[m-2]$, we get that $\langle \overline{z}, \Omega[r] \rangle$ contains no point of $\mathcal{S}$ (here we also use lemma 2.5.1 to prove that also the subspace $\langle \overline{z}, \Omega[r] \cap \Upsilon[m-2] \rangle$ contains no points of $\mathcal{S}$). Since $\overline{y} \in \langle \overline{z}, \Omega[r] \rangle$, we get that the subspaces $\langle \overline{y}, \Upsilon[m-2] \rangle$ and $\langle \overline{y}, \Omega[r] \rangle$ contain no point of $\mathcal{S}$.

If all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are contained in $\langle \overline{y}, \Omega[r] \rangle$ and $\langle \overline{y}, \Upsilon[m-2] \rangle$, then the lemma is proved. Assume therefore that there is a point $\overline{z}'$ of $\mathrm{P}(n,q) \setminus \mathcal{P}$ that does not belong to either $\langle \overline{y}, \Upsilon[m-2] \rangle$ or $\langle \overline{y}, \Omega[r] \rangle$. Clearly the line $\langle \overline{y}, \overline{z}' \rangle$ intersects $\Pi[m]$ in a point $\overline{x}'$ of $\mathcal{S}$. Through $\overline{x}'$ we can take a line $L_{\overline{x}'}$ of $\mathcal{S}$, $L_{\overline{x}'} \subset \Pi[m]$. Let $\overline{p}'$ be a point of $\Omega[r] \setminus \Upsilon[m-2]$. The plane $\langle \overline{p}', L_{\overline{x}'} \rangle$ $\pi_{\overline{x}'}$ contains two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, namely the point $\overline{p}'$ and a point $\overline{y}'$ of $\Upsilon[m-2] \setminus \Omega[r]$. Hence it is a plane of type II. From lemma 2.5.4, it follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the three dimensional space $\langle L_{\overline{x}'}, \overline{y}, \overline{p}' \rangle$ are either the points of two lines or the points of one line together with an extra point. However, this three dimensional space contains the two lines $\langle \overline{y}, \overline{y}' \rangle$ and $\langle \overline{y}, \overline{p}' \rangle$ that contain no point of $\mathcal{S}$, and the point $\overline{z}'$. This is a contradiction. So the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Pi'[m+1]$ are the points of the two subspaces $\langle \overline{y}, \Upsilon[m-2] \rangle$ and $\langle \overline{y}, \Omega[r] \rangle$. The subspace $\langle \overline{y}, \Upsilon[m-2] \rangle$

is $(m-1)$-dimensional, while $\langle \overline{y}, \Omega[r]\rangle$ has dimension less than or equal to $m-1$.

Assume next that $\overline{x}$ is a point of $\mathcal{S}$. Let $L_{\overline{x}}$ be a line of $\mathcal{S}$ through $\overline{x}$ in $\Pi[m]$. The plane $\langle \overline{y}, L_{\overline{x}}\rangle$ contains two points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, namely $\overline{y}$ and $\overline{z}$. So it is of type II. Using theorem 2.5.2, we see that all planes through $L_{\overline{x}}$ in $\Pi[m]$ contain an antiflag of $\mathcal{S}$. Let $\overline{p}'$ be a point of $\Omega[r]\setminus\Upsilon[m-2]$. Then $\langle \overline{p}', L_{\overline{x}}\rangle$ is a plane of type II. The three dimensional space $\langle \overline{p}', \overline{y}, L_{\overline{x}}\rangle$ contains two planes through $L_{\overline{x}}$ of type II, namely $\langle \overline{y}, L_{\overline{x}}\rangle$ and $\langle \overline{p}', L_{\overline{x}}\rangle$. By lemma 2.5.4, the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in this three dimensional space are the points of two lines or the points of one line together with an extra point. Since $\langle \overline{y}, \overline{z}\rangle$ contains points of $\mathcal{S}$, and $\langle \overline{y}, \Upsilon[m-2]\rangle$ contains no points of $\mathcal{S}$, the line $\langle \overline{z}, \overline{p}'\rangle$ has to contain $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Since $\overline{p}'$ was arbitrarily chosen in $\Omega[r]\setminus\Upsilon[m-2]$, we know that every line through $\overline{z}$ and a point of $\Omega[r]\setminus\Upsilon[m-2]$ contains no point of $\mathcal{S}$. Lemma 2.5.1 then tells us that $\langle \overline{z}, \Omega[r]\rangle$ contains no point of $\mathcal{S}$. If all points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ are contained in $\langle \overline{z}, \Omega[r]\rangle$ or $\langle \overline{y}, \Upsilon[m-2]\rangle$, then the lemma is proved. Assume now that there is a point $\overline{z}^*$ of $\mathrm{P}(n,q)\setminus\mathcal{P}$ that does not belong to $\langle \overline{y}, \Upsilon[m-2]\rangle$ or $\langle \overline{z}, \Omega[r]\rangle$. If $\langle \overline{y}, \overline{z}^*\rangle$ intersects $\Pi[m]$ in a point of $\Omega[r]$, then we can apply the previous part of the proof, replacing $\overline{z}^*$ by $\overline{z}$. It follows that $\langle y, \Omega[r]\rangle$ contains no point of $\mathcal{S}$. However, it also would follow that all points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Pi'[m+1]$ belong to $\langle \overline{y}, \Upsilon[m-2]\rangle$ and $\langle \overline{y}, \Omega[r]\rangle$, a contradiction since by assumption $\langle \overline{y}, \overline{z}\rangle$ intersects $\Pi[m]$ in a point of $\mathcal{S}$ and $\overline{z}\in\mathrm{P}(n,q)\setminus\mathcal{P}$. Hence we may assume that the line $\langle \overline{y}, \overline{z}^*\rangle$ intersects $\Pi[m]$ in a point $\overline{x}''\in\mathcal{P}$. In the same way as we did for $\overline{z}$, one can prove that $\langle \overline{z}^*, \Omega[r]\rangle$ contains no points of $\mathcal{S}$. Now $\langle \overline{z}, \Omega[r]\rangle$ and $\langle \overline{z}^*, \Omega[r]\rangle$ span an $(r+2)$-dimensional space $\Omega'[r+2]$. The space $\Omega'[r+2]$ intersects $\langle \overline{y}, \Upsilon[m-2]\rangle$ in a space of dimension at least r. Hence there are points of $\langle \overline{y}, \Upsilon[m-2]\rangle\setminus\Omega[r]$ contained in $\Omega'[r+2]$.

Suppose first that the $(r+1)$-dimensional space $\Omega'[r+2]\cap\Pi[m]$ is disjoint from $\Upsilon[m-2]\setminus\Omega[r]$. Let $\overline{y}_1$ be a point of $\langle \overline{y}, \Upsilon[m-2]\rangle\setminus\Omega[r]$ in $\Omega'[r+2]$. Then by the assumption $\overline{y}_1$ does not belong to $\Pi[m]$. A line through $\overline{y}_1$ in $\Omega'[r+2]$ disjoint from $\Omega[r]$ contains at most q points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, since it intersects $\Pi[m]$ in a point of $\mathcal{S}$. Moreover this line contains at least 3 points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, namely the point $\overline{y}_1$ and its intersection points with the spaces $\langle \overline{z}, \Omega[r]\rangle$ and $\langle \overline{z}', \Omega[r]\rangle$. This is in contradiction with lemma 2.5.1.

Suppose next that the $(r+1)$-dimensional space $\Omega'[r+2]\cap\Pi[m]$ is not disjoint from $\Upsilon[m-2]\setminus\Omega[r]$. Let $\overline{y}_2$ be a point of $\Upsilon[m-2]\setminus\Omega[r]$ in $\Omega'[r+2]\cap\Pi[m]$. Then every line through $\overline{y}_2$ in the space $\Omega'[r+2]$ that does not contain a point of $\Omega[r]$, contains at least 3 points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. By lemma 2.5.1, every such line has to contain $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Now let $M_{\overline{y}_2}$ be a line through $\overline{y}_2$ that intersects $\Omega[r]\setminus\Upsilon[m-2]$ in a point.

Then $M_{\overline{y}_2}$ contains $q-1$ points of $\mathcal{S}$. Take a plane through $M_{\overline{y}_2}$ in $\Omega'[r+2]$, that intersects $\Pi[m]$ in the line $M_{\overline{y}_2}$. The points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in this plane are the points of an affine plane together with two extra points. By lemma 2.5.1, such a plane can not exist. This proves that all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ have to be contained in $\langle \overline{y}, \Upsilon[m-2]\rangle$ or in $\langle \overline{z}, \Omega[r]\rangle$. So we have proved that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Pi'[m+1]$ are the points of two subspaces of dimension at most $m-1$ and that one of the subspaces has dimension $m-1$.

If $m = n+1$, then the lemma is proved. So assume from now on that $m < n-1$. Let $\Gamma[m+2]$ be an $(m+2)$-dimensional subspace containing $\Pi'[m+1]$ and a point of $\mathcal{S}$ not in $\Pi'[m+1]$. In the same way as above, we can prove that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Gamma[m+2]$ are the points of two subspaces, of dimension m resp. r'', for $0 \leq r'' \leq m$.

After a finite number of steps, we obtain that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\mathrm{PG}(n,q)$ are the points of two subspaces of $\mathrm{PG}(n,q)$ of dimension at most $n-2$ and that one of these subspaces has dimension $n-2$. □

Lemma 2.5.6 *Let $\mathcal{S}$ be a $(q-1,q)$-geometry, $q \neq 2$, fully embedded in $\mathrm{PG}(n,q)$. Assume that $\mathrm{PG}(n,q)$ contains a plane of type II. Let $\Pi[n-2]$ and $\Omega[r]$ be the two subspaces of points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, of dimension $n-2$ and r respectively, for $0 < r \leq n-2$. Let $\mathcal{B}$ be the set of lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathcal{S}$ and that do not belong to $\mathcal{S}$. Assume that $\mathcal{B} \neq \emptyset$. Then the elements of $\mathcal{B}$ in $\mathrm{PG}(n,q)$ through a point $u \in \mathcal{P}$ are contained in an l-dimensional subspace $\Psi_u[l]$ of $\mathrm{PG}(n,q)$, for $r+2 \leq l \leq n-2$, such that $\langle u, \Omega[r]\rangle \subseteq \Psi_u[l]$. Moreover $\Psi_u[l]$ contains no lines of $\mathcal{S}$.*

Proof. Let $\mathcal{S}$ be a $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q \neq 2$. Then from lemma 2.5.5 it follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of two subspaces $\Pi[n-2]$ and $\Omega[r]$ of dimension $n-2$ and r respectively, for $0 < r \leq n-2$. Let $\mathcal{B}$ be the set of lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathcal{S}$ but do not belong to $\mathcal{S}$. Assume that $\mathcal{B} \neq \emptyset$.

Let M_1 be a line of $\mathcal{B}$ in $\mathrm{PG}(n,q)$. Let $u \in M_1$. We will prove that the subspace $\langle M_1, \Omega[r]\rangle$ contains no lines of $\mathcal{S}$. Assume therefore that $\langle M_1, \Omega[r]\rangle$ contains a line $L \in \mathcal{L}$. From theorem 2.5.2 we know that every plane through L contains an antiflag of $\mathcal{S}$. Let y be a point of $\Omega[r] \setminus \Pi[n-2]$. Then the plane $\langle L, y\rangle$ contains an antiflag of $\mathcal{S}$ and two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, namely y and its intersection point with $\Pi[n-2]$. So $\langle L, y\rangle$ is a plane of type II. This implies that $\langle L, y\rangle$ does not contain a line of $\mathcal{B}$.

Assume first that $\Omega[r] \cap \Pi[n-2]$ is non-empty. Then $\Omega[r] \cap \Pi[n-2]$ is $(r-1)$-dimensional or $(r-2)$-dimensional. In the $(r+2)$-dimensional space

$\langle M_1, \Omega[r]\rangle$, the plane $\langle L, y\rangle$ intersects the subspace $\langle M_1, \Omega[r] \cap \Pi[n-2]\rangle$ in a point or in a line. This implies that there exists a point v of $\mathcal{S}$ in $\langle L, y\rangle$, $v \notin L$, such that the plane $\langle M_1, v\rangle$ contains a point of $\Omega[r] \setminus \Pi[n-2]$. Hence $\langle v, M_1\rangle$ contains the line M_1 of $\mathcal{B}$ and two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. It follows that there are no lines of $\mathcal{S}$ contained in the plane $\langle v, M_1\rangle$.

Let N be a line through v in the plane $\langle v, M_1\rangle$ not through a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Then N is a line of $\mathcal{B}$. Now we look at the three dimensional space spanned by N and $\langle L, y\rangle$. It intersects $\Omega[r]$ in a line through y. Since y is a point of $\Omega[r] \setminus \Pi[n-2]$, we can choose a point y' on this line that also belongs to $\Omega[r] \setminus \Pi[n-2]$. The plane $\langle y', N\rangle$ then contains two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ and the line N of $\mathcal{B}$. Hence it can not contain a line of $\mathcal{S}$. However, the plane $\langle y', N\rangle$ intersects the plane $\langle y, L\rangle$ in a line, since these two planes belong to a three dimensional space. This line does not contain the point y, since $y \notin \langle y', N\rangle$. If this line contains no point of $\Pi[n-2]$, then it is a line of $\mathcal{L}$. In that case we have found a contradiction, since $\langle y', N\rangle$ can not contain a line of $\mathcal{S}$. If this line contains a point of $\Pi[n-2]$, then we replace y' in the previous argument by a point y'' of $\Omega[r] \setminus \Pi[n-2]$ on the line $\langle y, y'\rangle$, $y \neq y'' \neq y'$. Note that y'' exists as by assumption $q \leq 3$. We get a plane $\langle y'', N\rangle$ that contains two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, the line N of $\mathcal{B}$ and a line of $\mathcal{S}$ on its intersection with the plane $\langle L, y\rangle$. This is a contradiction. We conclude that $\langle M_1, \Omega[r]\rangle$ cannot contain a line of $\mathcal{S}$.

Assume next that $\Omega[r] \cap \Pi[n-2] = \emptyset$. Then $\Omega[r]$ is a point or a line. If $\Omega[r]$ is a point, then the plane $\langle M_1, \Omega[r]\rangle$ contains two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ and the line M_1 of $\mathcal{B}$. Hence it does not contain a line of $\mathcal{S}$. If $\Omega[r]$ is a line, then $\langle M_1, \Omega[r]\rangle$ is a three dimensional space intersecting $\Pi[n-2]$ in a line. Hence this three dimensional space contains two lines with $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Every plane through M_1 contained in it then clearly contains two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Hence every line of $\langle M_1, \Omega[r]\rangle$ that intersects M_1 does not belong to $\mathcal{S}$. Now assume that $\langle M_1, \Omega[r]\rangle$ contains a line L of $\mathcal{S}$, L skew to M_1. Then for a point $z \in M_1$, it follows that $i(z, L) = 0$, which is clearly a contradiction. We conclude that also in this case $\langle M_1, \Omega[r]\rangle$ can not contain a line of $\mathcal{S}$.

So we have proved that $\langle M_1, \Omega[r]\rangle$ does not contain a line of $\mathcal{S}$. Now assume that $\Upsilon[d]$ is a d-dimensional subspace of $\mathrm{PG}(n,q)$ containing u and $\Omega[r]$, such that $\Upsilon[d]$ contains no lines of $\mathcal{S}$, for $r+2 \leq d \leq n-2$. If all elements of $\mathcal{B}$ through u are contained in $\Upsilon[d]$, then the lemma is proved, since u was an arbitrarily chosen point of $\mathcal{S}$. So assume that there is a line M_2 of $\mathcal{B}$ through u that is not contained in $\Upsilon[d]$. We will prove that the $(d+1)$-dimensional space $\langle \Upsilon[d], M_2\rangle$ cannot contain a line of $\mathcal{S}$.

From the first part of the proof, it follows that the $(r+2)$-dimensional space $\langle \Omega[r], M_2 \rangle$ contains no lines of $\mathcal{S}$. Let $\Gamma[r+2]$ be an $(r+2)$-dimensional subspace of $\Upsilon[d]$ that contains u and $\Omega[r]$. Then $\Gamma[r+2]$ intersects $\langle M_2, \Omega[r] \rangle$ in the $(r+1)$-dimensional space $\langle u, \Omega[r] \rangle$. Define $\Lambda[r+3] = \langle \Gamma[r+2], M_2 \rangle$. We will prove that $\Lambda[r+3]$ contains no lines of $\mathcal{S}$. Assume therefore that $\Lambda[r+3]$ contains a line L' of $\mathcal{S}$.

The space $\Pi[n-2]$ intersects $\Lambda[r+3]$ in an $(r+1)$-dimensional space, and it intersects $\Gamma[r+2]$ and $\langle M_2, \Omega[r] \rangle$ in an r-dimensional space. There are $(q^{r+2}-1)/(q-1)$ planes through L' in $\Lambda[r+3]$. At most $3(q^{r+1}-1)/(q-1) - 2(q^{r-1}-1)/(q-1)$ planes through L' in $\Lambda[r+3]$ contain a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$ on their intersection line with $\Gamma[r+2]$ or with $\langle M_2, \Omega[r] \rangle$. It follows that there are $(q^{r+2}-1)/(q-1) - 3(q^{r+1}-1)/(q-1) + 2(q^{r-1}-1)/(q-1)$ planes through L' in $\Lambda[r+3]$ that intersect both $\Gamma[r+2]$ and $\langle M_2, \Omega[r] \rangle$ in a line of $\mathcal{B}$. Since $(q^{r+2}-1)/(q-1) - 3(q^{r+1}-1)/(q-1) + 2(q^{r-1}-1)/(q-1) > 0$, there exists a plane containing an antiflag of $\mathcal{S}$ and two elements of $\mathcal{B}$, a contradiction. This proves that $\Lambda[r+3]$ can not contain a line of $\mathcal{S}$.

Now let $\Gamma'[r+3]$ be an $(r+3)$-dimensional subspace of $\Upsilon[d]$ that contains $\Gamma[r+2]$. Then $\Gamma'[r+3]$ intersects $\Lambda[r+3]$ in the $(r+2)$-dimensional subspace $\Gamma[r+1]$. Let $\Lambda'[r+4]$ be the $(r+4)$-dimensional subspace spanned by $\Lambda[r+3]$ and $\Gamma'[r+3]$. Assume that $\Lambda'[r+4]$ contains a line L'' of $\mathcal{S}$. There are $(q^{r+3}-1)/(q-1)$ planes through L'' in $\Lambda'[r+4]$. At most $2(q^{r+2}-1)/(q-1) + (q^{r+1}-1)/(q-1) - 2(q^{r-1}-1)/(q-1)$ planes through L'' in $\Lambda'[r+4]$ contain a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$ on their intersection line with $\Gamma'[r+3]$ or with $\Lambda[r+3]$. It follows that there are $(q^{r+3}-1)/(q-1) - 2(q^{r+2}-1)/(q-1) - (q^{r+1}-1)/(q-1) + 2(q^{r-1}-1)/(q-1)$ planes through L'' in $\Lambda'[r+4]$ that intersect both $\Gamma'[r+3]$ and $\Lambda[r+3]$ in a line of $\mathcal{B}$. Since $(q^{r+3}-1)/(q-1) - 2(q^{r+2}-1)/(q-1) - (q^{r+1}-1)/(q-1) + 2(q^{r-1}-1)/(q-1) > 0$, there exists a plane containing an antiflag of $\mathcal{S}$ and two elements of $\mathcal{B}$, a contradiction. This proves that $\Lambda'[r+4]$ cannot contain a line of $\mathcal{S}$. Continuing in this way, we get after a finite number of steps that the $(d+1)$-dimensional space $\langle \Upsilon[d], M_2 \rangle$ can not contain a line of $\mathcal{S}$.

Using induction on the dimension d, we get that all elements of $\mathcal{B}$ through u are contained in an l-dimensional subspace $\Psi_u[l]$ of $\mathrm{PG}(n,q)$ through u and $\Omega[r]$, for $r+2 \leq l \leq n-2$ and that this subspace contains no lines of $\mathcal{S}$. □

Remark. In the previous lemma we associate with every point u of $\mathcal{S}$ a subspace $\Psi[l]$ containing u. Note that the dimension l of $\Psi[l]$ is not necessarily the same for all points u of $\mathcal{S}$.

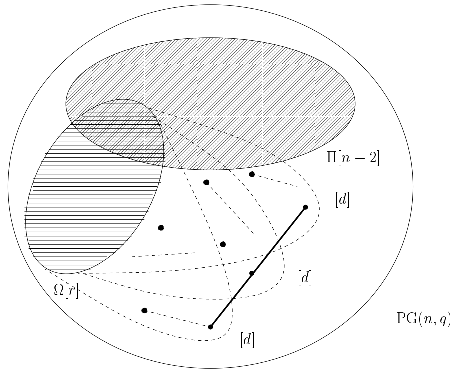

Figure 2.6: Fully embedded $(q-1,q)$-geometries in $\mathrm{PG}(n,q)$ with planes of type II

Theorem 2.5.7 *Let $\mathcal{S}$ be a $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q \neq 2$. Assume that there is a plane of type II. Then the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of two subspaces $\Pi[n-2]$ and $\Omega[r]$ of $\mathrm{PG}(n,q)$, of dimension $n-2$ and r respectively, for $1 \leq r \leq n-2$, with $\Omega[r] \cap \Pi[n-2]$ an $(r-2)$-dimensional space. The lines of $\mathcal{S}$ are either all lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathcal{S}$, or they are defined as follows. Let $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ be a partition of the points of $\mathcal{S}$, where $l = (q^{n-r}-1)/(q^{d-r}-1)$, and such that for $i = 1, \ldots, l$ we have that $\sigma_i = \Lambda_i[d] \setminus (\Pi[n-2] \cup \Omega[r])$, with $\Lambda_i[d]$ a d-dimensional subspace of $\mathrm{PG}(n,q)$ that contains $\Omega[r]$, and $r+2 \leq d \leq n-2$. The lines of $\mathcal{S}$ are the lines that intersect $q+1$ distinct elements of Σ in a point. A necessary and sufficient condition for such a partition to exist is that $(d-r)|(n-r)$. Further, if $(d-r)|(n-r)$ and $n-2 \geq d \geq r+2$, then this partition gives a $(q-1,q)$-geometry.*

Proof. Let $\mathcal{S}$ be a $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q \neq 2$. Assume that there is a plane of type II. In lemma 2.5.5 we have proved that the points of $\mathcal{S}$ are the points of $\mathrm{PG}(n,q)$ not contained in two subspaces $\Pi[n-2]$ and $\Omega[r]$ of $\mathrm{PG}(n,q)$, of dimension $n-2$ and r respectively, with $0 \leq r \leq n-2$. Now we want to determine which lines belong to $\mathcal{S}$. Let $\mathcal{B}$ be the set of lines of $\mathrm{PG}(n,q)$ that contain $q+1$ points of $\mathcal{S}$ but that do not belong to $\mathcal{S}$. We distinguish two cases.

Case 1. Assume that $\Omega[r]$ intersects $\Pi[n-2]$ in an $(r-1)$-dimensional space, $r \geq 0$.

Then $\langle \Omega[r], \Pi[n-2] \rangle$ is an $(n-1)$-dimensional space. We denote it by $\Upsilon[n-1]$. Through a point of $\mathcal{S}$ contained in $\Upsilon[n-1]$, exactly $(q^{n-1}-1)/(q-1)$

lines contain a point of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Through a point of $\mathcal{S}$ not contained in $\Upsilon[n-1]$, there are $(q^{n-1}-1)/(q-1)+(q^{r+1}-1)/(q-1)-(q^r-1)/(q-1)=(q^{n-1}-1)/(q-1)+q^r$ lines that contain a point of $\mathrm{P}(n,q)\setminus\mathcal{P}$. So it is clear that $\mathcal{B}\neq\emptyset$, as otherwise the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ would not be a constant.

Now we count the number of elements of $\mathcal{B}$ through a point $v\in\mathcal{P}$. From lemma 2.5.6 it follows that the elements of $\mathcal{B}$ through v are contained in some d-dimensional subspace $\Psi[d]$ containing v and $\Omega[r]$. It is clear that $\Psi[d]\not\subset\Upsilon[n-1]$, as otherwise there would be no elements of $\mathcal{B}$ through the points of $\mathcal{S}$ in $\Psi[d]$, and then $t+1$ would not be constant. Hence $\Upsilon[n-1]$ meets $\Psi[d]$ in a hyperplane of $\Psi[d]$. If $v\in(\Upsilon[n-1]\cap\Psi[d])$, then there are $(q^d-1)/(q-1)-(q^{d-1}-1)/(q-1)=q^{d-1}$ elements of $\mathcal{B}$ through v. If $v\in(\Psi[d]\setminus\Upsilon[n-1])$, then there are $(q^d-1)/(q-1)-(q^{d-1}-1)/(q-1)-(q^{r+1}-1)/(q-1)+(q^r-1)/(q-1)=q^{d-1}-q^r$ elements of $\mathcal{B}$ through v. Now, since $t+1$ is a constant, if through a point of $\Upsilon[n-1]$ there are c elements of $\mathcal{B}$, then through a point not contained in $\Upsilon[n-1]$ there are $c-q^r$ elements of $\mathcal{B}$. This implies that the dimension of the subspace of elements of $\mathcal{B}$ has to be the same for every point of $\mathcal{S}$. Hence the subspaces containing the elements of $\mathcal{B}$ through the points of $\mathcal{S}$ are the elements of a partition Σ of the points of $\mathcal{S}$, and every element of Σ has the same dimension d. Clearly $r+2\leq d\leq n-2$.

There are $q^n+q^{n-1}-q^r$ points of $\mathcal{S}$ in $\mathrm{PG}(n,q)$. Let $\Psi[d]$ be an arbitrary element of Σ. We count the number of elements of $\mathcal{P}$ that are contained in $\Psi[d]$. We know that $\Psi[d]$ is a d-dimensional subspace of $\mathrm{PG}(n,q)$ that intersects $\Pi[n-2]$ in a $(d-2)$-dimensional space. Indeed, $\Psi[d]$ contains lines of $\mathcal{B}$, and so the dimension of $\Pi[n-2]\cap\Psi[d]$ can not be more than $d-2$. Hence the points of $\mathcal{S}$ in $\Psi[d]$ are all points of $\Psi[d]$ that are not contained in $\Omega[r]$ and neither in $\Pi[n-2]\cap\Psi[d]$. So we get that there are

$$\frac{q^{d+1}-1}{q-1}-\frac{q^{d-1}-1}{q-1}-\frac{q^{r+1}-1}{q-1}+\frac{q^r-1}{q-1},$$

or thus $q^d+q^{d-1}-q^r$ points of $\mathcal{S}$ in $\Psi[d]$. Since $\Psi[d]$ was an arbitrary chosen element of $\mathcal{S}$, there are $q^d+q^{d-1}-q^r$ points of $\mathcal{S}$ contained in every element of Σ. Hence

$$|\Sigma|=\frac{q^n+q^{n-1}-q^r}{q^d+q^{d-1}-q^r}. \tag{2.1}$$

Every element of Σ intersects $\Upsilon[n-1]$ in a $(d-1)$-dimensional space. There are $q^{n-1}-q^r$ points of $\mathcal{S}$ in $\Upsilon[n-1]$. There are $q^{d-1}-q^r$ points of $\mathcal{S}$ in the

$(d-1)$-dimensional intersection of an element of Σ with $\Upsilon[n-1]$. It follows that

$$|\Sigma| = \frac{q^{n-1} - q^r}{q^{d-1} - q^r}. \tag{2.2}$$

From (2.1) and (2.2) it follows that $d = n$. This implies that there are no lines of $\mathcal{S}$ in $\mathrm{PG}(n,q)$, a contradiction. Hence if $\Omega[r]$ intersects $\Pi[n-2]$ in an $(r-1)$-dimensional space, then $\mathcal{S}$ can not be a $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$.

Case 2. Assume that $\Omega[r]$ intersects $\Pi[n-2]$ in an $(r-2)$-dimensional space, $r \geq 1$.

Then $\langle \Omega[r], \Pi[n-2] \rangle = \mathrm{PG}(n,q)$. Let v be a point of $\mathcal{S}$. We count the number of lines through v on which there are points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. The $(r+1)$-dimensional subspace $\langle v, \Omega[r] \rangle$ intersects $\Pi[n-2]$ in an $(r-1)$-dimensional space. Hence there are $(q^r-1)/(q-1)-(q^{r-1}-1)/(q-1) = q^{r-1}$ lines through v that contain 2 points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. All the other lines through v contain at most one point of $\mathrm{P}(n,q) \setminus \mathcal{P}$. It follows that there are $(q^{n-1}-1)/(q-1) + (q^{r+1}-1)/(q-1) - (q^{r-1}-1)/(q-1) - q^{r-1} = (q^{n-1}-1)/(q-1)+q^r$ lines through v on which there are points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Since v was an arbitrary point of $\mathcal{S}$, this is valid for every point of $\mathcal{S}$. The number of elements of $\mathcal{B}$ through a point of $\mathcal{S}$ equals the total number of lines through this point minus $t+1$ minus the number of lines on which there are points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. This implies that the number of elements of $\mathcal{B}$ through a point of $\mathcal{S}$ is also a constant.

If $\mathcal{B} = \emptyset$, then $t+1$ is a constant. Hence in this case $\mathcal{S}$ is a $(q-1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, and every line containing $q+1$ points of $\mathcal{S}$ is a line of $\mathcal{S}$.

If $\mathcal{B} \neq \emptyset$, then from lemma 2.5.6 it follows that the elements of $\mathcal{B}$ through a point v of $\mathcal{S}$ are contained in a d-dimensional subspace $\Psi[d]$ through v and $\Omega[r]$. Since we have proved that the number of elements of $\mathcal{B}$ through a point of $\mathcal{S}$ is a constant, it follows that the subspaces containing the elements of $\mathcal{B}$ through the points of $\mathcal{S}$ are the elements of a partition Σ of the points of $\mathcal{S}$, and that every element of Σ has the same dimension d.

There are $q^n - q^{n-1} - q^r - q^{r-1}$ points of $\mathcal{S}$ in $\mathrm{PG}(n,q)$. As in the first part of the proof, we can count that there are $q^d + q^{d-1} - q^r - q^{r-1}$ points of $\mathcal{S}$ contained in an element of Σ. Now the remainder of the division of $q^n - q^{n-1} - q^r - q^{r-1}$ by $q^d + q^{d-1} - q^r - q^{r-1}$ equals $q^{n-cd+cr} + q^{n-cd+cr-1} - q^r - q^{r-1}$, where $c \geq 0$ is a positive integer. This remainder will be equal to 0 if and only if $(d-r)|(n-d)$ or thus $(d-r)|(n-r)$.

Moreover if $(d-r)|(n-r)$, then the partition Σ always exists. Indeed, let $\Gamma[n-r-3]$ be an $(n-r-3)$-dimensional subspace of $\Pi[n-2]$ disjoint from $\Omega[r] \cap \Pi[n-2]$. Then $\langle \Omega[r], \Gamma[n-r-3] \rangle$ is an $(n-2)$-dimensional space. Since $\Omega[r]$ is contained in $\langle \Omega[r], \Gamma[n-r-3] \rangle$, the intersection of $\langle \Omega[r], \Gamma[n-r-3] \rangle$ and $\Pi[n-2]$ is an $(n-4)$-dimensional space. Let $\Pi^*[3]$ be a three dimensional space skew to this $(n-4)$-dimensional space. Then $\Pi[n-2]$ and $\langle \Omega[r], \Gamma[n-r-3] \rangle$ intersect $\Pi^*[3]$ each in a line M_1 resp. M_2 and these two lines are disjoint. In $\Pi^*[3]$ it is possible to take a third line M_3 disjoint from M_1 and from M_2. Then the $(n-r-1)$-dimensional space $\langle M_3, \Gamma[n-r-3] \rangle$ is skew to $\Omega[r]$ and it intersects $\Pi[n-2]$ in $\Gamma[n-r-3]$. Now let $\Lambda[d-r-3]$ be a $(d-r-3)$-dimensional subspace of $\Gamma[n-r-3]$ and let $\Lambda'[n-d+1]$ be an $(n-d+1)$-dimensional subspace of $\langle M_3, \Gamma[n-r-3] \rangle$ skew to $\Lambda[d-r-3]$. Then $\Lambda'[n-d+1]$ intersects $\Pi[n-2]$ in an $(n-d-1)$-dimensional space. Indeed, $\Lambda'[n-d+1] \cap \Pi[n-2]$ is contained in $\Gamma[n-r-3]$ so if its dimension would be greater than or equal to $n-d$, then it would intersect $\Lambda[d-r-3]$ in a point, a contradiction because we have chosen $\Lambda'[n-d+1]$ skew to $\Lambda[d-r-3]$. Now by [3] there exists a partial spread Σ_1 of lines of $\Lambda'[n-d+1] \setminus \Pi[n-2]$. Let Σ_2 be the set of $(d-r-1)$-dimensional spaces spanned by $\Lambda[d-r-3]$ and a line of Σ_1. Then the elements of the partition Σ are the d-dimensional spaces spanned by an element of Σ_2 and $\Omega[r]$.

Hence the lines of $\mathcal{S}$ are all the lines that contain $q+1$ points of $\mathcal{S}$ or they are the lines not contained in a partition Σ of the points of $\mathcal{S}$, where each element of Σ is d-dimensional and contains $\Omega[r]$. This partition exists if and only if $(d-r)|(n-r)$, and always gives a $(q-1,q)$-geometry. □

2.5.2 The case $\alpha = q - \sqrt{q}$

It is clear that in this subsection q will always be a square.

Assuming q to be odd, there exists no non-trivial maximal arc in a Desarguesian plane [1], i.e. there can not be $(q-\sqrt{q})$-planes contained in $\mathrm{PG}(n,q)$. As in the previous subsection, we distinguish three types of planes that contain an antiflag of $\mathcal{S}$.

- Type I are the q-planes.
- Type II are the planes in which the restriction of $\mathcal{S}$ is the closure of a net. Note that such planes contain $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$.
- Type III are the planes in which all points belong to $\mathcal{S}$ and lines of $\mathcal{S}$ are the secant lines to a unital or the tangent lines to a Baer subplane.

Remark. Let $\mathcal{S}$ be a $(q-\sqrt{q},q)$-geometry fully embeddable in $\mathrm{PG}(n,q)$, for $q = 2^{2h}$, $h \in \mathbb{N}$, $h > 1$. Then every plane containing an antiflag of $\mathcal{S}$ is a plane of type I, II or III as defined above. This follows in the same way as in the case $\alpha = q-1$. Note that also in this case α-planes can not exist. Indeed, if $q \neq 4$, then $q-\sqrt{q}$ does not divide q, so a maximal arc of degree $q-\sqrt{q}$ does not exist. However not everything what follows will be valid for q even, since sometimes theorems are used that are not true in case q is even. In those cases it will be mentioned explicitly that q has to be odd.

Lemma 2.5.8 *Let $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ be a $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q a square, $q \neq 4$. Every line of $\mathrm{PG}(n,q)$ contains 0, 1, $\sqrt{q}+1$ or $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$.*

Proof. One proves this lemma in the same way as lemma 2.5.1. □

Let $\Pi[n-m-1]$ be an $(n-m-1)$-dimensional subspace of $\mathrm{PG}(n,q)$. We define $\Pi[n-m-1]\mathcal{S}$ to be the cone with vertex $\Pi[n-m-1]$, projecting a $(q-\sqrt{q},q)$-geometry $\mathcal{S}$ fully embedded in an m-dimensional subspace of $\mathrm{PG}(n,q)$ skew to $\Pi[n-m-1]$.

Lemma 2.5.9 *Let $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ be a $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(m,q)$, with q a square. Let $\mathrm{PG}(n,q)$ be an n-dimensional projective space containing $\mathrm{PG}(m,q)$, $m < n$. Let $\mathcal{S}^* = (\mathcal{P}^*,\mathcal{L}^*,\mathrm{I}^*)$ be the incidence structure defined as follows: $\mathcal{P}^*$ is the set of points of the cone $\Pi[n-m-1]\mathcal{S}$ that are not contained in the vertex $\Pi[n-m-1]$, $\mathcal{L}^*$ is the set of lines that are contained in some plane $\langle x, L\rangle$, for any $x \in \Pi[n-m-1]$ and any $L \in \mathcal{L}$, and that do not contain x, I^* is the restriction of the incidence of $\mathrm{PG}(n,q)$ to $\mathcal{S}^*$. Then $\mathcal{S}^*$ is a $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$.*

Proof. It is immediately clear that the number of points of $\mathcal{S}^*$ on a line of $\mathcal{S}^*$ is equal to $q+1$. The number of lines through a point of $\mathcal{S}^*$ is also a constant, since this is true for the $(q-\sqrt{q},q)$-geometry $\mathcal{S}$. In particular, if $t+1$ is the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$, then the number of lines of $\mathcal{S}^*$ through a point of $\mathcal{S}^*$ equals $(t+1)q^{n-m}$. Now let $p \in \mathcal{P}^*$, $L \in \mathcal{L}^*$, such that $p \notin L$. Then either $\langle p, L\rangle$ contains a point of $\Pi[n-m-1]$ or it is skew to $\Pi[n-m-1]$. In the first case, $\langle p, L\rangle$ is a q-plane and hence $i(p,L) = q$. In the second case, $\langle p, L\rangle$ is contained in a base of the cone $\Pi[n-m-1]\mathcal{S}$, hence p and L both belong to a $(q-\sqrt{q},q)$-geometry fully embedded in an m-dimensional space skew to $\Pi[n-m-1]$. It follows that $i(p,L) = q-\sqrt{q}$ or $i(p,L) = q$. This proves that $\mathcal{S}^*$ is a $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$. □

We call a $(q-\sqrt{q}, q)$-geometry $\Pi[n-m-1]\mathcal{S}$ projecting a $(q-\sqrt{q}, q)$-geometry $\mathcal{S}$ fully embeddable in an m-dimensional subspace of $\mathrm{PG}(n, q)$ skew to $\Pi[n-m-1]$, $m<n$, a *degenerate* $(q-\sqrt{q}, q)$-geometry. It is clear that it suffices to classify all non-degenerate $(q-\sqrt{q}, q)$-geometries.

In the next two theorems, we consider the case in which no line of $\mathrm{PG}(n, q)$ contains $\sqrt{q}+1$ points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. We will prove that either $n=3$, or $\mathcal{S}$ is degenerate.

Theorem 2.5.10 *Let $\mathcal{S}$ be a $(q-\sqrt{q}, q)$-geometry that is fully embedded in $\mathrm{PG}(n, q)$, with q a square, $q \neq 4$. Assume that no line of $\mathrm{PG}(n, q)$ contains $\sqrt{q}+1$ points of $\mathrm{P}(n, q) \setminus \mathcal{P}$, and that $|\mathrm{P}(n, q) \setminus \mathcal{P}| \neq \emptyset$. Then $\mathcal{S}$ is a degenerate $(q-\sqrt{q}, q)$-geometry.*

Proof. Let $\mathcal{S}$ be a $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n, q)$, with q a square. Assume that no line of $\mathrm{PG}(n, q)$ contains $\sqrt{q}+1$ points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. From lemma 2.5.8 it follows that in this case the points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ are the points of a $(0,1, q+1)$-set, or in other words it are the points of an m-dimensional subspace $\Pi[m]$ of $\mathrm{PG}(n, q)$, $0 \leq m \leq n-2$.

Let $\mathcal{B}$ be the set of all lines of $\mathrm{PG}(n, q)$ that contain $q+1$ points of $\mathcal{S}$, but that do not belong to $\mathcal{S}$. It is clear that $\mathcal{B} \neq \emptyset$, as otherwise $\mathcal{S}$ would be a $(q, q+1)$-geometry. Let $N \in \mathcal{B}$. Then the $(m+2)$-dimensional space $\langle N, \Pi[m]\rangle$ does not contain a line of $\mathcal{S}$. Indeed, if there would be a line L of $\mathcal{S}$ in $\langle N, \Pi[m]\rangle$, L intersecting N, then $\langle L, N\rangle$ would be a plane containing an antiflag of $\mathcal{S}$, a line of $\mathcal{B}$ and a point of $\Pi[m]$, a contradiction as such a plane can not exist. This implies that $\langle N, \Pi[m]\rangle$ contains no lines of $\mathcal{S}$, as otherwise for such a line L' of $\mathcal{S}$, $i(z, L')=0$, for a point $z \in N$. Hence the $(m+2)$-dimensional spaces through $\Pi[m]$ in $\mathrm{PG}(n, q)$ either contain no lines of $\mathcal{S}$, or they contain no lines of $\mathcal{B}$.

Let $\Omega[n-m-1]$ be a subspace of $\mathrm{PG}(n, q)$ skew to $\Pi[m]$. Then each $(m+2)$-dimensional subspace of $\mathrm{PG}(n, q)$ that contains $\Pi[m]$, intersects $\Omega[n-m-1]$ in a line M. If $M \in \mathcal{B}$, then $\langle M, \Pi[m]\rangle$ contains no lines of $\mathcal{S}$. If $M \in \mathcal{L}$, then $\langle M, \Pi[m]\rangle$ contains no lines of $\mathcal{B}$. We will prove now that $\mathcal{S}$ intersects $\Omega[n-m-1]$ in a $(q-\sqrt{q}, q)$-geometry $\mathcal{S}'$. It is clear that every line of $\mathcal{S}$ in $\Omega[n-m-1]$ contains $q+1$ points of $\mathcal{S}$ and that for every antiflag (p, L) of $\mathcal{S}$ in $\Omega[n-m-1]$, we have that $i(p, L)=q-\sqrt{q}$ or q. So we only need to prove that the number of lines of $\mathcal{S}$ in $\Omega[n-m-1]$ through a point of $\mathcal{S}$ in $\Omega[n-m-1]$ is a constant. Let u be a point of $\Omega[n-m-1]$. If L_1 is a line of $\mathcal{S}$ through u in $\Omega[n-m-1]$, then $\langle L_1, \Pi[m]\rangle$ contains $1+(q-1) \frac{q^{m+1}-1}{q-1}=q^{m+1}$ lines of $\mathcal{S}$ through u. If L_2 is a line of $\mathcal{S}$ through u in $\Omega[n-m-1]$, $L_2 \neq L_1$, then $\langle L_1, \Pi[m]\rangle$ and $\langle L_2, \Pi[m]\rangle$ intersect in the $(m+1)$-dimensional space $\langle u, \Pi[m]\rangle$ that clearly contains no line of $\mathcal{S}$. Hence each line of $\mathcal{S}$ through u belongs to exactly one $(m+2)$-dimensional space $\langle L, \Pi[m]\rangle$ through $\langle u, \Pi[m]\rangle$. It follows that $t+1 = q^{m+1}(t_u+1)$, where t_u+1 is the number of lines of $\mathcal{S}$ through u contained in $\Omega[n-m-1]$. Since $t+1$ is a constant, it follows that also t_u+1 is a constant, independent of the choice of the point $u \in \Omega[n-m-1]$. This proves that $\mathcal{S}$ intersects $\Omega[n-m-1]$ in a $(q-\sqrt{q}, q)$-geometry $\mathcal{S}'$.

Hence $\mathrm{PG}(n,q)$ contains a cone $\Pi[m]\mathcal{S}'$, projecting a $(q-\sqrt{q}, q)$-geometry $\mathcal{S}'$ fully embedded in an $(n-m-1)$-dimensional subspace $\Omega[n-m-1]$ of $\mathrm{PG}(n,q)$ skew to $\Pi[m]$. Points of $\mathcal{S}$ are the points of this cone that are not contained in $\Pi[m]$, lines of $\mathcal{S}$ are the lines that contain $q+1$ points of $\mathcal{S}$ and are contained in a plane $\langle x, L\rangle$, for $x \in \Pi[m]$ and $L \in \mathcal{L}$. By definition $\mathcal{S}$ is a degenerate $(q-\sqrt{q}, q)$-geometry. □

Theorem 2.5.11 *If $n > 3$, then there is no $(q-\sqrt{q}, q)$-geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, fully embeddable in $\mathrm{PG}(n,q)$, q a square, $q \neq 4$, such that $\mathcal{P} = \mathrm{PG}(n,q)$.*

Proof. Let $\mathcal{S}$ be a $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q a square, $q \neq 4$ and $n > 3$. Let $\mathcal{P} = \mathrm{P}(n,q)$. It follows that every plane containing an antiflag of $\mathcal{S}$ is of type III. Let $\mathcal{B}$ be the set of lines containing $q+1$ points of $\mathcal{S}$ but not belonging to $\mathcal{L}$.

Let p be an arbitrary point of $\mathrm{PG}(n,q)$. Let $\Pi[n-1]$ be a hyperplane of $\mathrm{PG}(n,q)$ not through p. The elements of $\mathcal{B}$ through p intersect $\Pi[n-1]$ in a set of points, which we denote by $\mathcal{K}$. Since every plane through p contains 1, $\sqrt{q}+1$ or $q+1$ elements of $\mathcal{B}$ through p, the set $\mathcal{K}$ is a $(1, \sqrt{q}+1, q+1)$-set of $\Pi[n-1]$. If there is a line of $\Pi[n-1]$ containing $\sqrt{q}+1$ points of $\mathcal{K}$, then the points of $\mathcal{K}$ are the points of a Hermitian variety, a Baer subplane, a cone with base either a Hermitian variety or a Baer subplane, or $\sqrt{q}+1$ hyperplanes intersecting in an $(n-2)$-dimensional space ([35], theorems 23.5.1 and 23.5.19). If no line of $\Pi[n-1]$ intersects $\mathcal{K}$ in $\sqrt{q}+1$ points, then the points of $\mathcal{K}$ are the points of a hyperplane of $\Pi[n-1]$.

Assume that there is a plane π in $\Pi[n-1]$ for which the points of $\mathcal{K}$ in π are all the points that lie on $\sqrt{q}+1$ concurrent lines. We denote these lines by $L_1, \ldots, L_{\sqrt{q}+1}$. Let w be the intersection point of $L_1, \ldots, L_{\sqrt{q}+1}$. We look at the three dimensional space spanned by p and π. The plane $\langle p, L_i\rangle$, for $i \in \{1, \ldots, \sqrt{q}+1\}$, contains $q+1$ elements of $\mathcal{B}$ through p. Hence it can not contain a line of $\mathcal{S}$, as if there was a line L of $\mathcal{S}$ in this plane, then $i(p, L) = 0$, a contradiction. Suppose that there is a line L' of $\mathcal{S}$, $L' \subset \langle p, \pi\rangle$, such that L' is skew to the line $\langle p, w\rangle$. Let ρ be a plane through $\langle p, w\rangle$ in $\langle p, \pi\rangle$ different from $\langle p, L_i\rangle$, for all $i \in \{1, \ldots, \sqrt{q}+1\}$. Let $L' \cap \rho$ be the point x. The plane ρ contains an antiflag of $\mathcal{S}$, hence it is of type III. Through x we can take a line N of $\mathcal{B}$ such that $N \subset \rho$. The lines N and $\langle p, w\rangle$ intersect in a point y. Now we look at the plane $\langle L', y\rangle$. It contains the line $L' \in \mathcal{L}$ and at least $\sqrt{q}+2$ elements of $\mathcal{B}$ through y, namely the intersection lines of $\langle L', y\rangle$ with $\langle p, L_i\rangle$ $(i = 1, \ldots, \sqrt{q}+1)$ and the line N. So $i(y, L') \leq q - \sqrt{q} - 1$, a contradiction. We conclude that all the lines of $\mathcal{S}$ in $\langle p, \pi\rangle$ are contained in the $(q - \sqrt{q})$ planes through $\langle p, w\rangle$ and a line through w in π different from $L_1, \ldots, L_{\sqrt{q}+1}$. Now let M_1 and M_2 be two lines of $\mathcal{S}$ in $\langle p, \pi\rangle$ that contain p, such that the planes $\langle p, w, M_1\rangle$ and $\langle p, w, M_2\rangle$ are distinct. Then $\langle M_1, M_2\rangle$ is a plane containing an antiflag of $\mathcal{S}$. However, from the above it follows that all the lines of $\mathcal{S}$ in this plane contain the point p. This is a contradiction as $\alpha = q - \sqrt{q}$ and $\beta = q$. So no plane of $\Pi[n-1]$ can intersect $\mathcal{K}$ in $\sqrt{q}+1$ concurrent lines, that each contain $q+1$ points of $\mathcal{K}$.

Hence we have shown that the points of $\mathcal{K}$ have to be the points of a Baer subplane, a unital in some plane of $\Pi[n-1]$ or a hyperplane of $\Pi[n-1]$. Indeed, both a (possibly) singular Hermitian variety with $n > 3$ and a cone with base a Baer subplane contain planes that intersect it in $\sqrt{q}+1$ concurrent lines.

Assume first that $\mathcal{K}$ is the set of points of a unital or a Baer subplane of $\Pi[n-1]$. It follows that $n = 3$, since every line of $\Pi[n-1]$ contains at least one point of $\mathcal{K}$. This is a contradiction, as we assumed $n > 3$.

Assume next that $\mathcal{K}$ is the set of points of a hyperplane of $\Pi[n-1]$. Then clearly $t + 1 = q^{n-1}$. We denote the hyperplane containing the points of $\mathcal{K}$ by $\Gamma[n-2]$. The $(n-1)$-dimensional space $\langle p, \Gamma[n-2]\rangle$ contains no lines of $\mathcal{S}$. Indeed, if there would be a line L_p of $\mathcal{S}$ contained in this subspace, then $i(p, L_p) = 0$, a contradiction. Let p' be an arbitrary point of $\langle p, \Gamma[n-2]\rangle$. Since $t + 1 = q^{n-1}$, the lines of $\mathcal{S}$ through p' are all the lines through p' not in $\langle p, \Gamma[n-2]\rangle$. It follows that the lines of $\mathcal{S}$ in $\mathrm{PG}(n, q)$ are all lines of $\mathrm{PG}(n, q)$ not in $\langle p, \Gamma[n-2]\rangle$. However, this implies that for a point u' of $\mathrm{PG}(n, q)$, $u' \notin \langle p, \Gamma[n-2]\rangle$ there are $q+1$ lines of $\mathcal{S}$ intersecting each line of $\mathcal{S}$ not through u'. This is a contradiction since $\mathcal{S}$ is a $(q - \sqrt{q}, q)$-geometry. Hence the points of $\mathcal{K}$ can not be the points of a hyperplane of $\Pi[n-1]$.

We conclude that for $n \neq 3$ a $(q - \sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n, q)$, such that $\mathrm{P}(n, q) \setminus \mathcal{P} = \emptyset$, does not exist. □

Remark. The previous theorem does not give a classification of $(q-\sqrt{q}, q)$-geometries fully embedded in $\mathrm{PG}(3, q)$, for q a square and point set $\mathcal{P} = \mathrm{PG}(3, q)$. If there exists such a $(q - \sqrt{q}, q)$-geometry $\mathcal{S}$, then one of the following would hold.

- $t+1 = q\sqrt{q}+1$. The lines of $\mathcal{S}$ through a point p of PG$(3,q)$ intersect each plane not through p in the points not on a unital and the lines in each plane of PG$(3,q)$ are the lines that intersect a unital in that plane in $\sqrt{q}+1$ points.
- $t+1 = q+\sqrt{q}+1$. The lines of $\mathcal{S}$ through a point p of PG$(3,q)$ intersect each plane not through p in the points not contained in a Baer subplane and the lines in each plane of PG$(3,q)$ are the lines that are tangent to a Baer subplane in that plane.

It is not known to us whether such a $(q-\sqrt{q},q)$-geometry $\mathcal{S}$ fully embedded in PG$(3,q)$ does exist. If $\mathcal{S}$ exists, then from lemma 2.5.9 it follows that a cone $\Pi[n-4]\mathcal{S}$ gives rise to a $(q-\sqrt{q},q)$-geometry in PG(n,q).

Now let us look at the case in which PG(n,q) does contain a line on which there lie exactly $\sqrt{q}+1$ points of P$(n,q)\setminus\mathcal{P}$. If no line of PG$(n,q)$ contains $q+1$ points of P$(n,q)\setminus\mathcal{P}$, then from lemma 2.5.8 it follows that the points of P$(n,q)\setminus\mathcal{P}$ are the points of a $(0,1,\sqrt{q}+1)$-set. In the next two lemma's we will describe what a $(0,1,\sqrt{q}+1)$-set in PG(n,q) can look like. In lemma 2.5.15 it will then be proved that either $\mathcal{S}$ is non-degenerate and it contains no line on which there are $q+1$ points of P$(n,q)\setminus\mathcal{P}$, or it is degenerate and its base does not contain a line on which there are $q+1$ points of P$(n,q)\setminus\mathcal{P}$.
The results of lemma's 2.5.12 and 2.5.13 follow also immediately from a more general result of J. Ueberberg on regular $\{v,m\}$-arcs (a $\{v,m\}$-arc is a $(0,1,m)$-set with respect to lines that has v points) (corollary 1.5.8 and theorem 1.5.9). We included a proof of the lemma's, since our proof is slightly different from the one in [56]. Note that we only treat the special case $m=\sqrt{q}+1$, since this is the only case that we will need for the proof of the next theorem.

Lemma 2.5.12 *Let $\mathcal{K}$ be a set of type $(0,1,\sqrt{q}+1)$ with respect to lines in* PG(n,q)*, $n\geq 2$. Then each plane of* PG(n,q) *intersects $\mathcal{K}$ in 0 points, a singleton, $\sqrt{q}+1$ collinear points, a unital or a Baer subplane.*

Proof. Let $\mathcal{K}$ be a $(0,1,\sqrt{q}+1)$-set with respect to lines in PG(n,q), $n\geq 2$. Let π be a plane of PG(n,q). If π contains no line on which there are 0 points of $\mathcal{K}$, then the points of $\mathcal{K}$ in π form a $(1,\sqrt{q}+1)$-set in π. Hence $\mathcal{K}$ intersects π in a unital or a Baer subplane (see [34], theorem 12.17). So, from now on we may assume that π contains a line L on which there are no points of $\mathcal{K}$. We denote by m_0 (resp. m_1 and $m_{\sqrt{q}+1}$) the number of lines of π that contain 0 (resp. 1 and $\sqrt{q}+1$) points of $\mathcal{K}$. Then it follows that

$$\left\{\begin{array}{rcl} m_0+m_1+m_{\sqrt{q}+1} & = & q^2+q+1 \\ m_1+(\sqrt{q}+1)m_{\sqrt{q}+1} & = & (q+1)|\mathcal{K}| \\ \sqrt{q}(\sqrt{q}+1)m_{\sqrt{q}+1} & = & |\mathcal{K}|(|\mathcal{K}|-1). \end{array}\right. \tag{2.3}$$

The first equation we obtain by counting all lines of π, the second by counting pairs (p,M), with p a point of $\mathcal{K}$ in π, M a line of π and $p\in M$, the third equation we get by counting triples (p,p',M), with p and p' points of $\mathcal{K}$ in π, $p\neq p'$, M a line of π and $p,p'\in M$.

Let x be a point of $\mathcal{K}$, $x\in\pi$. Counting the points of $\mathcal{K}$ on the lines through x in π, we get that $|\mathcal{K}|=a\sqrt{q}+1$, where a is the number of lines through x in π on which there are $\sqrt{q}+1$ points of $\mathcal{K}$. It is clear that $0\le a\le q+1$. If $a=0$, then $\mathcal{K}$ is a single point. Assume from now on that $a>0$. From (2.3) it follows that

$$\left\{\begin{array}{rcl} m_{\sqrt{q}+1} & = & a^2+\frac{a-a^2}{\sqrt{q}+1} \\ m_1 & = & aq\sqrt{q}+a\sqrt{q}+q+1-a^2\sqrt{q}-a \\ m_0 & = & q^2-aq\sqrt{q}-a\sqrt{q}+a^2\sqrt{q}+a-a^2-\frac{a-a^2}{\sqrt{q}+1}. \end{array}\right.$$

Since m_0 and $m_{\sqrt{q}+1}$ have to be integers, we get that $\sqrt{q}+1\mid a^2-a$. Also, it is clear that $m_0\ge 0$. From the above we get that $m_0(\sqrt{q}+1)=q(a-q)(a-\sqrt{q}-1)$, so either $a\le 1+\sqrt{q}$, or $a\ge q$. Since by assumption $1\le a\le q+1$, there are the following cases to consider.

1. $a=q+1$. Then the condition $\sqrt{q}+1\mid a^2-a$ is not satisfied.

2. $a=q$. Then $|\mathcal{K}|=q\sqrt{q}+1$ and from (2.3) it follows that $m_0=0$, giving a contradiction, as we assumed that $m_0>0$.

3. $a=\sqrt{q}+1$. Then $|\mathcal{K}|=q+\sqrt{q}+1$. From (2.3) it follows that $m_0=0$, again a contradiction.

4. $a=1$. Then the points of $\mathcal{K}$ in the plane π are $\sqrt{q}+1$ collinear points.

5. $2\le a\le\sqrt{q}$. By definition, a is equal to the number of lines of π through a point x of $\mathcal{K}$ containing $\sqrt{q}+1$ points of $\mathcal{K}$. Now take a line through 2 points of $\mathcal{K}$ in π that does not contain x. This line contains at least 2 points of $\mathcal{K}$ and at most $\sqrt{q}$ such points, since at most $\sqrt{q}$ lines through x contain points of $\mathcal{K}$. This is a contradiction, since $\mathcal{K}$ is a $(0,1,\sqrt{q}+1)$-set. Hence this case does not occur.

This proves that every plane in $\mathrm{PG}(n,q)$ intersects $\mathcal{K}$ in 0 points, one point, $\sqrt{q}+1$ collinear points, a Baer subplane or a unital. $\square$

Lemma 2.5.13 *Let $\mathcal{K}$ be a set of type $(0,1,\sqrt{q}+1)$ with respect to lines in $\mathrm{PG}(n,q)$, $n \geq 3$ and $q \neq 4$. Assume that there is a line that contains $\sqrt{q}+1$ points of $\mathcal{K}$. Then $\mathcal{K}$ is a Baer subspace of $\mathrm{PG}(n,q)$, a Baer subspace of some subspace of $\mathrm{PG}(n,q)$, $\sqrt{q}+1$ collinear points, or a unital in a plane of $\mathrm{PG}(n,q)$.*

Proof. From lemma 2.5.12 it follows that every plane of $\mathrm{PG}(n,q)$ intersects $\mathcal{K}$ in 0 points, a singleton, $\sqrt{q}+1$ collinear points, a Baer subplane or a unital.

We will prove that if there exists a plane π_U in $\mathrm{PG}(n,q)$ that intersects $\mathcal{K}$ in a unital, then $\mathcal{K} \subset \pi_U$. So let π_U be a plane intersecting $\mathcal{K}$ in a unital, and assume that there is a point p of $\mathcal{K}$ that does not belong to π_U. Suppose first that $\langle p, \pi_U\rangle$ contains a line L that is exterior to $\mathcal{K}$. Let $L \cap \pi_U = \{x\}$. Let M be a line of π_U through x that intersects $\mathcal{K}$ in $\sqrt{q}+1$ points. The plane $\langle L, M\rangle$ contains $\sqrt{q}+1$ collinear points of $\mathcal{K}$ and a line exterior to $\mathcal{K}$. From lemma 2.5.12 it follows that $\langle L, M\rangle$ intersects $\mathcal{K}$ in $\sqrt{q}+1$ collinear points. Hence $p \notin \langle L, M\rangle$. In $\langle L, M\rangle$ there are q lines through x that are exterior to $\mathcal{K}$, while in π_U there are $q-\sqrt{q}$ lines through x that contain $\sqrt{q}+1$ points of $\mathcal{K}$. Hence in the three dimensional space $\langle p, \pi_U\rangle$ there is a plane ρ through the line $\langle p, x\rangle$ that contains an exterior line to $\mathcal{K}$ and $\sqrt{q}+1$ collinear points of $\mathcal{K}$ on its intersection line with π_U. So ρ contains an exterior line to $\mathcal{K}$ and at least $\sqrt{q}+2$ points of $\mathcal{K}$, a contradiction (lemma 2.5.12). It follows that every line of $\langle p, \pi_U\rangle$ contains a point of $\mathcal{K}$. Hence $\mathcal{K}$ intersects $\langle p, \pi_U\rangle$ is a $(1,\sqrt{q}+1)$-set. From ([35], theorem 23.5.1) we get a contradiction, as every $(1,\sqrt{q}+1,q+1)$-set in a three dimensional space contains lines on which there are $q+1$ points of $\mathcal{K}$. We conclude that if $\mathrm{PG}(n,q)$ contains a plane π_U that intersects $\mathcal{K}$ in a unital, then $\mathcal{K} \subset \pi_U$.

Now suppose that $\mathrm{PG}(n,q)$ does not contain a plane that intersects $\mathcal{K}$ in a unital. From lemma 2.5.12 it follows that every plane of $\mathrm{PG}(n,q)$ intersects $\mathcal{K}$ in 0 points, one point, $\sqrt{q}+1$ collinear points or a Baer subplane. We will prove that $\mathcal{K}$ has to be a Baer subspace of $\mathrm{PG}(n,q)$. From now on we call the lines that contain $\sqrt{q}+1$ points of $\mathcal{K}$, $\mathcal{K}$-lines. To prove that the points of $\mathcal{K}$ and the $\mathcal{K}$-lines are the points and lines of a projective geometry, we check whether the axioms of Dembowski hold (see [27]).

1. Through every 2 points of $\mathcal{K}$ there has to be exactly one $\mathcal{K}$-line. This follows immediately from the fact that $\mathcal{K}$ is a $(0,1,\sqrt{q}+1)$-set.

2. On every $\mathcal{K}$-line there have to be at least 3 points of $\mathcal{K}$. This is true, because every $\mathcal{K}$-line contains $\sqrt{q}+1 \geq 3$ points of $\mathcal{K}$.

3. Let L and M be two $\mathcal{K}$-lines that intersect in the point $p \in \mathcal{K}$. Let $u, v \in L$ and $w, z \in M$ be 4 different points of $\mathcal{K}$. Assume that $u \neq p \neq v$ and $w \neq p \neq z$. Then the lines $\langle u, w\rangle$ and $\langle v, z\rangle$ of $\mathcal{K}$ intersect in a point p' of $\mathcal{K}$. Indeed, two intersecting $\mathcal{K}$-lines span a plane that intersects $\mathcal{K}$ in a Baer subplane. Since p' lies on two $\mathcal{K}$-lines, which are Baer sublines, p' clearly belongs to the Baer subplane and hence also to $\mathcal{K}$.

Hence the points and lines of $\mathcal{K}$ are the points and lines of a projective geometry. Since there are $\sqrt{q}+1$ points of $\mathcal{K}$ on a line of $\mathcal{K}$, it now follows that $\mathcal{K}$ is a Baer subspace of $\mathrm{PG}(n,q)$, a Baer subspace of some subspace of $\mathrm{PG}(n,q)$, or $\sqrt{q}+1$ collinear points. □

Lemma 2.5.14 *Let $\mathcal{S}$ be a $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q a square, $q \neq 4$. Assume that there is a line that contains exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Then every line of $\mathrm{PG}(n,q)$ that contains $q+1$ points of $\mathcal{S}$, is a line of $\mathcal{S}$.*

Proof. Let $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ be a $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q a square, $q \neq 4$. Assume that there is a line M_z that contains exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Let $\mathcal{B}$ be the set of lines containing $q+1$ points of $\mathcal{S}$, but not belonging to $\mathcal{S}$. Assume that $\mathcal{B} \neq \emptyset$. Let $N \in \mathcal{B}$. Let z be a point of $\mathcal{S}$. The plane $\langle N, z\rangle$ is a plane of type III, or it contains no lines of $\mathcal{S}$. If $\langle N, z\rangle$ is a plane of type III, then it contains a line N_z of $\mathcal{B}$ through z, since in a plane of type III there is at least one line of $\mathcal{B}$ through every point of this plane. If $\langle N, z\rangle$ contains no lines of $\mathcal{S}$, then from lemma 2.5.12 it follows that $\langle N, z\rangle$ contains at most $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. So there is a line N_z through z in $\langle N, z\rangle$ that contains no point of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Hence also in this case there is a line N_z of $\mathcal{B}$ through z in $\langle N, z\rangle$. Let L_z be a line of $\mathcal{S}$ through z. The plane $\langle L_z, N_z\rangle$ contains an antiflag of $\mathcal{S}$ and a line of $\mathcal{B}$, hence it is a plane of type III. The plane $\langle L_z, M_z\rangle$ is a plane of type II. The plane $\langle N_z, M_z\rangle$ contains a line of $\mathcal{B}$ and points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, hence it can not contain lines of $\mathcal{S}$. Moreover from lemma 2.5.12 it follows that this plane contains exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, namely the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ on the line M_z. Each plane through N_z in the three dimensional space $\langle L_z, N_z, M_z\rangle$, different from the plane $\langle N_z, M_z\rangle$, intersects the plane $\langle L_z, M_z\rangle$ in a line of $\mathcal{S}$. Hence every such plane is a plane of type III. This implies that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle L_z, N_z, M_z\rangle$ are exactly the $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ on M_z.

Let y_1 and y_2 be points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ on M_z. Let N be a line of $\mathcal{B}$ in $\langle L_z, N_z\rangle$, such that $z \notin N$. Let L be a line of $\mathcal{S}$ in $\langle L_z, N_z\rangle$, such that

$z \notin L$. The plane $\langle N, y_1 \rangle$ contains a line of $\mathcal{B}$ and a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$, hence it contains no lines of $\mathcal{S}$. The plane $\langle L, y_2 \rangle$ is a q-plane, so it contains no lines of $\mathcal{B}$. However, these two planes belong to the three dimensional space $\langle N_z, L_z, M_z \rangle$, hence they intersect in a line. It is clear that this line contains $q+1$ points of $\mathcal{S}$, hence it is a line of $\mathcal{B}$ or a line of $\mathcal{S}$. In both cases we get a contradiction. This proves that $\mathcal{B} = \emptyset$. □

Lemma 2.5.15 *Let $\mathcal{S}$ be a $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q an odd square. If there is a line that contains $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, then the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of a cone with vertex an m-dimensional subspace $\Pi[m]$ (it is possible that $\Pi[m] = \emptyset$) and base a non-degenerate $(0,1,\sqrt{q}+1)$-set contained in an $(n-m-1)$-dimensional subspace of $\mathrm{PG}(n,q)$ skew to $\Pi[m]$.*

Proof. Let $\mathcal{S}$ be a $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q an odd square. Assume that $\mathrm{PG}(n,q)$ contains a line $M_{\sqrt{q}+1}$ on which there are $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. From lemma 2.5.14 it follows that every line that contains $q+1$ points of $\mathcal{S}$, is a line of $\mathcal{S}$. From lemma 2.5.8 it follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ form a $(0,1,\sqrt{q}+1,q+1)$-set. If $\mathrm{PG}(n,q)$ contains no line on which there are $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, then the lemma is proved. So we may assume that $\mathrm{PG}(n,q)$ contains such a line. A plane through $M_{\sqrt{q}+1}$ that contains an antiflag of $\mathcal{S}$ is a plane of type II. Let ρ_1 and ρ_2 be two planes of type II through $M_{\sqrt{q}+1}$ (there exist two planes through $M_{\sqrt{q}+1}$ containing an antiflag of $\mathcal{S}$ since $\mathcal{S}$ is not contained in a plane). We will prove that the points of $\mathrm{PG}(n,q) \setminus \mathcal{P}$ in the three dimensional space $\langle \rho_1, \rho_2 \rangle$ are the points of a three dimensional Baer subspace $\mathrm{PG}(3,\sqrt{q})$, of a Baer subplane, of a unital or of $\sqrt{q}+1$ concurrent lines in a plane.

Assume first that $\langle \rho_1, \rho_2 \rangle$ contains two points y_1, $y_2 \in \mathrm{P}(n,q) \setminus \mathcal{P}$, such that $\langle y_1, y_2 \rangle$ is skew to $M_{\sqrt{q}+1}$. The line $\langle y_1, y_2 \rangle$ intersects ρ_1 in a point of $\mathcal{S}$. From lemma 2.5.8 it follows that $\langle y_1, y_2 \rangle$ contains exactly $\sqrt{q}+1$ points $y_1, y_2, \ldots, y_{\sqrt{q}+1}$ of $\mathrm{P}(n,q) \setminus \mathcal{P}$. From lemma 2.5.12 and [34] theorem 19.4.4, it follows that each plane $\langle y_i, M_{\sqrt{q}+1} \rangle$ $(i = 1, \ldots, \sqrt{q}+1)$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the points of a unital or a Baer subplane, or in $\sqrt{q}+1$ lines through a point. Hence $\langle \rho_1, \rho_2 \rangle$ contains at least $(\sqrt{q}+1)q + \sqrt{q} + 1 = (q+1)(\sqrt{q}+1)$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Now let L be a line of $\mathcal{S}$ in ρ_1. Every plane through L is of type I, II or III. Hence each plane through L in $\langle \rho_1, \rho_2 \rangle$ contains at most $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. This proves that there are at most $(q+1)(\sqrt{q}+1)$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ contained in $\langle \rho_1, \rho_2 \rangle$. From these two inequalities we get that $\langle \rho_1, \rho_2 \rangle$ contains exactly $(q+1)(\sqrt{q}+1)$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ and hence that every plane $\langle y_i, M_{\sqrt{q}+1}\rangle$ $(i = 1,\dots \sqrt{q}+1)$ intersects $\mathrm{P}(n,q)\setminus\mathcal{P}$ in a Baer subplane, while the other planes through $M_{\sqrt{q}+1}$ in $\langle\rho_1,\rho_2\rangle$ intersect $\mathrm{P}(n,q)\setminus\mathcal{P}$ in the $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ on $M_{\sqrt{q}+1}$. This implies that no line of $\langle\rho_1,\rho_2\rangle$ contains $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Hence from lemma 2.5.13 it follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\rho_1,\rho_2\rangle$ are the points of a three dimensional Baer subspace in $\langle\rho_1,\rho_2\rangle$.

It is clear that not all the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\rho_1,\rho_2\rangle$ lie on the line $M_{\sqrt{q}+1}$. Indeed, otherwise $\langle\rho_1,\rho_2\rangle$ contains planes in which there is no point of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Such planes contain lines of $\mathcal{B}$, which is a contradiction with lemma 2.5.14.

Assume now that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\rho_1,\rho_2\rangle$ are contained in a plane $\langle y, M_{\sqrt{q}+1}\rangle$ through $M_{\sqrt{q}+1}$ and a point y of $\mathrm{P}(n,q)\setminus\mathcal{P}$, $y\notin M_{\sqrt{q}+1}$. There is no line L of $\mathcal{S}$ contained in $\langle y, M_{\sqrt{q}+1}\rangle$ (lemma 2.5.12). Hence the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\rho_1,\rho_2\rangle$ are the points of a $(1,\sqrt{q}+1,q+1)$-set contained in the plane $\langle y, M_{\sqrt{q}+1}\rangle$. From [35] theorem 19.4.4, it follows that such a set is a unital, a Baer subplane, or a set of $\sqrt{q}+1$ concurrent lines.

So we have proved that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\rho_1,\rho_2\rangle$ are the points of a three dimensional Baer subspace, of a Baer subplane, of a unital or of $\sqrt{q}+1$ concurrent lines in a plane. If $n=3$, then the lemma is proved. So assume that $n>3$. Let u be a point of $\mathcal{S}$, $u\notin\langle\rho_1,\rho_2\rangle$. Define $\Gamma[4]$ to be the four dimensional space spanned by u, ρ_1 and ρ_2. We will now describe the possibilities for the intersection of $\mathrm{P}(n,q)\setminus\mathcal{P}$ with $\Gamma[4]$.

Assume first that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\rho_1,\rho_2\rangle$ are the points of a three dimensional Baer subspace. If $\Gamma[4]$ contains no lines on which there are $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, then from lemma 2.5.8 it follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]$ are the points of a $(0,1,\sqrt{q}+1)$-set. So the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]$ are the points of a four dimensional Baer subspace (lemma 2.5.13). If $\Gamma[4]$ does contain a line M on which there are $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, then let $M\cap\langle\rho_1,\rho_2\rangle$ be the point x. Every line through x in $\langle\rho_1,\rho_2\rangle$ contains 1 or $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. From lemma 2.5.8 it follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in a plane through M are the points of a $(1,\sqrt{q}+1,q+1)$-set. Hence the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in each plane through M in $\Gamma[4]$ are either the points on M, or it are all the points of $\sqrt{q}+1$ concurrent lines (see [35], theorem 19.4.4).

Now suppose that there would be two planes π_1 and π_2 through M in $\Gamma[4]$, such that in π_1 the $\sqrt{q}+1$ lines containing no points of $\mathcal{S}$ intersect in the point x_1, while in π_2 these lines intersect in the point x_2, $x_1\neq x_2$. Then in the three dimensional space $\langle\pi_1,\pi_2\rangle$ every line contains a point of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Indeed, if there would be a line N in $\langle\pi_1,\pi_2\rangle$ containing no points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, then a plane through N in $\langle\pi_1,\pi_2\rangle$ that does not contain x_1

or x_2, contains at least $2\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. This is a contradiction with lemma 2.5.12. So the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\pi_1,\pi_2\rangle$ are the points of a set of type $(1,\sqrt{q}+1,q+1)$, which is a non-degenerate Hermitian variety $H(3,q)$ (see [35], section 23.5). However, $\langle\pi_1,\pi_2\rangle$ intersects $\langle\rho_1,\rho_2\rangle$ in a plane, and $\langle\rho_1,\rho_2\rangle$ contains no planes in which the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ are the points of a unital or of $\sqrt{q}+1$ concurrent lines. So we have found a contradiction.

Hence in each plane through M in $\Gamma[4]$ that contains $\sqrt{q}+1$ concurrent lines on which there are no points of $\mathcal{S}$, these lines intersect in the same point y of M. This proves that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]$ are the points of a cone with vertex y and base the three dimensional Baer subspace contained in $\langle\rho_1,\rho_2\rangle$.

Assume next that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\rho_1,\rho_2\rangle$ are contained in a plane τ. Note that from the above it follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\rho_1,\rho_2\rangle$ are the points of a Baer subspace, a unital or $\sqrt{q}+1$ concurrent lines in τ.

If $\Gamma[4]$ contains no line on which there are $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, then from lemma 2.5.8 it follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]$ are the points of a set of type $(0,1,\sqrt{q}+1)$. From lemma 2.5.13 and the fact that every plane of $\Gamma[4]$ contains at least one point of $\mathrm{P}(n,q)\setminus\mathcal{P}$, it follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]$ are the points of a four dimensional Baer subspace.

If $\Gamma[4]$ contains a line M on which there are $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, then we may assume that $M\not\subset\langle\rho_1,\rho_2\rangle$. Indeed, since every plane contains at least one point of $\mathrm{P}(n,q)\setminus\mathcal{P}$, there is a point x of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]\setminus\langle\rho_1,\rho_2\rangle$. If M would be contained in $\langle\rho_1,\rho_2\rangle$, then the plane $\langle M,x\rangle$ intersects $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\sqrt{q}+1$ concurrent lines, and $\sqrt{q}$ of them are not contained in $\langle\rho_1,\rho_2\rangle$. So we may assume from now on that $M\not\subset\langle\rho_1,\rho_2\rangle$. As in the previous paragraph, it follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in every plane through M in the three dimensional space $\langle M,\tau\rangle$ are the points of M or the points of $\sqrt{q}+1$ concurrent lines. This implies again that every line in $\langle M,\tau\rangle$ contains at least one point of $\mathrm{P}(n,q)\setminus\mathcal{P}$.

If not all lines containing $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle M,\tau\rangle$ intersect in the same point, then from [35] theorem 23.5.1, we know that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle M,\tau\rangle$ are the points of a non-degenerate Hermitian variety $H(3,q)$. If all such lines intersect in the same point $\tilde{y}$, then the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle M,\tau\rangle$ are the points of a cone with vertex the point $\tilde{y}$ and base the intersection of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in τ. If not all points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]$ would be contained in $\langle M,\tau\rangle$, then there would be a point y' of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]\setminus\langle M,\tau\rangle$. From the previous it follows that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$

in $\langle y,\tau\rangle$ are the points of a set of type $(1,\sqrt{q}+1,q+1)$. Let L be a line of $\mathcal{S}$ in $\langle\rho_1,\rho_2\rangle$. Then L intersects τ in a point u. Let $M^u_{\sqrt{q}+1}$ be a line through u in $\langle M,\tau\rangle$ containing exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ and such that $M^u_{\sqrt{q}+1}\notin\tau$. The plane $\langle L,M^u_{\sqrt{q}+1}\rangle$ intersects $\langle y',\tau\rangle$ in a line, and from the previous it follows that this line contains at least one point of $\mathrm{P}(n,q)\setminus\mathcal{P}$. This is a contradiction, since $\langle L,M^u_{\sqrt{q}+1}\rangle$ contains an antiflag of $\mathcal{S}$, so it cannot contain more than $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. So all points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]$ are contained in $\langle M,\tau\rangle$.

We conclude that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma[4]$ are the points of a four dimensional Baer subspace, a cone with vertex a point x and base a three dimensional Baer subspace, a cone with vertex a point x and base a Baer subplane, a cone with vertex a line M and base a Baer subline, a non-degenerate three dimensional Hermitian variety or a cone with vertex a point x and base a unital in a plane. If $n=4$, then the points of $\mathrm{P}(4,q)\setminus\mathcal{P}$ cannot be contained in a three dimensional subspace of $\mathrm{PG}(4,q)$. Indeed, we have proved in lemma 2.5.14 that $\mathrm{PG}(4,q)$ contains no lines of $\mathcal{B}$, and so if the points of $\mathrm{P}(4,q)\setminus\mathcal{P}$ are contained in a three dimensional subspace of $\mathrm{PG}(4,q)$, then the number $t+1$ of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ cannot be a constant. So if $n=4$, then the points of $\mathrm{P}(4,q)\setminus\mathcal{P}$ in $\mathrm{PG}(4,q)$ are the points of a four dimensional Baer subspace or the points of a cone with vertex a point x and base a three dimensional Baer subspace, and hence the lemma is proved.

Now assume that $n>4$. Let $\Gamma'[m]$ be an m-dimensional subspace of $\mathrm{PG}(n,q)$, $4\leq m\leq n-1$, such that $\Gamma[4]\subseteq\Gamma'[m]$. Assume that in the m-dimensional subspace $\Gamma'[m]$ of $\mathrm{PG}(n,q)$, the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ either span $\Gamma'[m]$ and are the points of a Baer subspace or of a cone with vertex a subspace and base a Baer subspace of dimension greater than or equal to 3; or these points span an $(m-1)$-dimensional subspace $\Upsilon[m-1]$ of $\Gamma'[m]$ and not $\Gamma'[m]$ and it are the points of a (possibly degenerate) Hermitian variety or a cone with vertex a subspace and base a Baer subplane or a Baer subline. Let $\Lambda[m+1]$ be an $(m+1)$-dimensional subspace of $\mathrm{PG}(n,q)$ through $\Gamma[m]$ and a point $u\in\mathcal{P}$, $u\notin\Gamma[m]$ (such a point exists since the points of $\mathcal{S}$ span $\mathrm{PG}(n,q)$)). We will describe how $\mathrm{P}(n,q)\setminus\mathcal{P}$ can intersect $\Lambda[m+1]$.

Assume first that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Gamma'[m]$ span $\Gamma'[m]$. If $\Lambda[m+1]$ would not contain a line on which there are $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, then the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Lambda[m+1]$ are the points of a set of type $(0,1,\sqrt{q}+1)$ (lemma 2.5.8), so it are the points of an $(m+1)$-dimensional Baer subspace (lemma 2.5.13). Now assume that $\Lambda[m+1]$ contains a line M on which there are $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. If $\Gamma'[m]$ intersects $\mathrm{P}(n,q)\setminus\mathcal{P}$ in an

m-dimensional Baer subspace, then as we did in the case $n = 4$, one proves that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Lambda[m+1]$ are the points of a cone with vertex a point and base the set of points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Gamma'[m]$. If $\Gamma'[m]$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in a cone with vertex an r-dimensional subspace $\Psi[r]$ and base an $(m-r-1)$-dimensional Baer subspace contained in a subspace $\Sigma[m-r-1]$, then we may assume that M is not contained in $\Gamma'[m]$. Indeed, let x be a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in a plane through a line L of $\mathcal{S}$ in $\Gamma'[m]$, such that $\langle L, x\rangle \not\subset \Gamma'[m]$. If M would be contained in $\Gamma'[m]$, then the plane $\langle M, x\rangle$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\sqrt{q}+1$ concurrent lines, and $\sqrt{q}$ or them are not contained in $\Gamma'[m]$. So we may assume that $M \not\subset \Gamma'[m]$.

Suppose first that no line of $\Lambda[m+1]$ on which there are $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, is skew to $\Psi[r]$. Then each $(m-r)$-dimensional subspace skew to $\Psi[r]$ in $\Lambda[m+1]$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in an $(m-r)$-dimensional Baer subspace. If there would be a line $\tilde{M}$, on which there are exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, that intersect $\Psi[r]$ in a point $\tilde{y}$, then let $\tilde{N}$ be a line in $\Gamma'[m]$ containing $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ and intersecting $\Sigma[m-r-1]$ in a point. The plane $\langle \tilde{N}, \tilde{M}\rangle$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in all the points of $\sqrt{q}+1$ concurrent lines intersecting in a point that is different from the point $\tilde{y}$. So there is a line containing $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ and skew to $\Psi[r]$, a contradiction with our assumption. This proves that every line intersecting $\Psi[r]$ contains 1 or $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Hence the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Lambda[m+1]$ are the points of a cone with vertex the subspace $\Psi[r]$ and base an $(m-r)$-dimensional Baer subspace contained in an $(m-r)$-dimensional subspace of $\Lambda[m+1]$ skew to $\Psi[r]$.

Suppose next that there is a line M in $\Lambda[m+1]$ on which there are $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, M skew to $\Psi[r]$. Each plane through M intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the points on M, the points on $\sqrt{q}+1$ lines, or all points of the plane. We will prove that no two planes π_1 and π_2 through M can intersect $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\sqrt{q}+1$ concurrent lines through the point x_1 and x_2 respectively, with $x_1 \neq x_2$. Assume therefore that there would exist such planes π_1 and π_2. As we did in the case $n = 4$, one proves that $\langle \pi_1, \pi_2\rangle$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the points of a non-degenerate Hermitian variety $H(3,q)$. Clearly $\langle \pi_1, \pi_2\rangle$ intersects $\Gamma'[m]$ in a plane ω containing $\sqrt{q}+1$ concurrent lines through a point x', on which there are $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Now let σ_1 be a three dimensional subspace of $\Gamma'[m]$ through ω intersecting $\mathrm{P}(n,q) \setminus \mathcal{P}$ in a cone with vertex the point x' and base a Baer subplane. Then σ_1 and $\sigma_2 = \langle \pi_1, \pi_2\rangle$ contain no lines of $\mathcal{S}$. Let $\sigma_1, \ldots \sigma_{q+1}$ be the three dimensional spaces through ω in $\langle \sigma_1, \sigma_2\rangle$. Assume that σ_i $(i \in \{3, \ldots, q+1\})$ contains a line L of $\mathcal{S}$. As in the case $n = 3$, it follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in σ_i are contained in ω. Let u be a point of $\mathcal{S}$ in ω. Let $M_u \not\subset \omega$ be a line through u in σ_2 containing $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Let π_u be a plane through M_u in $\langle \sigma_1, \sigma_2 \rangle$, $\pi_u \not\subset \sigma_2$. Then $\pi_u \cap \sigma_1$ contains at least one point of $\mathrm{P}(n,q) \setminus \mathcal{P}$, while $\pi_u \cap \sigma_i$ is a line of $\mathcal{S}$. So π_u contains at least $\sqrt{q}+2$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ and a line of $\mathcal{S}$, a contradiction with lemma 2.5.12. This proves that σ_i $(i = 3, \ldots, q+1)$ cannot contain a line of $\mathcal{S}$, and hence $\langle \sigma_1, \sigma_2 \rangle$ contains no lines of $\mathcal{S}$. So from lemma 2.5.8 it follows that $\mathrm{P}(n,q) \setminus \mathcal{P}$ intersects $\langle \sigma_1, \sigma_2 \rangle$ in a set of type $(1, \sqrt{q}+1, q+1)$. From [35] theorem 23.5.19, it follows that such a set is the set of all points of a Hermitian variety. This is a contradiction, since there are planes in $\langle \sigma_1, \sigma_2 \rangle$ intersecting $\mathrm{P}(n,q) \setminus \mathcal{P}$ in a Baer subspace. So there is a point $\overline{y} \in M$ such that in each plane through M, the lines containing $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ go through $\overline{y}$. Hence no line through $\overline{y}$ can contain exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Since M is skew to $\Psi[r]$ it follows that $\overline{y} \in \Lambda[m+1] \setminus \Gamma'[m]$. This proves that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Lambda[m+1]$ are the points of a cone with vertex the subspace $\langle \overline{y}, \Psi[r] \rangle$ and base the $(m-r)$-dimensional subspace of points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Sigma[m-r-1]$.

Assume next that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Gamma'[m]$ are contained in an $(m-1)$-dimensional subspace $\Psi[m-1]$ of $\Gamma'[m]$. By assumption $\Gamma'[m]$ contains a line M on which there are $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Let x be a point of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Lambda[m+1] \setminus \Gamma'[m]$. The plane $\langle x, M \rangle$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the points of $\sqrt{q}+1$ concurrent lines (lemma 2.5.8 and [35], theorem 19.4.4). Let $M' \subset \langle x, M \rangle$, $M' \neq M$, be a line containing no points of $\mathcal{S}$. Then $M' \not\subset \Gamma'[m]$. Let $\Psi[r]$ be the vertex of the cone of the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Gamma'[m]$ (it is possible that $\Psi[r] = \emptyset$).

If every line of $\langle x, \Upsilon[m-1] \rangle$ on which there are $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, intersects $\Psi[r]$ in a point, then as in the previous case it follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\langle x, \Upsilon[m-1] \rangle$ are the points of a cone with vertex $\Psi[r]$ and base a three dimensional Baer subspace, a Baer subplane or a unital. If there would be a point y of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Lambda[m+1] \setminus \langle x, \Upsilon[m-1] \rangle$, then let M_x be a line through x in $\langle x, \Upsilon[m-1] \rangle$ that contains $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. The plane $\langle M_x, y \rangle$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in all points of $\sqrt{q}+1$ concurrent lines. These $\sqrt{q}+1$ lines each contain a point of $\Psi[r]$, as otherwise $\langle M_x, y \rangle$ intersects $\Gamma'[m]$ in a line that contains $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, a contradiction since we have assumed that all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Gamma'[m]$ are contained in $\Upsilon[m-1]$. So $\langle y, \Upsilon[m-1] \rangle$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in a cone with vertex the subspace $\Psi[r]$ and base a unital or a Baer subplane or a three dimensional Baer subspace. It follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in each m-dimensional subspace through $\Upsilon[m-1]$ in $\Lambda[m+1]$ are either contained in $\Upsilon[m-1]$, or they are the points of a cone with vertex the subspace $\Psi[r]$

and base a unital or a Baer subplane or a three dimensional Baer subspace. This proves that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Lambda[m+1]$ are the points of a cone with vertex the subspace $\Psi[r]$ and base a Baer subspace or a unital. If all points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Lambda[m+1]$ are contained in a hyperplane of $\Lambda[m+1]$, then the base is a Baer subline, a Baer subplane or a unital (in this case this hyperplane cannot contain lines of $\mathcal{S}$). If the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Lambda[m+1]$ span $\Lambda[m+1]$ then the base is a Baer subspace of dimension at least three (since $\Gamma'[m]\subset\Lambda[m+1]$).

Assume next that $\Lambda[m+1]$ contains a line $\tilde{M}$ on which there are $q+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, such that $\tilde{M}$ is skew to $\Psi[r]$. It is clear that $\tilde{M}$ intersects $\Upsilon[m-1]$ in a point. Let $\tilde{M}\cap\Upsilon[m-1]$ be the point $\tilde{y}$. Let $M_{\sqrt{q}+1}$ be a line through $\tilde{y}$ in $\Upsilon[m-1]$ that contains exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Assume that $\langle\Psi[r],\tilde{M}\rangle$ would contain a line L of $\mathcal{S}$. Let $u\in M_{\sqrt{q}+1}$ be a point of $\mathcal{S}$. There is a line L_u of $\mathcal{S}$ through u intersecting L in a point. The plane $\langle L_u, M_{\sqrt{q}+1}\rangle$ is a plane of type II. From the first part of the proof (case $n=3$), we get that all the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle L_u, M_{\sqrt{q}+1},\tilde{M}\rangle$ are contained in a plane τ. This is a contradiction, as $\langle\tilde{M}, M_{\sqrt{q}+1}\rangle$ and $\langle L_u, M_{\sqrt{q}+1},\tilde{M}\rangle\cap\Upsilon[m-1]$ are two different planes in $\langle L_u, M_{\sqrt{q}+1},\tilde{M}\rangle$ that contain points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ not on $M_{\sqrt{q}+1}$. Hence $\langle\tilde{M},\Upsilon[m-1]\rangle$ contains no lines of $\mathcal{S}$. So the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\langle\tilde{M},\Upsilon[m-1]\rangle$ are the points of a set of type $(1,\sqrt{q}+1,q+1)$ (lemma 2.5.8), which is a (possibly degenerate) Hermitian variety or a cone with vertex a subspace and base a Baer subplane or a Baer subspace. If $\tilde{x}$ would be a point of $\mathrm{P}(n,q)\setminus\mathcal{P}$ that is contained in $\Lambda[m+1]\setminus\langle\tilde{M},\Upsilon[m-1]\rangle$, then the plane $\langle\tilde{x},\tilde{M}\rangle$ would intersect $\Gamma'[m]$ in a line $\tilde{N}$ containing $\sqrt{q}+1$ points of $\mathrm{P}(n,q)\setminus\mathcal{P}$, $\tilde{N}\not\subset\Upsilon[m-1]$. This is a contradiction, as by assumption $\Gamma'[m]\setminus\Upsilon[m-1]$ contains no points of $\mathrm{P}(n,q)\setminus\mathcal{P}$. Hence all points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Lambda[m+1]$ are contained in $\langle\tilde{M},\Upsilon[m-1]\rangle$.

We conclude that the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ in $\Lambda[m+1]$ either span $\Lambda[m+1]$ and are the points of a Baer subspace or a cone with vertex a subspace and base a Baer subspace of dimension at least three; or these points are contained in a hyperplane of $\Lambda[m+1]$ and are the points of a set of type $(1,\sqrt{q}+1,q+1)$. If $n=m+1$, then the points of $\mathrm{P}(n,q)\setminus\mathcal{P}$ span $\Lambda[m+1]$, as otherwise the number $t+1$ of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ would not be a constant. So in case $n=m+1$, the lemma is proved.

Continuing in this way, after a finite number of steps, the result of the theorem follows. □

So we have proved that if $\mathcal{S}$ is a non-degenerate $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, q an odd square, such that $\mathrm{PG}(n,q)$ contains a line on which there are exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, then the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of a $(0,1,\sqrt{q}+1)$-set in $\mathrm{PG}(n,q)$. We are now able to completely classify the non-degenerate $(q-\sqrt{q},q)$-geometries fully embedded in $\mathrm{PG}(n,q)$, such that $\mathrm{PG}(n,q)$ contains a line on which there are $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, with q an odd square.

Theorem 2.5.16 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$ that is non-degenerate, with q an odd square. Assume that there is a line contained in $\mathrm{PG}(n,q)$ on which there are exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Then $n = 3$ or $n = 4$ and there exists an n-dimensional Baer subspace $\mathrm{PG}(n,\sqrt{q})$ of $\mathrm{PG}(n,q)$ such that $\mathcal{P}$ is the set of all points of $\mathrm{PG}(n,q) \setminus \mathrm{PG}(n,\sqrt{q})$, $\mathcal{L}$ is the set of all lines not intersecting $\mathrm{PG}(n,\sqrt{q})$ and incidence is the one of $\mathrm{PG}(n,q)$.*

Proof. Let $\mathcal{S}$ be a non-degenerate $(q-\sqrt{q},q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q an odd square. Assume that $\mathrm{PG}(n,q)$ contains a line on which there are exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. From lemma 2.5.15 we know that no line of $\mathrm{PG}(n,q)$ can contain $q+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. From lemma 2.5.8 it then follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of a $(0,1,\sqrt{q}+1)$-set of $\mathrm{PG}(n,q)$. From lemma 2.5.13 we get that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of a Baer subspace of $\mathrm{PG}(n,q)$, of a Baer subspace of some subspace of $\mathrm{PG}(n,q)$, $\sqrt{q}+1$ collinear points, or a unital in some plane. From lemma 2.5.14 it follows that every line that contains $q+1$ points of $\mathcal{S}$, is a line of $\mathcal{S}$. This implies that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ span $\mathrm{PG}(n,q)$, as otherwise the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ cannot be constant. It follows that the points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ are the points of a Baer subspace $\mathrm{PG}(n,\sqrt{q})$ of $\mathrm{PG}(n,q)$. Every plane containing an antiflag of $\mathcal{S}$ is of type I or of type II, which implies that every plane containing an antiflag of $\mathcal{S}$ contains at least one point of $\mathrm{P}(n,q) \setminus \mathcal{P}$. For $n \geq 5$, it is clear that $\mathrm{PG}(n,q)$ contains planes skew to $\mathrm{PG}(n,\sqrt{q})$. For $n \leq 4$, there are no planes skew to $\mathrm{PG}(n,\sqrt{q})$ contained in $\mathrm{PG}(n,q)$. Since $\mathcal{B} = \emptyset$, $\mathcal{L}$ is the set of all lines of $\mathrm{PG}(n,q)$ not containing a point of $\mathrm{PG}(n,\sqrt{q})$. It follows immediately that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, with $\mathcal{P}$ the set of points of $\mathrm{PG}(n,q) \setminus \mathrm{PG}(n,\sqrt{q})$, $\mathcal{L}$ the set of lines of $\mathrm{PG}(n,q)$ not intersecting $\mathrm{PG}(n,\sqrt{q})$, is indeed a $(q-\sqrt{q},q)$-geometry for $n = 3$ or 4. $\square$

In the following theorem, the case in which $\mathrm{PG}(n,q)$ contains a line on which there are $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, and q is an odd square, is summarized.

Theorem 2.5.17 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with q an odd square. Assume that $\mathrm{PG}(n,q)$ contains a line on which there are exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Then $\mathrm{PG}(n,q)$ contains a cone $\Pi[n-m-1]\mathcal{S}'$, projecting a $(q-\sqrt{q}, q)$-geometry $\mathcal{S}'$ fully embedded in an m-dimensional subspace $\Gamma[m]$ skew to $\Pi[n-m-1]$, with $m = 3$ or $m = 4$. Points of $\mathcal{S}'$ are the points of $\Gamma[m]$ not contained in a Baer subspace $\mathrm{PG}(m, \sqrt{q})$ of $\Gamma[m]$, lines of $\mathcal{S}'$ are lines skew to $\mathrm{PG}(m, \sqrt{q})$. The points of $\mathcal{S}$ are the points of the cone $\Pi[n-m-1]\mathcal{S}'$ that are not contained in $\Pi[n-m-1]$, while $\mathcal{L}$ is the set of all lines that lie on the cone $\Pi[n-m-1]\mathcal{S}'$ and that contain $q+1$ points of $\mathcal{S}$.*

Proof. This follows immediately from theorem 2.5.16 and lemma 2.5.9. □

For q even, $q \neq 4$, lemma 2.5.15 also holds if we assume that no plane of $\mathrm{PG}(n,q)$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the points of a maximal arc of degree $\sqrt{q}$ union the $q+1$ points of a line exterior to this maximal arc. Hence we get the following theorem in the even order case.

Theorem 2.5.18 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q-\sqrt{q}, q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, with $q = 2^{2h}$, $h \in \mathbb{N}$ and $h > 1$. Assume that $\mathrm{PG}(n,q)$ contains a line on which there are exactly $\sqrt{q}+1$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$, and that no plane of $\mathrm{PG}(n,q)$ intersects $\mathrm{P}(n,q) \setminus \mathcal{P}$ in the points of a maximal arc of order $\sqrt{q}$ together with the $q+1$ points of a line exterior to this maximal arc. Then $\mathrm{PG}(n,q)$ contains a cone $\Pi[n-m-1]\mathcal{S}'$, projecting a $(q-\sqrt{q}, q)$-geometry $\mathcal{S}'$ fully embedded in an m-dimensional subspace $\Gamma[m]$ skew to $\Pi[n-m-1]$, with $m = 3$ or $m = 4$. Points of $\mathcal{S}'$ are the points of $\Gamma[m]$ not contained in a Baer subspace $\mathrm{PG}(m, \sqrt{q})$ of $\Gamma[m]$, lines of $\mathcal{S}'$ are lines skew to $\mathrm{PG}(m, \sqrt{q})$. The points of $\mathcal{S}$ are the points of the cone $\Pi[n-m-1]\mathcal{S}'$ that are not contained in $\Pi[n-m-1]$, while $\mathcal{L}$ is the set of all lines that lie on the cone $\Pi[n-m-1]\mathcal{S}'$ and that contain $q+1$ points of $\mathcal{S}$.*

2.5.3 The case $q - 1 \neq \alpha \neq q - \sqrt{q}$

Let $\mathcal{S}$ be a proper (α, q)-geometry fully embedded in $\mathrm{PG}(n,q)$, q odd and $\alpha > 1$, for which $q - 1 \neq \alpha \neq q - \sqrt{q}$. Then every plane that contains an antiflag of $\mathcal{S}$ is either a q-plane, or it intersects $\mathcal{S}$ in the closure of a net. Hence every plane containing an antiflag of $\mathcal{S}$ contains one point or $q+1-\alpha$ collinear points of $\mathrm{P}(n,q) \setminus \mathcal{P}$.

Lemma 2.5.19 *Let $\mathcal{S}$ be a proper (α, q)-geometry, q odd and $\alpha > 1$, fully embedded in* $\mathrm{PG}(n, q)$*, for which $q - 1 \neq \alpha \neq q - \sqrt{q}$. Then every line of* $\mathrm{PG}(n, q)$ *contains* 0, 1, $q + 1 - \alpha$ *or* $q + 1$ *points of* $\mathrm{P}(n, q) \setminus \mathcal{P}$.

Proof. One proves this lemma in exactly the same way as lemma 2.5.1. □

Theorem 2.5.20 *There exists no proper (α, q)-geometry fully embedded in* $\mathrm{PG}(n, q)$*, q odd and $\alpha > 1$, for which $q - 1 \neq \alpha \neq q - \sqrt{q}$.*

Proof. Let $\mathcal{S}$ be a proper (α, q)-geometry fully embedded in $\mathrm{PG}(n, q)$, q odd and $\alpha > 1$, for which $q - 1 \neq \alpha \neq q - \sqrt{q}$. Then every plane containing an antiflag of $\mathcal{S}$ is a q-plane or a plane intersecting $\mathrm{P}(n, q) \setminus \mathcal{P}$ in the closure of a net (these planes we will call planes of type II). It follows immediately that there are no lines of $\mathcal{B}$, since a plane containing an antiflag of $\mathcal{S}$ cannot contain a line of $\mathcal{B}$. Hence each plane contains at least one point of $\mathrm{P}(n, q) \setminus \mathcal{P}$.

There is at least one plane τ of type II, since $\mathcal{S}$ is proper. Let M be the line of τ that contains $q + 1 - \alpha$ points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. Let u be a point of $\mathcal{S}$, $u \notin \tau$. We will prove that all points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ in $\langle u, \tau \rangle$ are contained in a plane. Assume that this is not the case. Then at least two planes through M contain points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ that do not lie on M. From lemma 2.5.19 it follows that at least $q + 1 - \alpha$ planes through M contain points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ that do not lie on M. Since $\alpha \neq q - \sqrt{q}$, a plane through M and a point of $\mathrm{P}(n, q) \setminus \mathcal{P}$, intersects $\mathrm{P}(n, q) \setminus \mathcal{P}$ in the points of $q + 1 - \alpha$ concurrent lines (see [35], theorem 19.4.4). Hence $\langle u, \tau \rangle$ contains at least $(q - 1)(q + 1 - \alpha)^2 + 2(q + 1 - \alpha)$ points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. Now let L be a line of $\mathcal{S}$ in τ. Then every plane through L contains either 1 or $q + 1 - \alpha$ points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. It follows that $\langle u, \tau \rangle$ contains at most $(q + 1)(q + 1 - \alpha)$ points of $\mathrm{P}(n, q) \setminus \mathcal{P}$. So we have found the following inequality

$$\begin{aligned} (q - 1)(q + 1 - \alpha)^2 + 2(q + 1 - \alpha) &\leq (q + 1)(q + 1 - \alpha) \\ (q - \alpha)(q - 1) &\leq 0 \end{aligned}$$

This is a contradiction. It follows that all points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ in $\langle u, \tau \rangle$ are contained in a plane and it are the points of $q + 1 - \alpha$ concurrent lines.

Next, let $\Pi[m]$ be an m-dimensional subspace, $3 \leq m \leq n - 1$, containing $\langle u, \tau \rangle$, such that all points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ in $\Pi[m]$ are contained in an $(m - 1)$-dimensional subspace $\Upsilon[m - 1]$. Let $\Gamma[m + 1]$ be an $(m + 1)$-dimensional subspace through $\Pi[m]$ and a point $u \in \mathcal{P}$, $u \notin \Pi[m]$. Assume that not all points of $\mathrm{P}(n, q) \setminus \mathcal{P}$ are contained in an m-dimensional subspace of $\Gamma[m + 1]$. Then there is a line M in $\Gamma[m + 1]$ that is skew to $\Upsilon[m - 1]$ and that contains at least two points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. The line M intersects $\Pi[m]$ in a point of $\mathcal{S}$, hence from lemma 2.5.19 it follows that M contains $q+1-\alpha$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. Let M' be a line of $\Upsilon[m-1]$ that contains $q+1-\alpha$ points of $\mathrm{P}(n,q) \setminus \mathcal{P}$. (Note that M' exists, since $\langle u, \tau \rangle \subset \Pi[m]$). The three dimensional space spanned by M and M' intersects $\Pi[m]$ in a plane of type II through M'. So $\langle M, M' \rangle$ contains a plane of type II and not all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in this three dimensional space are contained in a plane. This is a contradiction with the first part of the proof. Hence all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\Gamma[m+1]$ are contained in an m-dimensional subspace of $\Gamma[m+1]$.

Continuing in this way, after a finite number of steps, we get that all points of $\mathrm{P}(n,q) \setminus \mathcal{P}$ in $\mathrm{PG}(n,q)$ are contained in an $(n-1)$-dimensional subspace of $\mathrm{PG}(n,q)$. This gives a contradiction, as it implies that the number $t+1$ of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ is not a constant. □

The reader is reminded that all of the results proved in the sections 2.3, 2.4 and this section were summarized in section 2.2.

Chapter 3

Full projective embeddings of proper (α, β)-geometries, for which $\alpha, \beta \in \{0, 1, q, q + 1\}$

In this chapter we will study proper (α, β)-geometries for some special values of α and β, that are fully embedded in $\mathrm{PG}(n, q)$. This topic has caught the attention of many people in the last few decades, although the concept of an (α, β)-geometry was not defined in general until recently. For the values $(\alpha, \beta) = (1, q + 1)$, $(\alpha, \beta) = (0, q)$ and $(\alpha, \beta) = (0, 1)$, results have been published by respectively F. Buekenhout and E. Shult [9], J. I. Hall [29] and P. J. Cameron [10]. We will include their results in this chapter. Next we will recall the classification of $(q, q + 1)$-geometries fully embeddable in a projective space, that was given in chapter 2 (see also [13]). The case of fully embeddable $(0, q + 1)$-geometries will be mentioned shortly, since such (α, β)-geometries can never be connected. In the last section, we will give a complete classification of fully embedded $(1, q)$-geometries in $\mathrm{PG}(n, q)$, for $q \neq 2$. Hence for all $\alpha, \beta \in \{0, 1, q, q + 1\}$, $\alpha \neq \beta$, a classification of the corresponding proper fully embeddable (α, β)-geometries is given in this chapter. The results of this chapter have been published in [11].

3.1 $(1, q + 1)$-geometries and $(0, q)$-geometries

The $(1, q + 1)$-geometries are a special class of the so-called *Shult spaces*. A Shult space is defined to be an incidence structure $\mathcal{S}$ of points and lines together with an incidence relation, such that for each point p and each line L, p not incident with L, p is collinear with either 1 or all points of L.

However, the number of lines of a Shult space that are incident with a point of this Shult space, is not necessarily the same for each point of the Shult space. This is the reason why not every Shult spaces is a $(1,q+1)$-geometry. Every $(1,q+1)$-geometry is a Shult space of order (q,t). A Shult space $\mathcal{S}$ is called *non-degenerate* if no point of $\mathcal{S}$ is collinear with all other points of $\mathcal{S}$. A *subspace* Π of a Shult space $\mathcal{S}$ is a set of pairwise collinear points such that each line of $\mathcal{S}$ that meets Π in at least two points, is completely contained in Π. A Shult space $\mathcal{S}$ is said to have *rank* n if n is the largest integer for which there is a chain $\Pi_0 \subset \Pi_1 \subset \ldots \subset \Pi_n$ of distinct subspaces $\Pi_0 = \emptyset, \Pi_1, \ldots, \Pi_n$.

The next theorem characterizes all non-degenerate Shult spaces, all of whose lines have cardinality at least three. To understand this theorem, we need to define what a polar space is. A *polar* space of rank n, $n \geq 3$, is a set $\mathcal{P}$ of elements called points together with a set of distinct subsets called subspaces, with the following properties.

1. A subspace together with the subspaces contained in it is a d-dimensional projective space, with $-1 \leq d \leq n-1$.

2. The intersection of any two subspaces is a subspace.

3. Given a subspace π of dimension $n-1$ and a point $p \in \mathcal{P} \setminus \pi$, there exists a unique subspace π' containing p such that the dimension of $\pi \cap \pi'$ is $n-2$. The subspace π' contains all points of π which are joined to p by some subspace of dimension 1.

4. There exist disjoint subspaces of dimension $n-1$.

A polar space of rank 2 is a generalized quadrangle.

Theorem 3.1.1 ([9]) *A non-degenerate Shult space of rank n, $n \geq 3$, all of whose lines have cardinality at least three, together with its subspaces, is a polar space of rank n. A Shult space of rank 2, all of whose lines have cardinality at least three and all of whose points are contained in at least three lines, is a generalized quadrangle.*

The full embeddings of Shult spaces of rank $n \geq 3$ in projective spaces have been classified in [8] and in [36]. It follows from this classification that each Shult space fully embedded in $\mathrm{PG}(n,q)$ has an order, and hence it is a $(1,q+1)$-geometry fully embedded in $\mathrm{PG}(n,q)$. In particular, the following theorem completely classifies fully embedded $(1,q+1)$-geometries in $\mathrm{PG}(n,q)$.

Theorem 3.1.2 ([8, 36]) *Let $\mathcal{S}$ be a proper $(1, q+1)$-geometry fully embedded in* $\mathrm{PG}(n, q)$*. Then one of the following holds.*

1. *The points and lines of $\mathcal{S}$ are the points and lines of a non-singular quadric of* $\mathrm{PG}(n, q)$*.*

2. *The points and lines of $\mathcal{S}$ are the points and lines of a non-singular Hermitian variety of* $\mathrm{PG}(n, q)$ *(in this case q is a square).*

3. *The points of $\mathcal{S}$ are all the points of* $\mathrm{PG}(n, q)$*, while the lines of $\mathcal{S}$ are the lines of* $\mathrm{PG}(n, q)$ *that are contained in the totally isotropic* $\left(\frac{n-1}{2}\right)$*-dimensional spaces with respect to a symplectic polarity of* $\mathrm{PG}(n, q)$ *(in this case n is odd).*

The $(0, q)$-geometries, for $q \in \mathbb{N}$, $q \geq 2$, have been studied extensively by J. I. Hall (see [29]). The remainder of this section recalls the results of [29]. A $(0, q)$-geometry has been called a *proper* Δ*-space* by Higman [33]. He observed that the property that for each point p and each line L either $i(p, L) = 0$ or $i(p, L) = q$, is more or less the converse of the defining property of a polar space. This is the reason why J. I. Hall calls a $(0, q)$-geometry a *copolar space* (see [29]).

The copolar spaces of order $(1, t)$ are graphs which contain no triangles. A copolar space of order $(2, t)$ is better known as a *cotriangle space*.

A copolar space $\mathcal{S}$ is called *indecomposable* if and only if $\mathcal{S}$ is not the union of two or more copolar spaces on disjoint point sets. A *reduced* copolar space is an indecomposable copolar space such that for all vertices x and y in the point graph of $\mathcal{S}$, $\Gamma(x) = \Gamma(y)$ implies $x = y$.

Remark that a semipartial geometry with parameters q, t, $\alpha = q$ is a copolar space of order (q, t). Of course the dual of a net is also a copolar space, and since there is no hope of classifying these partial geometries, we assume from now on that there exists at least one antiflag (x, L) such that $i(x, L) = 0$.

In [29] the finite reduced copolar spaces of order (q, t), $q \geq 2$, are classified up to isomorphism. It turns out that every reduced copolar space of order (q, t) is a semipartial geometry. In the next theorem, this classification is given. The semipartial geometries $U_{2,3}(n)$, $W(2n+1, q)$ and $\mathrm{NQ}^{\pm}(2n-1, 2)$ were defined in section 1.2.5. To define the semipartial geometry $\overline{M(k)}$, $k \in \{2, 3, 7, 57\}$, we first need to define the concept of a *Moore graph*. A Moore graph is a strongly regular graph with valency $k > 1$, with $\lambda = 0$, $\mu = 1$ and with the minimum number of vertices, which is $k^2 + 1$. Note that this definition implies that Moore graphs have neither 3-cycles nor 4-cycles, but they do have 5-cycles. It is known that necessarily $k \in \{2,3,7,57\}$. However a Moore graph with $k = 57$ is not known to exist. Now, with each Moore graph Γ there is associated a semipartial geometry, which we will denote by $\overline{M(k)}$, in the following way. The point set $\mathcal{P}$ is the set of vertices Γ, the line set $\mathcal{L}$ is the set $\{\Gamma(x) \| x \in P\}$, and I is the natural incidence relation. The semipartial geometry $(\mathcal{P},\mathcal{L},\mathrm{I})$ defined in this way, has parameters $q = t = \alpha = k-1$, $\mu = (k-1)^2$ (see for instance [24]), and is denoted by $\overline{M(k)}$.

Theorem 3.1.3 ([29]) *If $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$, is a finite reduced copolar space of order (s,t), $s \geq 2$, then $\mathcal{S}$ is isomorphic to one of the following semipartial geometries:*

1. $\overline{M(k)}$, $k \in \{2,3,7,57\}$,
2. $U_{2,3}(n)$,
3. $\overline{W(2n+1,q)}$,
4. $NQ^{\pm}(2n-1,2)$.

Remark. The cotriangle spaces were in fact classified by [44], an earlier version of which was proved by [43].

3.2 $(0,1)$-geometries and $(0,q+1)$-geometries

Full embeddings of $(0,1)$-geometries have not been classified yet, and their classification seems to be a very complicated problem. From each generalized quadrangle $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ a $(0,1)$-geometry can be constructed in the following way (see [6]). Let M be any line of $\mathcal{S}$, let $\mathcal{P}(M)$ be the set of the $q+1$ points of $\mathcal{S}$ on M and let $\mathcal{L}(M)$ be the set of lines of $\mathcal{S}$ intersecting M, including the line M itself. The incidence structure $\mathcal{S}_p = (\mathcal{P}_p,\mathcal{L}_p,\mathrm{I}_p)$ with $\mathcal{P}_p = \mathcal{P} \setminus \mathcal{P}(M)$, $\mathcal{L}_p = \mathcal{L} \setminus \mathcal{L}(M)$, and with I_p=I $\cap((\mathcal{P}_p \times \mathcal{L}_p) \cup (\mathcal{L}_p \times \mathcal{P}_p))$ is clearly a $(0,1)$-geometry of order $(s,t-1)$. If the generalized quadrangle $\mathcal{S}$ is fully embeddable in a projective space, then clearly also the $(0,1)$-geometry will be fully embeddable in this projective space.

A subclass of the $(0,1)$-geometries, the so-called partial quadrangles, have been studied by P. J. Cameron in [10]. Partial quadrangles are defined to be $(0,1)$-geometries with the property that for every two distinct points x and y of the partial quadrangle, there are exactly μ points collinear with both x and y. Hence their point graph is a strongly regular graph.

Also the classification of fully embedded partial quadrangles is an open, but very complicated problem. Recently, F. De Clerck, N. Durante and J. A. Thas have studied the embedding of dual partial quadrangles in a three dimensional projective space (see [18]).

A $(0, q+1)$-geometry $\mathcal{S}$ can never be connected. Indeed, let p be a point of $\mathcal{S}$ and L be a line of $\mathcal{S}$ such that $i(p, L) = 0$. For every line L' intersecting L, it also follows that $i(p, L) = 0$, as p is not collinear with the intersection point of L and L', so $i(p, L') \neq q+1$. Now let $\mathcal{P}'$ be the set of the points of L union all the points that lie on a line of $\mathcal{S}$ that intersects L. Let $\mathcal{L}'$ be the set of all lines of $\mathcal{S}$ that contain $q+1$ points of $\mathcal{P}'$. Let p_1 and p_2 be two points of $\mathcal{P}'$. If p_1 or p_2 lies on L, then by the definition of $\mathcal{P}'$ it follows that $\langle p_1, p_2\rangle \in \mathcal{L}'$. If p_1 and p_2 both are not contained in L, then let u be a point of L. By the definition of $\mathcal{P}'$, it follows that the lines $\langle u, p_1\rangle$ and $\langle u, p_2\rangle$ belong to $\mathcal{L}'$. Since p_1 is collinear with at least one point (namely u) of the line $\langle u, p_2\rangle$, we get that $i(p_1, \langle u, p_2\rangle) = q+1$ and hence that $\langle p_1, p_2\rangle$ is a line of $\mathcal{S}$. Moreover, since u is collinear with p_1, it follows that every point of the line $\langle p_1, p_2\rangle$ is collinear with the point $u \in L$, and hence all points of $\langle p_1, p_2\rangle$ belong to $\mathcal{P}'$. This implies that $\langle p_1, p_2\rangle \in \mathcal{L}'$. So we proved have that every two points of $\mathcal{P}'$ lie on a line of $\mathcal{L}'$. It follows that $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$, with I' the restriction of I to the points and lines of $\mathcal{S}'$, is a partial geometry $\mathrm{pg}(q, t, q+1)$. Hence every $(0, q+1)$-geometry consists of a disjoint union of a number of partial geometries $\mathrm{pg}(q, t, q+1)$. Since we do not study disconnected (α, β)-geometries, a $(0, q+1)$-geometry would be equal to a $\mathrm{pg}(q, t, q+1)$, and hence not a proper (α, β)-geometry. So this case is not very important for us.

3.3 $(q, q+1)$-geometries that are fully embeddable in PG(n, q)

In chapter 2 of this thesis, a complete classification of $(q, q+1)$-geometries fully embedded in PG(n, q), is obtained. We will repeat our two classification theorems here. The notation $\Pi[m]$ is used for an m-dimensional subspace of PG(n, q).

Theorem 3.3.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(q, q+1)$-geometry fully embedded in PG(n, q). Assume that every plane of PG(n, q) that contains an antiflag of $\mathcal{S}$ is a q-plane or a $(q+1)$-plane. Then $\mathcal{S}$ is the geometry $\mathrm{H}_q^{n,m}$ with point set $\mathcal{P}$ the set of points of $\mathrm{PG}(n, q) \setminus \Pi[m]$, for some $0 \leq m < n-2$, and line set $\mathcal{L}$ the set of the lines of PG(n, q) that are disjoint from $\Pi[m]$.*

Theorem 3.3.2 *Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a $(q,q+1)$-geometry fully embedded in $\mathrm{PG}(n,q)$, such that there is at least one mixed plane contained in $\mathrm{PG}(n,q)$. Then $\mathcal{S}$ is the geometry $\mathrm{SH}_q^{n,m}$ with point set $\mathcal{P}$ the set of points of $\mathrm{PG}(n,q)\setminus\Pi[m]$, with $-1\le m\le n-4$. Moreover there exists a partition of the points of $\mathcal{S}$ in m'-dimensional subspaces of $\mathrm{PG}(n,q)$ through $\Pi[m]$, such that each element of the partition contains $\Pi[m]$, $m+2\le m'\le n-2$. The lines of $\mathrm{SH}_q^{n,m}$ are the lines that intersect $q+1$ of these m'-dimensional spaces in a point. A necessary and sufficient condition for this partition to exist is that $(m'-m)|(n-m')$.*

3.4 $(1,q)$-geometries $(q>2)$ fully embeddable in $\mathrm{PG}(n,q)$

In [11], we have studied the case $(\alpha,\beta)=(1,q)$. We have obtained a complete classification of $(1,q)$-geometries fully embedded in $\mathrm{PG}(n,q)$, for $q\neq 2$. In the next subsection a new construction of $(1,q)$-geometries is described using fully embedded generalized quadrangles. In the last subsection it is then shown that these are the only $(1,q)$-geometries fully embeddable in $\mathrm{PG}(n,q)$, for $q\neq 2$.

Let $\Pi[n-m-1]$ be an $(n-m-1)$-dimensional subspace of $\mathrm{PG}(n,q)$. We define $\Pi[n-m-1]\mathrm{GQ}$ to be the cone with vertex $\Pi[n-m-1]$, projecting a generalized quadrangle GQ fully embedded in an m-dimensional subspace of $\mathrm{PG}(n,q)$ skew to $\Pi[n-m-1]$ $(m=3,4,5)$. We will prove the following main theorem which completely classifies $(1,q)$-geometries fully embeddable in $\mathrm{PG}(n,q)$, $q\neq 2$.

Main theorem ([11]) *Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a $(1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q\neq 2$. Then the points of $\mathcal{S}$ are the points that lie on a cone $\Pi[n-m-1]\mathrm{GQ}$, $(m=3,4,5)$, but that are not contained in the vertex $\Pi[n-m-1]$. The lines of $\mathcal{S}$ are the lines that contain $q+1$ points of $\mathcal{S}$.*

3.4.1 Construction of a $(1,q)$-geometry, $q\neq 2$, fully embedded in $\mathrm{PG}(n,q)$

In this subsection we construct a $(1,q)$-geometry $\mathcal{S}$, that is fully embedded in $\mathrm{PG}(n,q)$, $q>2$.

Theorem 3.4.1 *Let $\mathcal{P}$ be the set of points of $\mathrm{PG}(n,q)$, with $q>2$, that lie on a cone $\Pi[n-m-1]\mathrm{GQ}$, $(m=3,4,5)$, but that are not contained in*

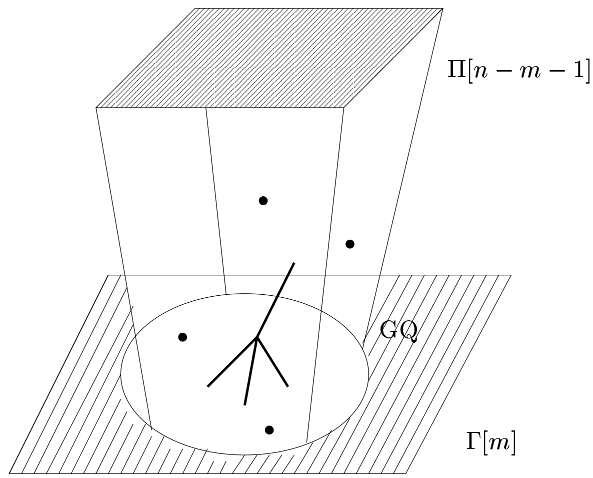

Figure 3.1: The cone $\Pi[n-m-1]\mathrm{GQ}$ $(m = 3, 4, 5)$

$\Pi[n-m-1]$. *Let* $\mathcal{L}$ *be the set of lines of* $\mathrm{PG}(n, q)$ *that lie on this cone and that contain* $q+1$ *points of* $\mathcal{S}$. *Let* I *be the incidence of* $\mathrm{PG}(n, q)$. *Then* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *is a* $(1, q)$*-geometry fully embedded in* $\mathrm{PG}(n, q)$.

Proof. Let $\Gamma[m]$ be an m-dimensional subspace of $\mathrm{PG}(n, q)$ that is skew to $\Pi[n-m-1]$. Then $\Gamma[m]$ intersects $\mathcal{S}$ in a generalized quadrangle.

From the definition of $\mathcal{S}$, it follows immediately that each line of $\mathcal{S}$ contains $q+1$ points of $\mathcal{S}$. It follows also that the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ is a constant. Indeed, since the points of the vertex $\Pi[n-m-1]$ of the cone $\Pi[n-m-1]\mathrm{GQ}$ do not belong to $\mathcal{S}$, it is clear that every point of $\mathcal{S}$ plays the same role. The number $t+1$ of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ can easily be counted as follows. Let u be a point of $\mathcal{S} \cap \Gamma[m]$. For each line M_u of $\mathcal{S}$ through u, M_u not contained in $\Gamma[m]$, there exists exactly one line L through u in $\Gamma[m]$, such that the plane $\langle L, M_u \rangle$ contains a point of the vertex $\Pi[n-m-1]$. Indeed, $\langle M_u, \Gamma[m] \rangle$ is an $m+1$-dimensional projective space, and since $\Gamma[m]$ is chosen to be skew to $\Pi[n-m-1]$, $\langle M_u, \Gamma[m] \rangle$ intersects $\Pi[n-m-1]$ in exactly one point x. By the definition of $\mathcal{S}$, the plane $\langle x, M_u \rangle$ intersects $\Gamma[m]$ in a line L of $\mathcal{S}$. Now assume that there exists a second line L' of $\mathcal{S}$ in $\Gamma[m]$ through u, such that the plane $\langle L', M_u \rangle$ intersects $\Pi[n-m-1]$ in a point x'. It is clear that $x \neq x'$. The planes $\langle L, M_u \rangle$ and $\langle L', M_u \rangle$ span a three dimensional space, and hence the plane $\langle L, L' \rangle$ intersects the line $\langle x, x' \rangle$ in a point. This is a contradiction, since $\langle L, L' \rangle \subset \Gamma[m]$ and $\langle x, x' \rangle \subseteq \Pi[n-m-1]$, and by assumption $\Gamma[m]$

and $\Pi[n-m-1]$ are disjoint subspaces of $\mathrm{PG}(n,q)$. Hence every line of $\mathcal{S}$ through u belongs to exactly one plane of the form $\langle L, x\rangle$, where L is a line of $\mathcal{S}$ through u in $\Gamma[m]$ and x a point of the vertex $\Pi[n-m-1]$. Now let L_u be a line of $\mathcal{S}$ through u in $\Gamma[m]$. Every plane through L_u and a point of the vertex $\Pi[n-m-1]$ contains q lines of $\mathcal{S}$ through u. Since $|\Pi[n-m-1]| = (q^{n-m}-1)/(q-1)$, in this way we have found $(q^{n-m}-1)+1 = q^{n-m}$ lines of $\mathcal{S}$ through u. So if $\Gamma[m] \cap \mathcal{S}$ is a generalized quadrangle of order (q, t'), then there are $q^{n-m}(t'+1)$ lines of $\mathcal{S}$ through u. This proves that $t+1 = q^{n-m}(t'+1)$. Now let p be a point of $\mathcal{S}$ and let L be a line of $\mathcal{S}$, $p \notin L$. The plane $\langle p, L\rangle$ intersects $\Pi[n-m-1]$ in a point or it is skew to $\Pi[n-m-1]$. In the first case, it follows from the definition of $\mathcal{S}$ that $\langle p, L\rangle$ is a q-plane and hence $i(p,L) = q$. In the second case, $\langle p, L\rangle$ is contained in an m-dimensional subspace of $\mathrm{PG}(n,q)$ intersecting $\mathcal{S}$ in a generalized quadrangle GQ. So $i(p,L) = 1$. Hence for every point p of $\mathcal{S}$ and every line L of $\mathcal{S}$ either $i(p,L) = 1$ or $i(p,L) = q$, and both cases occur. This proves that $\mathcal{S}$ is a $(1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$. □

Remark. This construction also holds for $q = 2$. However, in the case $q = 2$, not every plane containing an antiflag of $\mathcal{S}$ has to be a q-plane or a degenerate plane. Indeed, when $q = 2$, a plane containing an antiflag of $\mathcal{S}$ can intersect $\mathcal{S}$ in the points of 3 lines of $\mathcal{S}$ that are not concurrent. There do exist $(1,2)$-geometries fully embedded in $\mathrm{PG}(n,2)$ different from the one described in the theorem above. For example let $\mathcal{P}$ be the set of points of $\mathrm{PG}(n,2)$ not contained in an $(n-2)$-dimensional subspace of $\mathrm{PG}(n,2)$, and let $\mathcal{L}$ be the set of lines of $\mathrm{PG}(n,2)$ containing 3 points of $\mathcal{S}$, that are not contained in a spread Σ of the points of $\mathcal{S}$ in $\mathrm{PG}(n,2)$. Then $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a $(1,2)$-geometry fully embedded in $\mathrm{PG}(n,2)$. In fact, in the case $q = 2$, a $(1,q)$-geometry is the same as an $(q-1,q)$-geometry, which we have studied in section 2.5.

3.4.2 The intersection of a $(1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, $q \neq 2$, with a plane

To classify $(1,q)$-geometries fully embeddable in $\mathrm{PG}(n,q)$, $q \neq 2$, we first need to consider how a plane of $\mathrm{PG}(n,q)$, that contains an antiflag of $\mathcal{S}$, can intersect $\mathcal{S}$. Therefore let π be a plane of $\mathrm{PG}(n,q)$ that contains an antiflag of $\mathcal{S}$. Note that if π contains 3 lines of $\mathcal{S}$ that are not concurrent, then every point of $\mathcal{S}$ in π must be on either 1 or q lines of $\mathcal{S}$ in π.

Assume first that π contains at least two distinct points p_1 and p_2 through which there are q lines of $\mathcal{S}$ in π. We may assume that p_1 and p_2 are collinear in $\mathcal{S}$. Indeed, if p_1 and p_2 are not collinear then let p_3 be a point of π such that the lines $\langle p_1,p_3\rangle$ and $\langle p_2,p_3\rangle$ are lines of $\mathcal{S}$. Then clearly $p_3\notin\langle p_1,p_2\rangle$. Since there are at least two distinct lines of $\mathcal{S}$ through p_3 in π and since $\mathcal{S}$ is a $(1,q)$-geometry, there have to be q lines of $\mathcal{S}$ through p_3 in π. So in this case p_1 and p_3 are two collinear points of $\mathcal{S}$ in π through which there are q lines of $\mathcal{S}$ in π. Hence we may assume that p_1 is collinear with p_2 in $\mathcal{S}$. Let M_i $(i=1,2)$ be the line in π through p_i that does not belong to $\mathcal{S}$. Let x be the intersection point of M_1 and M_2 in π. As every point of π different from x lies on a line of $\mathcal{S}$, we may conclude that all the points of $\pi\setminus\{x\}$ belong to $\mathcal{S}$.

Suppose first that $x\notin\mathcal{S}$. We will show that in this case the points and lines of $\mathcal{S}$ in π are the points and lines of a partial geometry $\mathrm{pg}(q,q-1,q)$. We count the number of lines of $\mathcal{S}$ in π (which we denote by b_π) in two different ways. Let c be the number of points of M_2 through which there is exactly one line of $\mathcal{S}$ in π. Then, counting the number of lines of $\mathcal{S}$ in π intersecting M_2, we get that

$$b_\pi=c+(q-c)q=q^2+c(1-q). \tag{3.1}$$

Now let L be a line of $\mathcal{S}$ in π through p_1 different from $\langle p_1,p_2\rangle$. Let y be the intersection point of L and M_2. Every point of $L\setminus\{y\}$ is incident with at least two lines of $\mathcal{S}$ in π, and hence with q lines of $\mathcal{S}$ in π. Counting the number of lines of $\mathcal{S}$ in π that intersect L, we get that

$$b_\pi=q(q-1)+1=q^2-q+1 \qquad \text{or} \qquad b_\pi=q(q-1)+q=q^2,$$

depending on whether there are 1 or q lines of $\mathcal{S}$ through y in π. In the first case, from (3.1) it follows that $c=1$. This means that there is exactly one point (namely y) on M_2 through which there is one line of $\mathcal{S}$ in π. By assumption $q>2$, hence there is a line L' of $\mathcal{S}$ in π through p_1 different from L and $\langle p_1,p_2\rangle$. The line L' intersects M_2 in a point y' different from y, so through y' there are q lines of $\mathcal{S}$ in π. Counting the number of lines of $\mathcal{S}$ in π intersecting L', we get that $b_\pi=(q+1)(q-1)+1=q^2$, a contradiction. Hence we are left with the second case $b_\pi=q^2$ and from (3.1) it follows that in this case $c=0$. Hence through every point of $M_2\setminus\{x\}$ there are q lines of $\mathcal{S}$ in π. It follows immediately that through every point of $\mathcal{S}$ in π there are q lines of $\mathcal{S}$ in π. Hence the points and lines of $\mathcal{S}$ in π are the points and lines of a partial geometry $\mathrm{pg}(q,q-1,q)$.

Suppose next that $x\in\mathcal{S}$. We will obtain a contradiction. Let c be the number of points of M_2 through which there is exactly one line of $\mathcal{S}$ in π. Then, counting the number of lines of $\mathcal{S}$ in π intersecting M_2, we get that

$$b_\pi=c+(q+1-c)q=q^2+q+c(1-q). \tag{3.2}$$

Let L be a line of $\mathcal{S}$ in π through p_1 different from $\langle p_1, p_2\rangle$, and let y be the intersection point of L with M_2. Then, counting the lines of $\mathcal{S}$ in π intersecting L we get that $b_\pi = q^2 - q + 1$ or $b_\pi = q^2$, depending on the number of lines of $\mathcal{S}$ through y in π. Using (3.2), we get in the first case that $c = 2 + 1/(q-1)$ and in the second case $c = q/(q-1)$. Since $q \neq 2$, in both cases $c \notin \mathbb{N}$, a contradiction.

So if π contains at least two distinct points through which there are q lines of $\mathcal{S}$, then the points and lines of $\mathcal{S}$ in π form a partial geometry $\mathrm{pg}(q, q-1, q)$.

Now assume that π contains at most one point through which there are q lines of $\mathcal{S}$ in π. By assumption π contains an antiflag of $\mathcal{S}$, hence π contains two intersecting lines L_1 and L_2 of $\mathcal{S}$. Let $\{p\} = L_1 \cap L_2$. If there were a line L_3 of $\mathcal{S}$ in π not through p, then through p there would be q lines of $\mathcal{S}$ in π. Also, through the intersection point of L_3 and L_1 there would be q lines of $\mathcal{S}$ in π. So π contains two distinct points through which there are q lines of $\mathcal{S}$ in π, a contradiction with our assumption. Hence all lines of $\mathcal{S}$ in π contain p, i.e. the lines of $\mathcal{S}$ in π form a subset of a pencil of lines.

In the previous paragraphs, we have proved that there are two possible ways of intersection of $\mathcal{S}$ with a plane containing an antiflag of $\mathcal{S}$. Either $\mathcal{S}$ intersects a plane π of $\mathrm{PG}(n,q)$, π containing an antiflag of $\mathcal{S}$, in a partial geometry $\mathrm{pg}(q, q-1, q)$, in which case we call π a *q-plane* with *nucleus* the unique point in π that is not a point of $\mathcal{S}$, or $\mathcal{S}$ intersects π in a subset of a pencil of lines, and then we call π a *degenerate plane* with *center* the intersection point of the lines of $\mathcal{S}$ in π. Note that a degenerate plane contains at least two different lines of $\mathcal{S}$, since it contains an antiflag of $\mathcal{S}$.

3.4.3 Classification of $(1,q)$-geometries, $q \neq 2$, fully embedded in $\mathrm{PG}(n,q)$

The remainder of this chapter is devoted to the classification of $(1,q)$-geometries fully embedded in $\mathrm{PG}(n,q)$, $q \neq 2$.

Lemma 3.4.2 *There exists no $(1,q)$-geometry that is fully embedded in $\mathrm{PG}(3,q)$, $q \neq 2$.*

Proof. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a $(1,q)$-geometry fully embedded in $\mathrm{PG}(3,q)$, for $q \neq 2$. We have proved in the previous section that every plane of $\mathrm{PG}(3,q)$ that contains an antiflag of $\mathcal{S}$, is either a q-plane or a degenerate plane. Since $\mathcal{S}$ is not a generalized quadrangle, $\mathrm{PG}(3,q)$ contains at least one q-plane and at least one degenerate plane. From theorem 2.1.6 it follows that if $\mathrm{PG}(3,q)$ contains a q-plane as well as a degenerate plane, then $\mathcal{S}$ cannot exist. This proves that there exists no $(1,q)$-geometry fully embeddable in $\mathrm{PG}(3,q)$, for $q \neq 2$. □

Lemma 3.4.3 *Let* $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ *be a* $(1,q)$*-geometry fully embedded in* $\mathrm{PG}(n,q)$*, for* $q \neq 2$*. Then* $\mathrm{PG}(n,q)$ *contains a* q*-plane and a degenerate plane that intersect in a line of* $\mathcal{S}$*.*

Proof. Let $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ be a $(1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, for $q \neq 2$. Since $\mathcal{S}$ is not a generalized quadrangle, it follows that $\mathrm{PG}(n,q)$ contains a q-plane ρ and a degenerate plane π. Assume that π and ρ do not intersect in a line of $\mathcal{S}$. Then a plane τ_1 through a line L of $\mathcal{S}$ in π and a point u of $\mathcal{S}$ in ρ, $u \notin \pi$, is either a degenerate plane or a q-plane. If τ_1 is a q-plane, then π and τ_1 are a degenerate plane and a q-plane intersecting in a line of $\mathcal{S}$. If τ_1 is a degenerate plane, then let τ_2 be a plane through a line of $\mathcal{S}$ through u in τ_1 and a line of $\mathcal{S}$ through u in ρ. The plane τ_2 contains an antiflag of $\mathcal{S}$, hence it is a q-plane or a degenerate plane intersecting both the q-plane ρ and the degenerate plane τ_1 in a line of $\mathcal{S}$. So either τ_1 and τ_2 or τ_2 and ρ are a degenerate plane and a q-plane intersecting in a line of $\mathcal{S}$. This proves that $\mathrm{PG}(n,q)$ contains a q-plane and a degenerate plane intersecting in a line of $\mathcal{S}$. □

Lemma 3.4.4 *Let* $\mathcal{S}$ *be a* $(1,q)$*-geometry fully embedded in* $\mathrm{PG}(n,q)$*, for* $q \neq 2$*. Let* π *be a degenerate plane consisting of* r *lines of* $\mathcal{S}$ *through its center* p *and let* ρ *be a* q*-plane with nucleus* x*. Assume that* ρ *and* π *intersect in a line* M *of* $\mathcal{S}$*. Then the points of* $\mathcal{S}$ *in* $\Delta = \langle \pi, \rho \rangle$ *are the points different from* x *in* r *planes through* $\langle x, p \rangle$*, while the lines of* $\mathcal{S}$ *in* Δ *are all lines not through* x *contained in these planes.*

Proof. Since p is the center of π, it follows that $p \in M$. It is clear that $r > 1$, since π contains an antiflag of $\mathcal{S}$. There are two cases to consider, namely $r = q$ and $r \neq q$.

Suppose first that $r \neq q$*.* We will prove that every line of $\mathcal{S}$ contains a point of the line $\langle x, p \rangle$. Let y be a point of $\mathcal{S}$ in ρ, $y \notin \langle x, p \rangle$. Let y' be the intersection point of $\langle x, y \rangle$ and M (if $y \in M$ then $y = y'$). Then all lines through y' in π different from M contain r points of $\mathcal{S}$. Hence the q planes through $\langle x, y \rangle$ different from ρ in the three dimensional space Δ contain a line on which there are q points of $\mathcal{S}$ and a line on which there are $r \neq q$ points of $\mathcal{S}$. These planes can clearly not be q-planes and neither degenerate planes. Hence they contain no antiflag of $\mathcal{S}$ and in particular no lines of $\mathcal{S}$.

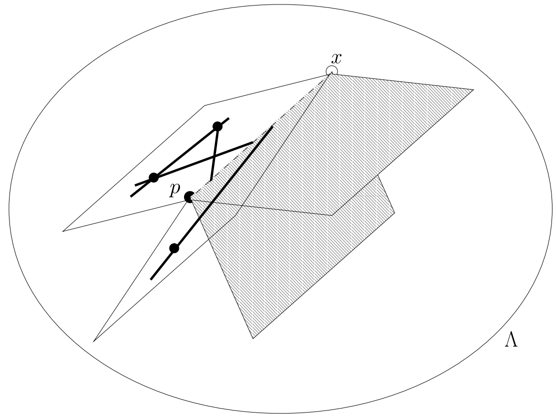

Figure 3.2: The intersection of $\mathcal{S}$ with Δ

This proves that all lines of $\mathcal{S}$ through y in Δ are contained in ρ and hence they intersect the line $\langle x,p\rangle$ in a point. As y was any point of $\mathcal{S}\cap\rho$ and as every line of $\mathcal{S}\cap\Delta$ has at least one point in common with ρ it follows that all lines of $\mathcal{S}$ intersect the line $\langle x,p\rangle$ in a point.

A plane through $\langle x,p\rangle$ in Δ intersects π in either a line of $\mathcal{S}$ or a line containing q points that do not belong to $\mathcal{S}$. The plane ρ is a q-plane through $\langle x,p\rangle$. A plane τ through $\langle x,p\rangle$ intersecting π in a line that does not belong to $\mathcal{S}$ can not contain an antiflag of $\mathcal{S}$. Indeed, it contains a line on which there are q points of $\mathcal{S}$ and a line on which there are q points which are not points of $\mathcal{S}$, but neither a q-plane nor a degenerate plane can contain both such lines. So τ contains no lines of $\mathcal{S}$. Further all points of $\mathcal{S}$ in τ are points of the line $\langle x,p\rangle$. Indeed, if there were a point w of $\mathcal{S}$ in τ, $w\notin\langle x,p\rangle$, then since all lines of $\mathcal{S}$ in Δ intersect $\langle x,p\rangle$, $i(w,L)$ would be 0 for a line L of $\mathcal{S}$ in ρ, a contradiction since $\mathcal{S}$ is a $(1,q)$-geometry. A plane τ' through $\langle x,p\rangle$ intersecting π in a line of $\mathcal{S}$ is clearly a q-plane. Indeed, if τ' were a degenerate plane, then through a point w' in $\tau'\setminus\langle x,p\rangle$, w' different from the center of τ', there would be exactly one line $L_{w'}$ of $\mathcal{S}$ in τ'. Let L be a line of $\mathcal{S}$ in ρ skew to $L_{w'}$. Then $i(w',L)=0$, a contradiction. So τ' is a q-plane. Hence the planes through $\langle x,p\rangle$ in Δ are either q-planes or planes that contain q^2+1 points that do not belong to $\mathcal{S}$. This proves that for $r\neq q$ the points of $\mathcal{S}$ in Δ are all points in r planes through $\langle x,p\rangle$, different from x, while the lines of $\mathcal{S}$ in Δ are all lines that do not contain x and are contained in these planes.

Suppose next that $r=q$. Let N be the line in π through p that does not belong to $\mathcal{S}$. We will prove that the points of $\mathcal{S}$ in the plane $\langle N,x\rangle$ are the points of $\langle x,p\rangle$ different from x. Clearly $\langle N,x\rangle$ contains no antiflag of $\mathcal{S}$ and hence no lines of $\mathcal{S}$. Assume that z is a point of $\mathcal{S}$ in $\langle N,x\rangle$, $z\notin\langle x,p\rangle$. The line $\langle z,x\rangle$ contains at least two points that do not belong to $\mathcal{S}$, namely x and the point of intersection of $\langle z,x\rangle$ with N. A plane through $\langle z,x\rangle$ and a line of $\mathcal{S}\cap\Delta$ through z intersects π and ρ both in a line containing q points of $\mathcal{S}$. So this plane is degenerate with center z and hence z is the only point of $\mathcal{S}$ on the line $\langle z,x\rangle$. It follows that on every line through x in $\langle N,x\rangle$ different from $\langle x,p\rangle$ there is at most one point of $\mathcal{S}$. Hence the plane $\langle N,x\rangle$ contains at most $2q$ points of $\mathcal{S}$. Now a line through z in $\langle N,x\rangle$ different from $\langle z,x\rangle$ and $\langle z,p\rangle$ contains q points of $\mathcal{S}$, since it is contained in a plane intersecting ρ in a line of $\mathcal{S}$ and π in a line containing q points of $\mathcal{S}$. So, counting the points of $\mathcal{S}$ in $\langle N,x\rangle$ on the lines through z we get that there are at least $(q-1)^2+1=q^2-2q+2$ points of $\mathcal{S}$ in $\langle N,x\rangle$. Assuming $q>3$, the inequality $q^2-2q+2\leq 2q$ gives a contradiction.

For $q=3$, let u, u', p and x be the points of $\langle x,p\rangle$, let w be the point of $\mathcal{S}$ on $\langle z,u\rangle$ and let w' be the point of $\mathcal{S}$ on $\langle z,u'\rangle$, $z\neq w\neq u$, $z\neq w'\neq u'$. Since there are at most $2q=6$ points of $\mathcal{S}$ in $\langle N,x\rangle$, the line $\langle z,p\rangle$ contains exactly two points of $\mathcal{S}$, i.e. z and p. Hence the line $\langle w,u'\rangle$ intersects $\langle z,p\rangle$ in a point that does not belong to $\mathcal{S}$. So $\langle w,u'\rangle$ is a line in $\langle N,x\rangle$ not through x or p that contains $q-1=2$ points of $\mathcal{S}$. This is a contradiction, since we already proved that every such line contains $q=3$ points of $\mathcal{S}$.

So we proved that $\langle N,x\rangle\setminus\langle x,p\rangle$ contains no points of $\mathcal{S}$. This implies that all lines of $\mathcal{S}\cap\Delta$ intersect the line $\langle x,p\rangle$ in a point. It immediately follows that all planes through $\langle x,p\rangle$ are q-planes, for otherwise they would contain a point w of $\mathcal{S}$ such that $i(w,L)=0$ for a line L of $\mathcal{S}$ in ρ. So also for $r=q$ we proved that the points of $\mathcal{S}\cap\Delta$ are the points different from x in r planes through $\langle x,p\rangle$, while the lines of $\mathcal{S}\cap\Delta$ are all lines not through x contained in these planes. □

Remark. We will denote the incidence structure described in the theorem above by $M^x(r)$, where $M=\langle x,p\rangle$.

Theorem 3.4.5 *Let* $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ *be a* $(1,q)$*-geometry fully embedded in* $\mathrm{PG}(n,q)$*, for* $q\neq 2$*. Let* Σ *be a four dimensional subspace of* $\mathrm{PG}(n,q)$ *that contains both a* q*-plane* ρ *and a degenerate plane* π*. Let* p *be the center of* π *and let* x *be the nucleus of* ρ*. Assume that* Σ *contains a line* R *of* $\mathcal{S}$ *that is skew to* $\langle x,p\rangle$*. Then the following cases can occur.*

1. *The points of* $\mathcal{S}$ *are the points of a cone* $\Pi[0]\mathrm{GQ}$*, different from* $\Pi[0]$*,*

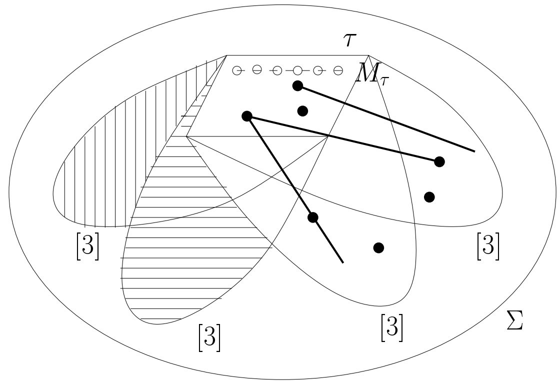

Figure 3.3: A possible intersection of $\mathcal{S}$ with Σ

while the lines of $\mathcal{S}$ are the lines that lie on this cone and that do not contain the vertex $\Pi[0]$.

2. *There is a plane τ that contains q^2 points of $\mathcal{S}$ and $q+1$ points that do not belong to $\mathcal{S}$ that lie on a line M_τ. The points of $\mathcal{S}$ are all points of some three dimensional spaces through τ, that do not lie on M_τ, while the lines of $\mathcal{S}$ are all lines in these three dimensional spaces that are skew to M_τ.*

Proof. From lemma 3.4.3 it follows that we may assume that π and ρ intersect in a line of $\mathcal{S}$. From lemma 3.4.4 it follows that the points of $\mathcal{S}$ in $\Delta=\langle\pi,\rho\rangle$ are the points of some planes through the line $\langle x,p\rangle$, while the lines of $\mathcal{S}$ in Δ are all lines not through x and contained in these planes. We denote the intersection point of R with Δ by u. By assumption $u\notin\langle x,p\rangle$. We may assume also that $u\notin\rho$, since Δ contains at least two q-planes through $\langle x,p\rangle$, so if R intersects ρ in a point, then we replace ρ by another q-plane through $\langle x,p\rangle$ in Δ.

First we will prove that every three dimensional space through ρ in Σ intersects $\mathcal{S}$ in either a partial geometry H_q^3 or in an incidence structure $M^x(r)$, with M a line of ρ that contains x. Let z be a point of R, $z\neq u$. If the three dimensional space $\langle z,\rho\rangle$ contains a degenerate plane, then from lemma 3.4.4 it follows that the points and lines of $\mathcal{S}$ in $\langle z,\rho\rangle$ define an incidence structure $M^x(r)$, where M is a line $\langle x,p'\rangle$, with p' a point of $\mathcal{S}$ in ρ. Assume now that $\langle z,\rho\rangle$ contains no degenerate plane. Then every plane in $\langle z,\rho\rangle$ that contains an antiflag (w,L_w) of $\mathcal{S}$ is a q-plane, and hence $i(w,L_w)=q$. Let w' be a point of $\mathcal{S}$ in $\langle z,\rho\rangle$. Then there is a q-plane in

$\langle z, \rho \rangle$ that does not contain w'. Indeed, if $w' \notin \rho$, then ρ is a q-plane not through w'. If $w' \in \rho$, then let L' be a line of $\mathcal{S}$ in ρ, $w' \notin L'$. The plane $\langle z, L' \rangle$ is a q-plane not through w'. So let τ be a q-plane in $\langle z, \rho \rangle$ not through w'. In τ, there are q^2 lines of $\mathcal{S}$. In the three dimensional space $\langle z, \rho \rangle$, every line through w' intersects τ in a point. Counting the lines of $\mathcal{S}$ through w' intersecting τ, we get that there are q^2 such lines. Since w' was an arbitrary chosen point of $\mathcal{S}$ in $\langle z, \rho \rangle$, it follows that through every point of $\mathcal{S}$ in $\langle z, \rho \rangle$ there are q^2 lines of $\mathcal{S}$ in $\langle z, \rho \rangle$. So the points and lines of $\mathcal{S}$ in $\langle z, \rho \rangle$ form a partial geometry $\mathrm{pg}(q, q^2 - 1, q)$. From [20] it follows that $\mathcal{S} \cap \langle z, \rho \rangle$ is the partial geometry H_q^3.

Let $\sigma_1, \ldots, \sigma_{q+1}$ be the three dimensional spaces through ρ in Σ, with $\sigma_1 = \langle \pi, \rho \rangle$. Now we distinguish two cases.

Case 1. Suppose that none of $\sigma_1, \ldots, \sigma_{q+1}$ intersects $\mathcal{S}$ in a partial geometry H_q^3. Then the incidence structure of points and lines of $\mathcal{S}$ in $\sigma_1, \ldots, \sigma_{q+1}$ is of the form ${M_i}^x(r_i)$, where x is the nucleus of ρ. We will call M_i (which is a line of ρ but not of $\mathcal{S}$) the *nuclear line* of σ_i.

We will prove now that each σ_i, $(i = 1, \ldots, q+1)$, has a distinct nuclear line. Assume therefore that some σ_i and σ_j both have $\langle x, v \rangle$ as nuclear line, with $v \in \rho \setminus \{x\}$. Let w be a point of ρ not on $\langle x, v \rangle$. We look at the plane $\langle w, R \rangle$. There are at least two lines through w in $\langle w, R \rangle$ that do not belong to $\mathcal{S}$, namely the intersection lines of $\langle w, R \rangle$ with σ_i respectively σ_j (since w does not belong to the nuclear line of σ_i and σ_j). Since $\mathcal{S}$ is a $(1, q)$-geometry, it follows that w is collinear with exactly one point of R. This implies that $\langle w, x \rangle$ is the nuclear line for exactly one σ_k, $i \neq k \neq j$. Since w was an arbitrary point of ρ not on $\langle x, v \rangle$, it follows that the q lines through x in ρ different from $\langle x, v \rangle$ are each of them nucleus for one of the three dimensional spaces $\sigma_1, \ldots, \sigma_{q+1}$. Since each σ_i $(i = 1, \ldots, q+1)$ has exactly one nuclear line, it follows that each line through x in ρ is the nuclear line of exactly one of the σ_i $(i = 1, \ldots, s+1)$. This is a contradiction with the assumption that σ_i and σ_j have the same nuclear line $\langle x, v \rangle$. It follows that each σ_i, $(i = 1, \ldots, q+1)$ has a distinct nuclear line through x in ρ.

Next we prove that for the incidence structures ${M_i}^x(r_i)$ in σ_i $(i = 1, \ldots, q+1)$ the parameter r_i is a constant r, i.e. for each σ_i the points and lines of $\mathcal{S}$ in this space lie in r q-planes through the nuclear line M_i. Let p_i be any point of $\mathcal{S}$ on M_i $(i = 1, \ldots, q+1)$. Consider the plane $\langle p_1, R \rangle$. It intersects σ_1 in a line of $\mathcal{S}$, while it intersects σ_i, $(i = 2, \ldots, q+1)$, in a line containing $q + 1 - r_i$ points that do not belong to $\mathcal{S}$. It follows that $\langle p_1, R \rangle$ is a degenerate plane. Hence $q + 1 - r_i = q + 1 - r$, for $i = 2, \ldots, q+1$ and r a constant. Considering the plane $\langle p_2, R \rangle$, in the same way as above it follows that $r_1 = r$. So in each σ_i there are r q-planes through the nuclear line $L_i = \langle p_i, x\rangle$, for $i = 1,\dots,q+1$.

Now let N be a line of $\mathcal{S}$ in ρ. Let $\Upsilon_N[3]$ be a three dimensional subspace of Σ containing N but not containing x. Then $\Upsilon_N[3]$ intersects σ_i, for $i = 1,\dots,q+1$ in a plane τ_i through N. The line N intersects the nuclear line M_i ($i = 1,\dots,q+1$) in one point. So in every plane τ_i all lines of $\mathcal{S}$ are concurrent. Since τ_i is a plane of σ_i, the lines of $\mathcal{S}$ in τ_i are r lines through the point $M_i \cap N$. Hence every τ_i is a degenerate plane. This implies that in $\Upsilon_N[3]$ there are $q+1$ degenerate planes through the line N of $\mathcal{S}$. From lemma 3.4.4 it follows that $\Upsilon_N[3]$ contains no q-planes. Hence $i(w,L) = 1$ for every antiflag (w,L) of $\mathcal{S}$ in $\Upsilon_N[3]$. Now we count the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ contained in $\Upsilon_N[3]$. If $z \in N$, then $z \in M_i$ for exactly one $i \in \{1,\dots,q+1\}$. So z is the center of exactly one of the degenerate planes through N, namely the plane τ_i. Hence there are r lines of $\mathcal{S}$ through z in $\Upsilon_N[3]$. If $z' \in \Upsilon_N[3] \setminus N$, then $z' \notin \rho$. So $z' \in \tau_j$ for exactly one $j \in \{1,\dots,q+1\}$. As $\Upsilon_N[3]$ contains no q-planes, z' is collinear with exactly one point of each of the $r-1$ lines of $\mathcal{S}$ different from N in τ_k, for $k \neq j$. In τ_j there is exactly one line of $\mathcal{S}$ through z'. So z' is incident with r lines of $\mathcal{S}$ in $\Upsilon_N[3]$. Hence the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ in $\Upsilon_N[3]$ is a constant r. It follows that $\mathcal{S}$ intersects $\Upsilon_N[3]$ in a generalized quadrangle of order $(q, r-1)$. Hence $\mathcal{S} \cap \Upsilon_N[3]$ is one of the following (see theorem 1.2.1): the points and lines of a hyperbolic quadric $Q^+(3,q)$, the points and lines of a Hermitian variety $\mathrm{H}(3,q)$ (in which case q is a square), or all the points and the totally isotropic lines of a symplectic polarity $W(3,q)$.

If L is a line of $\mathcal{S}$ in τ_i, for $i = 1,\dots,q+1$, then $\langle x, L\rangle$ is a q-plane. Let L' be a line of $\mathcal{S}$ in $\Upsilon_N[3]$, L' skew to N. Let $z_1,\dots,z_{q+1}$ be the points of L', with $z_i \in \tau_i$. Then $\langle M_i, z_i\rangle$ is a q-plane, for $i = 1,\dots,q+1$. In particular, the line $\langle x, z_i\rangle$ contains q points of $\mathcal{S}$. Hence the plane $\langle L', x\rangle$ contains exactly one point that does not belong to $\mathcal{S}$, namely the point x. So $\langle L', x\rangle$ is a q-plane. This proves that all the points of the cone with vertex x, projecting a generalized quadrangle contained in $\Upsilon_N[3]$, different from x, are points of $\mathcal{S}$. Suppose that there were a point v of $\mathcal{S}$ in Σ that does not belong to this cone. Then the line $\langle x, v\rangle$ intersects $\Upsilon_N[3]$ in a point that does not belong to $\mathcal{S}$. However from the above we know that every line through x contains either 0 or q points of $\mathcal{S}$. This implies that v can not exist. Hence the points different from x of the cone with vertex x projecting a generalized quadrangle in $\Upsilon_N[3]$, are the only points of $\mathcal{S}$ in Σ. We proved that for every line L of the generalized quadrangle in $\Upsilon_N[3]$, $\langle x, L\rangle$ is a q-plane. Hence every line that does not contain x and that lies on the cone with vertex x projecting the generalized quadrangle in $\Upsilon_N[3]$, is a line of $\mathcal{S}$.

These are the only lines of $\mathcal{S}$, since all other lines of Σ contain points that do not belong to $\mathcal{S}$. This proves that the points of $\mathcal{S}$ in Σ are the points of $\Lambda \setminus \{x\}$, where Λ is the set of points of the cone with vertex x projecting a generalized quadrangle in a three dimensional space skew to x, while the lines of $\mathcal{S}$ in Σ are all lines that lie on this cone and that do not contain x.
Case 2. Suppose now that at least one of the three dimensional spaces $\sigma_1, \ldots, \sigma_{q+1}$ intersects $\mathcal{S}$ in a partial geometry H_q^3. Let $\sigma_1 = \langle \pi, \rho \rangle$ and assume that σ_{q+1} intersects $\mathcal{S}$ in a partial geometry H_q^3. Let r be the number of q-planes through $M_1 = \langle x, p \rangle$ in σ_1. Let $y_1, \ldots, y_q, x$ be the points of the line H, or in other words the points of σ_{q+1} that do not belong to $\mathcal{S}$.

Assume first that $r \neq q$. As before, let R be a line of $\mathcal{S}$ skew to ρ. Then the planes $\langle R, y_i \rangle$, $(i = 1, \ldots, q)$, intersect ρ in a point of the line $\langle x, p \rangle$. Indeed, assume that the plane $\langle R, y_i \rangle$ would intersect ρ in a point $w \notin \langle x, p \rangle$. The plane $\langle R, y_i \rangle$ contains an antiflag of $\mathcal{S}$. Since w does not belong to the nuclear line of σ_1, the plane $\langle R, y_i \rangle$ intersects σ_1 in a line that contains exactly r points of $\mathcal{S}$. On the other hand $\langle R, y_i \rangle$ intersects σ_{q+1} in a line containing q points of $\mathcal{S}$. This is a contradiction, since neither a q-plane nor a degenerate plane can contain both of these lines. This proves that the plane $\langle R, y_i \rangle$ has to intersect ρ in a point of $\langle x, p \rangle$.

It follows that every line of $\mathcal{S}$ in Σ intersects the plane $\langle p, H \rangle$ in a point. Indeed, if there were a line M' of $\mathcal{S}$ that intersects σ_{q+1} in a point u, $u \notin \langle p, H \rangle$, then, replacing R by M' in the previous paragraph, we would get a contradiction.

Now we prove that σ_{q+1} is the only three dimensional space through ρ in Σ that intersects $\mathcal{S}$ in a partial geometry H_q^3. Let w' be a point of ρ, $w' \notin \langle x, p \rangle$. The plane $\langle R, w' \rangle$ intersects σ_{q+1} in a line of $\mathcal{S}$, and σ_1 in a line containing exactly $r \neq q$ points of $\mathcal{S}$. Hence $\langle R, w' \rangle$ is a degenerate plane with center different from w'. This implies that there are q lines through w' in $\langle R, w' \rangle$ that contain $r \neq q$ points of $\mathcal{S}$. Hence each σ_i, $(i = 1, \ldots, q)$, contains a line on which there are $r \neq q$ points of $\mathcal{S}$. This implies that σ_i, $(i = 1, \ldots, q)$ does not intersect $\mathcal{S}$ in a partial geometry H_q^3. So the points and lines of $\mathcal{S}$ in σ_i, $(i = 1, \ldots, q)$, are the points and lines of an incidence structure ${M_i}^x(r)$. Since w was an arbitrarily chosen point of $\rho \setminus \langle x, p \rangle$, it follows that the nuclear line M_i of σ_i $(i = 1, \ldots, q)$ is the same line $\langle p, x \rangle$ for each $i \in \{1, \ldots, q\}$.

Now we look at the three dimensional spaces through $\langle p, H \rangle$ in Σ. One of them is σ_{q+1}, with $\sigma_{q+1} \cap \mathcal{S}$ being a partial geometry H_q^3. Let L be a line of $\mathcal{S}$ in σ_1, L not contained in ρ. Then L intersects the line $\langle p, x \rangle$ in a point. The plane $\langle L, x \rangle$ is a q-plane. The planes $\langle L, y_i \rangle$, for $i = 1, \ldots, q$, are all q-planes, as $\langle x, p \rangle$ is the nuclear line of $\sigma_1, \ldots, \sigma_q$. So the three dimensional space $\langle L,H\rangle$ contains $q+1$ q-planes through L. Hence the points of $\langle L,H\rangle$ that do not belong to $\mathcal{S}$ are the points of the line H. So $\langle L,H\rangle$ can not contain a degenerate plane. Hence every plane in $\langle L,H\rangle$ that contains an antiflag of $\mathcal{S}$ is a q-plane. It follows that $\langle L,H\rangle$ intersects $\mathcal{S}$ in a partial geometry H_q^3. Now let N be a line in σ_1 that intersects $\langle x,p\rangle$ in a point z, $z \neq x$, $N \notin \mathcal{S}$. Then N contains q points that do not belong to $\mathcal{S}$. The plane $\langle N,x\rangle$ contains q^2+1 points that do not belong to $\mathcal{S}$. The planes $\langle N,y_i\rangle$, for $i=1,\ldots,q$, contain the line N with q points that do not belong to $\mathcal{S}$, and a line through y_i in σ_{q+1} that contains q points of $\mathcal{S}$. So the planes $\langle N,y_i\rangle$, for $i=1,\ldots,q$, can not contain an antiflag of $\mathcal{S}$. Since $z \in \langle x,p\rangle$, every line through z not in σ_{q+1} either belongs to $\mathcal{S}$ or it contains q points that do not belong to $\mathcal{S}$. It follows that all points of $\mathcal{S}$ in $\langle N,y_i\rangle$ lie on the line $\langle y_i,z\rangle$, $(i=1,\ldots,q)$. This implies that all points of $\mathcal{S}$ in the three dimensional space $\langle N,H\rangle$ are contained in the plane $\langle p,H\rangle$. This proves that the points of $\mathcal{S}$ in Σ are all the points of some three dimensional spaces through $\langle p,H\rangle$, not contained in the line H, while the lines of $\mathcal{S}$ in Σ in are the lines contained in these three dimensional spaces and skew to the line H.

Assume next that $r=q$. Then we may assume that each σ_i, $(i=1,\ldots,q+1)$, that contains a degenerate plane, has q q-planes through its nuclear line, for otherwise we can apply the previous case with σ_i instead of σ_1. Let $R \cap \sigma_{q+1}$ be the point z. We will prove that H is contained in the plane $\langle x,p,z\rangle$. Assume therefore that $y_i \notin \langle x,p,z\rangle$. Let $\langle z,y_i\rangle \cap \rho$ be the point z'. Then the intersection lines of $\langle R,z'\rangle$ with σ_1 and with σ_{q+1} each contain one point that does not belong to $\mathcal{S}$. It follows that $i(z',R)=1$, so $\langle z',R\rangle$ is a degenerate plane, with center different from z. Since $r=q$, $\langle z',R\rangle$ contains exactly q collinear points that do not belong to $\mathcal{S}$. Now let M_z be a line through z in $\langle z',R\rangle$ that contains exactly one point x' that does not belong to $\mathcal{S}$, with $x' \notin \sigma_1$ and $M_z \neq \langle z,y_i\rangle$. We count the points of Σ that do not belong to $\mathcal{S}$ in the planes through the line M_z.

Every line through x contains 1 or $q+1$ points that do not belong to $\mathcal{S}$. Since M_z contains exactly q points of $\mathcal{S}$, the plane $\langle M_z,x\rangle$ contains $q+1$ points that do not belong to $\mathcal{S}$, namely the points of the line $\langle x,x'\rangle$.

In the planes $\langle M_z,w\rangle$, for $w \in \rho$, $w \notin \langle p,x\rangle$ and $y_i \notin \langle z,w\rangle$, $(i=1,\ldots,q)$, the line $\langle w,z\rangle$ is a line of $\mathcal{S}$, while $\langle M_z,w\rangle$ intersects σ_1 in a line containing one point that does not belong to $\mathcal{S}$. Hence $\langle M_z,w\rangle$ contains an antiflag of $\mathcal{S}$ and at least two points that do not belong to $\mathcal{S}$. This implies that $\langle M_z,w\rangle$ is a degenerate plane that contains q points that do not belong to $\mathcal{S}$. Now there are q^2-q possible choices for the point w, so in total we find $q(q-1)^2+1$ points that do not belong to $\mathcal{S}$.

The planes $\langle M_z, v\rangle$, with $v \in \rho$, $y_j \in \langle z, v\rangle$, $(j = 1, \ldots, q)$, are contained in the three dimensional space $\langle R, x, v\rangle$, for we have chosen M_z to be a line of the plane $\langle R, y_i\rangle$. The plane $\langle R, x\rangle$ is a q-plane, while $\langle R, v\rangle$ is a degenerate plane. So from lemma 3.4.4, it follows that the points and lines of $\mathcal{S}$ in the three dimensional space $\langle R, x, v\rangle$ are the points and lines of an incidence structure $\tilde{M}^x(r)$ with nuclear line $\tilde{M}$ being contained in the plane $\langle R, x\rangle$. The line M_z is clearly skew to the nuclear line $\tilde{M}$, since it contains exactly one point that does not belong to $\mathcal{S}$ and this point is different from x. Hence the planes $\langle M_z, v\rangle$, with $v \in \rho$, $y_i \in \langle z, v\rangle$, are all degenerate, and they contain exactly q points that do not belong to $\mathcal{S}$. Since there are q such planes, there are $q(q-1)+1$ points that do not belong to $\mathcal{S}$ in these planes.

The planes $\langle M_z, p'\rangle$, for $p' \in \langle x, p\rangle \setminus \{x\}$ intersect σ_1 in a line of $\mathcal{S}$. So they contain an antiflag of $\mathcal{S}$ and hence they are either degenerate planes containing q points that do not belong to $\mathcal{S}$ or they are q-planes. Assume that c of these planes are q-planes. Then in these planes there are exactly $(q-c)(q-1)+1$ points that do not belong to $\mathcal{S}$.

So the number of points of Σ that do not belong to $\mathcal{S}$ equals

$$1 + q + q(q-1)^2 + q(q-1) + (q-c)(q-1) = q^3 + 1 - c(q-1). \qquad (3.3)$$

Now we count the points that do not belong to $\mathcal{S}$ in another way: in σ_1 there are q^2+1 such points, in σ_{q+1} there are $q+1$ such points and in σ_i, for $1 \neq i \neq q+1$, there are q^2+1 or $q+1$ such points. Assume that $(c'+1)$ three dimensional spaces through ρ do not intersect $\mathcal{S}$ in a partial geometry. Then the number of points of Σ that do not belong to $\mathcal{S}$ equals

$$1 + q^2 + q + c'q^2 + (q-1-c')q = 2q^2 + 1 + c'(q^2 - q). \qquad (3.4)$$

From (3.3) and (3.4) it follows that $c = q^2 - c'q - q^2/(q-1)$. Since $c \in \mathbb{N}$, it follows that $(q-1) \mid q^2$. This implies that $q = 2$, a contradiction with our assumption. This proves that $y_i \in \langle x, p, z\rangle$, for $i = 1, \ldots, q$. It follows also that every line of $\mathcal{S}$ in Σ intersects the plane $\langle p, H\rangle$ in a point, and that each σ_i, for $i = 2, \ldots, q$, that contains a degenerate plane, has the line $\langle x, p\rangle$ as nuclear line. Indeed, if σ_j, for $j \in \{1, \ldots, q\}$, would have a nuclear line $\langle x, p'\rangle$, with $p' \notin \langle x, p\rangle$, then in the same way as above we can prove that $y_i \in \langle x, p', z\rangle$, for $i = 1, \ldots, q$. However we proved above that the line H belongs to the plane $\langle z, x, p\rangle$. So $p' \in \langle x, p\rangle$, a contradiction. Hence $\langle x, p\rangle$ is the nuclear line for each σ_i, $(i = 1, \ldots, q)$.

Now we prove that each σ_i, $(i = 2, \ldots, q)$, contains a degenerate plane. Let $w \in \sigma_{q+1} \setminus \rho$ be a point of $\mathcal{S}$, $w \notin \langle p, H\rangle$. Let L_w be a line of $\mathcal{S}$ through

w in σ_{q+1} and skew to $\langle x,p\rangle$. Let $L_w \cap \rho$ be the point v. Let N_w be a line through w and a point of $\mathcal{S}$ in $\sigma_1 \setminus \rho$. Then N_w does not belong to $\mathcal{S}$, since it does not meet $\langle p,H\rangle$. The plane $\langle L_w,N_w\rangle$ contains an antiflag of $\mathcal{S}$ and two lines that do not belong to $\mathcal{S}$ through the point $N_w \cap \sigma_1$ (the second line being $\langle N_w,L_w\rangle \cap \sigma_1$). Hence it is a degenerate plane with center v' different from the point v. So in $\langle L_w,N_w\rangle$ there is one line $M_{v'}$ through v' on which there are q points that do not belong to $\mathcal{S}$. Let $M_{v'} \cap N_w$ be the point x^*. Then x^* lies in a σ_k, for $k \in \{2,\dots,q\}$. Now the q lines through v in $\langle L_w,N_w\rangle$, different from L_w, contain a point of $M_{v'}$ that does not belong to $\mathcal{S}$. Hence for $i=1,\dots,q$, each line $\sigma_i \cap \langle L_w,N_w\rangle$ contains a point that does not belong to $\mathcal{S}$. Now there are q^2-q possible choices for the line L_w of $\mathcal{S}$ through w in σ_{q+1}, L_w skew to $\langle x,p\rangle$. Hence each σ_i, $(i=2,\dots,q)$, $i \neq k$, contains at least q^2-q points that do not belong to $\mathcal{S}$. Now for $q>2$, $q^2-q>q+1$, which implies that no σ_i, $(i=2,\dots,q)$, $i\neq k$, intersects $\mathcal{S}$ in a partial geometry H_q^3. To prove that σ_k contains a degenerate plane, we argue in the same way, replacing N_w by a line N'_w through w and a point of $\mathcal{S}$ in $\sigma_k \setminus \rho$.

In the same way as in the case $r \neq q$, it can be shown that every three dimensional space in Σ through the plane $\langle p,H\rangle$ intersects $\mathcal{S}$ either in a partial geometry H_q^3, or in the points of the plane $\langle p,H\rangle$ not on the line H. Hence the points of $\mathcal{S}$ are all points of a set of three dimensional spaces through $\langle p,H\rangle$, not contained in the line H, while the lines of $\mathcal{S}$ are all lines in these three dimensional spaces skew to H.

So we have proved that $\mathcal{S}$ intersects Σ as follows.

1. The points of $\mathcal{S}$ are the points of a cone $x\mathrm{GQ}$, different from x, with GQ a generalized quadrangle fully embedded in a three dimensional space not containing x, while the lines of $\mathcal{S}$ are the lines that lie on this cone and that do not contain the vertex x.

2. There is a plane τ that contains q^2 points of $\mathcal{S}$ and $q+1$ points that do not belong to $\mathcal{S}$, that lie on a line M_τ. The points of $\mathcal{S}$ are all points of a set of three dimensional spaces through τ, that do not lie on M_τ, while the lines of $\mathcal{S}$ are all lines in these three dimensional spaces that are skew to M_τ.

□

Corollary 3.4.6 *Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a $(1,q)$-geometry fully embedded in $\mathrm{PG}(4,q)$, for $q\neq 2$. Then the points of $\mathcal{S}$ are the points of a cone $\Pi[0]\mathrm{GQ}$, different from the vertex $\Pi[0]$. The lines of $\mathcal{S}$ are the lines that lie on this cone and contain $q+1$ points of $\mathcal{S}$.*

Proof. Since $\mathcal{S}$ is not a partial geometry, $\mathrm{PG}(4,q)$ contains both a degenerate plane π with center p and a q-plane ρ with nucleus x. From lemma 3.4.3 it follows that we may assume that π and ρ intersect in a line of $\mathcal{S}$. Then from lemma 3.4.4 we get that the points and lines of $\mathcal{S}$ in $\langle \pi,\rho\rangle$ are the points and lines of an incidence structure $M^x(r)$, with nuclear line $M=\langle p,x\rangle$. The points of $\mathcal{S}$ span $\mathrm{PG}(4,q)$, so let u be a point of $\mathcal{S}$ in $\mathrm{PG}(4,q)\setminus\langle\pi,\rho\rangle$. Since $\mathcal{S}$ is a $(1,q)$-geometry there is a line R of $\mathcal{S}$ through u intersecting $\langle\pi,\rho\rangle$ in a point w. We may assume that $w\notin\langle x,p\rangle$. Indeed, through the points of $\mathcal{S}$ in $\langle\pi,\rho\rangle\setminus\langle x,p\rangle$ there are less lines of $\mathcal{S}$ in $\langle\pi,\rho\rangle$ than through the points of $\langle x,p\rangle\setminus\{x\}$. Since $\mathcal{S}$ is a $(1,s)$-geometry, the number $t+1$ of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ is a constant. Hence we can choose R such that $w\notin\langle x,p\rangle$. Now we can apply theorem 3.4.5. The result of this corollary then immediately follows, again since the number $t+1$ of lines of $\mathcal{S}$ through a point of $\mathcal{S}$, has to be a constant. $\square$

Let $\mathcal{S}$ be a $(1,q)$-geometry fully embedded in $\mathrm{PG}(n,q)$, $q\neq 2$. Let $\Upsilon[d]$ be a d-dimensional subspace of $\mathrm{PG}(n,q)$, $4\leq d<n$. Assume that the incidence structure of points and lines of $\mathcal{S}$ in $\Upsilon[d]$ is one of the following two incidence structures.

1. Let $\Lambda[d-2]$ be a $(d-2)$-dimensional subspace of $\Upsilon[d]$ and let $\Psi[d-3]$ be a hyperplane of $\Lambda[d-2]$. The points of $\mathcal{S}$ in $\Upsilon[d]$ are all points not contained in $\Psi[d-3]$ of some $(d-1)$-dimensional spaces through $\Lambda[d-2]$ in $\Upsilon[d]$. The lines of $\mathcal{S}$ in $\Upsilon[d]$ are all lines in these $(d-1)$-dimensional spaces skew to $\Psi[d-3]$. We say that this incidence structure $\Upsilon[d]\cap\mathcal{S}$ is of type I, and call $\Psi[d-3]$ its *nuclear subspace*.

2. The points of $\mathcal{S}$ in $\Upsilon[d]$ are all points of a cone $\Omega[d-m-1]\mathrm{GQ}$, that are not contained in the vertex $\Omega[d-m-1]$. The lines of $\mathcal{S}$ are all lines that lie on this cone and that contain no point of $\Omega[d-m-1]$. In this case we say that $\Upsilon[d]\cap\mathcal{S}$ is of type II.

Lemma 3.4.7 *Let* $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ *be a* $(1,q)$*-geometry fully embedded in* $\mathrm{PG}(n,q)$*, for* $q>2$ *and* $n>4$*. Assume that for every* l*-dimensional subspace* $\Upsilon[l]$ *of* $\mathrm{PG}(n,q)$*, with* $4\leq l<n$*,* $\Upsilon[l]$ *containing a* q*-plane and a degenerate plane,* $\Upsilon[l]\cap\mathcal{S}$ *is of type* I *or of type* II*. Let* $\Gamma[l+1]$ *be an* $(l+1)$*-dimensional space containing* $\Upsilon[l]$ *and a point* u *of* $\mathcal{S}$*,* $u\notin\Upsilon[l]$*. Then* $\Gamma[l+1]\cap\mathcal{S}$ *is also of type* I *or of type* II*.*

Proof. Let $\Upsilon[l]$ be an l-dimensional subspace of $\mathrm{PG}(n,q)$ that contains both a degenerate plane and a q-plane. Let $u\in\mathcal{S}$, $u\notin\Upsilon[l]$. Let $\Gamma[l+1]=\langle\Upsilon[l],u\rangle$. Now we determine how $\mathcal{S}$ intersects $\Gamma[l+1]$.

Assume first that $\Upsilon[l] \cap \mathcal{S}$ *is of type* II. Then the points of $\mathcal{S}$ in $\Upsilon[l]$ are all the points of a cone $\Omega[l-m-1]$GQ, $(m = 3, 4, 5)$, that are not contained in the vertex $\Omega[l-m-1]$. The lines of $\mathcal{S}$ are all lines that lie on this cone and that contain no point of $\Omega[l-m-1]$. We consider first the case $m = 3$. If $\Gamma[l+1]$ contains a q-plane ρ' that intersects $\Upsilon[l]$ in a line M of $\mathcal{S}$, then $\Gamma[l+1]$ contains an l-dimensional subspace $\Upsilon'[l]$ for which $\Upsilon'[l] \cap \mathcal{S}$ is of type I. Indeed, let y be the point of ρ' that does not belong to $\mathcal{S}$. Let $x \in \Omega[l-4]$. Then $\langle x, \rho' \rangle$ is a three dimensional space that contains two q-planes $\langle x, M \rangle$ and ρ' that intersect in the line M of $\mathcal{S}$. From lemma 3.4.4, it follows that $\langle x, \rho' \rangle$ contains no degenerate planes. So $\langle x, \rho' \rangle$ intersects $\mathcal{S}$ in a partial geometry H_q^3. Hence the line $\langle x, y \rangle$ contains no point of $\mathcal{S}$. Since $x \in \Omega[l-4]$ was arbitrarily chosen, it follows that $\langle y, \Omega[l-4] \rangle$ contains no point of $\mathcal{S}$ and moreover all points of $\langle \rho', \Omega[l-4] \rangle$ that do not belong to $\mathcal{S}$ are contained in $\langle y, \Omega[l-4] \rangle$. So $\langle \rho', \Omega[l-4] \rangle$ contains no degenerate plane. Hence $\langle \rho', \Omega[l-4] \rangle$ intersects $\mathcal{S}$ in a partial geometry H_q^{l-1}. Now let $\Upsilon'[l]$ be an l-dimensional subspace of $\Gamma[l+1]$ that contains $\langle \rho', \Omega[l-4] \rangle$ and a line of the generalized quadrangle GQ intersecting $\langle \rho', \Omega[l-4] \rangle$ in one point. Then $\Upsilon'[l]$ contains a degenerate plane, since it intersects GQ in a plane. Hence $\Upsilon'[l] \cap \mathcal{S}$ is of type I or II. Since $\Upsilon'[l]$ contains an $(l-1)$-dimensional space intersecting $\mathcal{S}$ in a partial geometry, $\Upsilon'[l] \cap \mathcal{S}$ is of type I. This proves that $\Gamma[l+1]$ contains an l-dimensional subspace $\Upsilon'[l]$ for which $\Upsilon'[l] \cap \mathcal{S}$ is of type I. We will treat this case later. So we may assume for now that $\Gamma[l+1]$ does not contain a q-plane that intersects $\Upsilon[l]$ in a line of $\mathcal{S}$.

Let $\Pi[3]$ be a three dimensional subspace of $\Upsilon[l]$ skew to $\Omega[l-4]$. Then $\Pi[3] \cap \mathcal{S}$ is a generalized quadrangle. Let $u \in \mathcal{S}$ again be a point of $\Gamma[l+1] \setminus \Upsilon[l]$. Then $\langle u, \Pi[3] \rangle$ is a four dimensional space containing no s-planes (since by assumption no q-plane in $\Gamma[l+1]$ intersects $\Upsilon[l]$ in a line of $\mathcal{S}$ and since no line in $\Pi[3]$ contains exactly q points of $\mathcal{S}$). Hence for every antiflag (z, M) of $\mathcal{S}$ in $\langle u, \Pi[3] \rangle$, we have that $i(z, M) = 1$. It follows immediately that the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ in $\langle u, \Pi[3] \rangle$ is a constant. Hence $\mathcal{S}$ intersects $\langle u, \Pi[3] \rangle$ in a generalized quadrangle. Since u was an arbitrary point of $\Gamma[l+1] \cap \mathcal{S}$, every four dimensional subspace of $\Gamma[l+1]$ skew to $\Omega[l-4]$ intersects $\mathcal{S}$ in a generalized quadrangle.

Let M be a line of $\mathcal{S}$ in $\Gamma[l+1]$. Let $x \in \Omega[l-4]$. Assume that $\langle x, M \rangle$ is not a q-plane. Then clearly M does not belong to $\Upsilon[l]$. So $\langle x, M \rangle$ intersects $\Upsilon[l]$ in a line on which there are q points of $\mathcal{S}$. Hence $\langle x, M \rangle$ contains an antiflag of $\mathcal{S}$ and so it is a degenerate plane. It follows that in $\langle x, M \rangle$ there is exactly one line through x on which there are q points that do not belong to $\mathcal{S}$. Let N' be a line of $\langle x, M \rangle$ that contains a point that does not belong to $\mathcal{S}$, $x \notin N'$. We proved above that a four dimensional space through

N' skew to $\Omega[l-4]$ intersects $\mathcal{S}$ in a generalized quadrangle. However, in a generalized quadrangle no line contains exactly one point that does not belong to $\mathcal{S}$, a contradiction. Hence for every line M of $\mathcal{S}$ in $\Gamma[l+1]$ and every $x \in \Omega[l-4]$, the plane $\langle x, M\rangle$ is a q-plane. This proves that every point of the cone with vertex $\Omega[l-4]$, projecting a generalized quadrangle contained in a four dimensional space skew to $\Omega[l-4]$, not contained in $\Omega[l-4]$, belongs to $\mathcal{S}$. It follows that $\Gamma[l+1]\cap\mathcal{S}$ is of type II, with a four dimensional generalized quadrangle as base of the cone.

Consider next the case $m=4$. Assume first that $\Gamma[l+1]$ contains a q-plane ρ' that intersects $\Upsilon[l]$ in a line of $\mathcal{S}$. Then as in the previous case we can prove that $\langle\rho', \Omega[l-5]\rangle$ intersects $\mathcal{S}$ in a partial geometry H_q^{l-2}. Let $\Upsilon'[l]$ be an l-dimensional subspace of $\Gamma[l+1]$ containing $\langle\rho', \Omega[l-5]\rangle$ and a line of $\mathcal{S}$ in GQ, intersecting $\langle\rho', \Omega[l-5]\rangle$ in a point. Then $\Upsilon'[l]$ contains a degenerate plane and hence $\Upsilon'[l]\cap\mathcal{S}$ is of type I or II. Since $\Upsilon'[l]$ contains the $(l-2)$-dimensional space $\langle\rho', \Omega[l-5]\rangle$ intersecting $\mathcal{S}$ in a partial geometry H_q^{l-2}, it follows that $\Upsilon'[l]\cap\mathcal{S}$ is of type II with a three dimensional generalized quadrangle as base of the cone, or $\Upsilon'[l]\cap\mathcal{S}$ is of type I. The first case we dealt with above and the second case we will deal with later. If $\Gamma[l+1]$ contains no s-plane that intersects $\Upsilon[l]$ in a line of $\mathcal{S}$, then as in the previous paragraph, one proves that $\Gamma[l+1]\cap\mathcal{S}$ is of type II with a five dimensional generalized quadrangle as base of the cone.

In the case $m=5$, as in the previous cases, it follows that either $\Gamma[l+1]$ contains an l-dimensional subspace $\Upsilon'[l]$, such that $\Upsilon'[l]\cap\mathcal{S}$ is of type II with a four dimensional generalized quadrangle as base of the cone, or $\Upsilon'[l]\cap\mathcal{S}$ is of type I (the first case we dealt with above and the second case we will deal with later), or $\Gamma[l+1]\cap\mathcal{S}$ is of type II with a six dimensional quadrangle as base of the cone, a contradiction since there exists no generalized quadrangle that is fully embedded in a six dimensional projective space and that is not contained in a five dimensional subspace.

So if $\Upsilon[l]\cap\mathcal{S}$ is of type II, then $\Gamma[l+1]\cap\mathcal{S}$ is either also of type II, or $\Gamma[l+1]$ contains an l-dimensional subspace $\Upsilon'[l]$, for which $\Upsilon'[l]\cap\mathcal{S}$ is of type I.

Assume next that $\Upsilon[l]\cap\mathcal{S}$ *is of type* I. Then we proceed in the same way as we did in theorem 3.4.5. Let $T[l-1]$ be an $(l-1)$-dimensional subspace of $\Upsilon[l]$ that contains $\Lambda[l-2]$ and such that $T[l-1]\cap\mathcal{S}$ is a partial geometry H_q^{l-1}. Let $\sigma_1,\ldots,\sigma_{q+1}$ be the l-dimensional subspaces of $\Gamma[l+1]$ containing $T[l-1]$. Assume that $\sigma_1=\Upsilon[l]$. Then as in theorem 3.4.5, one can prove that for $i=1,\ldots,q+1$, σ_i intersects $\mathcal{S}$ either in a partial geometry H_q^l or that all points of $\mathcal{S}\cap\sigma_i$ are contained in some $(l-1)$-dimensional subspaces around $\langle p_i, \Psi[l-3]\rangle$, with $p_i \in T[l-1]\setminus\Psi[l-3]$, while the lines of $\mathcal{S}\cap\sigma_i$

are the lines contained in these subspaces that contain $q+1$ points of $\mathcal{S}$. In the last case, we call $\langle p_i, \Psi[l-3]\rangle$ the *nuclear subspace* of σ_i.

Case 1. Suppose that none of $\sigma_1, \ldots, \sigma_{q+1}$ intersects $\mathcal{S}$ in a partial geometry H_q^l. Then $\sigma_1, \ldots, \sigma_{q+1}$ intersect $\mathcal{S}$ in a number of $(l-1)$-dimensional spaces containing $\langle p_i, \Psi[l-3]\rangle$, for $p_i \in T[l-1] \setminus \Psi[l-3]$, which intersect $\mathcal{S}$ in a partial geometry H_q^{l-1}.

As in theorem 3.4.5 one can prove that $\sigma_1, \ldots, \sigma_{q+1}$ each have a different nuclear subspace $\langle p_i, \Psi[l-3]\rangle$, $(i = 1, \ldots, q)$, and that each σ_i has a constant number r of $(l-1)$-dimensional spaces through its nuclear subspace $\langle p_i, \Psi[l-3]\rangle$.

Now we count the number of lines of $\mathcal{S}$ through a point in $\Gamma[l+1]$. Let u_1 be a point of $\mathcal{S}$ in $T[l-1]$. Then $u_1 \in \langle p_i, \Psi[l-3]\rangle$ for just one $i \in \{1, \ldots, q+1\}$. So all lines of $\mathcal{S}$ in $\Gamma[l+1]$ through u_1 are contained in σ_i. Hence the number of lines of $\mathcal{S}$ through u_1 in $\Gamma[l+1]$ is rq^{l-2}. Let u_2 be a point of $\mathcal{S}$ not in $T[l-1]$. Then $u_2 \in \sigma_i$ for some $i \in \{1, \ldots, q+1\}$. Let L be a line of $\mathcal{S}$ in σ_j, for $j \neq i$, such that L does not belong to $T[l-1]$. The plane $\langle u_2, L\rangle$ intersects σ_k, for $k \neq j$, in a line containing $s+1-q$ points that do not belong to $\mathcal{S}$. Hence it is a degenerate plane and in particular $i(u_2, L) = 1$. Since L was an arbitrarily chosen line of $\sigma_j \setminus T[l-1]$, $i(u_2, L) = 1$ for each line L of $\mathcal{S}$ in $\sigma_j \setminus T[l-1]$. Now let c be the number of points of $\sigma_j \setminus T[l-1]$ collinear with u_2. We count in two different ways the flags (z, L_z), for $z \sim u_2$ and L_z a line of $\sigma_j \setminus T[l-1]$. We get that $q^{l-2}c = 1q^{2l-4}(r-1)$ or thus $c = q^{l-2}(r-1)$. In σ_i there are q^{l-2} lines of $\mathcal{S}$ through u_2. Hence in total there are rq^{l-2} lines of $\mathcal{S}$ through u_2 in $\Gamma[l+1]$. This proves that the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ in $\Gamma[l+1]$ is a constant. Hence the points and lines of $\mathcal{S}$ in $\Gamma[l+1]$ form a $(1, q)$-geometry in $\Gamma[l+1]$.

In the same way as in theorem 3.4.5, one proves that the $(1, q)$-geometry $\mathcal{S} \cap \Gamma[l+1]$ has points the points of a cone with vertex $\Psi[l-3]$, projecting a generalized quadrangle in a three dimensional subspace of $\Gamma[l+1]$ skew to $\Psi[l-3]$, not contained in $\Psi[l-3]$, and its lines are the lines on this cone that contain $q+1$ points of $\mathcal{S}$.

Case 2. Suppose that at least one σ_i, for $i = 1, \ldots, q+1$, intersects $\mathcal{S}$ in a partial geometry H_q^l. Let $\langle y, \Psi[l-3]\rangle$ be the $(l-2)$-dimensional subspace of points that do not belong to $\mathcal{S}$ in one such a σ_i. Then in the same way as in theorem 3.4.5, it follows that the points of $\mathcal{S}$ in $\Gamma[l+1]$ are the points of some l-dimensional spaces through $\langle \Lambda[l-2], y_1\rangle$ not contained in $\langle \Psi[l-3], y_1\rangle$, while the lines of $\mathcal{S}$ in $\Gamma[l+1]$ are the lines in these l-dimensional spaces that are skew to $\langle \Psi[l-3], y_1\rangle$.

So we proved that $\Gamma[l+1] \cap \mathcal{S}$ is also of type I or of type II. This proves the lemma. $\square$

Theorem 3.4.8 *Let* $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ *be a* $(1,q)$*-geometry fully embedded in* $\mathrm{PG}(n,q)$*, for* $q\neq 2$*. Then the points of* $\mathcal{S}$ *are the points of a cone* $\Pi[n-m-1]\mathrm{GQ}$*,* $(m=3,4,5)$*, that are not contained in the vertex* $\Pi[n-m-1]$*. The lines of* $\mathcal{S}$ *are the lines that lie on this cone and contain* $q+1$ *points of* $\mathcal{S}$*.*

Proof. From theorem 3.4.5 it follows that a four dimensional subspace Σ of $\mathrm{PG}(n,q)$, that contains both a q-plane and a degenerate plane, intersects $\mathcal{S}$ in one of the following.

1. The points of $\mathcal{S}$ are the points of a cone $x\mathrm{GQ}$, different from x, while the lines of $\mathcal{S}$ are the lines that lie on this cone and that do not contain the vertex.

2. There is a plane τ that contains q^2 points of $\mathcal{S}$ and $q+1$ points that do not belong to $\mathcal{S}$, that lie on on a line M_τ. The points of $\mathcal{S}$ are all points of some three dimensional spaces through τ, that do not lie on M_τ, while the lines of $\mathcal{S}$ are all lines in these three dimensional spaces that are skew to M_τ.

Applying lemma 3.4.7 a finite number of times we get that $\mathrm{PG}(n,q)$ intersects $\mathcal{S}$ in one of the following.

1. The points of $\mathcal{S}$ are the points of a cone $\Pi[n-m-1]\mathrm{GQ}$, not contained in $\Pi[n-m-1]$, while the lines of $\mathcal{S}$ are the lines that lie on this cone and contain $q+1$ points of $\mathcal{S}$.

2. There is an $(n-2)$-dimensional subspace $\Pi'[n-2]$ and the points of $\mathcal{S}$ in $\Pi'[n-2]$ are the points of the affine space $\Pi'[n-2]\setminus\Psi'[n-3]$, with $\Psi'[n-3]$ an $(n-3)$-dimensional subspace of $\Pi'[n-2]$. The points of $\mathcal{S}$ in $\mathrm{PG}(n,q)$ are all points of a set of $(n-1)$-dimensional spaces through $\Pi'[n-2]$, not contained in $\Psi'[n-3]$. The lines of $\mathcal{S}$ are all lines in these $(n-1)$-dimensional spaces, that are skew to $\Psi'[n-3]$.

The second case can not occur, since the number of lines of $\mathcal{S}$ through a point of $\mathcal{S}$ is not a constant there. This proves the theorem. $\square$

Chapter 4

Characterizations of some (α, β)-geometries

In the previous chapters we have discovered some classes of (α, β)-geometries that are fully embeddable in $\mathrm{PG}(n, q)$. For three of these classes of (α, β)-geometries we will give a characterization theorem in this chapter. One of them is the one we have denoted as $\mathrm{H}_q^{n,m}$. This $(q, q+1)$-geometry is closely related to the partial geometry H_q^n. A characterization of the $(q, q+1)$-geometry $\mathrm{SH}_q^{n,m}$ will follow out of the same theorem as the one that gives a characterization of $\mathrm{H}_q^{n,m}$. The third class is the $(\frac{q-1}{2}, \frac{q+1}{2})$-geometry $\mathrm{NQ}^+(3, q)$, q odd, that has point set the points of $\mathrm{PG}(3, q)$ that are not contained in a non-degenerate hyperbolic quadric $Q^+(3, q)$ of $\mathrm{PG}(3, q)$, and line set the lines that are exterior to this quadric $Q^+(3, q)$.

4.1 The $(q, q+1)$-geometries $\mathrm{H}_q^{n,m}$ and $\mathrm{SH}_q^{n,m}$

4.1.1 Description of $\mathrm{H}_q^{n,m}$ and $\mathrm{SH}_q^{n,m}$

In [13, 12] (and in chapter 2) we introduced the two (α, β)-geometries $\mathrm{H}_q^{n,m}$ and $\mathrm{SH}_q^{n,m}$, for $0 \leq m \leq n-2$. Both of them are fully embeddable in a projective space $\mathrm{PG}(n, q)$. The $(q, q+1)$-geometry $\mathrm{H}_q^{n,m}$ has points the points of $\mathrm{PG}(n, q)$ that are not contained in an m-dimensional subspace $\Pi[m]$ of $\mathrm{PG}(n, q)$, while its lines are the lines of $\mathrm{PG}(n, q)$ that are skew to $\Pi[m]$. Note that $\mathrm{H}_q^{n,m}$ will only be a proper (α, β)-geometry if $0 \leq m < n-2$, since for $m = -1$ and $m = n-2$ we get a partial geometry. The partial geometry $\mathrm{H}_q^{n,n-2}$ is usually denoted by H_q^n. The $(q, q+1)$-geometry $\mathrm{SH}_q^{n,m}$ has the same point set as $\mathrm{H}_q^{n,m}$, while its lines are the lines of $\mathrm{PG}(n, q)$

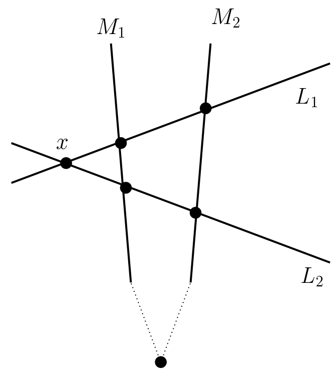

Figure 4.1: The axiom of Pasch

that are skew to $\Pi[m]$ and not contained in an element of a partition of the points of $\mathrm{PG}(n,q)\setminus\Pi[m]$ into m'-dimensional spaces that pairwise intersect in $\Pi[m]$, for $m+2\le m'\le n-1$.

4.1.2 A characterization of the partial geometry H_q^n

In [54], J. A. Thas and F. De Clerck gave a characterization of the partial geometry H_q^n. In this characterization of H_q^n, the axiom of Pasch appears, which is also known as the axiom of Veblen or the axiom of Veblen-Young. We will define the axiom of Pasch for any (α,β)-geometry, since we will need it later. So let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be an (α,β)-geometry. If the points $x,y\in\mathcal{P}$ are collinear in $\mathcal{S}$, then we write $x\sim y$. If the lines $L,M\in\mathcal{L}$ are concurrent in $\mathcal{S}$, then we write $L\sim M$. An (α,β)-geometry $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ satisfies *the axiom of Pasch* if

$$\forall L_1,L_2,M_1,M_2\in\mathcal{L}, L_1\neq L_2, L_1\ \mathrm{I}\ x\ \mathrm{I}\ L_2, x\notin M_1, x\notin M_2,\\ L_i\sim M_j \text{ for all } i,j\in\{1,2\} : M_1\sim M_2.$$

Note that for $\alpha=\beta=1$ and for $\alpha=\beta=t+1$, the axiom of Pasch is trivially satisfied.

In their characterization of H_q^n, J. A. Thas and F. De Clerck introduced the notion of a *regular* partial geometry. To define what is a regular partial geometry, first some other definitions are needed.

Let $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ be a partial geometry satisfying the axiom of Pasch, with $\alpha\notin\{1,t+1\}$. Let L and M, $L\neq M$, be two concurrent lines of $\mathcal{S}$ with intersection point x. Then the substructure $S(L,M)=(\mathcal{P}^*,\mathcal{L}^*,\mathrm{I}^*)$ of $\mathcal{S}$ is defined as follows: $\mathcal{L}^*$ is the set of the $s(\alpha - 1)$ lines N, such that $x \notin N$ and $L \sim N \sim M$, together with the set of the α lines through x that are concurrent with at least one of these $s(\alpha-1)$ lines; $\mathcal{P}^*$ is the set of points of $\mathcal{S}$ that lie on the lines of $\mathcal{L}^*$ and $\mathrm{I}^*{=}\mathrm{I} \cap((\mathcal{P}^* \times \mathcal{L}^*) \cup (\mathcal{L}^* \times \mathcal{P}^*))$. Since $\mathcal{S}$ satisfies the axiom of Pasch, it follows that $S(L, M) = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ is a $\mathrm{pg}(s, \alpha-1, \alpha)$. Note that for N_1, $N_2 \in \mathcal{L}^*$, $N_1 \neq N_2$, the substructures $S(N_1, N_2)$ and $S(L, M)$ coincide. Also note that for any pair (x, N), $x \in \mathcal{P}$, $N \in \mathcal{L}$, $x \notin N$, there is exactly one substructure $S(L, M)$ that contains both x and N. This substructure we will denote by $S(x, N)$.

We assume from now on that $\alpha \notin \{1, s+1, t+1\}$. Let x and y be two non-collinear points of $\mathcal{S}$. It follows from the previous paragraph that there are $(t+1)/\alpha$ subgeometries $S(L, M)$ of $\mathcal{S}$ that contain both x and y. We denote these subgeometries by $S_i^* = (\mathcal{P}_i^*, \mathcal{L}_i^*, \mathrm{I}_i^*)$, for $i = 1, \ldots, (t+1)/\alpha$. The *line of the second type* $\langle x, y\rangle$ is defined to be the set $\mathcal{P}_1^* \cap \ldots \cap \mathcal{P}_{(t+1)/\alpha}^*$. From the construction it follows that no two distinct points of the line $\langle x, y\rangle$ are collinear in $\mathcal{S}$. Indeed, suppose that z_1, $z_2 \in \langle x, y\rangle$, $z_1 \neq z_2$, are two collinear points of $\mathcal{S}$. The line L of $\mathcal{S}$ through z_1 and z_2 has to be an element of $\mathcal{L}_i^*$ for each $i \in \{1, \ldots, (t+1)/\alpha\}$. If $x \notin L$, then $S(x, L)$ is the only substructure containing x and L. However, x and L are contained in the $(t+1)/\alpha > 1$ different substructures $\mathcal{S}_i^*$, for $i = 1, \ldots, (t+1)/\alpha$. This is a contradiction. So $x \in L$. In the same way one proves that $y \in L$. Hence x is collinear with y in $\mathcal{S}$, a contradiction with the assumption. This proves that no two distinct points of $\langle x, y\rangle$ are collinear in $\mathcal{S}$. It follows immediately that for z_1, $z_2 \in \langle x, y\rangle$, the lines $\langle z_1, z_2\rangle$ and $\langle x, y\rangle$ coincide. Since $\langle x, y\rangle$ is a set of two by two non-collinear points of the partial geometry $\mathcal{S}_i^*$, $i \in \{1, \ldots, (t+1)/\alpha\}$, $|\langle x, y\rangle|$ is at most the number of points of an ovoid of $\mathcal{S}_i^*$. Hence $|\langle x, y\rangle| \leq s + 1 - s/\alpha$. If $|\langle x, y\rangle| = s + 1 - s/\alpha$ for all $x, y \in \mathcal{S}$, x not collinear with y, then the partial geometry $\mathcal{S}$ is called *regular*.

It is easy to check that H_q^n is a $\mathrm{pg}(q, q^{n-1} - 1, q)$. Since H_q^n is fully embedded in $\mathrm{PG}(n, q)$, it satisfies the axiom of Pasch. Moreover it is the only known partial geometry that satisfies the axiom of Pasch in a non-trivial way. J. A. Thas and F. De Clerck have proved the following theorem.

Theorem 4.1.1 ([54]) *The partial geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with parameters (s, t, α), such that $\alpha \neq 1$, $t+1$, $s+1$, is isomorphic to an H_q^n if and only if*

1. *$\mathcal{S}$ satisfies the axiom of Pasch;*
2. *$\mathcal{S}$ is regular;*
3. *$2s > s^4 - \alpha s^3 + \alpha^2 s^2 + \alpha^3 s - 2\alpha^4$.*

Remark. The third condition of the theorem is in fact a very strong condition. If $\alpha \neq s$, this condition is almost never satisfied.

4.1.3 (α, β)-geometries satisfying the axiom of Pasch

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper (α, β)-geometry of order (s, t), satisfying the axiom of Pasch, with $1 < \alpha < \beta < t + 1$. Let L and M be two distinct concurrent lines of $\mathcal{S}$, with $L \cap M = \{x\}$. Then a substructure $S(L, M) = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ of $\mathcal{S}$ can be defined exactly in the same way as is done for a partial geometry $\mathcal{S}$ in the previous paragraph. However, since $\mathcal{S}$ is an (α, β)-geometry, for every point p of $\mathcal{S}^*$ and every line L_p of $\mathcal{S}^*$, such that p is not incident with L_p, there are either α or β points on L_p that are collinear with p. Hence it is possible that $\mathcal{S}^*$ contains a point z_1 through which there are α lines of $\mathcal{S}^*$ and a point z_2 through which there are β lines of $\mathcal{S}^*$. If this is the case, then $\mathcal{S}^*$ is clearly not an (α, β)-geometry, as the number of lines through a point in an (α, β)-geometry has to be a constant. If the number of lines of $\mathcal{S}^*$ through a point of $\mathcal{S}^*$ is a constant, then it follows that $S(L, M)$ is either a $\mathrm{pg}(s, \alpha - 1, \alpha)$ or a $\mathrm{pg}(s, \beta - 1, \beta)$. This is easy to prove, using the assumption that $\mathcal{S}$ satisfies the axiom of Pasch. If the number of lines of $\mathcal{S}^*$ through a point of $\mathcal{S}^*$ is not a constant, then $S(L, M)$ is not a partial geometry. A substructure $S(L, M)$ that is a $\mathrm{pg}(s, \alpha - 1, \alpha)$ we call an *α-substructure* or an *α-subgeometry*, and a substructure $S(L, M)$ that is a $\mathrm{pg}(s, \beta - 1, \beta)$ we call a *β-substructure* or a *β-subgeometry*. A substructure $S(L, M)$ that is not a partial geometry, we call a *mixed substructure*. Note that also in this case for every two distinct elements N_1 and N_2 of $\mathcal{L}^*$, we have that $S(N_1, N_2) = S(L, M)$. Moreover for any pair (x, N), $x \in \mathcal{P}$, $N \in \mathcal{L}$, x not incident with N, there is exactly one substructure $S(L, M)$ containing x as a point and N as a line. This substructure will be denoted by $S(x, N)$. In the following, we will denote a substructure $S(L, M)$ sometimes as π, ρ or σ.

Since the number of lines through a point x in a substructure $S(L, M)$ through x can be either α or β, the number of substructures through two distinct non-collinear points of $\mathcal{S}$ is not necessarily a constant. In the following lemma we will prove that this number is a constant in the case that $\beta = s + 1$.

Lemma 4.1.2 *Let $\mathcal{S}$ be a proper $(\alpha, s+1)$-geometry of order (s, t), $\alpha \neq 1,\ t + 1,\ s + 1$, that satisfies the axiom of Pasch. Then the number of substructures $S(L, M)$ through two distinct non-collinear points of $\mathcal{S}$ is a constant.*

Proof. Assume that $\mathcal{S}$ is a proper $(\alpha, s+1)$-geometry of order (s,t), $\alpha \neq 1$, $t+1$, $s+1$, that satisfies the axiom of Pasch. Let x and y be distinct non-collinear points of $\mathcal{S}$. Let L be a line of $\mathcal{S}$ through x. Since x and y are not collinear in $\mathcal{S}$ and $\beta = s+1$, it follows that exactly α points of L are collinear with y. Hence, counting the points of $\mathcal{S}$ collinear with both x and y, we get that there are $\mu = (t+1)\alpha$ such points. So there are exactly $(t+1)/\alpha$ substructures $S(L,M)$ that contain both x and y. Since x and y were arbitrarily chosen distinct non-collinear points of $\mathcal{S}$, it follows that there are exactly $(t+1)/\alpha$ substructures $S(L,M)$ through every two distinct non-collinear points of $\mathcal{S}$. This proves the lemma. □

Let $\mathcal{S}$ be a proper $(\alpha, s+1)$-geometry of order (s,t), that satisfies the axiom of Pasch, and for which $1 < \alpha < s+1 < t+1$. From lemma 4.1.2 it follows that through every two distinct non-collinear points of $\mathcal{S}$ there is a constant number $c = (t+1)/\alpha$ of substructures $S(L,M)$. Now we define a *line of the second type* through two distinct non-collinear points x and y of $\mathcal{S}$ as the intersection of all substructures $S(L,M)$ containing both x and y. Note that there are at least two distinct substructures through x and y, since $t+1 > \alpha$. We denote the line of the second type through x and y by $\langle x,y\rangle$.

It immediately follows that any two distinct points of $\langle x,y\rangle$ are non-collinear in $\mathcal{S}$. Indeed, let $\mathcal{S}_i^* = (\mathcal{P}_i^*, \mathcal{L}_i^*, \mathrm{I}_i^*)$ $(i = 1,\ldots,c)$ be the substructures $S(L,M)$ of $\mathcal{S}$ containing x and y. Suppose that there would be two points z_1 and z_2 contained in $\langle x,y\rangle$, with $z_1 \neq z_2$ and $z_1 \sim z_2$. Let L be the line of $\mathcal{S}$ that is incident with z_1 and z_2. Then L is an element of $\mathcal{L}_i^*$, for all $i = 1,\ldots,c$. If x is not incident with L, then there is exactly one substructure $S(L,M)$ through x and L. Since x and L are contained in $c > 1$ such substructures, we get a contradiction. Hence $x \mathrm{I} L$. In the same way one proves that $y \mathrm{I} L$. So $x \sim y$, a contradiction with the assumption. This proves that every two distinct points of the line $\langle x,y\rangle$ of the second type are non-collinear in $\mathcal{S}$. Moreover from lemma 4.1.2, it follows that for two distinct points z_1 and z_2 of the line $\langle x,y\rangle$, the lines $\langle z_1,z_2\rangle$ and $\langle x,y\rangle$ coincide.

Assume that there is an α-subgeometry $S(L,M)$ contained in $\mathcal{S}$. Then $S(L,M)$ is a $\mathrm{pg}(s, \alpha-1, \alpha)$. Let x and y be two non-collinear points of $\mathcal{S}$ in $S(L,M)$. Note that $S(L,M)$ contains non-collinear points since $\alpha < s+1$. The points of the line $\langle x,y\rangle$ of the second type are two by two non-collinear points of $S(L,M)$. It follows that $|\langle x,y\rangle| \leq s+1-s/\alpha$.

An $(\alpha, s+1)$-geometry $\mathcal{S}$ of order (s,t), with $1 < \alpha < s+1 < t+1$, that satisfies the axiom of Pasch, is called *regular with respect to non-collinear* *points* if and only if every line of the second type and every line of $\mathcal{S}$ that are both contained in a substructure $S(L, M)$, intersect in at least one point. Note that it follows immediately that every line of $\mathcal{S}$ and every line of the second type, that are both contained in a substructure $S(L, M)$, intersect in exactly one point, for a line of the second type can not contain two points that are collinear in $\mathcal{S}$.

Now assume that $\mathcal{S}$ is a proper $(\alpha, s+1)$-geometry of order (s, t), for which $1 < \alpha < s+1 < t+1$, satisfying the axiom of Pasch and being regular with respect to non-collinear points. Assume moreover that $\mathcal{S}$ contains an α-substructure $S(L, M)$. Since $\alpha < s+1$, the substructure $S(L, M)$ contains two non-collinear points. Let x, $y \in S(L, M)$, x not collinear with y. Since $\mathcal{S}$ is regular with respect to non-collinear points, the points of the line $\langle x, y \rangle$ form an ovoid in the partial geometry $S(L, M)$, which is a $\text{pg}(s, \alpha - 1, \alpha)$. We have defined ovoids of partial geometries in 1.4. It follows that $|\langle x, y \rangle| = s + 1 - s/\alpha$. Hence $\alpha | s$.

In order to prove our characterization theorem for $\text{H}_q^{n,m}$ and $\text{SH}_q^{n,m}$, we will prove in the next lemma that for a proper $(\alpha, s+1)$-geometry $\mathcal{S}$ of order (s, t), $1 < \alpha < s+1 < t+1$, satisfying the axiom of Pasch and regularity with respect to non-collinear points, it follows that $\alpha = s$.

Remark. The definition of regularity with respect to non-collinear points can be given also in the case $\beta \neq s+1$. Let $\mathcal{S}$ be a proper (α, β)-geometry of order (s, t), for which $1 < \alpha < \beta < t+1$, that satisfies the axiom of Pasch, that is regular with respect to non-collinear points and such that there is no mixed substructure $S(L, M)$. If for every two distinct non-collinear points x and y of $\mathcal{S}$, the number μ of points of $\mathcal{S}$ that are collinear with both x and y, is a constant, then we say that $\mathcal{S}$ satisfies the *μ-condition*. Now assume that $\mathcal{S}$ does satisfy the μ-condition. Then it immediately follows that $\beta = s+1$. Indeed, let x and y be two distinct non-collinear points of $\mathcal{S}$. If there is an α-subgeometry $S(L, M)$ containing both x and y, then from the regularity with respect to non-collinear points it follows that $|\langle x, y \rangle| = s + 1 - s/\alpha$. If there is a β-subgeometry through x and y, then in the same way we get that $|\langle x, y \rangle| = s + 1 - s/\beta$. Since by assumption $\alpha \neq \beta$, this implies that x and y cannot be contained in both an α-subgeometry and a β-subgeometry. We are also assuming that there is no mixed substructure. Since $\mathcal{S}$ is proper, it follows that there is an α-subgeometry $S(L, M)$. Since $\alpha < \beta \leq s+1$, $S(L, M)$ contains two distinct non-collinear points x_1 and y_1. From the above it follows that every substructure through both x_1 and y_1 is an α-subgeometry. Hence on every line of $\mathcal{S}$ through x_1, there are exactly α points which are collinear with the point y_1. It follows that $\mu = (t+1)\alpha$.

However, again since we assumed that there is no mixed substructure and since $\mathcal{S}$ is proper, it follows that there is a β-substructure $S'(L, M)$. If $\beta \neq s+1$, then $S'(L, M)$ contains two distinct non-collinear points x_2 and y_2. Again, it follows that every substructure that contains both x_2 and y_2 is a β-substructure and hence $\mu = (t+1)\beta$. Since $\alpha \neq \beta$, we have found a contradiction. This proves that $\beta = s+1$. So it is not necessary to assume that $\beta = s+1$ from the beginning, since the conditions are chosen such that this follows immediately from them. However, to keep the formulation more readable and simple, we will assume $\beta = s+1$ in what follows.

Lemma 4.1.3 *Let $\mathcal{S}$ be a proper $(\alpha, s+1)$-geometry of order (s, t), such that $1 < \alpha < s+1 < t+1$, satisfying the following conditions:*

1. *Pasch axiom,*
2. *regularity with respect to non-collinear points,*
3. *there is at least one α-subgeometry,*

then $\alpha = s$.

Proof. Let $\mathcal{S}$ be a proper $(\alpha, s+1)$-geometry of order (s, t), $1 < \alpha < s+1 < t+1$, satisfying the conditions of the lemma. Assume first that there is no mixed substructure $S(L, M)$. Then every substructure $S(L, M)$ of $\mathcal{S}$ is an α-subgeometry or an $(s+1)$-subgeometry. Since $\mathcal{S}$ is a proper $(\alpha, s+1)$-geometry, both an α-subgeometry and an $(s+1)$-subgeometry exist.

We have to prove that $\alpha = s$. Let $S(L, M)$ be an α-subgeometry, and let $\langle x, y \rangle$ be a line of the second type contained in $S(L, M)$. Then every subgeometry through $\langle x, y \rangle$ is an α-subgeometry and since $\alpha < t+1$, there are at least two distinct subgeometries $\pi_1 = S(L, M)$ and π_2 through $\langle x, y \rangle$. Let p be a point of $\mathcal{S}$ contained in π_1, $p \notin \langle x, y \rangle$. Let N be a line of $\mathcal{S}$ contained in π_2. Since $\mathcal{S}$ is an $(\alpha, s+1)$-geometry, there are either α or $s+1$ lines through p intersecting N in a point. Since $\alpha > 1$, there exists a line L_1 of $\mathcal{S}$ through p intersecting N in a point, such that L_1 is not contained in π_1. Let L_2 be a line of $\mathcal{S}$ contained in π_1 and not incident with p. Then L_2 intersects the line $\langle x, y \rangle$ (and hence also π_2) in a point. Define $\mathcal{P}'$ to be the set of points of $\cup_{i=1}^{s+1} S(x_i, L_2)$, with x_i a point of L_1, for $i = 1, \ldots, s+1$. Since $\pi_1 = S(p, L_2)$, and $p \in L_1$, all points of π_1 belong to $\mathcal{P}'$. In particular every point of $\langle x, y \rangle$ belongs to $\mathcal{P}'$.

Now we prove that every line of $\mathcal{S}$ containing at least two points of $\mathcal{P}'$, is entirely contained in $\mathcal{P}'$. Let z and z' be points of $\mathcal{P}'$, $z \neq z'$. Suppose that

$z \sim z'$. We denote the line of $\mathcal{S}$ containing z and z' by M. We need to show that all points of M are points of $\mathcal{P}'$. Assume first that $z \mathrm{~I~} L_2$. Since $z' \in \mathcal{P}'$, z' is contained in $S(L_2, x_i)$, for a point x_i of L_1. Clearly $z \in S(L_2, x_i)$. So the line M spanned by z and z' is also contained in $S(L_2, x_i)$. This proves that every point of M belongs to $\mathcal{P}'$. If $M = L_1$, or if $z' \in S(z, L_2)$, then the result follows immediately. So from now on we suppose that $z' \notin S(z, L_2)$, $M \neq L_1$ and that z is not incident with L_2. We distinguish two cases:

1. $L_1 \sim M$. Let w be the point of L_1 that is contained in the substructure $S(z', L_2)$. Let $z'' \in M$, $z \neq z'' \neq z'$. We have to prove that $z'' \in \mathcal{P}'$. If $z'' \in L_1$, then clearly $z'' \in \mathcal{P}'$. So suppose that z'' is not incident with L_1. Since w, $z' \in S(z', L_2)$, the line $\langle w, z' \rangle$ (which can be either a line of $\mathcal{S}$ or a line of the second type) has a point u in common with L_2. The line $\langle u, z'' \rangle$ (which can be either a line of $\mathcal{S}$ or a line of the second type) has a point w' in common with L_1, since both $\langle u, z'' \rangle$ and L_1 belong to the substructure $S(L_1, M)$. All the points of this line $\langle u, z'' \rangle = \langle u, w' \rangle$ are elements of $S(w', L_2)$. Hence z'' is a point of $S(w', L_2)$, $w' \in L_1$, and so $z'' \in \mathcal{P}'$.

2. $L_1 \not\sim M$. Let M' be a line that does not belong to π_1, such that $z \mathrm{~I~} M'$, $L_1 \sim M'$ and M' skew to L_2. (Note that M' exists, since there are either α or β lines through z intersecting L_1 and since $p \notin L_2$ at most one of these lines can contain a point of L_2). From the previous case it follows that the $s+1$ points of M' are contained in $\mathcal{P}'$. Moreover, the $s+1$ substructures $S(x_i, L_2)$, for $x_i \mathrm{~I~} L_1$ $(i = 1, \ldots, s+1)$, coincide with the $s+1$ substructures $S(x_i', L_2)$, for $x_i' \mathrm{~I~} M'$ $(i = 1, \ldots, s+1)$. Hence from the previous case, it now follows that any point of the line M is contained in one of the substructures $S(x_i', L_2)$, for $x_i' \in M'$ $(i = 1, \ldots, s+1)$. We conclude that every point of M is contained in $\mathcal{P}'$.

Hence every point of the line spanned by z and z' belongs to $\mathcal{P}'$. Now define $\mathcal{L}'$ to be the set of lines of $\mathcal{S}$ containing at least two distinct points of $\mathcal{P}'$. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$, with I' the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. We distinguish two cases.

Assume that there is an $(s+1)$-subgeometry ρ through L_1 in $\mathcal{S}'$. By assumption $\beta = s+1$, so we know that every two points of $\mathcal{S}$ in ρ are collinear in $\mathcal{S}$. Let u be a point of ρ not incident with L_1. The point u is contained in $\mathcal{S}'$, hence $u \in S(x', L_2)$ for a point $x' \mathrm{~I~} L_1$. Since u and x' belong to ρ, the line $\langle x', u \rangle$ is a line of $\mathcal{S}$ contained in $S(x', L_2)$. Every two lines of $\mathcal{S}$ in a substructure of $\mathcal{S}$ intersect, hence $\langle x', u \rangle$ intersects L_2

in a point v. So ρ contains the point v of L_2. Since $L_2 \subset \pi_1$, $v \in \pi_1$. Moreover ρ contains the point p that is the intersection point of L_1 and π_1. It follows that ρ intersects π_1 in the line $\langle v, p\rangle$. Since $\langle v, p\rangle$ belongs to ρ, it is a line of $\mathcal{S}$. In π_1 the line $\langle v, p\rangle$ intersects the line $\langle x, y\rangle$ of the second type in a point w (here we use the regularity of $\mathcal{S}$ with respect to non-collinear points). Let $L_1^w = \langle v, p\rangle, \ldots, L_{s+1}^w$ $(i = 1, \ldots, s+1)$, be the $s+1$ lines through w in ρ. Every substructure $S(L_i^w, x)$ $(i = 1, \ldots, s+1)$ contains the line $\langle x, y\rangle$ of the second type, and hence it is an α-substructure. So the substructures $S(x, L_i^w)$ $(i = 1, \ldots, s+1)$ are $(s+1)$ α-substructures through $\langle x, y\rangle$ in $\mathcal{S}'$. These substructures contain all points of $\mathcal{S}'$. Through w there are $(s+1)\alpha$ lines of $\mathcal{S}$ in $\mathcal{S}'$, namely α lines in each substructure $S(x, L_i^w)$ $(i = 1, \ldots, s+1)$. Now we count the lines of $\mathcal{S}$ through w in $\mathcal{S}'$ in another way. Let therefore N_w be a line of $\mathcal{S}$ through w in π_1, with $N_w \neq \langle p, v\rangle$. Then in each substructure through N_w in $\mathcal{S}'$ there are α or β lines of $\mathcal{S}$ through w. By the previous paragraph, we know that ρ intersects π_1 in the line $\langle p, v\rangle$. Since $w \in \langle p, v\rangle$, the line N_w intersects ρ in the point w. The $s+1$ lines L_i^w $(i = 1, \ldots, s+1)$ through w in ρ give $s+1$ substructures $S(N_w, L_i^v)$ through N_w in $\mathcal{S}'$. Now counting the lines of $\mathcal{S}$ through w in the substructures $S(N_w, L_i^w)$ $(i = 1, \ldots, s+1)$, we get that there are $c(\beta-1)+(s+1-c)(\alpha-1)+1$ lines of $\mathcal{S}$ through w in $\mathcal{S}'$, for $c \in \mathbb{N}$, $0 \leq c \leq s+1$. It follows that $(s+1)\alpha = c(\beta-1)+(s+1-c)(\alpha-1)+1$, or $c = s/(\beta-\alpha)$. From the previous we know that $\beta = s+1$. So $(s+1-\alpha)|s$, and since $\alpha \neq 1$, it follows that $s/2+1 \leq \alpha$. Since $\mathcal{S}$ is regular with respect to non-collinear points and there exists an α-substructure, it follows that $\alpha|s$. This proves that $\alpha = s$.

Assume next that there is no β-subgeometry through L_1 in $\mathcal{S}'$. Then either $\mathcal{S}'$ contains a β-subgeometry not through L_1, or $\mathcal{S}'$ contains no β-subgeometry. Assume first that there is a β-subgeometry contained in $\mathcal{S}'$. This β-subgeometry in $\mathcal{S}'$ can not contain the line $\langle x, y\rangle$ so it contains a line N of $\mathcal{S}$ that is skew to $\langle x, y\rangle$. Then N is not contained in π_1. If N is contained in one of the substructures $S(x_i, L_2)$, where x_i I L_1 for $i = 1, \ldots, s+1$, then clearly N intersects L_2 and hence also π_1 in a point. If N is not contained in any of the substructures $S(x_i, L_2)$, where x_i I L_1 for $i = 1, \ldots s+1$, then it contains at most one point of each $S(x_i, L_2)$. Indeed, if N would contain two points of $S(x_j, L_2)$ for a $j \in \{1, \ldots, s+1\}$, then by definition of a substructure $S(L, M)$ it follows that N is contained in $S(x_j, L_2)$, a contradiction with our assumption. Hence each of the $s+1$ points of N is contained in a different substructure $S(x_i, L_2)$, for x_i I L_1 and $i = 1, \ldots, s+1$. Since $\pi_1 = S(p, L_2)$, with p a point of the line L_1, it follows that N intersects π_1 in a point. Let N' be a line of $\mathcal{S}$ in π_1, such that N' is skew to N. The $s+1$ substructures $S(x_i, L_2)$, for x_i I L_1, for $i = 1, \ldots, s+1$, coincide with the $s+1$ substructures $S(x'_i, N')$, for x'_i I N, for $i = 1, \ldots, s+1$. So, replacing L_1 by N and L_2 by N' in the previous paragraph, we get that $\alpha = s$. Assume next that there is no β-subgeometry contained in $\mathcal{S}'$. Since $\mathcal{S}$ is a proper $(\alpha, s+1)$-geometry and there are no mixed $S(L, M)$, $\mathcal{S}$ contains a β-substructure ρ'. Let $L_{w'}$ be a line of $\mathcal{S}$ through x intersecting ρ' in a point w'. The substructure $S(L_{w'}, y)$ contains the line $\langle x, y\rangle$ of the second type, hence it is an α-subgeometry. In $S(L_{w'}, y)$ there are $s+1-\alpha$ lines of the second type through w'. Let $\langle w', u'\rangle$ be such a line of the second type through w'. Let M_1 and M_2 be two lines of $\mathcal{S}$ through w' in ρ'. Then $S(u', M_1)$ and $S(u', M_2)$ are both α-subgeometries intersecting ρ' in the lines M_1 and M_2 of $\mathcal{S}$. Let p' be a point of $S(u', M_2)$, $p' \notin \langle u', w'\rangle$. Let $L_{p'}$ be a line of $\mathcal{S}$ through p' intersecting M_1 in a point different from w'. Let M_3 be a line of $\mathcal{S}$ in $S(u', M_2)$, M_3 skew to $L_{p'}$. Let $\mathcal{P}^*$ be the set of all the points of $\mathcal{S}$ contained in the substructures $S(x_i, M_3)$, for $x_i \in L_{p'}$ $(i = 1, \ldots, s+1)$. Let $\mathcal{L}^*$ be the set of all lines intersecting $\mathcal{P}^*$ in at least two points. As before, it follows that all points of $\mathcal{S}$ on the lines of $\mathcal{S}^*$ are contained in $\mathcal{S}^*$. Let I^* be the restriction of I to $(\mathcal{P}^* \times \mathcal{L}^*) \cup (\mathcal{L}^* \times \mathcal{P}^*)$. Then, replacing $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ by $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$, the result follows in the same way as in the previous paragraph. This proves that $s = \alpha$.

From the previous paragraphs it follows that, if there is no mixed substructure, then $\alpha = s$, and hence in this case the lemma is proved. Now assume that there is a mixed substructure σ in $\mathcal{S}$. Let p be a point of σ through which there are α lines of $\mathcal{S}$ in σ. Let L be a line of $\mathcal{S}$ in σ not through p. Since $\alpha < s+1$, L contains a point p' that is not collinear in $\mathcal{S}$ with p. Hence $\langle p, p'\rangle$ is a line of the second type through p. Let u be a point of σ through which there are $\beta = s+1$ lines of $\mathcal{S}$. Let L' be a line of σ not through u. Then every point of L' is collinear with u. Since $\mathcal{S}$ is regular with respect to non-collinear points, a line of the second type through u in σ has to contain a point of L', a contradiction since a line of the second type can not contain collinear points. Hence there is no line of the second type through u in σ. In particular $u \notin \langle p, p'\rangle$, and so $\langle p, p'\rangle$ contains exactly one point of each of the $s+1$ lines through u in σ. It follows that $|\langle p, p'\rangle| = s+1$. Now counting the lines of $\mathcal{S}$ in σ that intersect $\langle p, p'\rangle$, we get that there are $\alpha(s+1)$ such lines. These lines are all the lines of $\mathcal{S}$ in σ, since $\mathcal{S}$ is regular with respect to non-collinear points. Counting the lines of $\mathcal{S}$ in σ intersecting the line L of $\mathcal{S}$ in σ in a point, it follows that there are $c(\alpha-1) + (s+1-c)s + 1$ such lines, for a $c \in \mathbb{N}$, $0 \le c \le s+1$. Since every two lines of $\mathcal{S}$ in σ intersect, these are all the lines of $\mathcal{S}$ in σ. Hence

$$\begin{aligned}
\alpha(s+1) &= c(\alpha-1)+(s+1-c)s+1 \\
s\alpha+\alpha &= s^2+s+1+c(\alpha-s-1) \\
c(s+1-\alpha) &= s^2+s+1-s\alpha-\alpha \\
c &= s+\frac{1-\alpha}{s+1-\alpha} \\
c &= s+1-\frac{s}{s+1-\alpha}
\end{aligned}$$

It follows that either $(s+1-\alpha)|s$. So $s+1-\alpha=s$ or $s+1-\alpha \leq s/2$. In the first case, we get that $\alpha=1$, a contradiction. In the second case we get that $s/2+1 \leq \alpha$.

By assumption there is an α-subgeometry contained in $\mathcal{S}$. It follows from the above that an α-subgeometry in $\mathcal{S}$ contains a line of the second type on which there are $s+1-s/\alpha$ points of $\mathcal{S}$. Hence $\alpha|s$. So $\alpha=s$ or $\alpha \leq s/2$. Combining these conditions with the one of the previous paragraph, we get that $\alpha=s$. So also in the case that there is a mixed substructure, the result of the lemma follows. □

Remark. The previous lemma together with the remark that precedes it, have the following corollary.

Let $\mathcal{S}$ be a proper (α,β)-geometry of order (s,t), such that $1<\alpha<\beta<t+1$, satisfying the following conditions:

1. Pasch axiom,
2. μ-condition,
3. regularity with respect to non-collinear points,
4. there is no mixed substructure,

then $\beta=s+1$ and $\alpha=s$.

4.1.4 A characterization of $\mathrm{H}_q^{n,m}$ and $\mathrm{SH}_q^{n,m}$

Let $\mathcal{S}$ be a proper $(\alpha,s+1)$-geometry of order (s,t), that satisfies the axiom of Pasch, that is regular with respect to non-collinear points and such that there is at least one α-subgeometry. From lemma 4.1.3 it follows that $\alpha=s$. We will now count the number of points and lines of $\mathcal{S}$ in an $(s+1)$-substructure, an s-substructure and a mixed substructure, assuming that such substructures would exist (it is not necessarily the case that each of these substructures do appear).

By definition, the points and lines of $\mathcal{S}$ in an $(s+1)$-substructure form a partial geometry $\mathrm{pg}(s, s, s+1)$. Therefore we will use from now on the term *projective plane* instead of $(s+1)$-substructure, which will simplify the notation. In a projective plane ρ, every two points of $\mathcal{S}$ are collinear in $\mathcal{S}$. It follows that ρ contains s^2+s+1 points of $\mathcal{S}$ and s^2+s+1 lines of $\mathcal{S}$.

The points and lines of $\mathcal{S}$ in an s-substructure are the points and lines of a $\mathrm{pg}(s, s-1, s)$. Hence we will use the term *dual affine plane* instead of s-substructure, again to simplify the notation. In a dual affine plane π, there are s^2+s points of $\mathcal{S}$ and s^2 lines of $\mathcal{S}$. Through each point p of $\mathcal{S}$ in π there is exactly one line of the second type (lines of the second type were defined in section 4.1.3). Indeed, let L be a line of $\mathcal{S}$ in π, L not incident with p. Then the s lines of $\mathcal{S}$ in π each intersect L in a point. Let z be the point of L that is not collinear with p. Then $\langle p, z\rangle$ is a line of the second type through p in π. Moreover it is the unique line of the second type through p in π, since every line of the second type in π has to contain a point of L. Since $\mathcal{S}$ is regular with respect to non-collinear points, it follows that a line of the second type in π contains s points of $\mathcal{S}$.

A mixed substructure contains exactly one line of the second type. Indeed, let σ be a mixed substructure and let y be a point of $\mathcal{S}$ in σ through which there are s lines of $\mathcal{S}$ in σ. Then σ contains a line $\langle y, z\rangle$ of the second type through y. Let p be a point of σ through which there are $s+1$ lines of $\mathcal{S}$. By regularity with respect to non-collinear points, a line of the second type in σ has exactly one point in common with each line of $\mathcal{S}$ in σ. So $\langle y, z\rangle$ has exactly one point in common with each of the $s+1$ lines of $\mathcal{S}$ through p in σ. Hence $|\langle y, z\rangle| = s+1$. Now let L be a line of $\mathcal{S}$ in σ. Counting the lines of $\mathcal{S}$ in σ that intersect L, we get that there are $1+cs+(s+1-c)(s-1) = s^2+c$ such lines, where c is the number of points of L through which there are $s+1$ lines of $\mathcal{S}$ in σ. Since every two lines of $\mathcal{S}$ in σ intersect, it follows that σ contains s^2+c lines of $\mathcal{S}$. Now also every line of $\mathcal{S}$ in σ intersects $\langle y, z\rangle$ in exactly one point (since $\mathcal{S}$ is regular with respect to non-collinear points). Through each point of $\langle y, z\rangle$ there are s lines of $\mathcal{S}$ in σ. Hence counting the lines of $\mathcal{S}$ in σ that intersect $\langle y, z\rangle$, we have that there are $(s+1)s$ such lines. It follows that $c = s$ or thus L contains exactly one point through which there are s lines of $\mathcal{S}$ in σ. Since $\mathcal{S}$ is regular with respect to non-collinear points, this proves that σ contains exactly one line of the second type, since every line of the second type in σ has to intersect L in a point. It follows that σ contains s^2+s+1 points of $\mathcal{S}$ and s^2+s lines of $\mathcal{S}$. From now on we will speak of a *punctured affine plane* instead of a mixed substructure.

Theorem 4.1.4 *Let $\mathcal{S}$ be a proper $(\alpha, s+1)$-geometry of order (s,t), such that $1 < \alpha < s+1 < t+1$, satisfying the following conditions:*

1. *Pasch axiom,*
2. *regularity with respect to non-collinear points,*
3. *there is at least one α-subgeometry,*

then $\mathcal{S}$ is isomorphic to $\mathrm{H}_q^{n,m}$ or $\mathrm{SH}_q^{n,m}$.

Proof. Let $\mathcal{S}$ be a proper $(\alpha, s+1)$-geometry of order (s,t), $1 < \alpha < s+1 < t+1$, that satisfies the conditions of the theorem. Then from lemma 4.1.3 we know that $\alpha = s$.

Let $\langle x, y\rangle$ and $\langle x, z\rangle$ be two different lines of the second type. Then y is not collinear with z. Indeed, if $y \sim z$ then on the line $\langle y, z\rangle$ of $\mathcal{S}$ there are at most $s-1$ points collinear with x, a contradiction since $\mathcal{S}$ is an $(s, s+1)$-geometry. In other words, non-collinearity is transitive. On $\langle x, y\rangle$ there are either s or $s+1$ points of $\mathcal{S}$. Also on $\langle x, z\rangle$ there are either s or $s+1$ points of $\mathcal{S}$. We consider the different possibilities separately.

1. The case $|\langle x, y\rangle| = s$ and $|\langle x, z\rangle| = s$.

Let L be a line of $\mathcal{S}$ through x. Then $S(y, L)$ and $S(z, L)$ are dual affine planes. Let M be a line of $\mathcal{S}$ through y in $S(y, L)$. Let N be a line of $\mathcal{S}$ through z intersecting M in a point, such that N is not contained in $S(z, L)$. Then N is skew to L. Let $\mathcal{P}'$ be the set of points of the substructures $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$. Then as in lemma 4.1.3 we can prove that every line containing at least two points of $\mathcal{P}'$ is contained in $\mathcal{P}'$. Let $\mathcal{L}'$ be the set of lines intersecting $\mathcal{P}'$ in at least two points. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$, with I' the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$.

We will first prove that all substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, are dual affine planes. Let w be a point of the line $\langle y, z\rangle$ of the second type. If $w \sim x$, then there would be at most $s-1$ lines of $\mathcal{S}$ through y intersecting $\langle x, w\rangle$, a contradiction since $\mathcal{S}$ is an $(s, s+1)$-geometry. Hence w is not collinear with x. Since w was an arbitrarily chosen point of $\langle y, z\rangle$, no point of $\langle y, z\rangle$ is collinear with x. So if $|\langle y, z\rangle| = s+1$, then each of the substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, contains a line of the second type through x. Assume now that $|\langle y, z\rangle| = s$. Then each of the s substructures $S(L, z_i)$, $z_i \in N$, that contains a point of $\langle y, z\rangle$, contains a line of the second type through x. Let $S(L, \tilde{z})$ be the remaining substructure through L and a point $\tilde{z}$ of N. Denote the intersection point of M and L

by u. Then $y, z, u \in S(M, N)$. Clearly $u \notin \langle y, z \rangle$, since $u \sim y$ and $\langle y, z \rangle$ is a line of the second type. Moreover it follows that $S(M, N)$ is a dual affine plane, since by assumption $|\langle y, z \rangle| = s$. By regularity with respect to non-collinear points, we know that every line of $\mathcal{S}$ through u in $S(M, N)$ contains exactly one point of the line $\langle y, z \rangle$. So the s lines of $\mathcal{S}$ through u in $S(M, N)$ intersect $\langle y, z \rangle$ in a point. Hence $S(L, \tilde{z})$ intersects $S(M, N)$ in the line of the second type through u. So every substructure $S(L, z_i)$, for $z_i \in N$ $(i = 1, \dots, s+1)$, contains a line of the second type, where s of these lines are incident with x and one of them is incident with x or u.

Assume first that $s \neq 2$. Let $x' \in L$, $x \neq x' \neq u$. In the dual affine plane $S(y, L)$ and $S(z, L)$, there is a line of the second type through x'. We denote these lines by $\langle x', y' \rangle$ and $\langle x', z' \rangle$ respectively. Since non-collinearity is transitive, the line $\langle y', z' \rangle$ is also a line of the second type. Let M' be a line of $\mathcal{S}$ through y' in $S(y, L)$ such that $u, x \notin M'$ (there is such a line since $s > 2$). Let N' be a line through z' intersecting M' in a point, such that N' is skew to L. Then in the same way as we did above (replace x, M, N, y, z by x', M', N', y', z'), it follows that either all $S(z_i, L)$, for $z_i \in N$ $(i = 1, \dots, s+1)$, contain a line of the second type through x', or s of them contain a line of the second type through x' and the remaining one contains a line of the second type through u', for a point $u' \in L$, $x \neq u' \neq u$ and $u' \neq x'$. In either case, all $S(z_i, L)$, for $z_i \in N$ $(i = 1, \dots, s+1)$, contain at least two different lines of the second type and hence they are all dual affine planes.

Assume next that $s = 2$. We know that $S(y, L)$ and $S(z, L)$ are dual affine planes. The other substructure $S(z_i, L)$, for $z_i \in N$, we denote as $S(p, L)$. Assume that there are 3 lines of $\mathcal{S}$ through x in $S(p, L)$. We denote them by L, L' and L''. Then $S(y, L)$, $S(y, L')$ and $S(y, L'')$ contain the line $\langle x, y \rangle$ of the second type, with $|\langle x, y \rangle| = 2$. Hence they are all dual affine planes. A dual affine plane contains $s^2 + s = 6$ points, so we get that $|\mathcal{P}'| = 14$. Now $S(y, L)$ and $S(z, L)$ contain together 9 different points of $\mathcal{P}'$. So $S(p, L)$ has to contain $5 + 3 = 8$ points of $\mathcal{S}$. This is a contradiction, since a substructure $S(L, M)$ contains at most $s^2 + s + 1 = 7$ points of $\mathcal{S}$. So $S(p, L)$ contains a line of the second type through x. Let x' be a point of L, $x' \neq x$. In the same way as above we prove that there is a line of the second type through x' in $S(p, L)$. So $S(p, L)$ contains at least two lines of the second type. This proves that $S(p, L)$ is a dual affine plane. Hence every substructure $S(L, z_i)$, for $z_i \in N$ $(i = 1, 2, 3)$, is a dual affine plane.

Now we will prove that all the substructures contained in $\mathcal{S}'$ are dual affine planes.

Assume first that there would be a projective plane ρ contained in $\mathcal{S}'$.

From the previous part of the proof, it follows that ρ does not contain the line L. Since ρ is contained in $\mathcal{S}'$, every point of ρ is contained in the substructures $S(L,z_i)$, for $z_i \in N$ $(i=1,\ldots,s+1)$. So ρ contains at least two points of a $S(L,\tilde{z})$, for $\tilde{z} \in N$, since there are s^2+s+1 points of $\mathcal{S}$ in ρ contained in the $s+1$ substructures $S(L,z_i)$, for $z_i \in N$ $(i=1,\ldots,s+1)$. The line through these two points intersects L in a point x''. So $\rho \cap L$ is the point x''. Now let w be a point of L, $w \neq x''$. In ρ there are $s+1$ lines of $\mathcal{S}$ through x''. Since every substructure $S(L,z_i)$, for $z_i \in N$ $(i=1,\ldots,s+1)$, is a dual affine plane, on each of the lines of $\mathcal{S}$ through x'' in ρ there are s points collinear with w. Hence counting the lines of $\mathcal{L}'$ through w, we get that there are $(s+1)(s-1)+1=s^2$ such lines. Since ρ is a projective plane, we know that there are s^2+s+1 lines of $\mathcal{S}$ in ρ. Assume that c of these lines contain s points collinear with w. Then

$$cs+(s^2+s+1-c)(s+1)=(s+1)s^2,$$

since there are s^2 lines through w intersecting ρ. It follows that $c=s^2+2s+1$, a contradiction since c has to be less than or equal to the number of lines of $\mathcal{S}$ in ρ, which is s^2+s+1. This proves that there is no projective plane contained in $\mathcal{S}'$.

Assume next that there is a punctured affine plane σ contained in $\mathcal{S}'$. Then σ contains exactly one line $\langle w_1,w_2\rangle$ of the second type, with $|\langle w_1,w_2\rangle| = s+1$. From the previous we know that L is not contained in σ. It is clear that σ contains a point u of L. Indeed, σ contains s^2+s+1 points of $\mathcal{S}$, hence there are two distinct points of σ that are contained in the same substructure $S(L,z_j)$, for $z_j \in N$ $i \in \{1,\ldots,s+1\}$. The line through these two points intersects L in a point. Let $\tilde{x}$ be a point of L, $\tilde{x} \neq u$. Through $\tilde{x}$ there are s^2 lines of $\mathcal{S}$ contained in $\mathcal{S}'$, namely s in each dual affine plane $S(L,z_i)$, for $z_i \in N$ $(i=1,\ldots,s+1)$. Clearly there are $s+1$ lines through u in σ, since $\langle w_1,w_2\rangle$ can not be contained in a dual affine plane. Each of these lines is contained in a different $S(L,z_i)$, for $z_i \in N$ $(i=1,\ldots,s+1)$. Now all lines through $\tilde{x}$ in $\mathcal{S}'$ are contained in the $S(L,z_i)$, for $z_i \in N$ $i=1,\ldots,s+1$, and so they intersect a line of σ through u in a point. Hence all lines through $\tilde{x}$ intersect σ. Now there are s^2+s lines of $\mathcal{S}$ in σ. Let c be the number of lines of σ on which there are s points collinear with $\tilde{x}$. Then we get that

$$\begin{aligned} cs+(s^2+s-c)(s+1) &= as+(s^2-a)(s+1) \\ c &= s^2+s+a, \end{aligned}$$

where a is the number of points of σ, collinear with x, through which there are s lines of $\mathcal{S}$ in σ. Now $c \leq s^2+s$, and thus $a=0$. This implies that through every point of σ, that is collinear with $\tilde{x}$, there are $s+1$ lines of $\mathcal{S}$ in σ. Hence every point of the line $\langle w_1, w_2 \rangle$ of the second type is not collinear with $\tilde{x}$. Now let $\overline{x}$ be a point of L, $u \neq \overline{x} \neq \tilde{x}$. Then in the same way as above we get that every point of $\langle w_1, w_2 \rangle$ is not collinear with $\overline{x}$. Since $\langle w_1, w_2 \rangle$ is contained in $\mathcal{S}'$, it follows that $w_1 \in \mathcal{P}'$. So $w_1 \in S(L, z_k)$, for a $z_k \in N$. Now $S(L, z_k)$ is a dual affine plane, containing two lines of the second type through w_1, namely $\langle w_1, \tilde{x} \rangle$ and $\langle w_1, \overline{x} \rangle$. This is a contradiction, since in a dual affine plane there is exactly one line of the second type through every point. This proves that there can not be a punctured affine plane contained in $\mathcal{S}'$.

We conclude that every substructure contained in $\mathcal{S}'$ is a dual affine plane. There are $s+1$ dual affine planes through L. In every dual affine plane there is one line of the second type through x containing s points, so in total there are $(s+1)(s-1) = s^2 - 1$ points non collinear with x in $\mathcal{S}'$. Now let $\mathcal{P}^*$ be a set of s^2 points of $\mathcal{S}'$, containing x and the $s^2 - 1$ points of $\mathcal{S}'$ that are not collinear with x. Since non-collinearity is transitive, every two points of $\mathcal{P}^*$ are non-collinear. From transitivity of the non-collinearity, it follows that every line of the second type containing at least two points of $\mathcal{P}^*$, contains s points of $\mathcal{P}^*$. Let $\mathcal{L}^*$ be the set of lines of the second type containing at least two points of $\mathcal{P}^*$. Let I^* be the natural incidence relation. Then $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ is a $2-(s^2, s, 1)$ design, i.e. an affine plane of order s. Since an affine plane is generated by any of its triangles, it follows that $\mathcal{S}^*$ is independent of the choice of $\mathcal{S}'$.

2. The case $|\langle x, y \rangle| = s+1$ and $|\langle x, z \rangle| = s+1$.

Let L be a line of $\mathcal{S}$ through x. Then $S(y, L)$ and $S(z, L)$ are punctured affine planes. Let M be a line of $\mathcal{S}$ through y in $S(L, y)$. Let $M \cap L$ be the point u. Let N be a line of $\mathcal{S}$ through z intersecting M in y', $y' \neq u$. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be the incidence structure defined as follows: $\mathcal{P}'$ is the set of points of the substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, $\mathcal{L}'$ is the set of lines of $\mathcal{L}$ containing at least two points of $\mathcal{P}'$, and I' is the restriction of I to $((\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}'))$. Then every point of $\langle y, z \rangle$ is contained in a different substructure $S(L, z_i)$, for a $z_i \in N$ $(i = 1, \ldots, s+1)$, since $\langle y, z \rangle$ has no point in common with L. Every point of $\langle y, z \rangle$ is not collinear with x because of the transitivity of non-collinearity. Hence at least s substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, contain a line of the second type through x. If at least two substructures $S(L, z_1)$ and $S(L, z_2)$, for $z_1, z_2 \in N$, are dual affine planes, then as in the first case that was treated above, we can prove that all $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, are dual affine planes. This is clearly a contradiction since $S(L, y)$ and $S(L, z)$ are punctured affine planes. Hence at most one of the $S(z_i, L)$, for $z_i \in N$, is a dual affine plane, and at least $(s-1)$ of the substructures $S(z_i, L)$, for $z_i \in N$, $i \in \{1, \ldots, s+1\}$, are punctured affine planes. Also we know that at most one of the substructures $S(z_i, L)$, for $z_i \in N$, is a projective plane, since at least s of these substructures contain a line of the second type through x. We deal with each of the remaining possibilities separately.

(2a) *Assume that exactly one of the substructures $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, is a dual affine plane and exactly one of the $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, is a projective plane.* In a dual affine plane there are s^2+s points of $\mathcal{S}$, in a projective plane and a punctured affine plane there are s^2+s+1 points of $\mathcal{S}$. It follows that

$$\begin{aligned} |\mathcal{P}'| &= (s-1)s^2 + s^2 - 1 + s^2 + s + 1 \\ &= s^3 + s^2 + s. \end{aligned}$$

The line $\langle x, y\rangle$ is a line of the second type contained in a punctured affine plane through L. Let $N_1, \ldots, N_{s+1} = L$ be the $s+1$ lines of $\mathcal{S}$ through x in the projective plane through L in $\mathcal{S}'$. Then $S(y, N_i)$ $(i = 1, \ldots, s+1)$, are punctured affine planes, since they contain the line $\langle x, y\rangle$ of the second type, with $|\langle x, y\rangle| = s+1$. Clearly every point of $S(y, N_i)$ $(i = 1, \ldots, s+1)$, is contained in $\mathcal{S}'$. Now counting again the points of $\mathcal{S}'$, we get that $|\mathcal{P}'| \geq (s+1)s^2 + s + 1 = s^3 + s^2 + s + 1$, clearly a contradiction.

(2b) *Assume that exactly one of the substructures $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, is a projective plane and none of them is a dual affine plane.* Then $|\langle y, z\rangle| = s$. Indeed, if $|\langle y, z\rangle| = s+1$, then the line $\langle y, z\rangle$ would have a point $\tilde{y}$ in common with the projective plane through L in $\mathcal{S}'$. Hence $\langle x, \tilde{y}\rangle$ would be a line of the second type contained in a projective plane, a contradiction since in a projective plane every two points of $\mathcal{S}$ are collinear in $\mathcal{S}$. Now let L_z be a line of $\mathcal{S}$ through z in $S(z, L)$, with $L_z \cap L = \{u\}$. Then $S(y, L_z)$ is a dual affine plane, since it contains the line $\langle y, z\rangle$ of the second type with $|\langle y, z\rangle| = s$. So through u there are s lines of $\mathcal{S}$ in $S(y, L_z)$ and each of these lines contains a point of $\langle y, z\rangle$. The line of the second type through u in $S(y, L_z)$ then has to be contained in the projective plane through L. This is clearly a contradiction, as a projective plane contains no lines of the second type.

(2c) *Assume that none of the substructures* $S(z_i, L)$, *for* $z_i \in N$ $(i = 1, \ldots, s+1)$, *is a dual affine plane or a projective plane.* Then all $S(z_i, L)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, are punctured affine planes. Counting the points of $\mathcal{S}'$ we get that

$$|\mathcal{P}'| = (s+1)s^2 + s + 1 = s^3 + s^2 + s + 1.$$

Let $L_1 = L, \ldots L_s$ be the s lines of $\mathcal{S}$ through x in $S(L, z)$. Then $S(y, L_i)$ $(i = 1, \ldots, s)$, are s punctured affine planes. Together they contain $s(s^2) + s + 1 = s^3 + s + 1$ points of $\mathcal{S}'$. Hence there are exactly s^2 points not contained in one of the s punctured affine planes $S(y, L_i)$ $(i = 1, \ldots, s)$. Let $\mathcal{P}^*$ be the set of these s^2 points, together with the points of the line $\langle x, y\rangle$. Then $|\mathcal{P}^*| = s^2 + s + 1$. All lines of the second type through x and a point of $\langle y, z\rangle$ different from y, are not contained in the s punctured affine planes $S(y, L_i)$ $(i = 1, \ldots, s)$, since $\langle x, y\rangle$ is the only line of the second type contained in these s punctured affine planes. If a point $p \in \mathcal{P}^*$, $p \neq y$, would be collinear with y in $\mathcal{S}$, then $S(x, \langle y, p\rangle)$ would be a punctured affine plane through $\langle x, y\rangle$. Since $|\mathcal{P}^*| = s^2 + s + 1$, $S(x, \langle y, p\rangle)$ contains all the points of $\mathcal{P}^*$. Hence $z \in S(x, \langle y, p\rangle)$ and thus there are at least two lines of the second type through x in $S(x, \langle y, p\rangle)$, namely the lines $\langle x, y\rangle$ and $\langle x, z\rangle$. This is clearly a contradiction. Hence all points of $\mathcal{P}^*$ are non-collinear with y. By transitivity of the non-collinearity, it follows that every two points of $\mathcal{P}^*$ are non-collinear in $\mathcal{S}$. So the points of $\mathcal{P}^*$ are $s^2 + s + 1$ pairwise non-collinear points. Note that $\mathcal{P}^*$ is the set of all points of $\mathcal{S}'$ not collinear with x, union x, since in each $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, there are exactly s points that are not collinear with x.

Assume first that $\mathcal{S}'$ contains a dual affine plane π. Then the $s^2 + s$ points of $\mathcal{S}'$ in π are contained in the substructures $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$. So there is a substructure $S(L, z_k)$, for $z_k \in N$, that contains two points of π. The line through these two points intersects L in a point u. Hence π intersects L in the point u. Since π is a dual affine plane, we know that there is a line $\langle u, w\rangle$ of the second type through u, with $|\langle u, w\rangle| = s$. Since $u \in L$, $\langle u, w\rangle$ is contained in an $S(L, z_j)$, for a $z_j \in N$. This is a contradiction, since by assumption $S(L, z_j)$ is a punctured affine plane and hence it can not contain the line $\langle u, w\rangle$ of the second type, for which $|\langle u, w\rangle| = s$. This proves that $\mathcal{S}'$ does not contain a dual affine plane. It follows that every line of the second type contained in $\mathcal{S}'$, contains $s + 1$ points of $\mathcal{S}'$.

Assume next that $\mathcal{S}'$ contains a projective plane ρ. As before, one proves that ρ contains a point v of L. Since $|\rho| = s^2 + s + 1$, and $|\rho \cap S(L, z_i)| \leq s+1$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, it follows that ρ intersects $S(L, y)$ in a line N through v. Since $N \subset \rho$, N is a line of $\mathcal{S}$. In $S(L, y)$, N has to intersect the line $\langle x, y \rangle$ of the second type. So ρ contains a point u of $\langle x, y \rangle$. Through u there are $s+1$ lines of $\mathcal{S}$ in ρ. We denote them by $L_1, \ldots, L_{s+1}$. The substructures $S(L_i, \langle x, y \rangle)$ $(i = 1, \ldots, s+1)$ are $s+1$ punctured affine planes. These substructures contain all points of $\mathcal{S}'$. Hence z is contained in a punctured affine plane through $\langle x, y \rangle$. This is clearly a contradiction, since in a punctured affine plane there can not be two lines of the second type through x. This proves that there is no projective plane contained in $\mathcal{S}'$.

We conclude that every substructure contained in $\mathcal{S}'$ is a punctured affine plane. Now we define $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ as follows: $\mathcal{P}^*$ is the set of points of $\mathcal{S}'$ containing x and the points of $\mathcal{S}'$ that are not collinear with x, $\mathcal{L}^*$ is the set of lines of the second type containing at least two points of $\mathcal{P}^*$ (and hence all points of $\mathcal{S}$ on the lines of $\mathcal{L}^*$ are points of $\mathcal{P}^*$), and I^* is the natural incidence relation. Then it is easy to see that $\mathcal{S}^*$ is a $2-(s^2+s+1, s+1, 1)$-design, hence $\mathcal{S}^*$ is a projective plane. Since a projective plane is defined by any three of its points, we know that $\mathcal{S}^*$ does not depend on the choice of $\mathcal{S}'$.

(2d) *Assume that exactly one of the substructures $S(z_i, L)$, for $z_i \in N$, $(i = 1, \ldots, s+1)$ is a dual affine plane and none of them is a projective plane.* Let π be the dual affine plane through L. Through every point of L there is a line of the second type in π. All these lines of the second type contain s points of $\mathcal{S}$, hence they are not contained in projective planes or punctured affine planes. Let σ be a punctured affine plane $S(L, z_i)$ through L, for a $z_i \in N$. Let $\langle u_1, u_2 \rangle$ be the line of the second type contained in σ. Then $\langle u_1, u_2 \rangle$ intersects L in a point v. Let $\langle w_1, w_2 \rangle$ be a line of the second type in π not through v. Let $\langle w_1, w_2 \rangle \cap L$ be the point v'. Through v' there are $s+1$ lines of $\mathcal{S}$ in σ. Hence through $\langle w_1, w_2 \rangle$ there are $s+1$ dual affine planes in $\mathcal{S}'$. This proves that every point of $\mathcal{S}'$ is contained in a dual affine plane through $\langle w_1, w_2 \rangle$. Hence through every point of $\mathcal{S}'$ there is a line of the second type containing s points of $\mathcal{S}$. Assume that there would be two lines of the second type containing s points of $\mathcal{S}$ through a point $p \in \mathcal{S}'$. Let L_p be a line of $\mathcal{S}'$ through p. It follows that through L_p there are at least two dual affine planes contained in $\mathcal{S}'$. As in the first part of the proof, we can prove that then all substructures contained in $\mathcal{S}'$

are dual affine planes, a contradiction since by assumption there are punctured projective planes in $\mathcal{S}'$. Hence every point of $\mathcal{S}'$ is contained in exactly one line of the second type that contains s points of $\mathcal{S}'$.

Now we define an incidence structure $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ as follows: $\mathcal{P}^*$ is the set of all the points of $\mathcal{S}'$ that are not collinear with x, $\mathcal{L}^*$ is the set of all lines of the second type containing at least two points of $\mathcal{P}^*$, and I^* is the natural incidence relation. Since there are s punctured affine planes and one dual affine plane through L, it follows that there are s lines of the second type through x containing $s+1$ points of $\mathcal{S}$, while the remaining line of the second type through x contains s points of $\mathcal{S}$. Note that the line of the second type in each of the punctured affine planes through L is incident with x. For the punctured affine planes containing a point of $\langle y, z\rangle$, this is immediately clear. Moreover if $|\langle y, z\rangle| = s$, and there is a punctured affine plane $S(L, z_i)$, for $z_i \in N$, that contains no point of $\langle y, z\rangle$, then in the unique dual affine plane $S(L, z_k)$, for $z_k \in N$, the line through x and the intersection point u of $\langle y, z\rangle$, contains s points of $\mathcal{S}$. Hence through u there are two lines containing s points of $\mathcal{S}$, a contradiction with the previous paragraph. Hence $|\mathcal{P}^*| = s^2 + s$. By transitivity of non-collinearity, we know that every two points of $\mathcal{P}^*$ are non-collinear. Moreover through every point of $\mathcal{P}^*$ there is exactly one line of $\mathcal{L}^*$ on which there are s points of $\mathcal{S}$ by the previous argument. Hence the lines of $\mathcal{L}^*$ containing s points of $\mathcal{S}$ partition the points of $\mathcal{P}^*$. Now we add a new point w to $\mathcal{P}^*$, and we define w to be incident with every line of $\mathcal{L}^*$ that contains s points of $\mathcal{S}$. Then clearly $\mathcal{S}^* = (\mathcal{P}^* \cap \{w\}), \mathcal{L}^*, \mathrm{I}^*$ is a $2-(s^2+s+1, s+1, 1)$ design, hence a projective plane. Since w was not a point of $\mathcal{S}$, it follows that $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ is a dual affine plane. A projective plane is uniquely defined by any three of its points, hence $\mathcal{S}^*$ is independent of the choice of $\mathcal{S}'$.

3. The case $|\langle x, y\rangle| = s$ and $|\langle x, z\rangle| = s+1$.

Let L be a line of $\mathcal{S}$ through x. Then $S(L, y)$ is a dual affine plane, and $S(L, z)$ is a punctured affine plane. Let N be a line of $\mathcal{S}$ intersecting both $S(L, y)$ and $S(L, z)$ in a point not on L. If there are two dual affine planes or two punctured affine planes $S(L, z_i)$ and $S(L, z_j)$, for z_i, $z_j \in N$, $z_i \neq z_j$, then we can apply one of the previous paragraphs. So we may assume that there is exactly one dual affine plane through L and a point of N (containing $\langle x, y\rangle$) and exactly one punctured affine plane (containing $\langle x, z\rangle$). All other substructures through L and a point of N are projective planes. Let ρ be a projective plane through L. Since $x \in L$, there are $s+1$ lines of $\mathcal{S}$ through x in ρ. We denote them by $L_1 = L, \ldots, L_{s+1}$. The substructures $\langle y, L_i \rangle$, $(i = 1, \ldots, s+1)$, contain the line $\langle x, y \rangle$ of the second type, with $|\langle x, y \rangle| = s$ and so they are all dual affine planes. Hence $|\mathcal{P}'| = (s+1)s^2 + s = s^3 + s^2 + s$. Now counting the points of $\mathcal{P}'$ in the $s+1$ punctured affine planes $S(z, L_i)$, $(i = 1, \ldots, s+1)$, we get that $|\mathcal{P}'| = (s+1)s^2 + s + 1 = s^3 + s^2 + s + 1$. This gives us a contradiction. Hence this case does not occur.

We have now studied all the different possibilities.

From now on, we call the incidence structures $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$, with $\mathcal{P}^*$ a set of non-collinear points as defined above, planes of type IV, V and VI, when $\mathcal{S}^*$ is respectively a projective plane, an affine plane and a dual affine plane.

Next we will define parallelism among the lines of the second type containing s points of $\mathcal{S}$. Two lines of the second type containing s points of $\mathcal{S}$ are parallel if they coincide or if they are disjoint subsets of either a dual affine plane or a plane of type V or VI. Clearly the parallelism defined in this way is reflexive and symmetric. It remains to prove that it is also transitive. Let therefore $\langle x, y \rangle$, $\langle u, v \rangle$ and $\langle p, w \rangle$ be three lines of the second type containing s points of $\mathcal{S}$. Suppose that $\langle x, y \rangle$ is parallel with $\langle u, v \rangle$, that $\langle x, y \rangle$ is parallel with $\langle p, w \rangle$, and that no two of them coincide. From the definition of parallelism, it follows that the lines $\langle x, y \rangle$ and $\langle u, v \rangle$ are both contained in a dual affine plane, a plane of type V or a plane of type VI. In the same way, we get that the lines $\langle x, y \rangle$ and $\langle p, w \rangle$ are both contained in a dual affine plane, a plane of type V or a plane of type VI. We have to consider three cases.

1. *Assume that both the plane containing $\langle x, y \rangle$ and $\langle u, v \rangle$ and the one containing $\langle x, y \rangle$ and $\langle p, w \rangle$ are dual affine planes.*

Let π_1 (respectively π_2) be the dual affine plane containing $\langle x, y \rangle$ and $\langle u, v \rangle$ (respectively $\langle x, y \rangle$ and $\langle p, w \rangle$). If $\pi_1 = \pi_2$, then $\langle u, v \rangle$ and $\langle p, w \rangle$ are contained in the dual affine plane π_1. By definition of parallelism it follows that $\langle u, v \rangle$ and $\langle p, w \rangle$ are parallel. So we can assume that $\pi_1 \neq \pi_2$. In this case the lines $\langle u, v \rangle$ and $\langle p, w \rangle$ are clearly disjoint, since they are contained in different dual affine planes through $\langle x, y \rangle$ and so they are both skew to $\langle x, y \rangle$.

Let M be a line of $\mathcal{S}$ in π_1 and let N be a line of $\mathcal{S}$ in π_2 skew to M. Then we can define an incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ as follows: $\mathcal{P}'$ is the set of points contained in the $S(M, z_i)$, with $z_i \in N$ $(i = 1, \ldots, s+1)$; $\mathcal{L}'$ is the set of lines containing at least two (and hence $s+1$) points of $\mathcal{P}'$, I' is the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. Now let L_p be a line of $\mathcal{S}$ through p in π_2. Then L_p intersects $\langle x, y \rangle$ in a point w. Let L_w be a line of $\mathcal{S}$ in π_1 through w'. The substructure $S(L_p, L_{w'})$ is a dual affine plane, a projective plane or a punctured affine plane. So it contains either s or $s+1$ lines of $\mathcal{S}$ through p and at most one line of the second type. We denote these lines by $N_1 = L_p, \ldots, N_{s+1}$, where N_{s+1} can be a line of $\mathcal{S}$ or a line of the second type. The substructures $S(w, N_i)$ $(i = 1, \ldots, s)$ are s dual affine planes, since each of them contains the line $\langle p, q \rangle$. The substructure $S(q, N_{s+1})$, is either a dual affine plane, or it is a plane of type V or VI. These planes intersect π_1 each in a line of the second type. Indeed, if they would intersect π_1 in a line L of $\mathcal{S}$, then L would intersect $\langle x, y \rangle$ in a point u', hence $\langle p, q \rangle$ intersects $\langle x, y \rangle$ in the point u', a contradiction with the assumption. Now π_1 contains exactly $s+1$ lines of the second type. Hence $\langle u, v \rangle$ is one of the lines of the second type contained in either $S(w, N_k)$, for a $k \in \{1, \ldots, s\}$, or it is contained in the plane $S(w, N_{s+1})$, which can be a dual affine plane, or a plane of type V or VI. By definition of parallelism, it follows in every case that $\langle p, w \rangle$ is parallel with $\langle u, v \rangle$.

2. *Assume that the plane containing $\langle x, y \rangle$ and $\langle u, v \rangle$ is a dual affine plane but the plane containing $\langle x, y \rangle$ and $\langle p, w \rangle$ is not a dual affine plane.*

Then $\langle x, y \rangle$ and $\langle p, w \rangle$ are contained in a plane of type V or type VI. We call this plane of type V or VI the plane ω, while we call the dual affine plane through $\langle x, y \rangle$ and $\langle u, v \rangle$ the plane π. Let N be a line of $\mathcal{S}$ in π. Let M be a line of $\mathcal{S}$ through the point $p \in \omega$ that intersects N in a point. Let M' be a line of $\mathcal{S}$ intersecting both $S(M, x)$ and $S(M, y)$ in a point not on M. Let $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ be the incidence structure defined as follows: $\mathcal{P}'$ is the set of points of $\mathcal{S}$ contained in the substructures $S(M, x_i)$, for x_i on M' $(i = 1, \ldots, s+1)$, $\mathcal{L}'$ is the set of lines of $\mathcal{S}$ containing at least two points of $\mathcal{P}'$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. Then $\mathcal{S}'$ contains both π and ω, since it contains three points of each of them.

The line $\langle x, p \rangle$ is a line of the second type through x in ω. Let N_1 be a line of $\mathcal{S}$ through x in π. Then $S(p, N_1)$ is a dual affine plane or a punctured affine plane. So p is collinear with the s points of N_1 different from x. Let $M_1, \ldots M_s$ be the s lines of $\mathcal{S}$ through p and a point of N_1. Then $S(w, M_i)$ $(i = 1, \ldots, s)$, are s dual affine planes containing the line $\langle p, w \rangle$. Clearly they can not intersect π in a line of $\mathcal{S}$, for otherwise this line would have a point in common with $\langle p, w \rangle$ and so $\langle x, y \rangle$ and $\langle p, w \rangle$ would have a point in common, a contradiction. Hence they each intersect π in a line of the second type different from $\langle x, y \rangle$. Now in the dual affine plane π there are exactly $s+1$ lines of the second type. So $\langle u, v\rangle$ has to be one of the lines contained in a dual affine plane $S(w, M_i)$, for a $i \in \{1, \dots, s\}$. By definition it now follows that $\langle w, q\rangle$ is parallel with $\langle u, v\rangle$.

3. *Assume that none of the two planes containing $\langle x, y\rangle$ and $\langle u, v\rangle$ respectively containing $\langle x, y\rangle$ and $\langle p, w\rangle$ is a dual affine plane.*

In this case the points of the lines $\langle x, y\rangle$, $\langle p, w\rangle$ and $\langle u, v\rangle$ belong to an equivalence class C of non-collinear points of $\mathcal{S}$. Suppose that $w \notin C$. Then w' is collinear with every point of C. In particular w' is collinear with x. The plane $S(\langle w', x\rangle, y)$ is a dual affine plane. It contains a line of the second type through w' that is parallel to $\langle x, y\rangle$. We denote this line by $\langle w', z\rangle$. From the preceding case it follows that $\langle w', z\rangle$ is parallel to both $\langle u, v\rangle$ and $\langle p, w\rangle$.

The plane containing $\langle x, y\rangle$ and $\langle p, w\rangle$ is not a dual affine plane by assumption. It follows that the line $\langle x, p\rangle$ is a line of the second type. So $S(\langle w', x\rangle, p)$ is a dual affine plane or a punctured affine plane. Let L_p be a line of $\mathcal{S}$ through p in $S(\langle w', x\rangle, p)$. Let L_y be a line of $\mathcal{S}$ through y in $S(\langle w', x\rangle, y)$, such that L_y is skew to L_p. Then we can again define an incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ as follows: $\mathcal{P}'$ is the set of points contained in the $S(L_p, z_i)$, with $z_i \in L_y$, $(i = 1, \dots, s+1)$; $\mathcal{L}'$ is the set of lines containing at least two (and hence $s+1$) points of $\mathcal{P}'$, I' is the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. Since the plane containing $\langle x, y\rangle$ and $\langle p, w\rangle$ is a plane of type V or VI, and since from the previous we know that there is exactly one plane of type V or VI in $\mathcal{S}'$ through the point p, it follows that the plane containing $\langle w', z\rangle$ and $\langle p, w\rangle$ is a dual affine plane. In the same way one proves that $\langle w', z\rangle$ and $\langle u, v\rangle$ are contained in a dual affine plane. From the first case it now follows that $\langle p, w\rangle$ and $\langle u, v\rangle$ are parallel.

So we proved that the parallelism defined above is also transitive, and hence it is an equivalence relation. Note that every parallel class is a partition of the point set of $\mathcal{S}$. The parallel classes are called points of the second type, and the set of these classes is denoted by $\mathcal{P}^*$.

Now we will define parallelism among the planes of type V. Suppose therefore that ω is a plane of type V. Let x be a point of $\mathcal{S}$, $x \notin \omega$. Since we proved that the parallel classes of lines partition the points of $\mathcal{S}$, we know that through x there are $s+1$ lines $\langle x, y_1\rangle, \dots, \langle x, y_{s+1}\rangle$ parallel to lines of ω. We will prove that the lines $\langle x, y_1\rangle, \dots, \langle x, y_{s+1}\rangle$ are contained in a plane of type V.

First we assume that there is a line L of $\mathcal{S}$ containing x and a point u of ω. Let $\langle u, u'\rangle$ and $\langle u, u''\rangle$ be two lines of the second type through u in ω. Then

$|\langle u, u' \rangle| = s$ and $|\langle u, u'' \rangle| = s$. Hence $S(L, u')$ and $S(L, u'')$ are two dual affine planes through L. Let N be a line of $\mathcal{S}$ intersecting $S(L, u')$ and $S(L, u'')$ in a point not on L. Then we define an incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ as follows: $\mathcal{P}'$ is the set of points of $\mathcal{S}$ contained in $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, $\mathcal{L}'$ is the set of lines of $\mathcal{S}$ containing at least two points of $\mathcal{P}'$ and I' is the restriction of I to $(\mathcal{P}' \times \mathcal{L}') \cup (\mathcal{L}' \times \mathcal{P}')$. Clearly x belongs to $\mathcal{S}'$. Moreover $S(L, u')$ and $S(L, u'')$ belong to $\mathcal{S}'$ and hence $\mathcal{S}'$ contains three distinct points u, u' and u'' of ω. So ω is contained in $\mathcal{S}'$. From the previous part of the proof, it follows that every substructure $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, is a dual affine plane. The lines $\langle x, y_1 \rangle, \ldots, \langle x, y_{s+1} \rangle$, are each contained in a different dual affine plane $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$. Indeed, these lines are lines through x parallel to lines of ω and in each of the dual affine planes $S(L, z_i)$, for $z_i \in N$ $(i = 1, \ldots, s+1)$, there is exactly one line of the second type containing s points of $\mathcal{S}$ through x that is parallel with a line of ω. By definition of parallelism of the lines of the second type containing s points of $\mathcal{S}$, we know that there can not be two different lines through x parallel to the same line of $\mathcal{S}$ in ω. Let $\mathcal{P}''$ be the set of points of $\mathcal{S}'$ that are not collinear with x, union x. Let $\mathcal{L}''$ be the set of lines of the second type containing at least two points of $\mathcal{P}''$. It follows that all the points of $\mathcal{S}$ on a line of $\mathcal{L}''$ are contained in $\mathcal{P}''$. Let I'' be the restriction of I' to $(\mathcal{P}'' \times \mathcal{L}'') \cup (\mathcal{L}'' \times \mathcal{P}'')$. Then $\mathcal{S}'' = (\mathcal{P}'', \mathcal{L}'', \mathrm{I}'')$ contains each of the lines $\langle x, y_i \rangle$, for $(i = 1, \ldots, s+1)$. From the previous part of the proof it follows that $\mathcal{S}''$ is a plane of type V.

Next we assume that there is no line of $\mathcal{S}$ through x and a point of ω. In this case x and the points of ω belong to an equivalence class C of non-collinear points in $\mathcal{S}$. Suppose that $v \notin C$. Then v is collinear with x and with each point of ω. Let $\langle v, z_1 \rangle, \ldots, \langle v, z_{s+1} \rangle$ be the lines containing v and parallel to $\langle x, y_1 \rangle, \ldots, \langle x, y_{s+1} \rangle$. From the preceding case it follows that the points on these lines are the points of a plane of type V. Indeed, $\langle v, z_1 \rangle, \ldots, \langle v, z_{s+1} \rangle$ are parallel to lines of ω and v is collinear in $\mathcal{S}$ with every point of ω. As $\langle v, x \rangle$ is a line of $\mathcal{S}$, the same argument shows that the points on $\langle x, y_1 \rangle, \ldots, \langle x, y_{s+1} \rangle$ are points of a plane of type V.

So we have proved that the points of the lines $\langle x, y_1 \rangle, \ldots, \langle x, y_{s+1} \rangle$ are the points of a plane of type V. Now we define parallelism between planes of type V as follows: two planes ω and ω' of type V are parallel if some line of the second type in ω is parallel to a line of the second type in ω'. The so defined parallelism is an equivalence relation, which follows immediately from the definition of parallelism for lines of the second type containing s points of $\mathcal{S}$. Each parallel class of planes of type V partitions the point set of $\mathcal{S}$. The parallel classes are called lines of the third type, and the set of these classes is denoted by $\mathcal{L}^*$.

Next we introduce a new incidence structure $\overline{\mathcal{S}} = (\overline{\mathcal{P}}, \overline{\mathcal{L}}, \overline{\mathrm{I}})$, with $\overline{\mathcal{P}} = \mathcal{P} \cup \mathcal{P}^*$, $\overline{\mathcal{L}}$ is the set of all lines of $\mathcal{S}$, all lines of the second type and all lines of the third type and with incidence relation $\overline{\mathrm{I}}$ defined as follows:

1. for $x \in \mathcal{P}$ and $L \in \mathcal{L}$: $x \ \overline{\mathrm{I}} \ L \iff x \ \mathrm{I} \ L$;
2. for $x \in \mathcal{P}$ and $\langle y, z\rangle$ a line of the second type, $|\langle y, z\rangle| = s$ or $s+1$: $x \ \overline{\mathrm{I}} \ \langle y, x\rangle \iff x \in \langle y, z\rangle$;
3. for $x \in \mathcal{P}$ and $[\omega] \in \mathcal{L}^*$: x is not incident with $[\omega]$;
4. for $[\langle y, z\rangle] \in \mathcal{P}^*$ and $L \in \mathcal{L}$: $[\langle y, z\rangle]$ is not incident with L;
5. for $[\langle y, z\rangle] \in \mathcal{P}^*$ and $\langle u, v\rangle$ a line of the second type, $|\langle u, v\rangle| = s$: $[\langle y, z\rangle] \ \overline{\mathrm{I}} \ \langle u, v\rangle \iff \langle u, v\rangle \in [\langle y, z\rangle]$;
6. for $[\langle y, z\rangle] \in \mathcal{P}^*$ and $\langle u, v\rangle$ a line of the second type, $|\langle u, v\rangle| = s+1$: $[\langle y, z\rangle]$ is not incident with $\langle u, v\rangle$;
7. $[\langle y, z\rangle] \in \mathcal{P}^*$ and $[\omega] \in \mathcal{L}^*$: $[\langle y, z\rangle] \ \overline{\mathrm{I}} \ [\omega]$ if and only if the line $\langle y, z\rangle$ of the second type is parallel to a line of the second line that is contained in the plane ω.

It remains to prove that $\overline{\mathcal{S}}$ is the design of points and lines of a projective space. We first prove that every two different points of $\overline{\mathcal{S}}$ are incident with exactly one line of $\overline{\mathcal{S}}$. We distinguish three cases.

1. Assume that p_1, $p_2 \in \mathcal{P}$, $p_1 \neq p_2$. Then either p_1 is collinear in $\mathcal{S}$ with p_2, in which case there is exactly one line of $\mathcal{S}$ through p_1 and p_2 and no other line of $\overline{\mathcal{S}}$ contains p_1 and p_2, or p_1 and p_2 are not collinear in which case they are on a line of the second type and on no other line of $\overline{\mathcal{S}}$.

2. Assume that $p_1 \in \mathcal{P}$, $[\langle x, y\rangle] \in \mathcal{P}^*$. In this case the unique line of $\overline{\mathcal{S}}$ through p_1 and $[\langle x, y\rangle]$ is the line of the second type through p_1 that belongs to the parallel class of $\langle x, y\rangle$.

3. Assume that $[\langle x, y\rangle], [\langle u, v\rangle] \in \mathcal{P}^*$, $[\langle x, y\rangle] \neq [\langle u, v\rangle]$. If $\langle x, y\rangle$ and $\langle u, v\rangle$ have a point in common, then x, y, u, v are contained in a plane ω of type V. Note that it cannot be a plane of type VI, because in a plane of type VI all lines of the second type containing s points of $\mathcal{S}$ belong to the same parallel class. So clearly $[\langle x, y\rangle]$ and $[\langle u, v\rangle]$ are two points of the line $[\omega]$ and this is the only line through them. If $\langle x, y\rangle$ and $\langle u, v\rangle$ have no point in

common, then we can always choose a line in $[\langle x,y\rangle]$ that does have a point in common with $\langle u,v\rangle$ (namely the line through u and $[\langle x,y\rangle]$). So the same argument as before shows that $[\langle x,y\rangle]$ and $[\langle u,v\rangle]$ are on exactly one line of $\overline{\mathcal{S}}$.

Next we prove that every three distinct points of $\overline{\mathcal{S}}$, that are not incident with a common element of $\overline{\mathcal{L}}$, generate a projective plane. From the definition of $\overline{\mathcal{S}}$ it follows that an dual affine plane, a projective plane and a punctured affine plane induce projective planes. Also planes of type IV, V and VI are projective planes, containing no lines of $\mathcal{S}$ but lines of the second type and lines of $\mathcal{L}^*$. Now we consider the following cases.

1. Assume that p_1, p_2, $p_3 \in \mathcal{P}$. Then clearly there is either a dual affine plane, a projective plane, a punctured affine plane or a plane of type IV, V or VI containing p_1, p_2 and p_3. Hence in any case p_1, p_2 and p_3 are in a projective plane.

2. Assume that p_1, $p_2 \in \mathcal{P}$ and $[\langle x,y\rangle] \in \mathcal{P}^*$. The lines $\langle p_1,[\langle x,y\rangle]\rangle$ and $\langle p_2,[\langle x,y\rangle]\rangle$ are lines of the second type containing s points of $\mathcal{S}$. If $\langle p_1,p_2\rangle$ is a line of $\mathcal{S}$, then $S(\langle p_1,p_2\rangle,[\langle x,y\rangle])$ is a dual affine plane, and hence a projective plane. If $\langle p_1,p_2\rangle$ is a line of the second type containing s points of $\mathcal{S}$, then let L_{p_1} be a line of $\mathcal{S}$ through p_1. The substructures $S(L_{p_1},p_2)$ and $S(L_{p_1},[\langle x,y\rangle])$ are both dual affine planes. Let M be a line of $\mathcal{S}$ contained in $S(L_{p_1},p_2)$. Let N be a line of $\mathcal{S}$ contained in $S(L_{p_1},[\langle x,y\rangle])$, N skew to M. Then we define an incidence structure $\mathcal{S}'=(\mathcal{P}',\mathcal{L}',\mathrm{I}')$ as follows. Let $\mathcal{P}'$ be the set of points of $\mathcal{S}$ contained in $S(M,z_i)$, for z_i a point of N, let $\mathcal{L}'$ be the set of lines of $\mathcal{S}$ containing at least two points of $\mathcal{P}'$ and let I' is the restriction of I to $(\mathcal{P}'\times\mathcal{L}')\cup(\mathcal{L}'\times\mathcal{P}')$. Then $\mathcal{S}'$ contains the dual affine planes $S(L_{p_1},p_2)$ and $S(L_{p_1},[\langle x,y\rangle])$. Since $\langle p_1,p_2\rangle$ is a line of the second type containing s points of $\mathcal{S}$, it follows that there is a plane of type V through p_1, p_2 and $[\langle x,y\rangle]$. If $\langle p_1,p_2\rangle$ is a line of the second type containing $s+1$ points of $\mathcal{S}$, then as in the previous case we can define an incidence structure $\mathcal{S}'$. It then follows that there is a plane of type VI through p_1, p_2 and $[\langle x,y\rangle]$.

3. Assume that $p\in\mathcal{P}$ and $[\langle x,y\rangle]$, $[\langle u,v\rangle]\in\mathcal{P}^*$. The lines $\langle p,[\langle x,y\rangle]\rangle$ and $\langle p,[\langle u,v\rangle]\rangle$ are lines of the second type that contain s points of $\mathcal{S}$. We have proved above that there is a line $[\omega]$ of the third type, that contains both $[\langle x,y\rangle]$ and $\langle u,v\rangle$. We have also proved that every point of $\mathcal{S}$ belongs to a plane ω' of type V with parallel class $[\omega]$. Hence there is a plane ω_p of type V through p with parallel class $[\omega]$. The plane ω_p is a projective plane containing the points p, $[\langle x,y\rangle]$ and $\langle u,v\rangle$.

4. Assume that $[\langle x,y\rangle]$, $[\langle u,v\rangle]$, $[\langle p,q\rangle] \in \mathcal{P}^*$. Let w be a point of $\mathcal{P}$. The line through w and $[\langle x,y\rangle]$ (respectively $[\langle p,q\rangle]$ and $[\langle u,v\rangle]$) is a line of the second type that contains s points of $\mathcal{S}$. Let r_1 (respectively r_2 and r_3) be a point of $\mathcal{S}$ on this line, that is different from w. Let L be a line of $\mathcal{S}$ through w. The substructures $S(L, r_1)$ and $S(L, r_2)$ are both dual affine planes, since they contain a line of the second type on which there are s points of $\mathcal{S}$. As we did before, we can prove that the points w, r_1 and r_2 are contained in a plane ω of type V. Hence $\langle r_1, r_2\rangle$ is a line of the second type containing s points of $\mathcal{S}$, while $[\langle x,y\rangle]$ and $[\langle p,q\rangle]$ are both contained in the line $[\omega]$ that is an element of $\mathcal{L}^*$. Moreover for every point z of the line $\langle r_1, r_2\rangle$, $z \in \mathcal{P}$, the line $\langle w, z\rangle$ is a line of the second type containing s points of $\mathcal{S}$ and the point $[\langle w,z\rangle]$ belongs to the line $[\omega]$. The plane through r_1, r_2 and r_3 is a projective plane (since r_1, r_2 and r_3 are points of $\mathcal{P}$, this follows from the previous). We denote this plane by π. The line $\langle r_3, r_1\rangle$ intersects $\langle r_1, r_2\rangle$ in the point r_1 point of $\mathcal{S}$. The line $\langle w, r_3\rangle$ is a line of the second type containing s points of $\mathcal{S}$. Hence the substructures $S(L, r_1)$ and $S(L, r_3)$ are both dual affine planes. As we did before, we can prove that the plane through w, r_1 and r_3 is a plane ω' of type V. It follows that the line $[\omega']$ is an element of $\mathcal{L}^*$, and it intersects $[\omega]$ in the point $[\langle w, r_1\rangle]$.

Let $\rho = (\mathcal{P}_w, \mathcal{L}_w, \mathrm{I}_w)$ be the incidence structure defined as follows. The point set $\mathcal{P}_w$ is the set of points of $\mathcal{P}^*$ that lie on a line of $\mathcal{L}^*$ that intersects both $[\omega]$ and $[\omega']$, together with the points of the lines of $\mathcal{L}^*$ through $[\langle x,y\rangle]$ that intersect a line $[\omega'']$ in a point, $[\omega'']$ being a line of $\mathcal{L}^*$ that intersects both $[\omega]$ and $[\omega']$ in a different point, $\mathcal{L}_w$ is the set of all lines of $\mathcal{L}^*$ containing two points of $\mathcal{P}^*$ (and from the definition of $\mathcal{P}_w$ it follows that every point of such a line belongs to $\mathcal{P}_w$), and I_w is the restriction of I^*.

We will now prove that this plane ρ is a projective plane. Let z' be an arbitrary point of $\mathcal{S}$ in the plane π. If z' is a point of $\langle r_1, r_2\rangle$, then we know that the line $\langle w, z'\rangle$ contains a point of the line $[\omega]$, which is a point of $\mathcal{P}_w$. Assume now that $z' \notin \langle r_1, r_2\rangle$. The line $\langle z', r_3\rangle$ either intersects $\langle r_1, r_2\rangle$ in a point of $\mathcal{S}$, or it is parallel to $\langle r_1, r_2\rangle$, in which case it is a line of the second type containing s points of $\mathcal{S}$. Assume first that $\langle z', r_3\rangle$ intersects $\langle r_1, r_2\rangle$ in a point z'' of $\mathcal{S}$. Then z'' is a point of the plane ω of type V, hence $\langle w, z''\rangle$ is a line of the second type containing s points of $\mathcal{S}$. Also $\langle w, r_3\rangle$ is a line of the second type containing s points of $\mathcal{S}$. Hence the substructures $S(L, z'')$ and $S(L, r_3)$ are both dual affine planes.

As we did before, we can prove that the plane through w, z'' and r_3 is a plane ω^* of type V. It follows that $\langle w, z'\rangle$ is a line of the second type containing s points of $\mathcal{S}$. The line $[\omega^*]$ intersects $[\omega]$ in the point $[\langle w, z''\rangle]$, and it intersects $[\omega']$ in the point $[\langle w, r_3\rangle]$. Hence $[\omega^*]$ is a line of $\mathcal{L}_w$. The point $[\langle w, z'\rangle]$ lies on this line, hence it is a point of $\mathcal{P}_w$. Assume next that $\langle z', r_3\rangle$ and $\langle r_1, r_2\rangle$ are parallel. Then these two lines both contain the point $[\langle z', r_3\rangle]$ of $\mathcal{P}^*$. The line through w and $[\langle z', r_3\rangle]$ is a line of the second type containing s points of $\mathcal{S}$. Let $\tilde{z}$ be a point of $\mathcal{S}$ on this line, $\tilde{z} \neq w$. The substructures $S(L, \tilde{z})$ and $S(L, r_3)$ are both dual affine planes. It follows that the plane containing w, r_3 and $\tilde{z}$ is a plane $\tilde{\omega}$ of type V. The line $[\tilde{\omega}]$ is an element of $\mathcal{L}^*$. This line intersects $[\omega]$ in the point $[\langle z', r_3\rangle]$, while it intersects $[\omega']$ in the point $[\langle w, r_3\rangle]$. Hence $[\tilde{\omega}]$ is an element of $\mathcal{L}_w$, and the point $[\langle w, z'\rangle]$ is an element of $\mathcal{P}_w$. So with each point z of the plane $\langle r_1, r_2, r_3\rangle$ there corresponds a point $[\langle w, z\rangle]$ of ρ, and this point is unique. This proves that ρ is isomorphic with the projective plane $\langle r_1, r_2, r_3\rangle$. It follows that ρ is a projective plane. So $[\langle x, y\rangle]$, $[\langle p, q\rangle]$ and $[\langle u, v\rangle]$ generate a projective plane.

Hence $\overline{\mathcal{S}}$ is the design of points and lines of a projective space $\mathrm{PG}(n, s)$. Since every two distinct points of $\mathcal{P}^*$ generate a line of $\mathcal{L}^*$, it follows that $\mathcal{S}^* = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ is the design of points and lines of a projective subspace $\Psi[m]$ of $\mathrm{PG}(n, s)$. Since not every line of $\overline{\mathcal{S}}$ contains a point of $\mathcal{P}^*$, it is clear that $m \leq n - 2$.

Assume that there is no punctured affine plane. The lines of $\overline{\mathcal{S}}$ are the lines of $\mathcal{S}$, the lines of the second type containing s points of $\mathcal{S}$ and a point of $\mathcal{S}^*$, and the lines of the form $[\omega]$, with ω a plane of type V. So $\mathcal{P}$ is the set of all points of $\mathrm{PG}(n, s) \setminus \Psi[m]$, $\mathcal{L}$ is the set of all lines skew to $\Psi[m]$ and I is the incidence of $\mathrm{PG}(n, s)$. This proves that $\mathcal{S}$ is isomorphic to $\mathrm{H}_q^{n,m}$. If $m = 0$ or $m = n - 2$, then $\mathcal{S}$ is a partial geometry, a contradiction. Hence $0 < m < n - 2$.

Assume next that there is a punctured affine plane. Then $\overline{\mathcal{S}}$ contains lines on which there are $s + 1$ points of $\mathcal{S}$, that are not lines of $\mathcal{S}$. Let $\mathcal{B}$ be the set of all these lines. Since $\mathcal{S}$ is a $(q, q+1)$-geometry of order (s, t), the number of lines of $\mathcal{S}$ through a point is a constant $t + 1$. The points of $\mathcal{S}^*$ are the points of a subspace $\mathrm{PG}(m, s)$ of $\mathrm{PG}(n, s)$, hence the number of lines through a point of $\mathcal{S}$ that contain a point of $\mathcal{S}^*$, is also a constant. It follows that the number of lines of $\mathcal{B}$ through a point of $\mathcal{S}$ is a constant. A punctured affine plane contains exactly one line of $\mathcal{B}$. Neither a dual affine plane, nor a projective plane can contain a line of $\mathcal{B}$. Hence a plane that contains two lines of $\mathcal{B}$, can not contain a line of $\mathcal{S}$. This proves that the lines of $\mathcal{B}$ through a point x of $\mathcal{S}$ are the lines through x in an r-dimensional subspace $\Pi_x[r]$ of $\mathrm{PG}(n, s)$, and this subspace contains no lines of $\mathcal{S}$ through x. It immediately follows that the subspace $\Pi_x[r]$ can not contain lines of

$\mathcal{S}$, since on such a line L of $\mathcal{S}$ there would be no points that are collinear with x, a contradiction since $\mathcal{S}$ is an $(s, s+1)$-geometry. Let y be a point of $\Pi_x[r]$, y different from x. Then the subspace $\Pi_y[r']$ coincides with $\Pi_x[r]$. Indeed, all lines through y in $\Pi_x[r]$ are lines that do not belong to $\mathcal{S}$, so surely $\Pi_x[r] \subset \Pi_y[r']$. Now assume that $\Pi_y[r']$ is not a subspace of $\Pi_x[r]$. Then $\Pi_y[r]$ would contain a line L of $\mathcal{S}$ through x. By definition of $\Pi_y[r']$, it follows that no point of L is collinear with y in $\mathcal{S}$. This is a contradiction, since $\mathcal{S}$ is an $(s, s+1)$-geometry. This proves that $\Pi_y[r'] = \Pi_x[r]$. Hence for every point z of $\mathcal{S}$, the dimension of $\Pi_z[r'']$ is r. Now $\Psi[m] \subset \Pi_z[r]$, for every point z of $\mathcal{S}$. Indeed, if $\Psi[m]$ would not be contained in $\Pi_z[r]$, then let N be a line of $\mathcal{B}$ through z and let $\langle z, y\rangle$ be a line through z that contains a point y of $\Psi[m]$, such that the plane through y and N contains a line of $\mathcal{S}$ through z. This gives us a contradiction. Indeed, since $N \in \mathcal{B}$, it would follow that $\langle y, N\rangle$ is a punctured affine plane and a punctured affine plane does not contain a point of $\Psi[m]$. This proves that $\Psi[m] \subset \Pi_z[r]$, for each point z of $\mathcal{S}$. Hence $r \geq m+2$. Two subspaces $\Pi_{p_1}[r]$ and $\Pi_{p_2}[r]$ either coincide, or they have no point of $\mathcal{S}$ in common. Hence the subspaces $\Pi_z[r]$, for $r \in \mathcal{S}$, partition the points of $\mathrm{PG}(n, s) \setminus \Psi[m]$. So $\mathcal{P}$ is the set of all points of $\mathrm{PG}(n, s) \setminus \Psi[m]$, for $0 \leq m \leq n-2$, $\mathcal{L}$ is the set of all lines of $\mathrm{PG}(n, s)$ skew to Ψ and not contained in a partitioning Σ of the points of $\mathrm{PG}(n, s) \setminus \Psi[m]$ into r-dimensional, for $m+2 \leq r \leq n-1$. We conclude that $\mathcal{S}$ is isomorphic to $\mathrm{SH}_q^{n,m}$. $\square$

Note that the $(q, q+1)$-geometry $\mathrm{SH}_q^{n,m}$ contains no dual affine plane in case $m = -1$. So for the case $m = -1$, $\mathrm{SH}_q^{n,m}$ is not characterized by the previous theorem. In fact also in chapter 2 the case $m = -1$ was excluded, as in this case every plane of $\mathrm{PG}(n, q)$ is a punctured affine plane.

Remark. If we do not assume $\beta = s+1$ from the beginning, then the previous theorem and the remarks given above prove the following theorem.

Let $\mathcal{S}$ be a proper (α, β)-geometry of order (s, t), $1 < \alpha < \beta < t+1$, satisfying the following conditions:

1. Pasch axiom,

2. μ-condition,

3. regularity with respect to non-collinear points,

4. there exists no punctured affine plane.

Then $\mathcal{S}$ is isomorphic to $\mathrm{H}_q^{n,m}$.

4.2 The $(\frac{q-1}{2}, \frac{q+1}{2})$-geometry $\mathrm{NQ}^+(3, q)$, q odd

4.2.1 Description of $\mathrm{NQ}^+(3, q)$, for q odd

The $(\frac{q-1}{2}, \frac{q+1}{2})$-geometry $\mathrm{NQ}^+(3, q)$ was constructed by J. A. Thas (personal communication). It is described as follows. Let $\mathcal{P}$ be the set of all points of the projective space $\mathrm{PG}(3, q)$, q odd, that are not contained in a hyperbolic quadric $Q^+(3, q)$. Let $\mathcal{L}$ be the set of the lines of $\mathrm{PG}(3, q)$ that are skew to $Q^+(3, q)$. Let I be the incidence of $\mathrm{PG}(3, q)$ restricted to $(\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$. Then it is easy to check that the incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a $(\frac{q-1}{2}, \frac{q+1}{2})$-geometry. This geometry is denoted as $\mathrm{NQ}^+(3, q)$. Note that this notation is also used for the semipartial geometry $\mathrm{NQ}^+(2n-1, 2)$, that is defined in a similar way.

4.2.2 A useful lemma

Lemma 4.2.1 *Let $\mathcal{S}$ be a $(\frac{q-1}{2}, \frac{q+1}{2})$-geometry fully embedded in $\mathrm{PG}(n, q)$, q odd and $q > 3$. Let π be a plane of $\mathrm{PG}(n, q)$ that contains an antiflag of $\mathcal{S}$. Then the points of $\mathcal{S}$ in π are the points not contained in a conic C of π, and the lines of $\mathcal{S}$ in π are the lines in π that contain no point of C.*

Proof. Let $\mathcal{S}$ be a $(\frac{q-1}{2}, \frac{q+1}{2})$-geometry fully embedded in $\mathrm{PG}(n, q)$, q odd. Let π be a plane of $\mathrm{PG}(n, q)$ that contains an antiflag of $\mathcal{S}$. Through every point of $\mathcal{S}$ in π, there are either $\frac{q-1}{2}$ or $\frac{q+1}{2}$ lines of $\mathcal{S}$ in π. We will prove now that π contains at least one point through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π. Assume therefore that π contains no points of $\mathcal{S}$ through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π. Then the restriction of $\mathcal{S}$ to π is a partial geometry $\mathrm{pg}(q, \frac{q-1}{2}, \frac{q+1}{2})$. From [20] it follows that the points and lines of a partial geometry fully embedded in a projective plane, are the points not contained in a maximal arc $\mathcal{K}$ of order $\frac{q+1}{2}$, and the lines that contain no point of $\mathcal{K}$. However, for q odd, a non-trivial maximal arc does not exist. It follows that π does contain points through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π. In the same way one can prove that π contains at least one point through which there are $\frac{q+1}{2}$ lines of $\mathcal{S}$ in π.

Assume first that π contains a line N on which there are $q+1$ points of $\mathcal{S}$, but that is not a line of $\mathcal{S}$. Then we count the number of lines of $\mathcal{S}$ in π that intersect N. Let therefore c be the number of points of N through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π. Then the number of lines of $\mathcal{S}$ in π equals $c\frac{q-1}{2} + (q+1-c)\frac{q+1}{2} = \frac{(q+1)^2}{2} - c$. Now let L be a line of $\mathcal{S}$ in π. Let a be the number of points of L through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π. Then the number of lines of $\mathcal{S}$ in π equals $1 + a\frac{q-3}{2} + (q+1-a)\frac{q-1}{2} = \frac{q^2+1}{2} - a$. So we get that

$$\begin{aligned} \frac{(q+1)^2}{2} - c &= \frac{q^2+1}{2} - a \\ c &= q + a \end{aligned}$$

Now $1 \leq a \leq q$, since a is a constant for every line L of $\mathcal{S}$ in π and π contains points through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π and points through which there are $\frac{q+1}{2}$ lines of $\mathcal{S}$ in π. We also know that $0 \leq c \leq q+1$, since N has $s+1$ points of $\mathcal{S}$. So from $c = q + a$, it follows that $a = 1$ and $c = q + 1$. In other words, through every point of N there are $\frac{q-1}{2}$ lines of $\mathcal{S}$. Assume now that π would contain a point p, $p \notin N$, through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$. Then every line through p in π intersects N in a point. Hence every such line contains two points through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$. However, we have proved above that every line of $\mathcal{S}$ in π contains exactly one point through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π. So there can be no lines of $\mathcal{S}$ through p in π, a contradiction. We conclude that through every point of π not on N there are $\frac{q+1}{2}$ lines of $\mathcal{S}$. Now we define a new incidence structure $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ as follows: $\mathcal{P}' = \mathcal{P} \cap \pi$, $\mathcal{L}' = (\mathcal{L} \cap \pi) \cup \{N\}$, and I' the restriction of I to $\mathcal{S}'$. Then $\mathcal{S}'$ is a partial geometry $\mathrm{pg}(q, \frac{q-1}{2}, \frac{q+1}{2})$ fully embedded in the projective plane π. This is again a contradiction, as by assumption q is odd and hence there exists no non-trivial maximal arc in π [1], which implies that also a partial geometry $\mathrm{pg}(q, \frac{q-1}{2}, \frac{q+1}{2})$ does not exist. This proves that π can not contain a line N on which there are $q+1$ points of $\mathcal{S}$, but that does not belong to $\mathcal{S}$.

Let v_α (respectively v_β) be the number of points of π through which there are $\frac{q-1}{2}$ (respectively $\frac{q+1}{2}$) lines of $\mathcal{S}$ in π. Let v_π and b_π be the number of points and lines of $\mathcal{S}$ in π. Then clearly $v_\alpha + v_\beta = v_\pi$. Moreover we have the following equations.

$$\frac{q-1}{2}v_\alpha + \frac{q+1}{2}v_\beta = (q+1)b_\pi \tag{4.1}$$

$$\binom{\frac{q-1}{2}}{}\left(\frac{q-3}{2}\right)v_\alpha + \left(\frac{q+1}{2}\right)\left(\frac{q-1}{2}\right)v_\beta = b_\pi(b_\pi - 1) \tag{4.2}$$

The first equation is obtained by counting the flags (p, L), for p and L a point and a line of $\mathcal{S}$ in π, in two different ways. The second equation we obtain by counting triples (p, L, L'), for p, L and L' a point and two lines of $\mathcal{S}$ in π, p incident with both L and L'.

From (4.1) it follows that

$$v_\alpha = 2\frac{q+1}{q-1}(b_\pi - \frac{v_\beta}{2}). \tag{4.3}$$

Substituting (4.3) in (4.2), we get that

$$v_\beta = \frac{2b_\pi(b_\pi - 1)}{q+1} - (q-3)b_\pi. \tag{4.4}$$

Now substituting (4.4) in (4.3), it follows that

$$v_\alpha = (q+1)b_\pi - \frac{2b_\pi(b_\pi - 1)}{q-1}. \tag{4.5}$$

From the equation $v_\alpha + v_\beta = v_\pi$, we obtain

$$v_\pi = 4b_\pi \frac{q^2 - b_\pi}{q^2 - 1}. \tag{4.6}$$

Now let L be a line of $\mathcal{S}$ in π. Assume that L contains a points through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π. Then again $1 \leq a \leq q$, since we have proved above that π contains points through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π as well as points through which there are $\frac{q+1}{2}$ lines of $\mathcal{S}$ in π. Counting the lines of $\mathcal{S}$ in π that intersect L, we get that

$$\begin{aligned} b_\pi &= 1 + a\frac{q-3}{2} + (q+1-a)\frac{q-1}{2} \\ &= \frac{q^2+1}{2} - a \end{aligned}$$

Substituting this result into (4.6), it follows that

$$v_\pi = q^2 + 1 + \frac{4a(1-a)}{q^2-1} \tag{4.7}$$

The number of points of π that do not belong to $\mathcal{S}$ clearly equals $q^2+q+1-v_\pi$. So there are $q+\frac{4a(a-1)}{q^2-1}$ such points. Since $a \geq 1$, we get that $q+\frac{4a(a-1)}{q^2-1} \geq q$. From $a \leq q$, it follows that

$$\frac{4a(a-1)}{q^2-1} \leq \frac{4q(q-1)}{q^2-1} = \frac{4q}{q+1}.$$

Since $q/(q+1) < 1$, we get that

$$q + \frac{4a(a-1)}{q^2-1} < q + 4.$$

We conclude that the number of points of π that do not belong to $\mathcal{S}$, equals q, $q+1$, $q+2$ or $q+3$.

Assume that π contains a line M on which there are q points of $\mathcal{S}$. We will prove that in this case there are exactly $q+1$ points in π that do not belong to $\mathcal{S}$. Let c be the number of points of M through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π. Then $0 \leq c \leq q$. Counting the lines of $\mathcal{S}$ in π through the points of M, we get that

$$\begin{aligned} b_\pi &= c\frac{q-1}{2} + (q-c)\frac{q+1}{2} \\ &= \frac{q^2+q}{2} - c. \end{aligned}$$

Let L be a line of $\mathcal{S}$ in π. Let a be the number of points of L through which there are $\frac{q-1}{2}$ lines of $\mathcal{S}$ in π. Then $1 \leq a \leq q$. Counting the lines of $\mathcal{S}$ through the points of L, it follows that

$$b_\pi = \frac{q^2+1}{2} - a.$$

From these two equations it follows that

$$c = \frac{q-1}{2} + a. \tag{4.8}$$

Now we prove that $a \geq \frac{q+1}{2}$. Assume therefore that $a < \frac{q+1}{2}$. Then

$$\begin{aligned} a(a-1) &< \left(\frac{q+1}{2}\right)\left(\frac{q-1}{2}\right) \\ \Longleftrightarrow 4a(a-1) &< q^2-1 \\ \Longleftrightarrow \frac{4a(a-1)}{q^2-1} &< 1. \end{aligned}$$

However, from (4.7) we know that $\frac{4a(a-1)}{q^2-1} \in \mathbb{N}$. This is a contradiction. So $a \geq \frac{q+1}{2}$. Substituting this in (4.8), we get that $c \geq q$. By definition, $c \leq q$. So $c = q$ and $a = \frac{q+1}{2}$. Substituting this value for a in (4.7), we obtain $v_\pi = q^2$. This proves that there are exactly $q+1$ points of π that do not belong to $\mathcal{S}$, if π contains a line on which there are exactly q points of $\mathcal{S}$.

Assume first that the number of points of π that do not belong to $\mathcal{S}$, equals q. Let x be a point of π that does not belong to $\mathcal{S}$. The $q+1$ lines through x in π contain the $q-1$ remaining points of π that do not belong to $\mathcal{S}$. It follows that there is a line through x in π on which there lies no other point that does not belong to $\mathcal{S}$. Hence π contains a line on which there are exactly q points of $\mathcal{S}$. From the previous paragraph it follows that $v_\pi = q^2$, a contradiction since we assumed that π contains exactly q points that do not belong to $\mathcal{S}$.

Assume next that π contains $q+2$ points that do not belong to $\mathcal{S}$. Since q is odd, a hyperoval does not exist. Hence there is a line M in π that contains r points of $\mathcal{S}$, $0 \neq r \neq 2$. If $r = 1$, then π contains a line on which there lie exactly q points of $\mathcal{S}$. As in the previous paragraph, this gives us a contradiction. If $r \geq 3$, then let x be a point of M that does not belong to $\mathcal{S}$. The q lines through x in π different from M contain at most $q-1$ points that do not belong to $\mathcal{S}$. Hence also in this case there is a line through x that contains q points of $\mathcal{S}$. It follows that $v_\pi = q^2$, a contradiction since we assumed that π contains $q+2$ points that do not belong to $\mathcal{S}$.

Assume finally that π contains $q+3$ points that do not belong to $\mathcal{S}$. Then on each line of π there are at most 3 points that do not belong to $\mathcal{S}$, as otherwise there would be a line containing exactly s points of $\mathcal{S}$, which would give us a contradiction as in the previous paragraph. Let m_1, m_2 and m_3 be the number of lines of π that contain respectively 1, 2 and 3 points that do not belong to $\mathcal{S}$. Counting the lines of π, the flags (x, M) for x a point of π that does not belong to $\mathcal{S}$ and M a line of π, and the triples (x, y, M) for x and y points of π that do not belong to $\mathcal{S}$ and M a line of π, such that x, $y \in M$, we get the following equations.

$$\begin{cases} b_\pi + m_1 + m_2 + m_3 &= q^2 + q + 1 \\ m_1 + 2m_2 + 3m_3 &= (q+1)(q+3) \\ 2m_2 + 6m_3 &= (q+3)(q+2). \end{cases}$$

If $m_1 > 0$, then from the previous it would follow that $v_\pi = q^2$, a contradiction since we assumed that π contains $q+3$ points that do not belong to $\mathcal{S}$. Hence $m_1 = 0$. So we get that

$$\begin{cases} b_\pi &= \frac{q^2}{2} - \frac{5q}{6} \\ m_3 &= \frac{q+3}{3} \\ m_2 &= \frac{q(q+3)}{2}. \end{cases}$$

We will now count the number b_π of lines of $\mathcal{S}$ in π in another way. From (4.7) it follows that $v_\pi = q^2 + 1 + \frac{4a(1-a)}{q^2-1}$. By assumption the plane π contains $q+3$ points that do not belong to $\mathcal{S}$. So $v_\pi = q^2 - 2$. Hence

$$\frac{4a(a-1)}{q^2-1} = 3,$$

so

$$4a^2 - 4a - 3q^2 + 3 = 0.$$

Solving this quadratic equation in a, we get that $a = \frac{1+\sqrt{3q^2-2}}{2}$. Substituting this in the equation $b_\pi = \frac{q^2+1}{2} - a$, we get that $b_\pi = \frac{q^2-\sqrt{3q^2-2}}{2}$. So

$$\begin{aligned} \frac{q^2}{2} - \frac{5q}{6} &= \frac{q^2}{2} - \frac{\sqrt{3q^2-2}}{2} \\ \frac{5q}{3} &= \sqrt{3q^2-2} \\ 25q^2 &= 9(3q^2-2) \\ q &= 3. \end{aligned}$$

Hence $q = 3$. This is a contradiction, since we assumed $q > 3$. Hence π can not contain $q+3$ points that do not belong to $\mathcal{S}$.

From the previous paragraphs it follows that the number of points that do not belong to $\mathcal{S}$, equals $q+1$. So $a = \frac{q+1}{2}$, and $b_\pi = \frac{q(q-1)}{2}$. If π would contain a line M on which there are at least 3 points that do not belong to $\mathcal{S}$, then counting the lines of $\mathcal{S}$ through the points of π, we get that $b_\pi \leq (q-2)\frac{q+1}{2} = \frac{q^2-q-2}{2}$. This is a contradiction, since we proved above that $b_\pi = \frac{q(q-1)}{2}$. Hence every line of π contains 0, 1 or 2 points that do not belong to $\mathcal{S}$. We conclude that the points of π that do not belong to $\mathcal{S}$ are the points of a conic C. It also follows from the previous that the lines of $\mathcal{S}$ in π are the lines that contain no point of C. This proves the lemma. □

4.2.3 Characterization of $\mathrm{NQ}^+(3,q)$, for q odd

Theorem 4.2.2 *Let $\mathcal{S}$ be a $(\frac{q-1}{2}, \frac{q+1}{2})$-geometry of order (q,t) fully embeddable in* $\mathrm{PG}(3,q)$*, for q odd and $q > 3$. Then $\mathcal{S} = \mathrm{NQ}^+(3,q)$.*

Proof. Let $\mathcal{S}$ be a $(\frac{q-1}{2}, \frac{q+1}{2})$-geometry of order (q,t) fully embeddable in $\mathrm{PG}(3,q)$, for q odd and $q > 3$. From lemma 4.2.1 it follows that every plane that contains an antiflag of $\mathcal{S}$, intersects $\mathcal{S}$ in the points not on a conic C and the lines that contain no point of C.

We will first prove that $\mathrm{PG}(3,q)$ contains exactly $(q+1)^2$ points that do not belong to $\mathcal{S}$. If $\mathrm{PG}(3,q)$ does not contain two lines of $\mathcal{S}$ that are disjoint, then every two lines of $\mathcal{S}$ in $\mathrm{PG}(3,q)$ intersect. It follows that the lines of $\mathcal{S}$ in $\mathrm{PG}(3,q)$ are contained in a plane, or that they are a subset of a pencil of lines through a point p. Both of these possibilities give a contradiction, as the number of lines through a point would not be constant. So we may assume that $\mathrm{PG}(3,q)$ contains two lines L and L' of $\mathcal{S}$ that are skew to each other. Now every plane through L in $\mathrm{PG}(3,q)$ contains an antiflag of $\mathcal{S}$. From lemma 4.2.1 it follows that every plane through L in $\mathrm{PG}(3,q)$ contains $q+1$ points that do not belong to $\mathcal{S}$. Since L is a line of $\mathcal{S}$, it contains no points that do not belong to $\mathcal{S}$. Hence, counting the points of $\mathrm{PG}(3,q)$ that do not belong to $\mathcal{S}$ in the planes through L, we get that there are exactly $(q+1)^2$ such points.

We will now prove that the points of $\mathrm{PG}(3,q)$ that do not belong to $\mathcal{S}$, are the points of a non-singular quadratic set, as defined in section 1.4. We first prove that every line of $\mathrm{PG}(3,q)$ contains 0, 1, 2 or $q+1$ points that do not belong to $\mathcal{S}$. Assume therefore that M is a line on which there are r points of $\mathcal{S}$, $3 \leq r \leq q-1$. Then M contains two points u_1 and u_2 of $\mathcal{S}$. Let L be a line of $\mathcal{S}$ through u_1. The plane $\langle M, L\rangle$ contains an antiflag of $\mathcal{S}$. From lemma 4.2.1 it follows that the points of $\langle M, L\rangle$ that do not belong to $\mathcal{S}$, are the points of a conic. This is a contradiction, since M contains r points of $\mathcal{S}$, $3 \leq r \leq q-1$. Hence $\mathrm{PG}(3,q)$ can not contain a line on which there are r points that do not belong to $\mathcal{S}$, $3 \leq r \leq q-1$. Assume now that $\mathrm{PG}(3,q)$ contains a line M' on which there are q points that do not belong to $\mathcal{S}$. If there is a plane through M' that contains an antiflag of $\mathcal{S}$, then we get again a contradiction. So every plane through M' contains no antiflag of $\mathcal{S}$. Now let u be the point of M' that belongs to $\mathcal{S}$. Since $t+1 \geq 2$, there are at least two lines of $\mathcal{S}$ through u in $\mathrm{PG}(3,q)$. A plane through M and a line of $\mathcal{S}$ through u contains q^2 points that do not belong to $\mathcal{S}$ (since it contains no antiflag of $\mathcal{S}$). It follows that $\mathrm{PG}(3,q)$ contains at least $2(q^2-q)+q = 2q^2-q$ points that do not belong to $\mathcal{S}$. We have proved in the previous paragraph that $\mathrm{PG}(3,q)$ contains exactly $(q+1)^2$ points that do not belong to $\mathcal{S}$. So $(q+1)^2 \geq 2q^2-q$, or thus $q \leq 3$. Since q is assumed to be odd, it follows that $q=3$. However, we assumed that $q>3$. Hence no line of $\mathrm{PG}(3,q)$ contains q points that do not belong to $\mathcal{S}$. We conclude that every line of $\mathrm{PG}(3,q)$ contains 0, 1, 2 or $q+1$ points that do not belong to $\mathcal{S}$.

Let $\mathcal{K}$ be the set of all points of $\mathrm{PG}(3,q)$ that do not belong to $\mathcal{S}$. We call a line of $\mathrm{PG}(3,q)$ a tangent line of $\mathcal{K}$ if it contains either 1 or $q+1$ points of $\mathcal{K}$. We prove now that the tangent lines through a point x of $\mathcal{K}$, are the $(q+1)$ lines through x contained in a plane through x.

From the previous we know that $|\mathcal{K}| = (q+1)^2$. So on the q^2+q+1 lines through x in $\mathrm{PG}(3,q)$, there lie q^2+2q points of $\mathcal{K}$. Since every line through x in $\mathrm{PG}(3,q)$ contains 0, 1 or q points of $\mathcal{K} \setminus \{x\}$, it follows that there is a line N through x on which there are $q+1$ points of $\mathcal{K}$. Now let a

(respectively c) be the number of lines through x on which there are $q+1$ (respectively 2) points of $\mathcal{K}$. Then $aq+c1 = q^2+q$ and $a+c \leq q^2+q+1$. If $a=1$, then $c = q^2+q$ and hence $a+c = q^2+q+1$. It follows that there is no line through x that contains exactly one point of $\mathcal{K}$. This is a contradiction, since the plane $\langle x, L\rangle$ intersects $\mathcal{K}$ in a conic and hence it contains a line through x that contains no point of $\mathcal{K} \setminus \{x\}$. This proves that there are at least two lines through x that contain $q+1$ points of $\mathcal{K}$.

So let N' be a line through x containing $q+1$ points of $\mathcal{K}$, $N' \neq N$. Every line through x in the plane $\langle N, N'\rangle$ is a tangent line of $\mathcal{K}$. Indeed, if not there would be a line through x in $\langle N, N'\rangle$ containing exactly two points x and y of $\mathcal{K}$. Every line through y in $\langle N, N'\rangle$ different from $\langle x, y\rangle$, contains at least 3 points of $\mathcal{K}$, namely y and its intersection point with N and N'. Hence every line through y in $\langle N, N'\rangle$, different from $\langle x, y\rangle$, contains $q+1$ points of $\mathcal{K}$. Let u be a point of $\langle x, y\rangle$, $x \neq u \neq y$. Then the lines through u in $\langle N, N'\rangle$, different from $\langle x, y\rangle$, contain exactly q points of $\mathcal{K}$. This is a contradiction, since we have proved that every line of $\mathrm{PG}(3,q)$ contains 0, 1, 2 or $q+1$ points of $\mathcal{K}$. This proves that every line through x in $\langle N, N'\rangle$ is a tangent line of $\mathcal{K}$.

Assume that $\langle N, N'\rangle$ contains a line N'' through x, $N \neq N'' \neq N'$, on which there are $q+1$ points of $\mathcal{K}$. Then every line of $\langle N, N'\rangle$ not through x contains at least 3 points of $\mathcal{K}$. Hence each such line has to contain $q+1$ points of $\mathcal{K}$. It follows that all points of $\langle N, N'\rangle$ belong to $\mathcal{K}$. This is a contradiction, as every line of $\mathrm{PG}(3,q)$ intersect the plane $\langle N, N'\rangle$ in a point, so it would follow that $\mathrm{PG}(3,q)$ contains no lines of $\mathcal{S}$, which is a contradiction. We conclude that the points of $\mathcal{K}$ in $\langle N, N'\rangle$ are the points of the lines N and N'.

Now assume that there would be a line M through x, M not contained in $\langle N, N'\rangle$, that is tangent to $\mathcal{K}$. Then M contains either 1 or $q+1$ points of $\mathcal{K}$. Assume first that M contains exactly 1 point of $\mathcal{K}$. Let L be a line of $\mathcal{S}$ that intersects M in a point. The plane $\langle L, M\rangle$ contains an antiflag of $\mathcal{S}$. From lemma 4.2.1 it follows that the points of $\mathcal{K}$ in $\langle L, M\rangle$ are the points of a conic. Now the intersection line of $\langle L, M\rangle$ and $\langle N, N'\rangle$ is a tangent line through x different from M. Hence $\langle L, M\rangle$ contains two tangent lines through x at the conic C, which is clearly a contradiction. Assume next that M contains $q+1$ points of $\mathcal{K}$. From the above it follows that every line through x that is not contained in $\langle N, N'\rangle$, contains 2 or $q+1$ points of $\mathcal{K}$, while the plane $\langle N, N'\rangle$ contains $2q+1$ points of $\mathcal{K}$. So in total $\mathcal{K}$ contains at least $1 + 3q + (q^2 - 1) = q^2 + 3q$ points. This is a contradiction, since we have proved above that $|\mathcal{K}| = q^2 + 2q + 1$. This proves that all lines through x that are not contained in $\langle N, N'\rangle$, contain 2 points of $\mathcal{K}$.

In the previous paragraphs we proved that $\mathcal{K}$ is a non-singular quadratic set. From [35] (theorem 22.10.23) it follows that $\mathcal{K}$ is either a quadric, or a $(q+1)$-arc in a plane, or an ovaloïd of $\mathrm{PG}(3,q)$, $q>2$. Since $\mathrm{PG}(3,q)$ contains lines on which there are $q+1$ points of $\mathcal{K}$, it follows that $\mathcal{K}$ is a quadric. Since $|\mathcal{K}|=(q+1)^2$, it is clear that $\mathcal{K}$ is a hyperbolic quadric $Q^+(3,q)$. We conclude that the points of $\mathcal{S}$ in $\mathrm{PG}(3,q)$ are the points that do not lie on a hyperbolic quadric $Q^+(3,q)$, and the lines of $\mathcal{S}$ are the lines that contain no point of $Q^+(3,q)$. Hence $\mathcal{S}=\mathrm{NQ}^+(3,q)$. □

Theorem 4.2.3 *Let $\mathcal{S}$ be a $(\frac{q-1}{2},\frac{q+1}{2})$-geometry of order (q,t), fully embedded in $\mathrm{PG}(n,q)$, for q odd and $q>3$. Then $\mathcal{S}=\mathrm{NQ}^+(3,q)$.*

Proof. Let $\mathcal{S}$ be a $(\frac{q-1}{2},\frac{q+1}{2})$-geometry of order (q,t), fully embedded in $\mathrm{PG}(n,q)$, for q odd and $q>3$. From theorem 4.2.3 it follows that if $n=3$, $\mathcal{S}=\mathrm{NQ}^+(3,q)$. So we assume from now on that $n>3$.

Let $\mathcal{K}$ be the set of points of $\mathrm{PG}(n,q)$ that do not belong to $\mathcal{S}$. We will prove that every line of $\mathrm{PG}(n,q)$ contains 0, 1, 2 or $q+1$ points of $\mathcal{K}$.

Assume therefore that there is a line M containing r points of $\mathcal{K}$, $3\leq r\leq q-1$. Then M contains two points u_1 and u_2 of $\mathcal{S}$. Let L be a line of $\mathcal{S}$ through u_1, $L\neq M$. The plane $\langle L,M\rangle$ contains an antiflag of $\mathcal{S}$. From lemma 4.2.1 it follows that the points of $\mathcal{K}$ in $\langle L,M\rangle$ are the points of a conic. This is a contradiction, since M contains r points of $\mathcal{K}$, $3\leq r\leq q-1$. Hence no line of $\mathrm{PG}(n,q)$ contains r points of $\mathcal{K}$, $3\leq r\leq q-1$.

Assume next that there is a line M' that contains q points of $\mathcal{K}$. Let u be the point of $\mathcal{S}$ on M'. Let L_1 be a line of $\mathcal{S}$ through u. The plane $\langle L_1,M'\rangle$ can not contain an antiflag of $\mathcal{S}$, as otherwise the points of $\mathcal{K}$ in $\langle L_1,M'\rangle$ would be the points of a conic, a contradiction since M' contains q points of $\mathcal{K}$. Hence $\langle L_1,M'\rangle$ contains q^2 points of $\mathcal{K}$. Let L_2 be a line of $\mathcal{S}$ through u, $L_2\neq L_1$. Then the plane $\langle L_2,M'\rangle$ also contains q^2 points of $\mathcal{K}$. Let p be a point of the three dimensional space $\langle M',L_1,L_2\rangle$, $p\notin\langle L_1,L_2\rangle$. If $p\in\mathcal{P}$, then there are at least $(q-1)/2$ lines of $\mathcal{S}$ through p that intersect L_1. Let L_p be a line of $\mathcal{S}$ through p intersecting L_1 in a point different from u. Then L_p intersects the plane $\langle L_2,M'\rangle$ in a point p' that is not contained in L_2. Hence $p'\in\mathcal{K}$. This is a contradiction, as the line L_p of $\mathcal{S}$ can not contain points of $\mathcal{K}$. So $p\notin\mathcal{P}$. It follows that every point of $\mathcal{S}$ in the three dimensional space $\langle M',L_1,L_2\rangle$, is contained in the plane $\langle L_1,L_2\rangle$.

Let L_3 be a line of $\mathcal{S}$ intersecting $\langle L_1,L_2\rangle$ in a point z. It is clear that L_3 exists, since $\mathcal{S}$ is connected and the points of $\mathcal{S}$ span $\mathrm{PG}(n,q)$. From the previous it follows that L_3 is not contained in $\langle M',L_1,L_2\rangle$. Every plane through L_3 and a line M_z through z in $\langle M',L_1,L_2\rangle$, M_z not contained in

$\langle L_1, L_2\rangle$, contains q^2 points of $\mathcal{K}$ (since M_z contains q points of $\mathcal{K}$). Hence all points of $\mathcal{S}$ in the four dimensional space $\langle M', L_1, L_2, L_3\rangle$ are contained in the three dimensional space $\langle L_1, L_2, L_3\rangle$. Continuing in this way, it follows that all points of $\mathcal{S}$ in $\mathrm{PG}(n, q)$ are contained in an $(n-1)$-dimensional subspace of $\mathrm{PG}(n, q)$. This is a contradiction, since the points of $\mathcal{S}$ span $\mathrm{PG}(n, q)$. So $\mathrm{PG}(n, q)$ can not contain a line M' on which there are q points of $\mathcal{K}$. This proves that every line of $\mathrm{PG}(n, q)$ contains 0, 1, 2 or $q+1$ points of $\mathcal{K}$.

Let L be a line of $\mathcal{S}$ in $\mathrm{PG}(n, q)$. Since no line of $\mathrm{PG}(n, q)$ contains q points of $\mathcal{S}$, every plane through L contains an antiflag of $\mathcal{S}$. From lemma 4.2.1 it follows that every plane through L contains $q+1$ points of $\mathcal{K}$. So, counting the points of $\mathcal{K}$ in the planes through L, we get that

$$|\mathcal{K}| = \frac{q^{n-1} - 1}{q - 1}(q + 1).$$

Let x be a point of $\mathcal{K}$. A line N of $\mathrm{PG}(n, q)$ is called a tangent line at $\mathcal{K}$ in x, if N contains x and 0 or q other points of $\mathcal{K}$. We will prove that the tangent lines through x at $\mathcal{K}$ are the lines through x in a hyperplane of $\mathrm{PG}(n, q)$ through x. Let therefore N_1 and N_2 be two lines of $\mathrm{PG}(n, q)$ through x that are tangent to $\mathcal{K}$. Then N_1 contains either 1 or $q+1$ points of $\mathcal{K}$. Also N_2 contains either 1 or $q+1$ points of $\mathcal{K}$. We will deal with each of the possibilities separately.

Assume first that N_1 and N_2 both contain $q+1$ points of $\mathcal{K}$. Assume that $\langle N_1, N_2\rangle$ would contain a line M through x that contains exactly one other point y of $\mathcal{K}$. Then every line through y in $\langle N_1, N_2\rangle$, different from $\langle x, y\rangle$, contains at least 3 points of $\mathcal{K}$, namely y and the intersection points of this line with N_1 and N_2. Hence every such line contains $q+1$ points of $\mathcal{K}$. It follows that the points of $\mathcal{S}$ in $\langle N_1, N_2\rangle$ all lie on the line $\langle x, y\rangle$. Let $u \in \langle x, y\rangle$ be a point of $\mathcal{S}$. Then there is a line through u in $\langle N_1, N_2\rangle$ that contains q points of $\mathcal{K}$. This is a contradiction, since we have proved in the previous paragraph that each line contains 0, 1, 2 or $q+1$ points of $\mathcal{K}$. Hence every line through x in $\langle N_1, N_2\rangle$ is a tangent line to $\mathcal{K}$.

Assume next that N_1 contains $q+1$ points of $\mathcal{K}$, while N_2 contains exactly one point of $\mathcal{K}$. Assume that the plane $\langle N_1, N_2\rangle$ contains a line M through x on which there are exactly two points x and y of $\mathcal{K}$. Let u be a point of $\mathcal{S}$, $u \in N_2$. Let L_u be a line of $\mathcal{S}$ through u. Then it is clear that L_u is not contained in the plane $\langle N_1, N_2\rangle$. Since we have proved above that no line can contain q points of $\mathcal{K}$, it follows that every plane through L_u contains an antiflag of $\mathcal{S}$, and hence every such plane contains exactly $q+1$ points of $\mathcal{K}$ (see lemma 4.2.1). So the three dimensional space $\langle N_1, N_2, L_u\rangle$ contains exactly $(q+1)^2$ points of $\mathcal{K}$. It follows that $\langle N_1, N_2, L_u\rangle$ contains a line N_3 through x on which there are $(q+1)$ points of $\mathcal{K}$. Indeed, otherwise on each of the q^2+q lines through x in $\langle N_1, N_2, L_u\rangle$, different from N_1, there would lie at most one point of $\mathcal{K}\setminus\{x\}$. On the line N_2 there lie no points of $\mathcal{K}\setminus\{x\}$. So it would follow that $|\mathcal{K}\cap\langle N_1, N_2, L_u\rangle| \leq (q+1)+q^2+q-1$. This is a contradiction, since we have proved above that $|\mathcal{K}\cap\langle N_1, N_2, L_u\rangle| = (q+1)^2$. So there is a line N_3 through x in $\langle N_1, N_2, L_u\rangle$ that contains $q+1$ points of $\mathcal{K}$, $N_3 \neq N_1$. From the previous paragraph it follows that the plane $\langle N_1, N_3\rangle$ contains $q+1$ tangent lines through x at $\mathcal{K}$. The plane $\langle N_1, N_3\rangle$ clearly contains points of $\mathcal{S}$, as it is contained in the three dimensional space $\langle N_1, N_2, L_u\rangle$, that contains the line L_u of $\mathcal{S}$. Hence the points of $\mathcal{K}$ in $\langle N_1, N_3\rangle$ are the points of the lines N_1 and N_3. Now assume that there would be a tangent line N_4 through x in $\langle N_1, N_2, L_u\rangle$, such that N_4 is not contained in the plane $\langle N_1, N_3\rangle$ and such that N_4 contains q points of $\mathcal{S}$. Through a point w of $\mathcal{S}$ on N_4, there is a line L_w of $\mathcal{S}$ that intersects L_u in a point (since $\mathcal{S}$ is a $(q, q+1)$-geometry). The line L_w is contained in the three dimensional space $\langle N_1, N_2, L_u\rangle$, hence it intersects the plane $\langle N_1, N_3\rangle$ in a point w' of $\mathcal{S}$. The plane $\langle N_4, w'\rangle$ is a plane containing an antiflag of $\mathcal{S}$ and two tangent lines through x at $\mathcal{K}$ (namely N_4 and the intersection line of $\langle N_4, w'\rangle$ and $\langle N_1, N_3\rangle$). This is in contradiction with lemma 4.2.1. Hence there can not be a tangent line N_4 through x at $\mathcal{K}$ in $\langle N_1, N_2, L_u\rangle$, that is not contained in the plane $\langle N_1, N_2\rangle$ and that contains q points of $\mathcal{S}$. From $|\mathcal{K}\cap\langle N_1, N_2, L_u\rangle| = (q+1)^2$, it now follows that all tangent lines through x in $\langle N_1, N_2, L_u\rangle$ are contained in the plane $\langle N_1, N_3\rangle$. However, the plane $\langle N_1, N_2\rangle$ is contained in $\langle N_1, N_2, L_u\rangle$ and it contains two tangent lines N_1 and N_2 through x at $\mathcal{K}$. Hence $\langle N_1, N_2\rangle = \langle N_1, N_3\rangle$, a contradiction since $\langle N_1, N_2\rangle$ contains by assumption the line M on which there lie exactly two points x and y of $\mathcal{K}$. It follows that the plane $\langle N_1, N_2\rangle$ can not contain a line through x on which there lie exactly two points of $\mathcal{K}$. Hence every line through x in the plane $\langle N_1, N_2\rangle$ is a tangent line to $\mathcal{K}$.

Assume finally that N_1 and N_2 both contain exactly one point of $\mathcal{K}$. From lemma 4.2.1 it follows that the plane $\langle N_1, N_2\rangle$ can not contain an antiflag of $\mathcal{S}$, since it contains two tangent lines through x at $\mathcal{K}$. Hence every line of the plane $\langle N_1, N_2\rangle$ has to contain a point of $\mathcal{K}$. We have proved above that every line contains 0, 1, 2 or $q+1$ points of $\mathcal{K}$. It follows that the plane $\langle N_1, N_2\rangle$ contains a line N_3 through x on which there lie $q+1$ points of $\mathcal{K}$. Hence we can apply the previous part of the proof to the plane $\langle N_1, N_2\rangle$. This proves that every line through x in $\langle N_1, N_2\rangle$ is a tangent line to $\mathcal{K}$.

In the previous paragraphs we have proved that every plane spanned by two tangent lines through x at $\mathcal{K}$, contains $q+1$ tangent lines through x at $\mathcal{K}$. Hence the tangent lines through x at $\mathcal{K}$ are all lines through x in a subspace Υ_x of $PG(n,q)$. Since every plane of $PG(n,q)$ through x that contains an antiflag of $\mathcal{S}$, contains exactly one tangent line through x, it follows that Υ_x is $(n-1)$-dimensional.

Hence $\mathcal{K}$ is a non-singular quadratic set. From [35] it follows that $\mathcal{K}$ is the set of points of a quadric in $PG(n,q)$. However, since $n > 3$, in $PG(n,q)$ there are planes that contain exactly one point of the quadric $\mathcal{K}$. Such planes contain an antiflag of $\mathcal{S}$, hence from lemma 4.2.1 we get a contradiction. It follows that for $n > 3$, $\mathcal{S}$ can not exist. Hence $n = 3$, and from theorem 4.2.3 we get that $\mathcal{S} = NQ^+(3,q)$. □

Remark. In theorem 4.2.2 and theorem 4.2.3, we use the result of theorem 1.5.1 on quadratic sets. However, as J. A. Thas noted, the proofs could be shortened using the results of G. Tallini (theorem 1.5.2 and 1.5.3) instead.

Appendix A

Nederlandstalige samenvatting

A.1 Inleiding

In de inleiding worden eerst grafen gedefinieerd, en daarna ook het speciaal geval van de sterk reguliere grafen. Dat doen we omdat met elke (α, β)-metkunde een graaf kan geassocieerd worden, dat we het puntgraaf van deze (α, β)-meetkunde zullen noemen, en er dus verbanden bestaan tussen de theorie van de grafen en de theorie van de (α, β)-meetkunden. Een *graaf* $\Gamma = (V, E)$ bestaat uit een niet-ledige (eindige) verzameling V van toppen, samen met een verzameling E van bogen, zodanig dat elke boog precies twee verschillende toppen bevat en elke twee toppen in hoogstens één boog bevat zijn. Een graaf Γ wordt *verbonden* genoemd als er voor elke twee verschillende toppen x en y van Γ er toppen $z_1 = x, z_2, \ldots, z_r = y$ bestaan, voor $r \in \mathbb{N}$, zo dat z_i adjacent is met z_{i+1} $(i = 1, \ldots, r-1)$. Het *complement* van een graaf Γ is het graaf Γ^C, dat dezelfde toppenverzameling heeft dan Γ, maar zo dat twee verschillende toppen x en y adjacent zijn als en slechts als x en y niet-adjacent zijn in Γ. Een *sterk regulier graaf* $\mathrm{srg}(v, k, \lambda, \mu)$ is een graaf Γ met v toppen, zodanig dat elke top in precies k verschillende bogen is bevat en zo dat ook nog aan de volgende twee voorwaarden is voldaan.

1. Voor elke twee toppen x en y, die tot eenzelfde boog van Γ behoren, zijn er precies λ toppen z waarvoor zowel (x, z) als (y, z) bogen van Γ zijn.

2. Voor elke twee verschillende toppen x en y, die niet tot eenzelfde boog van Γ behoren, zijn er precies μ toppen z waarvoor zowel (x, z) als

(y, z) bogen van Γ zijn.

Aangezien we geen niet-verbonden grafen of hun complementen willen beschouwen, veronderstellen we verder dat $0 < \mu < k < v - 1$.

Daarna definiëren we het begrip partiële lineaire ruimte. Een *partiële lineaire ruimte* van de orde (s, t) is een incidentiestructuur $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, met $\mathcal{P} \neq \emptyset$ een verzameling van punten, $\mathcal{L}$ een verzameling van rechten en I een incidentierelatie die aan de volgende voorwaarden voldoet.

1. Elke twee punten zijn incident met hoogstens één rechte.
2. Elke rechte is incident met precies $s + 1$ punten, $s > 1$.
3. Elk punt is incident met precies $t + 1$ rechten, $t > 1$.

Twee punten x en y van $\mathcal{S}$, worden *collineair* genoemd als er een rechte $L \in \mathcal{L}$ bestaat zodanig dat $x \mathrm{~I~} L$ en $y \mathrm{~I~} L$. Twee rechten L_1 en L_2 van $\mathcal{S}$, worden *concurrent* genoemd als er een punt $p \in \mathcal{P}$ bestaat zo dat $p \mathrm{~I~} L_1$ en $p \mathrm{~I~} L_2$. Een *antivlag* van $\mathcal{S}$ is een koppel (p, L), met p een punt van $\mathcal{S}$, L een rechte van $\mathcal{S}$, en zo dat p niet incident is met L. Het *incidentiegetal* $i(p, L)$ van een antivlag (p, L) is het aantal punten dat incident is met de rechte L en collineair met het punt p. Een (α, β)*-meetkunde* is een partiële lineaire ruimte van de orde (s, t) die voldoet aan het volgende axioma: voor elke antivlag (p, L) geldt er dat $i(p, L) = \alpha$ of $i(p, L) = \beta$, en beide komen voor. Een *echte* (α, β)-meetkunde voldoet aan $0 < \alpha < \beta$. Het *puntgraaf* van een (α, β)-meetkunde is het graaf met als toppen de punten van de (α, β)-meetkunde, waarbij twee toppen tot een boog behoren als de twee overeenkomstige punten van de (α, β)-meetkunde tot een rechte van de (α, β)-meetkunde behoren. Een (α, β)-meetkunde wordt *sterk regulier* genoemd als zijn puntgraaf een sterk regulier graaf is. Een (α, β)-meetkunde $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is *volledig ingebed* in een projectieve ruimte $\mathrm{PG}(n, q)$, als $\mathcal{P}$ een deelverzameling is van de puntenverzameling van $\mathrm{PG}(n, q)$, als $\mathcal{L}$ een deelverzameling is van de rechtenverzameling van $\mathrm{PG}(n, q)$, als I de restrictie van de incidentie van $\mathrm{PG}(n, q)$ is en als $s = q$. We veronderstellen ook dat de punten van $\mathcal{S}$ niet bevat zijn in een hypervlak van $\mathrm{PG}(n, q)$.

In het verleden werden een aantal (α, β)-meetkunden met "speciale" parameters bestudeerd. In het bijzonder zijn dit de polaire ruimten of $(1, q+1)$-meetkunden, de copolaire ruimten of $(0, q)$-meetkunden, de partiële meetkunden (waarvoor $\alpha = \beta \neq 1$), de veralgemeende vierhoeken (waarvoor $\alpha = \beta = 1$), en de semipartiële meetkunden (dit zijn $(0, \beta)$-meetkunden met sterk regulier puntgraaf). Al deze incidentiestructuren vormen een deelverzameling van de verzameling van alle (α, β)-meetkunden. Van de meeste ervan zijn de volledige inbeddingen in $\mathrm{PG}(n,q)$ bestudeerd en vaak ook geclassificeerd. In stellingen 1.2.2, 1.2.1, 1.2.3, 1.2.4, 1.2.5 van de Engelstalige tekst worden deze gekende classificaties dan ook vermeld, zonder bewijs.

In een volgend deel van de inleiding wordt er aandacht besteed aan sterk reguliere (α,β)-meetkunden. Deze zijn interessant omdat ze aanleiding geven tot meetkundig mooie structuren en een meer directe veralgemening zijn van partiële meetkunden en semipartiële meetkunden, die immers ook een sterk regulier puntgraaf hebben. In stelling 1.3.1 worden allereerst een aantal bestaansvoorwaarden voor sterk reguliere (α,β)-meetkunden gegeven. In sectie 1.3.2 wordt het begrip (α,β)-regulus gedefinieerd aan de hand waarvan sterk reguliere (α,β)-meetkunden kunnen worden geconstrueerd. Er wordt ook een voorbeeld van een dergelijke constructie van een sterk reguliere (α,β)-meetkunde gegeven, namelijk het volgende. Zij $Q^{\pm}(2n+1,q)$ een niet-ontaarde hyperbolische of elliptische kwadriek in $\mathrm{PG}(2n+1,q)$. Onderstel dat er een partitie Σ bestaat van de punten van $\mathrm{PG}(2n+1,q)\setminus Q^{\pm}(2n+1,q)$ in rechten. Elk vlak door een element L van Σ snijdt de kwadriek $Q^{\pm}(2n+1,q)$ in een punt of in een kegelsnede. Er volgt dat elk dergelijk vlak ofwel q^2-q-1 ofwel q^2-1 punten van $\mathrm{PG}(2n+1,q)\setminus(Q^{\pm}(2n+1,q)\cup L)$ bevat. Zij nu $\mathrm{PG}(2n+2,q)$ een $(2n+2)$-dimensionale projectieve ruimte die $\mathrm{PG}(2n+1,q)$ als hypervlak $\Pi[2n+1]$ bevat. Zij $\mathcal{P}$ de verzameling van de punten van $\mathrm{PG}(2n+2,q)\setminus\Pi[2n+1]$. Zij $\mathcal{L}$ de verzameling van alle vlakken in $\mathrm{PG}(2n+2,q)$ die $\Pi[2n+1]$ in een element van Σ snijden, en die niet in $\Pi[2n+1]$ bevat zijn. Zij I de natuurlijke incidentie. Dan is $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ een sterk reguliere (q^2-q-1,q^2-1)-meetkunde.

Tenslotte worden enkele definities gegeven van wiskundige objecten die, hoewel ze niet het onderwerp van dit proefschrift zijn, in latere hoofdstukken zullen voorkomen. Een *maximale boog* $\mathcal{K}$ in een projectief vlak π is een $(0,d)$-verzameling m.b.t. rechten, d.w.z. $\mathcal{K}$ is een verzameling van punten van π zodanig dat elke rechte van π 0 of d punten van $\mathcal{K}$ bevat. Een maximale boog wordt triviaal genoemd als $d\in\{0,1,q,q+1\}$, met andere woorden als $\mathcal{K}=\emptyset$, $\mathcal{K}$ bestaat uit één enkel punt, $\mathcal{K}$ is de verzameling van de punten van een affien vlak of $\mathcal{K}$ is de verzameling van alle punten van een projectief vlak. Een *Baer deelruimte* van een n-dimensionale projectieve ruimte van de orde q, q een kwadraat, is een m-dimensionale projectieve deelruimte van de orde $\sqrt{q}$, voor $1\le m\le n$. Een *unitaal* $\mathcal{U}$ in een projectief vlak π, q een kwadraat, is een verzameling van $q\sqrt{q}+1$ punten, zo dat elke rechte van π precies 1 of $\sqrt{q}+1$ punten van $\mathcal{U}$ bevat. Een *duaal net* van de orde $s+1$ en met afwijking $t+1-s$ (>0) is een partiële meetkunde waarvoor

$\alpha = s$. De orde van het duaal net is het aantal punten dat op een rechte is gelegen. Verder is de afwijking het aantal rechten door een punt p dat niet concurrent is met een gegeven rechte L, waarbij p niet incident is met L.

A.2 Volledige inbeddingen van (α, β)-meetkunden in $\mathrm{PG}(n, q)$, onder welbepaalde voorwaarden

In het tweede hoofdstuk van dit proefschrift worden volledige inbeddingen van (α, β)-meetkunden in projectieve ruimten bestudeerd. Allereerst merken we op dat echte (α, β)-meetkunden niet volledig ingebed kunnen zijn in een projectief vlak $\mathrm{PG}(2, q)$, vermits in een vlak elke twee rechten snijden, en hieruit zou volgen dat $\alpha = \beta = t + 1$.

Het geval $\alpha = 1$ wordt afzonderlijk behandeld in het eerste deel van dit hoofdstuk. Als $\alpha = 1$, dan kan een vlak de $(1, \beta)$-meetkunde snijden in een (deel van) een rechtenwaaier. Een dergelijk vlak noemen we ontaard. We bestuderen enkel de inbeddingen van $(1, \beta)$-meetkunden in $\mathrm{PG}(3, q)$, omdat, althans met de door ons gebruikte methodes, de classificatie in algemene n-dimensionale projectieve ruimten zeer omslachtig zou zijn en er wellicht een betere bewijsmethode bestaat. We veronderstellen ook dat er zowel een β-vlak als een ontaard vlak bevat is in $\mathrm{PG}(3, q)$. We bewijzen dan de volgende stelling.

Stelling A.2.1 (Stelling 2.1.6) *Er bestaat geen $(1, \beta)$-meetkunde, $\beta \neq 1$, die volledig ingebed is in $\mathrm{PG}(n, q)$, onder de voorwaarde dat $\mathrm{PG}(n, q)$ zowel een ontaard vlak als een β-vlak bevat.*

Beschouwen we nu het geval $\alpha > 1$. We gaan na hoe een (α, β)-meetkunde een vlak π kan snijden. Daarvoor zijn er verschillende mogelijkheden. Als π geen antivlag van $\mathcal{S}$ bevat, dan bevat π geen punten van $\mathcal{S}$, ofwel bevat π een aantal punten van $\mathcal{S}$, maar geen rechten van $\mathcal{S}$, ofwel bevat π één rechte van $\mathcal{S}$, en alle punten van $\mathcal{S}$ in π liggen op die rechte. Als π wel een antivlag van $\mathcal{S}$ bevat, dan is $\pi \cap \mathcal{S}$ een partiële meetkunde $\mathrm{pg}(s, t, \alpha)$, een partiële meetkunde $\mathrm{pg}(s, t, \beta)$, of π bevat twee antivlaggen (p_1, L_1) en (p_2, L_2) van $\mathcal{S}$, zodanig dat $i(p_1, L_1) = \alpha$ en $i(p_2, L_2) = \beta$. In het eerste geval noemen we π een α-vlak, in het tweede geval wordt π een β-vlak genoemd, en in het laatste geval spreken we van een *gemengd vlak*. Het doel van de rest van dit hoofdstuk is om een classificatie van volledig ingebedde (α, β)-meetkunden te bekomen, voor q oneven en $\alpha \neq 1$, onder de voorwaarde dat $\mathrm{PG}(n, q)$ een α-vlak of een β-vlak bevat.

De veronderstelling q oneven is nodig voor het volgende. Een partiële meetkunde ingebed in een projectief vlak heeft als punten de punten niet bevat in een maximale boog $\mathcal{K}$, en als rechten alle rechten die geen punten van $\mathcal{K}$ bevatten. Voor q oneven bestaan er echter geen niet-triviale maximale bogen [1]. Daaruit volgt dat de enige mogelijke β-vlakken (en natuurlijk ook α-vlakken) de volgende zijn: alle punten en rechten van het vlak behoren tot $\mathcal{S}$, en dan is $\beta = q + 1$; er is precies één punt van het vlak dat niet tot $\mathcal{S}$ behoort en de rechten van $\mathcal{S}$ zijn alle rechten van het vlak die dit punt niet bevatten, in dit geval is $\beta = q$ (of $\alpha = q$). Met onze veronderstellingen hebben we dus de volgende mogelijkheden: $\alpha = q$ en $\beta = q + 1$, of $\alpha < q$ (en dan zijn er geen α-vlakken) en $\beta = q + 1$ of $\beta = q$. In het geval q even bestaan er wel degelijk niet-triviale maximale bogen en dus gaat deze redenering niet op. Wel blijven veel van onze stellingen ook gelden in geval q even, maar ze maken geen deel uit van een complete classificatie.

In wat volgt worden de notaties $\Pi[m]$, $\Omega[m]$ en $\Lambda[m]$ gebruikt voor een (vast gekozen) m-dimensionale deelruimte van PG(n,q). Verder is $\Pi[m]\mathcal{S}$ een kegel met als top de deelruimte $\Pi[m]$ van PG(n,q) en als basis de (α,β)-meetkunde $\mathcal{S}$, volledig ingebed in een $(n-m-1)$-dimensionale deelruimte disjunct met $\Pi[m]$.

Onze classificatie van volledig ingebedde (α,β)-meetkunden in PG(n,q), q oneven en $\alpha \neq 1$, onder de voorwaarde dat PG(n,q) een α-vlak of een β-vlak bevat, is een complete classificatie, behalve voor één specifiek geval, namelijk het geval van een $(q-\sqrt{q},q)$-meetkunde $\mathcal{S}$ volledig ingebed in PG$(3,q)$ zodanig dat alle punten van PG$(3,q)$ ook punten van de meetkunde zijn. Als een dergelijke $(q-\sqrt{q},q)$-meetkunde zou bestaan, dan zou $\mathcal{S}' = (\mathcal{P}',\mathcal{L}',\mathrm{I}')$ een $(q-\sqrt{q},q)$-meetkunde volledig ingebed in PG(n,q) zijn, met $\mathcal{P}'$ de verzameling van de punten van de kegel $\Pi[n-4]\mathcal{S}$, die niet tot $\Pi[n-4]$ behoren, $\mathcal{L}'$ de verzameling van de rechten die $q+1$ punten van $\mathcal{P}'$ bevatten, en I' de restrictie van de incidentie van PG(n,q). Een dergelijke $(q-\sqrt{q},q)$-meetkunde $\mathcal{S}'$ noemen we ontaard. In wat volgt citeren we onze classificatie en we vermelden daarbij ook de resultaten die geldig blijven voor q even.

Stelling A.2.2 (Sectie 2.2, Stelling 2.3.2, Stelling 2.4.1, Stelling 2.5.3, Stelling 2.5.7, Stelling 2.5.17)
Zij $\mathcal{S} = (\mathcal{P},\mathcal{L},\mathrm{I})$ een echte (α,β)-meetkunde, volledig ingebed in PG(n,q)*, met $\alpha \neq 1$ en q oneven. Onderstel dat* PG(n,q) *minstens één α-vlak of één β-vlak bevat. Dan is $\mathcal{S}$ één van de volgende (α,β)-meetkunden.*

1. *$\mathcal{S}$ is een $(q,q+1)$-meetkunde, met als punten de punten die bevat zijn in* PG$(n,q)\setminus\Pi[m]$*, voor $0 \leq m < n-2$, en als rechten de rechten die disjunct zijn met $\Pi[m]$. We noteren $\mathcal{S}$ als $\mathrm{H}_q^{n,m}$.*

2. *$\mathcal{S}$ is een $(q, q+1)$-meetkunde, met als punten de punten die bevat zijn in $\mathrm{PG}(n, q) \setminus \Pi[m]$, voor $0 \leq m < n-3$, en waarvoor de rechten als volgt zijn gedefinieerd. Zij $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ een partitie van de punten van $\mathcal{S}$, met $l = (q^{n-m}-1)/(q^{m'-m}-1)$, zodanig dat er voor $i = 1, \ldots, l$ geldt dat $\sigma_i = \Omega_i[m'] \setminus \Pi[m]$, waarbij $\Omega_i[m']$ een m'-dimensionale deelruimte is van $\mathrm{PG}(n, q)$ die $\Pi[m]$ bevat, en $m+2 \leq m' \leq n-2$. De rechten van $\mathcal{S}$ zijn dan de rechten die $q+1$ verschillende elementen van Σ in een punt snijden. Een nodige en voldoende voorwaarde opdat de partitie en dus ook de $(q, q+1)$-meetkunde zou bestaan is dat $(m'-m) \mid (n-m')$. We noteren $\mathcal{S}$ als $\mathrm{SH}_q^{n,m}$.*

3. *$\mathcal{S}$ is een $(q-1, q)$-meetkunde, met als punten de punten die bevat zijn in $\mathrm{PG}(n, q) \setminus \Pi[n-2]$, en waarvoor de rechten als volgt zijn gedefinieerd. Zij $\Sigma = \{\sigma_1, \ldots, \sigma_{n-r}\}$ een partitie van de punten van $\mathcal{S}$, zodanig dat er voor $i = 1, \ldots, n-r$ geldt dat $\sigma_i = \Omega_i[r] \setminus \Pi[n-2]$, waarbij $\Omega_i[r]$ een r-dimensionale deelruimte is van $\mathrm{PG}(n, q)$ die $\Pi[n-2]$ snijdt in een $(r-2)$-dimensionale deelruimte, voor $1 \leq r \leq n-2$. De rechten van $\mathcal{S}$ zijn de rechten die $q+1$ verschillende elementen van Σ in een punt van $\mathcal{S}$ snijden. Een dergelijke partitie Σ bestaat voor elke $1 \leq r \leq n-2$ en geeft een $(q-1, q)$-meetkunde.*

4. *$\mathcal{S}$ is een $(q-1, q)$-meetkunde, met als punten de punten van $\mathrm{PG}(n, q)$ die niet bevat zijn in twee deelruimten $\Pi[n-2]$ en $\Omega[r]$ van $\mathrm{PG}(n, q)$ van dimensie respectievelijk $n-2$ en r $(1 \leq r \leq n-2)$, en zodanig dat $\Pi[n-2] \cap \Omega[r]$ een $(r-2)$-dimensionale ruimte is. De rechten van $\mathcal{S}$ zijn ofwel alle rechten die $q+1$ punten van $\mathcal{S}$ bevatten, ofwel zijn ze als volgt gedefinieerd. Zij $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ een partitie van de punten van $\mathcal{S}$, met $l = (q^{n-r}-1)/(q^{d-r}-1)$, zodanig dat er voor $i = 1, \ldots, l$ geldt dat $\sigma_i = \Lambda_i[d] \setminus (\Pi[n-2] \cup \Omega[r])$, waarbij $\Lambda_i[d]$ een d-dimensionale deelruimte is van $\mathrm{PG}(n, q)$ die $\Omega[r]$ bevat, en $r+2 \leq d \leq n-2$. De rechten van $\mathcal{S}$ zijn de rechten die $q+1$ verschillende elementen van Σ elk in een punt van $\mathcal{S}$ snijden. Een nodige en voldoende voorwaarde opdat een dergelijke partitie, en dus ook de $(q-1, q)$-meetkunde, zou bestaan, is dat $(d-r) \mid (n-r)$.*

5. *$\mathcal{S}$ is een $(q-\sqrt{q}, q)$-meetkunde met als punten de punten van een kegel $\Pi[m]\mathcal{S}'$ die niet bevat zijn in de top $\Pi[m]$, voor $m = n-4$ of $m = n-5$, en als basis een $(q-\sqrt{q}, q)$-meetkunde $\mathcal{S}'$ die volledig is ingebed in een $(n-m-1)$-dimensionale deelruimte $\Omega[n-m-1]$ van $\mathrm{PG}(n, q)$ die disjunct is met $\Pi[m]$, en waarbij $\mathcal{S}'$ als volgt kan worden beschreven. De punten van $\mathcal{S}'$ zijn de punten van $\Omega[n-m-1]$ die niet bevat zijn*

in een $(n-m-1)$-dimensionale Baerdeelruimte van $\Omega[n-m-1]$, en de rechten van $\mathcal{S}'$ zijn de rechten van $\Omega[n-m-1]$ die geen punt van deze Baerdeelruimte bevatten.

6. *$\mathcal{S}$ is een $(q-\sqrt{q}, q)$-meetkunde volledig ingebed in $\mathrm{PG}(3, q)$, met als punten alle punten van $\mathrm{PG}(3, q)$, en de rechten zo gekozen dat voor elk punt $p \in \mathcal{P}$ de rechten van $\mathcal{S}$ door p elk vlak niet door p snijden in de punten niet bevat in een unitaal, en zo dat de rechten van $\mathcal{S}$ in elk vlak de rechten zijn die een unitaal in dat vlak in $\sqrt{q}+1$ punten snijden. We weten niet of een dergelijke $(q-\sqrt{q}, q)$-meetkunde al dan niet bestaat. Als zo een $(q-\sqrt{q}, q)$-meetkunde bestaat, dan is $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ een $(q-\sqrt{q}, q)$-meetkunde volledig ingebed in $\mathrm{PG}(n, q)$, met $\mathcal{P}'$ de verzameling van de punten van de kegel $\Pi[n-4]\mathcal{S}$, die niet bevat zijn in de top $\Pi[n-4]$, en $\mathcal{L}'$ de verzameling van de rechten die $q+1$ punten van $\mathcal{S}'$ bevatten, waarbij $\mathcal{S}$ de $(q-\sqrt{q}, q)$-meetkunde is die hiervoor werd beschreven.*

7. *$\mathcal{S}$ is een $(q-\sqrt{q}, q)$-meetkunde volledig ingebed in $\mathrm{PG}(3, q)$, met als punten alle punten van $\mathrm{PG}(3, q)$, en de rechten zo gekozen dat voor elk punt $p \in \mathcal{P}$ de rechten van $\mathcal{S}$ door p elk vlak niet door p snijden in de punten die niet bevat zijn in een Baerdeelvlak, en zo dat de rechten van $\mathcal{S}$ in elk vlak de rechten zijn die een Baerdeelvlak in dat vlak in één punt snijden. We weten niet of een dergelijke $(q-\sqrt{q}, q)$-meetkunde al dan niet bestaat. Als deze meetkunde bestaat, dan is $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ een $(q-\sqrt{q}, q)$-meetkunde volledig ingebed in $\mathrm{PG}(n, q)$, met $\mathcal{P}'$ de verzameling van de punten van de kegel $\Pi[n-4]\mathcal{S}$, die niet bevat zijn in de top $\Pi[n-4]$, en $\mathcal{L}'$ de verzameling van de rechten die $q+1$ punten van $\mathcal{S}$ bevatten, waarbij $\mathcal{S}$ de $(q-\sqrt{q}, q)$-meetkunde is die hiervoor werd beschreven.*

Voor q even verkregen we de volgende resultaten.

Stelling A.2.3 (Sectie 2.2, Stelling 2.3.3) *Zij $\mathcal{S}$ een $(q, q+1)$-meetkunde volledig ingebed in $\mathrm{PG}(n, q)$, voor q even. Veronderstel dat elk vlak van $\mathrm{PG}(n, q)$ dat een antivlag van $\mathcal{S}$ bevat, een q-vlak of een $(q+1)$-vlak is. Dan heeft $\mathcal{S}$ als punten de punten van $\mathrm{PG}(n, q) \setminus \Pi[m]$, voor $0 \leq m < n-2$, en als rechten de rechten die geen punt met $\Pi[m]$ gemeen hebben.*

Stelling A.2.4 (Sectie 2.2, Stelling 2.4.2) *Zij $\mathcal{S}$ een $(q, q+1)$-meetkunde volledig ingebed in $\mathrm{PG}(n, q)$, voor q even. Veronderstel dat $\mathrm{PG}(n, q)$ een gemengd vlak bevat. Dan heeft $\mathcal{S}$ als punten de punten die bevat zijn in*

$\mathrm{PG}(n,q) \setminus \Pi[m]$, *voor* $0 \leq m < n-3$, *en dan zijn de rechten van* $\mathcal{S}$ *als volgt gedefinieerd. Zij* $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ *een partitie van de punten van* $\mathcal{S}$, *met* $l = (q^{n-m}-1)/(q^{m'-m}-1)$, *zodanig dat er voor* $i = 1, \ldots, l$ *geldt dat* $\sigma_i = \Omega_i[m'] \setminus \Pi[m]$, *waarbij* $\Omega_i[m']$ *een* m'*-dimensionale deelruimte is van* $\mathrm{PG}(n,q)$ *die* $\Pi[m]$ *bevat, en* $m+2 \leq m' \leq n-2$. *De rechten van* $\mathcal{S}$ *zijn dan de rechten die* $q+1$ *verschillende elementen van* Σ *in een punt snijden. Een nodige en voldoende voorwaarde opdat deze partitie, en dus ook de* $(q,q+1)$*-meetkunde, zou bestaan is dat* $(m'-m) \mid (n-m')$.

Stelling A.2.5 (Sectie 2.2, Stelling 2.5.3) *Zij* $\mathcal{S}$ *een* $(q-1,q)$*-meetkunde volledig ingebed in* $\mathrm{PG}(n,q)$, *voor* q *even,* $q \neq 2$. *Veronderstel dat er geen vlak is dat een antivlag van* $\mathcal{S}$ *bevat en precies twee punten die niet tot* $\mathcal{S}$ *behoren. Dan heeft* $\mathcal{S}$ *als punten de punten van* $\mathrm{PG}(n,q) \setminus \Pi[n-2]$, *en dan zijn de rechten van* $\mathcal{S}$ *als volgt gedefinieerd. Zij* $\Sigma = \{\sigma_1, \ldots, \sigma_{n-r}\}$ *een partitie van de punten van* $\mathcal{S}$, *zodanig dat er voor* $i = 1, \ldots, n-r$ *geldt dat* $\sigma_i = \Omega_i[r] \setminus \Pi[n-2]$, *waarbij* $\Omega_i[r]$ *een* r*-dimensionale deelruimte is van* $\mathrm{PG}(n,q)$ *die* $\Pi[n-2]$ *snijdt in een* $(r-2)$*-dimensionale deelruimte, voor* $1 \leq r \leq n-2$. *De rechten van* $\mathcal{S}$ *zijn de rechten die* $q+1$ *verschillende elementen van* Σ *in een punt van* $\mathcal{S}$ *snijden. Een dergelijke partitie* Σ *bestaat voor elke* $1 \leq r \leq n-2$ *en geeft een* $(q-1,q)$*-meetkunde.*

Stelling A.2.6 (Sectie 2.2, Stelling 2.5.7) *Zij* $\mathcal{S}$ *een* $(q-1,q)$*-meetkunde volledig ingebed in* $\mathrm{PG}(n,q)$, *voor* q *even,* $q \neq 2$. *Veronderstel dat er een vlak is dat een antivlag van* $\mathcal{S}$ *bevat en precies twee verschillende punten* y_1 *en* y_2 *die niet tot* $\mathcal{S}$ *behoren. Dan zijn de punten van* $\mathcal{S}$ *de punten van* $\mathrm{PG}(n,q)$ *die niet bevat zijn in twee deelruimten* $\Pi[n-2]$ *en* $\Omega[r]$ *van* $\mathrm{PG}(n,q)$ *van dimensie respectievelijk* $n-2$ *en* r $(1 \leq r \leq n-2)$, *zodanig dat* $\Pi[n-2] \cap \Omega[r]$ *een* $(r-2)$*-dimensionale ruimte is. De rechten van* $\mathcal{S}$ *zijn ofwel alle rechten die* $q+1$ *punten van* $\mathcal{S}$ *bevatten, ofwel zijn ze als volgt gedefinieerd. Zij* $\Sigma = \{\sigma_1, \ldots, \sigma_l\}$ *een partitie van de punten van* $\mathcal{S}$, *met* $l = (q^{n-r}-1)/(q^{d-r}-1)$, *zodanig dat er voor* $i = 1, \ldots, l$ *geldt dat* $\sigma_i = \Lambda_i[d] \setminus (\Pi[n-2] \cup \Omega[r])$, *waarbij* $\Lambda_i[d]$ *een* d*-dimensionale deelruimte is van* $\mathrm{PG}(n,q)$ *die* $\Omega[r]$ *bevat, en* $r+2 \leq d \leq n-2$. *De rechten van* $\mathcal{S}$ *zijn de rechten die* $q+1$ *verschillende elementen van* Σ *elk in een punt van* $\mathcal{S}$ *snijden. Een nodige en voldoende voorwaarde opdat een dergelijke partitie, en dus ook de* $(q-1,q)$*-meetkunde, zou bestaan, is dat* $(d-r) \mid (n-r)$.

A.3 (α, β)-meetkunden ingebed in PG(n, q), voor $\alpha, \beta \in \{0, 1, q, q+1\}$

In dit derde hoofdstuk worden volledige inbeddingen van (α, β)-meetkunden in PG(n, q) verder onderzocht, maar deze keer voor "extreme" waarden van α en β. Voor $(\alpha, \beta) = (1, q+1)$ en $(\alpha, \beta) = (0, q)$ is een volledige classificatie bekend (het gaat hier om respectievelijk de polaire en de copolaire ruimten). De $(0, 1)$-meetkunden of partiële vierhoeken zijn nog niet geclassificeerd, en de $(0, q+1)$-meetkunden zijn in feite disjuncte copiën van partiële meetkunden met $\alpha = q+1$. De $(q, q+1)$-meetkunden werden in het vorige hoofdstuk bestudeerd en geclassificeerd.

Wij slaagden erin een volledige classificatie te bekomen van de $(1, q)$-meetkunden, voor $q \neq 2$, wat het volgende "extreme" geval lijkt te zijn. In de volgende stelling wordt deze classificatie gegeven. Er wordt gebruik gemaakt van de volgende definitie. Zij $\Pi[n-m-1]$ een $(n-m-1)$-dimensionale deelruimte van PG(n, q). Zij GQ een veralgemeende vierhoek die volledig ingebed is in een m-dimensionale deelruimte van PG(n, q), die disjunct is met $\Pi[n-m-1]$. Dan is $\Pi[n-m-1]$GQ bij definitie de kegel met basis de ruimte $\Pi[n-m-1]$, die de veralgemeende vierhoek GQ projecteert.

Stelling A.3.1 (Stelling 3.4.8) *Zij* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *een* $(1, q)$*-meetkunde die volledig ingebed is in* PG(n, q)*, voor* $q \neq 2$*. Dan zijn de punten van* $\mathcal{S}$ *de punten van een kegel* $\Pi[n-m-1]$GQ *(*$m = 3, 4, 5$*), die niet bevat zijn in de top* $\Pi[n-m-1]$*. De rechten van* $\mathcal{S}$ *zijn de rechten die* $q+1$ *punten van* $\mathcal{S}$ *bevatten en op deze kegel gelegen zijn.*

In het geval $q = 2$ is onze classificatie niet geldig. De voorgaande stelling geeft wel een voorbeeld van een $(1, 2)$-meetkunde in PG$(n, 2)$, maar dit is zeker niet het enige voorbeeld. Dat blijkt namelijk uit het feit dat een $(1, 2)$-meetkunde in PG$(n, 2)$ ook een $(q-1, q)$-meetkunde is, en van $(q-1, q)$-meetkunden zijn er andere voorbeelden gegeven in het voorgaande hoofdstuk. We halen hier een dergelijk voorbeeld aan uit het vorige hoofdstuk. Punten van $\mathcal{S}$ zijn de punten van PG$(n, 2)$ die niet bevat zijn in een $(n-2)$-dimensionale deelruimte van PG$(n, 2)$. Verder is er een partiële spread Σ van de punten van $\mathcal{S}$ in rechten. De rechten van $\mathcal{S}$ zijn dan de rechten die 3 punten van $\mathcal{S}$ bevatten en niet tot Σ behoren.

A.4 Karakterisaties van bepaalde klassen van volledig ingebedde (α, β)-meetkunden

In dit laatste hoofdstuk van dit proefschrift worden een aantal karakterisaties gegeven van in de vorige hoofdstukken besproken (α, β)-meetkunden.

In het eerste deel van dit hoofdstuk wordt een karakterisatie gegeven van twee klassen van $(q, q+1)$-meetkunden. We beginnen met de definities te geven van deze $(q, q+1)$-meetkunden. Zij daartoe H een m-dimensionale deelruimte van $\mathrm{PG}(n, q)$, voor $0 \leq m \leq n-3$. De $(q, q+1)$-meetkunde $\mathrm{H}_q^{n,m}$ is dan als volgt gedefinieerd. De punten van $\mathrm{H}_q^{n,m}$ zijn de punten van $\mathrm{PG}(n, q)$ die niet bevat zijn in de m-dimensionale deelruimte H. De rechten van $\mathrm{H}_q^{n,m}$ zijn de rechten van $\mathrm{PG}(n, q)$ die $q+1$ punten van $\mathcal{S}$ bevatten. Voor $m = -1$ en $m = n-2$ zou deze constructie een partiële meetkunde opleveren. De partiële meetkunde $\mathrm{H}_q^{n-2,n}$ wordt traditioneel als H_q^n genoteerd. Een andere $(q, q+1)$-meetkunde, die toch heel sterk op $\mathrm{H}_q^{n,m}$ gelijkt, is de meetkunde $\mathrm{SH}_q^{n,m}$, die als volgt is gedefinieerd. De punten van $\mathrm{SH}_q^{n,m}$ zijn opnieuw de punten van $\mathrm{PG}(n, q) \setminus \mathrm{H}$. Verder is er een partitie S die de punten van $\mathrm{SH}_q^{n,m}$ partitioneert in m'-dimensionale ruimten, voor $m+2 \leq m' \leq n-2$, en zo dat elk element van de partitie S de ruimte H bevat. Merk op dat hieruit volgt dat de dimensie m van H nu strikt kleiner moet zijn dan $n-3$. De rechten van $\mathrm{SH}_q^{n,m}$ zijn de rechten die $q+1$ punten van $\mathcal{S}$ bevatten en niet bevat zijn in een element van de partitie S.

Voor de partiële meetkunde H_q^n bestaat al geruime tijd een classificatie aan de hand van het axioma van Pasch [54]. Het axioma van Pasch voor een partiële meetkunde $\mathrm{pg}(s, t, \alpha)$ is het volgende.

$$\begin{aligned} &\forall L_1, L_2, M_1, M_2 \in \mathcal{L}, L_1 \neq L_2, L_1 \mathrm{~I~} x \mathrm{~I~} L_2, x \notin M_1, x \notin M_2, \\ &\qquad L_i \sim M_j \text{ for all } i, j \in \{1, 2\} : M_1 \sim M_2. \end{aligned}$$

Verder werd ook gebruik gemaakt van een nieuw ingevoerd begrip, namelijk *regulariteit*. Daartoe worden eerst deelstructuren van $\mathcal{S}$ als volgt gedefinieerd. Zij L en M twee verschillende rechten van $\mathcal{S}$, die elkaar snijden in het punt x. Zij $\mathcal{L}^*$ de verzameling van de $s(\alpha-1)$ rechten die zowel L als M snijden in een punt verschillend van x, samen met de α rechten door x die minstens één van deze $s(\alpha-1)$ rechten snijden. Zij $\mathcal{P}^*$ de verzameling van de punten die op een van de rechten van $\mathcal{L}^*$ zijn gelegen. Zij I^* de restrictie van de incidentie van $\mathcal{S}$ tot $(\mathcal{P}^* \times \mathcal{L}^*) \cup (\mathcal{L}^* \times \mathcal{P}^*)$. Dan kan men, gebruik makend van het axioma van Pasch bewijzen dat $S(L, M) = (\mathcal{P}^*, \mathcal{L}^*, \mathrm{I}^*)$ een $\mathrm{pg}(s, \alpha-1, \alpha)$ is. Zij nu x en y twee niet-collineaire punten van $\mathcal{S}$, $x \neq y$. De (nieuwe) rechte door x en y (ook rechte *van de tweede soort* genoemd) wordt dan gedefinieerd als de verzameling van punten die in de doorsnede zit van alle deelmeetkunden van $\mathcal{S}$ die zowel x als y bevatten. Deze nieuwe rechte door x en y is een verzameling van punten die twee aan twee niet-collinear zijn in $\mathcal{S}$ en bevat hoogstens $q + 1 - q/\alpha$ punten. De partiële meetkunde $\mathcal{S}$, die aan het axioma van Pasch voldoet, wordt *regulier* genoemd als en slechts als elk zulke rechte door twee niet collineaire punten van $\mathcal{S}$ precies $q + 1 - q/\alpha$ punten bevat.

Stelling A.4.1 ([54]) *De partiële meetkunde $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ met parameters (s, t, α), zo dat $\alpha \neq 1, t + 1, s + 1$, is isomorf met H_q^n als en slechts als*

1. *$\mathcal{S}$ voldoet aan het axioma van Pasch;*
2. *$\mathcal{S}$ is regulier;*
3. *$2s > s^4 - \alpha s^3 + \alpha^2 s^2 + \alpha^3 s - 2\alpha^4$.*

Merk op dat de laatste voorwaarde in stelling A.4.1 een zeer sterke voorwaarde is. In de praktijk is er aan deze voorwaarde bijna nooit voldaan als $\alpha \neq s$. Voor het geval dat $\alpha = s$ bestaat er de volgende stelling.

Stelling A.4.2 ([54]) *Zij $\mathcal{S}$ een duaal net van de orde $s+1$ en met deficiëntie $t+1-s$ (> 0). Als $\mathcal{S}$ aan het axioma van Pasch voldoet, dan is $\mathcal{S}$ isomorf met H_q^n.*

Wij hebben deze classificatie veralgemeend tot een classificatie voor $\mathrm{H}_q^{n,m}$ en $\mathrm{SH}_q^{n,m}$. Het axioma van Pasch blijft natuurlijk gelden voor een willekeurige (α, β)-meetkunde. Het begrip regulariteit moet enigszins worden uitgebreid. Een (α, β)-meetkunde wordt regulier t.o.v. rechten van de tweede soort (dit zijn nieuwe rechten gedefinieerd door twee niet-collineaire punten, op dezelfde manier als in het geval van partiële meetkunden) genoemd als en slechts als elke rechte van de tweede soort en elke rechte van $\mathcal{S}$, die tot eenzelfde deelmeetkunde van $\mathcal{S}$ behoren, elkaar snijden. Met deze definities verkregen we de volgende classificatie voor $\mathrm{H}_q^{n,m}$ en $\mathrm{SH}_q^{n,m}$.

Stelling A.4.3 (Stelling 4.1.4) *Zij $\mathcal{S}$ een echte $(\alpha, s+1)$-meetkunde van de orde (s, t), zo dat $1 < \alpha < s + 1$ en $\alpha < t + 1$, die aan de volgende voorwaarden voldoet:*

1. *het axioma van Pasch;*
2. *regulariteit t.o.v. rechten van de tweede soort;*

3. er is minstens één α-deelstructuur bevat in $\mathcal{S}$.

Dan is $\mathcal{S}$ isomorf met $\mathrm{H}_q^{n,m}$ of $\mathrm{SH}_q^{n,m}$.

Een tweede klasse van (α, β)-meetkunde waarvan een classificatie wordt gegeven in dit hoofdstuk is de $((q-1)/2, (q+1)/2)$-meetkunde $\mathrm{NQ}^+(3,q)$, q oneven, die als volgt is gedefinieerd. Zij $Q^+(3,q)$ een driedimensionale hyperbolische kwadriek in $\mathrm{PG}(3,q)$. De punten van $\mathrm{NQ}^+(3,q)$ zijn de punten van $\mathrm{PG}(3,q) \setminus Q^+(3,q)$. De rechten van $\mathrm{NQ}^+(3,q)$ zijn de rechten die $q+1$ punten van $\mathrm{NQ}^+(3,q)$ bevatten.

Voor deze $((q-1)/2, (q+1)/2)$-meetkunde bekwamen wij de volgende classificatie.

Stelling A.4.4 (Stelling 4.2.3) *Zij $\mathcal{S}$ een $(\frac{q-1}{2}, \frac{q+1}{2})$-meetkunde volledig ingebed in $\mathrm{PG}(n,q)$, voor q oneven. Dan is $\mathcal{S}$ isomorf met $\mathrm{NQ}^+(3,q)$.*

Bibliography

[1] S. Ball, A. Blokhuis, and F. Mazzocca. Maximal arcs in Desarguesian planes of odd order do not exist. *Combinatorica*, 17(1):31–41, 1997.

[2] A. Barlotti. Sui $\{k;n\}$-archi di un piano lineare finito. *Boll. Un. Mat. Ital.*, 11:553–556, 1956.

[3] A. Beutelspacher. Partial spreads in finite projective spaces and partial designs. *Math. Z.*, 145(3):211–229, 1975.

[4] R. C. Bose. Strongly regular graphs, partial geometries and partially balanced designs. *Pacific J. Math.*, 13:389–419, 1963.

[5] A. E. Brouwer, A. M. Cohen, and A. Neumaier. *Distance-Regular Graphs*, volume 18 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1989.

[6] F. Buekenhout, editor. *Handbook of Incidence Geometry. Buildings and foundations*. North-Holland, Amsterdam, 1995.

[7] F. Buekenhout and C. Lefèvre. Generalized quadrangles in projective spaces. *Arch. Math. (Basel)*, 25:540–552, 1974.

[8] F. Buekenhout and C. Lefèvre. Semi-quadratic sets in projective spaces. *J. Geometry*, 7(1):17–42, 1976.

[9] F. Buekenhout and E. Shult. On the foundations of polar geometry. *Geom. Dedicata*, 3:155–170, 1974.

[10] P. J. Cameron. Partial quadrangles. *Quart. J. Math. Oxford Ser. (2)*, 26:61–73, 1975.

[11] S. Cauchie. Classification of $(1,s)$-geometries fully embedded in $\mathrm{PG}(n,s)$, for $s \neq 2$. To appear in *J. Combin. Theory, Ser. A*.

[12] S. Cauchie. Full embeddings of (α, β)-geometries in projective spaces. Part II. *In proceedings of the 18th British Combinatorial Conference.* To appear in *Discrete Mathematics.*

[13] S. Cauchie, F. De Clerck, and N. Hamilton. Full embeddings of (α, β)-geometries in projective spaces. *Europ. J. Comb.*, 23(6):635–646, 2002.

[14] A. M. Cohen. A new partial geometry with parameters $(s, t, \alpha) = (7, 8, 4)$. *J. Geom.*, 16(2):181–186, 1981.

[15] F. De Clerck. Partial and semipartial geometries: an update. *In proceedings of Combinatorics 2002, Gaeta.* To appear in Discrete Mathematics.

[16] F. De Clerck. New partial geometries constructed from old ones. *In proceedings of Finite geometry and combinatorics (Deinze, 1997). Bull. Belg. Math. Soc. Simon Stevin*, 5(2-3):255–263, 1998.

[17] F. De Clerck, M. Delanote, N. Hamilton, and R. Mathon. Perp-systems and partial geometries. *Adv. Geom.*, 2(1):1–12, 2002.

[18] F. De Clerck, N. Durante, and J. A. Thas. Dual partial quadrangles embedded in $\mathrm{PG}(3, q)$. To appear in Adv. Geom.

[19] F. De Clerck, R. H. Dye, and J. A. Thas. An infinite class of partial geometries associated with the hyperbolic quadric in $\mathrm{PG}(4n - 1, 2)$. *European J. Combin.*, 1(4):323–326, 1980.

[20] F. De Clerck and J. A. Thas. Partial geometries in finite projective spaces. *Arch. Math.*, 30:537–540, 1978.

[21] F. De Clerck and J. A. Thas. The embedding of $(0, \alpha)$-geometries in $\mathrm{PG}(n, q)$. I. In *Combinatorics '81 (Rome, 1981)*, pages 229–240. North-Holland, Amsterdam, 1983.

[22] F. De Clerck and H. Van Maldeghem. On linear representations of (α, β)-geometries. *European J. Combin.*, 15:3–11, 1994.

[23] I. Debroey and J. A. Thas. Semipartial geometries in $\mathrm{AG}(2, q)$ and $\mathrm{AG}(3, q)$. *Simon Stevin*, 51(4):195–209, 1977/78.

[24] I. Debroey and J. A. Thas. On semipartial geometries. *J. Combin. Theory Ser. A*, 25(3):242–250, 1978.

[25] I. Debroey and J. A. Thas. Semipartial geometries in PG(2, q) and PG(3, q). *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. (8)*, 64(2):147–151, 1978.

[26] M. Delanote. A new semipartial geometry, *In proceedings of the Second Pythagorean Conference (Pythagoreion, 1999). J. Geom.*, 67(1-2):89–95, 2000.

[27] P. Dembowski. *Finite Geometries.* Springer-Verlag, Berlin, 1968.

[28] R. H. F. Denniston. Some maximal arcs in finite projective planes. *J. Combin. Theory*, 6:317–319, 1969.

[29] J. I. Hall. Classifying copolar spaces and graphs. *Quart. J. Math. Oxford Ser. (2)*, 33(132):421–449, 1982.

[30] J. I. Hall. Linear representations of cotriangular spaces. *Linear Algebra Appl.*, 49:257–273, 1983.

[31] N. Hamilton and R. Mathon. More maximal arcs in desarguesian projective planes and their geometric structure. To appear in *Adv. Geom.*

[32] N. Hamilton and R. Mathon. Strongly regular (α, β)-geometries. *J. Combin. Theory Ser. A*, 95(2):234–250, 2001.

[33] D. G. Higman. δ-spaces. Lecture notes, unpublished.

[34] J. W. P. Hirschfeld. *Projective Geometries over Finite Fields.* The Clarendon Press Oxford University Press, New York, second edition, 1998.

[35] J. W. P. Hirschfeld and J. A. Thas. *General Galois Geometries.* Oxford Science Publications, Oxford, 1991.

[36] C. Lefèvre-Percsy. Sur les semi-quadriques en tant qu'espaces de Shult projectifs. *Acad. Roy. Belg. Bull. Cl. Sci. (5)*, 63(2):160–164, 1977.

[37] C. Lefèvre-Percsy. An extension of a theorem of G. Tallini. *J. Combin. Theory Ser. A*, 29(3):297–305, 1980.

[38] C. Lefèvre-Percsy. Ensembles de classe (0, 1, q, $q+1$) de PG(d, q). *J. Geom.*, 15(1):93–98, 1981.

[39] C. Lefèvre-Percsy. Quadriques hermitiennes et ensembles de classe $(0, 1, n, q+1)$ de PG(d, q). *Geom. Dedicata*, 12:209–213, 1982.

[40] C. Lefèvre-Percsy. Copolar spaces fully embedded in projective spaces. In *Combinatorics '81 (Rome, 1981)*, volume 18 of *Ann. Discrete Math.*, pages 553–566. North-Holland, Amsterdam, 1983.

[41] R. Mathon. New maximal arcs in Desarguesian planes. *J. Combin. Theory Ser. A*, 97(2):353–368, 2002.

[42] S. E. Payne and J. A. Thas. *Finite Generalized Quadrangles*, volume 110 of *Research Notes in Mathematics*. Pitman (Advanced Publishing Program), Boston, MA, 1984.

[43] J. J. Seidel. *On Two-graphs and Shult's Characterization of Symplectic and Orthogonal Geometries over GF(2)*. Department of Mathematics Technological University Eindhoven, Eindhoven, 1973. T.H.-Report, No. 73-WSK-02.

[44] E. E. Shult. Groups, polar spaces and related structures. In *Combinatorics, Part 3: Combinatorial group theory (Proc. Advanced Study Inst., Breukelen, 1974)*, pages 130–161. Math. Centre Tracts, No. 57. Math. Centrum, Amsterdam, 1974.

[45] G. Tallini. Sulle k-calotte di uno spazio lineare finito. *Ann. Mat. Pura Appl. (4)*, 42:119–164, 1956.

[46] G. Tallini. Caratterizzazione grafica delle quadriche ellittiche negli spazi finiti. *Rend. Mat. e Appl. (5)*, 16:328–351, 1957.

[47] J. A. Thas. Construction of partial geometries. *Simon Stevin*, 46:95–98, 1972/73.

[48] J. A. Thas. Construction of maximal arcs and partial geometries. *Geom. Dedicata*, 3:61–64, 1974.

[49] J. A. Thas. Partial geometries in finite affine spaces. *Math. Z.*, 158(1):1–13, 1978.

[50] J. A. Thas. Construction of maximal arcs and dual ovals in translation planes. *European J. Combin.*, 1(2):189–192, 1980.

[51] J. A. Thas. Some results on quadrics and a new class of partial geometries. *Simon Stevin*, 55(3):129–139, 1981.

[52] J. A. Thas. Semipartial geometries and spreads of classical polar spaces. *J. Combin. Theory Ser. A*, 35(1):58–66, 1983.

[53] J. A. Thas. SPG systems and semipartial geometries. *Adv. Geom.*, 1(3):229–244, 2001.

[54] J. A. Thas and F. De Clerck. Partial geometries satisfying the axiom of Pasch. *Simon Stevin*, 51(3):123–137, 1977/78.

[55] J. A. Thas, I. Debroey, and F. De Clerck. The embeddings of $(0, \alpha)$-geometries in $\mathrm{PG}(n, q)$: Part II. *Discrete Math.*, 51:283–292, 1984.

[56] J. Ueberberg. On regular $\{v, n\}$-arcs in finite projective spaces. *J. Combin. Des.*, 1(6):395–409, 1993.

[57] J. H. van Lint and A. Schrijver. Construction of strongly regular graphs, two-weight codes and partial geometries by finite fields. *Combinatorica*, 1(1):63–73, 1981.

List of Notations

For a graph Γ

$x \sim y$	the vertex x is adjacent with the vertex y
$x \not\sim y$	the vertex x is not adjacent with the vertex y
Γ^C	the complement of Γ
$\Gamma(x)$	the set of all vertices of Γ adjacent with x

For a projective space

$\mathrm{PG}(n,q)$	the n-dimensional projective space over the field $\mathrm{GF}(q)$, q a prime power
$\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$	an incidence structure, with $\mathcal{P}$ a set of elements called points, $\mathcal{L}$ a set of elements called lines, and $\mathrm{I} \subset (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$ an incidence relation
π^D	the dual plane of π
$\Pi[m]$	an m-dimensional subspace of $\mathrm{PG}(n,q)$
$Q(n,q)$	parabolic quadric in $\mathrm{PG}(n,q)$, n even
$Q^+(n,q)$	hyperbolic quadric in $\mathrm{PG}(n,q)$, n odd
$Q^-(n,q)$	elliptic quadric in $\mathrm{PG}(n,q)$, n odd
$H(n,q)$	Hermitian variety in $\mathrm{PG}(n,q)$, q a square

For (α, β)-geometries

$x \sim y$	the point x is collinear with the point y
$x \not\sim y$	the point x is not collinear with the point y
$L \sim M$	the line L is concurrent with the line M
$L \not\sim M$	the line L is not concurrent with the line M
$i(x,L)$	the number of lines through the point x that intersect the line L

$\mathrm{pg}(s,t,\alpha)$	a partial geometry with parameters s,t and α
H_q^n	the partial geometry fully embedded in $\mathrm{PG}(n,q)$ with points the points not contained in an $(n-2)$-dimensional subspace $\Pi[n-2]$, and lines the lines skew to $\Pi[n-2]$
$\mathrm{H}_q^{n,m}$	the $(q,q+1)$-geometry fully embedded in $\mathrm{PG}(n,q)$ with points the points not contained in an m-dimensional subspace $\Pi[m]$, $0 \leq m \leq n-3$, and lines the lines skew to $\Pi[m]$
$\mathrm{SH}_q^{n,m}$	the $(q,q+1)$-geometry fully embedded in $\mathrm{PG}(n,q)$ with points the points not contained in an m-dimensional subspace $\Pi[m], -1 \leq m \leq n-4$, and lines the lines skew to $\Pi[m]$ that are not contained in a partitioning Σ of the points of $\mathrm{PG}(n,q) \setminus \Pi[m]$ into m'-dimensional spaces, each of them containing $\Pi[m]$
$\mathrm{NQ}^-(2n-1,2)$	the semipartial geometry with points the points not on $Q^-(2n-1,2)$ and lines the lines skew to $Q^-(2n-1,2)$
$\mathrm{NQ}^+(2n-1,2)$	the semipartial geometry with points the points not on $Q^+(2n-1,2)$ and lines the lines skew to $Q^+(2n-1,2)$
$\mathrm{NQ}(4,2)$	the semipartial geometry with points the points not on $Q(4,2)$ and lines the lines skew to $Q(4,2)$
$\mathrm{NQ}^+(3,q)$	the $(\frac{q-1}{2},\frac{q+1}{2})$-geometry, q odd, with points the points not on $Q^+(3,q)$ and lines the lines skew to $Q^+(3,q)$
$\Pi[m]\mathcal{S}$	cone with vertex $\Pi[m]$, projecting an (α,β)-geometry $\mathcal{S}$ fully embedded in an $(n-m-1)$-dimensional subspace of $\mathrm{PG}(n,q)$ skew to $\Pi[m]$
GQ	a generalized quadrangle fully embedded in $\mathrm{PG}(n,q)$

For a set of points $\mathcal{K}$ in $\mathrm{PG}(n,q)$

$(r_1,r_2,\ldots,r_s)$-set	a set of points of $\mathrm{PG}(n,q)$ such that for each line L of $\mathrm{PG}(n,q)$ we have that $\vert L \cap \mathcal{K}\vert \in \{r_1,r_2,\ldots,r_s\}$
$\mathcal{K}^C$	the complement of the set $\mathcal{K}$, or in other words, the set of points of $\mathrm{PG}(n,q) \setminus \mathcal{K}$

Index