On Q-polynomial regular near $2d$-gons

B. De Bruyn and F. Vanhove

Ghent University, Department of Mathematics, Krijgslaan 281 (S22), B-9000 Gent, Belgium,
E-mail: bdb@cage.ugent.be/fvanhove@cage.ugent.be

Abstract

We discuss thick regular near $2d$-gons with a Q-polynomial collinearity graph. For $d \geq 4$, we show that apart from Hamming near polygons and dual polar spaces there are no thick Q-polynomial regular near polygons. We also show that no regular near hexagons exist with parameters (s, t_2, t) equal to $(3, 1, 34)$, $(8, 4, 740)$, $(92, 64, 1314560)$, $(95, 19, 1027064)$ or $(105, 147, 2763012)$. Such regular near hexagons are necessarily Q-polynomial. All these nonexistence results imply the nonexistence of distance-regular graphs with certain classical parameters. We also discuss some implications for the classification of dense near polygons with four points per line.

Keywords: regular near polygon, distance-regular graph, Q-polynomial
MSC2010: 05E30, 05B25, 51E12

1 Introduction

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a finite point-line geometry with nonempty point set $\mathcal{P}$ and (possibly empty) line set $\mathcal{L}$, where each line is a set containing at least two points. We say that $\mathcal{S}$ has *order* (s, t) if every line contains exactly $s + 1$ points, and every point is in exactly $t + 1$ lines. If every line of $\mathcal{S}$ contains at least three points, then $\mathcal{S}$ will be called *thick*. Distances $\mathrm{d}(\cdot, \cdot)$ between points of $\mathcal{S}$ will always be measured in the collinearity graph.

The point-line geometry $\mathcal{S}$ is called a *near polygon* if there is at most one line through any two distinct points, its collinearity graph is connected, and for every point x and every line L, there exists a unique point y on L for which $\mathrm{d}(x, y) = \mathrm{d}(x, L) := \min\{\mathrm{d}(x, z) \,|\, z \in L\}$. A near polygon for which the collinearity graph has diameter $d \in \mathbb{N}$ is also called a *near $2d$-gon*. A near 0-gon is a point, a near 2-gon is a line with at least two points, and near quadrangles are usually called *generalized quadrangles*.

A near $2d$-gon $\mathcal{S}$ with $d \geq 2$ is called *regular* if there exist constants s, t and t_i, $i \in \{0, 1, \ldots, d\}$, such that $\mathcal{S}$ has order (s, t), and for every two points x and y at distance i

from each other, there are precisely t_i+1 lines through y containing a (necessarily unique) point at distance $i-1$ from x. If this holds, then $(t_0, t_1, t_d) = (-1, 0, t)$ and we say that $\mathcal{S}$ is *regular with parameters* $(s, t_2, t_3, \ldots, t_{d-1}, t)$. If $t_1 = \ldots = t_{d-1} = 0$, then $\mathcal{S}$ is a so-called *generalized* $2d$*-gon*.

Let Γ be a finite undirected connected graph without loops and multiple edges whose diameter d is at least 2. The graph Γ is called *distance-regular* if there exist nonnegative integers a_i, b_i, c_i ($i \in \{0, 1, \ldots, d\}$), known as the *intersection numbers*, such that for any two vertices x and y at distance i from each other, we have $|\Gamma_i(x) \cap \Gamma_1(y)| = a_i$, $|\Gamma_{i+1}(x) \cap \Gamma_1(y)| = b_i$ and $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = c_i$. The distance-regular graph Γ is said to have *classical parameters* (d, b, α, β) where $\alpha, \beta \in \mathbb{R}$ and $b \in \mathbb{R} \setminus \{0\}$ if

$$\begin{aligned} b_i &= \left(\begin{bmatrix} d \\ 1 \end{bmatrix}_b - \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right)\left(\beta - \alpha \begin{bmatrix} i \\ 1 \end{bmatrix}_b\right), \\ c_i &= \begin{bmatrix} i \\ 1 \end{bmatrix}_b \left(1 + \alpha \begin{bmatrix} i-1 \\ 1 \end{bmatrix}_b\right), \end{aligned}$$

for every $i \in \{0, 1, \ldots, d\}$. Here, $\begin{bmatrix} i \\ 1 \end{bmatrix}_b := i$ if $b = 1$ and $\begin{bmatrix} i \\ 1 \end{bmatrix}_b := \frac{b^i - 1}{b-1}$ if $b \neq 1$.

The finite regular near $2d$-gons with $d \geq 2$ are precisely the finite near $2d$-gons whose collinearity graph is distance-regular. If $\mathcal{S}$ is a regular near $2d$-gon with parameters s, t and t_i, $i \in \{0, 1, \ldots, d\}$, then the collinearity graph of $\mathcal{S}$ is a distance-regular graph of diameter d whose intersection numbers satisfy $a_i = (s-1)(t_i+1)$, $b_i = s(t - t_i)$ and $c_i = t_i + 1$ for every $i \in \{0, 1, \ldots, d\}$.

An important class of distance-regular graphs are the so-called *Q-polynomial distance-regular graphs* (see Subsection 2.2). All distance-regular graphs with classical parameters are Q-polynomial (Brouwer et al. [5, Corollary 8.4.2]), in particular also the so-called Hamming graphs and dual polar graphs ([5, Section 6.1 or Chapter 9]). We call a regular near polygon *Q-polynomial* if its collinearity graph is a Q-polynomial distance-regular graph.

If $\mathcal{S}$ is a finite regular near hexagon with parameters (s, t_2, t) where $s \geq 2$, then an inequality first derived by Mathon [19] states that $t + 1 \leq (s^2 - s + 1)(t_2 + s + 1)$. We call this inequality the *Mathon bound*. If the Mathon bound is attained, then we call $\mathcal{S}$ a *maximal regular near hexagon*. Maximal regular near hexagons are thick by definition. The Mathon bound for thick regular near hexagons is one of the so-called Krein conditions of the associated distance-regular graph, see Brouwer and Wilbrink [6, p. 162]. If $\mathcal{S}$ is a (necessarily thick) maximal regular near hexagon with parameters (s, t_2, t), then the collinearity graph of $\mathcal{S}$ has classical parameters $(d, b, \alpha, \beta) = (3, -s, -\frac{s+t_2}{s-1}, s(t_2 + s + 1))$ and is thus Q-polynomial. If $t_2 = 0$, then the Mathon bound reduces to the well-known *Haemers-Roos inequality* $t \leq s^3$ for generalized hexagons of order (s, t), $s \geq 2$, see [16]. In the following proposition, we collect some inequalities and divisibility conditions that need to be satisfied by the parameters of a maximal regular near hexagon. These conditions follow from known restrictions, and will be proved in Proposition 2.1.

Proposition 1. *Let $\mathcal{S}$ be a finite regular near hexagon whose parameters (s, t_2, t) satisfy $s \geq 2$ and $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$. Then the following hold:*

(1) *either $t_2 \in \{0, 1\}$ or $\sqrt{s} \leq t_2 \leq s^2$;*

(2) *$s + t_2$ is a divisor of $s^2(s^2 - 1)$;*

(3) *if $t_2 \neq 0$, then t_2 is a divisor of s^3;*

(4) *$t_2 + 1$ is a divisor of $s(s - 1)(s^2 + 1)(s^2 - s + 1)$;*

(5) *if $t_2 \neq s^2$, then $s^2 + st_2 \geq t_2^2 + 2t_2$;*

(6) *$(t_2+1)(2s^2+st_2-t_2)$ is a divisor of $s^2(s+1)(s^2-s+1)^2(s+t_2+1)(s^3+t_2s^2-t_2s+t_2)$.*

The above restrictions are always satisfied when $t_2 = 0$ or $t_2 = s^2$. If $t_2 = 0$ then the near hexagon is a generalized hexagon of order (s, s^3). If $t_2 = s^2$, then $t = s^4 + s^2$ and it is known that $\mathcal{S}$ must then be a dual polar space (see discussion at the end of Section 5), necessarily isomorphic to $DH(5, s^2)$ (and so, s must be a prime power). We have done an exhaustive computer search to find all pairs (s, t_2) of positive integers $s \in \{2, 3, \dots, 10^7\}$ and $t_2 \notin \{0, s^2\}$ satisfying the conditions of Proposition 1. We have found the following 25 possibilities for (s, t_2): $(2, 1)$, $(2, 2)$, $(3, 1)$, $(3, 3)$, $(4, 1)$, $(5, 1)$, $(5, 5)$, $(8, 1)$, $(8, 4)$, $(10, 5)$, $(11, 1)$, $(18, 1)$, $(23, 1)$, $(32, 1)$, $(33, 11)$, $(53, 1)$, $(92, 64)$, $(95, 19)$, $(105, 147)$, $(129, 27)$, $(158, 1)$, $(221, 169)$, $(285, 75)$, $(558, 216)$, $(2093, 91)$. We do not know whether additional feasible parameter sets exist for $s > 10^7$. Only three of the above 25 possibilities had already been settled, namely the cases $(2, 1)$, $(2, 2)$ and $(3, 3)$.

Up to isomorphism, there exists a unique regular near hexagon with parameters $(s, t_2, t) = (2, 1, 11)$. It was first constructed by Shult and Yanushka [24, Section 2.5] using the extended ternary Golay code and its uniqueness was shown by Brouwer [2]. Next, up to isomorphism, there exists a unique regular near hexagon with parameters $(s, t_2, t) = (2, 2, 14)$. It was first constructed by Shult and Yanushka [24, Section 2.5] using the Steiner system $S(5, 8, 24)$ and its uniqueness was shown by Brouwer [3]. Finally, there exists no regular near hexagon with parameters $(s, t_2, t) = (3, 3, 48)$, see Shad and Shult [23, p. 70] (or alternatively, see De Bruyn [9]).

In the present paper, we settle the (non)existence problem for five additional possibilities for (s, t_2).

Theorem 1. *There are no regular near hexagons with parameters (s, t_2, t) where $(s, t_2) \in \{(3, 1), (8, 4), (92, 64), (95, 19), (105, 147)\}$ and $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$.*

The nonexistence of regular near hexagons with parameters (s, t_2, t) where $(s, t_2) \in \{(8, 4), (92, 64), (95, 19), (105, 147)\}$ and $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$ will be proved in Section 3 (Corollary 3.8). The case $(s, t_2, t) = (3, 1, 34)$ will be treated in Subsection 4.3 (Proposition 4.7). The nonexistence of regular near hexagons with parameters $(s, t_2, t) =$

$(3, 1, 34)$ will have some implications for the classification of the so-called dense near polygons with four points per line. This will be discussed in Subsection 4.4.

In the present paper, we will also show the nonexistence of the class of Q-polynomial regular near $2d$-gons mentioned in the following theorem. This theorem will be proved in Section 5 (Proposition 5.3).

Theorem 2. *There are no finite regular near $2d$-gons whose parameters s, t and t_i, $i \in \{0, 1, \ldots, d\}$ satisfy $d \geq 4$, $s \geq 2$ and $t_i = s^3 \cdot \frac{(s^{i-2}-(-1)^i)(s^{i-1}+(-1)^i)}{(s^2-1)(s+1)}$ for every $i \in \{0, 1, \ldots, d\}$.*

Theorems 1 and 2 imply the following.

Corollary 1. *There are no distance-regular graphs with classical parameters* $(3, -3, -2, 15)$, $(3, -8, -\frac{12}{7}, 104)$, $(3, -92, -\frac{12}{7}, 14444)$, $(3, -95, -\frac{57}{47}, 10925)$, $(3, -105, -\frac{63}{26}, 26565)$ *or* $(d, -s, -\frac{s}{s-1}, s + s^2\frac{(-s)^{d-1}-1}{s^2-1})$ *where $d \geq 4$ and $s \geq 2$.*

Proof. Suppose Γ is a distance-regular graph with classical parameters as stated above. Then results of Terwilliger [26, Theorem 2.12] and Brouwer et al. [5, Theorem 6.4.1] imply that Γ is the collinearity graph of a regular near $2d$-gon (see also Proposition 2.3 in this context). However, such a regular near $2d$-gon cannot exist by Theorems 1 and 2. $\square$

Combining Theorem 2 with the work of several other authors, we will prove the following classification result for Q-polynomial regular near polygons in Section 5.

Theorem 3. *Suppose $\mathcal{S}$ is a thick Q-polynomial regular near $2d$-gon $\mathcal{S}$ with $d \geq 3$. Then precisely one of the following cases occurs.*

(1) *$\mathcal{S}$ is a Hamming near $2d$-gon.*

(2) *$\mathcal{S}$ is a dual polar space, necessarily isomorphic to $DW(2d-1, q)$, $DQ(2d, q)$ (q odd), $DQ^-(2d+1, q)$, $DH(2d-1, q^2)$ or $DH(2d, q^2)$ for some prime power q.*

(3) *$\mathcal{S}$ is a generalized hexagon of order (s, s^3), $s \geq 2$.*

(4) *$\mathcal{S}$ is the regular near hexagon derived from the extended ternary Golay code.*

(5) *$\mathcal{S}$ is the regular near hexagon derived from the Steiner system $S(5, 8, 24)$.*

(6) *$\mathcal{S}$ is a regular near hexagon with parameters (s, t_2, t), with $s \geq 3$, $1 \leq t_2 \leq s^2 - s$ and $t + 1 = (s^2 - s + 1)(t_2 + s + 1)$.*

No example of a near hexagon is known for which case (6) of Theorem 3 occurs. In this case, s and t_2 must satisfy the restrictions stated in Proposition 1. By Theorem 1 and [23, p. 70], we moreover know that $(s, t_2) \notin \{(3, 1), (3, 3), (8, 4), (92, 64), (95, 19), (105, 147)\}$.

The only known generalized hexagons of order (s, s^3), $s \geq 2$, are the *dual twisted triality hexagons* constructed by Tits [27], see also Van Maldeghem [28, pp. 71-72]. Each dual twisted triality hexagon has order (q, q^3) for some prime power q and is related to the group ${}^3D_4(q)$. It follows from Cohen and Tits [8, Theorem 2] that there is a unique generalized hexagon of order $(2, 8)$. For $s > 2$, the classification of the generalized hexagons of order (s, s^3) is still open.

We refer to the literature (e.g. Brouwer et al. [5]) for definitions of the other regular near polygons mentioned in Theorem 3. If q is even, then the dual polar space $DQ(2d, q)$ is isomorphic to $DW(2d-1, q)$, and for this reason, we have omitted this possibility in Theorem 3. The collinearity graphs of the dual polar spaces $DW(2d-1, q)$, $DQ(2d, q)$, $DQ^-(2d+1, q)$, $DH(2d-1, q^2)$ and $DH(2d, q^2)$ are also denoted by $C_d(q)$, $B_d(q)$, ${}^2D_{d+1}(q)$, ${}^2A_{2d-1}(q)$ and ${}^2A_{2d}(q)$, respectively. It follows from Brouwer and Cohen [4, Corollary 2, p. 195], Brouwer and Wilbrink [6, Theorem 7] and De Bruyn [10] that any thick regular near $2d$-gon with $d \geq 4$ and $t_2 \geq 2$ is a dual polar space.

2 Preliminaries

2.1 Near polygons

Suppose $\mathcal{S} = (\mathcal{P}, \mathcal{L})$ is a near polygon. If $x \in \mathcal{P}$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points at distance i from x. If $\emptyset \neq X \subseteq \mathcal{P}$ and $i \in \mathbb{N}$, then $\Gamma_i(X)$ denotes the set of points at distance i from X, i.e. the set of all points $y \in \mathcal{P}$ for which $\mathrm{d}(y, X) := \min\{\mathrm{d}(y, x) \mid x \in X\} = i$, and $X^\perp$ denotes the set of all points of $\mathcal{S}$ that lie at distance at most 1 from every point of X. An *ovoid* of $\mathcal{S}$ is a set of points intersecting each line in one element.

A near polygon $\mathcal{S}$ is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbors. By Shult and Yanushka [24, Proposition 2.5], every two points x and y of a dense near polygon at distance 2 from each other are contained in a unique convex subspace $Q(x, y)$ of diameter 2, called a *quad*. The subgeometry $\widetilde{Q(x, y)}$ induced on $Q(x, y)$ by those lines of the near polygon that are completely contained in $Q(x, y)$ is a generalized quadrangle. Note that distances between points in $\widetilde{Q(x, y)}$ are the same as in $\mathcal{S}$. If $\mathcal{S}$ is regular with parameters $(s, t_2, \ldots, t)$, then the generalized quadrangle $\widetilde{Q(x, y)}$ has order (s, t_2).

Suppose $\mathcal{S}$ is a dense near polygon and Q is a quad of $\mathcal{S}$. By Shult and Yanushka [24, Proposition 2.6], exactly one of the following two cases occurs for a point x of $\mathcal{S}$:

- there exists a unique point $\pi_Q(x) \in Q$ at distance $\mathrm{d}(x, Q)$ from x, and $\mathrm{d}(x, y) = \mathrm{d}(x, \pi_Q(x)) + \mathrm{d}(\pi_Q(x), y)$ for every point $y \in Q$;

- the set of points in Q at distance $\mathrm{d}(x, Q)$ from x is an ovoid O_x of $\widetilde{Q}$, and all remaining points in Q are at distance $\mathrm{d}(x, Q) + 1$ from x.

The point x is called *classical* or *ovoidal with respect to* Q depending on whether the former or the latter case occurs. For every $i \in \mathbb{N}$, let $\Gamma_{i,C}(Q)$ denote the set of all $x \in \Gamma_i(Q)$ which are classical with respect to Q and let $\Gamma_{i,O}(Q)$ denote the set of all $x \in \Gamma_i(Q)$ which are ovoidal with respect to Q. If $x \in \Gamma_{i,O}(Q)$, then we refer to $O_x := \Gamma_i(x) \cap Q$ as *the ovoid of* $\widetilde{Q}$ *subtended by* x.

Suppose L is a line of $\mathcal{S}$ at distance i from Q. By Brouwer and Wilbrink [6, Section (b)], exactly one of the following cases occurs:

(1) $L \subseteq \Gamma_{i,C}(Q)$;

(2) L contains a unique point of $\Gamma_{i,C}(Q)$ and all remaining points are contained in $\Gamma_{i+1,C}(Q)$;

(3) L contains a unique point of $\Gamma_{i,C}(Q)$ and all remaining points are contained in $\Gamma_{i+1,O}(Q)$;

(4) $L \subseteq \Gamma_{i,O}(Q)$;

(5) L contains a unique point of $\Gamma_{i,O}(Q)$ and all remaining points are in $\Gamma_{i+1,O}(Q)$.

2.2 Distance-regular graphs

Let Γ be a distance-regular graph with vertex set Ω and diameter $d \geq 2$. For every $i \in \{0, 1, \ldots, d\}$, let A_i be the real matrix whose rows and columns are indexed (in the same way) by the vertices of Γ such that the entry $(A_i)_{(p,p')}$ of A_i is equal to 1 if $\mathrm{d}(p, p') = i$ and equal to 0 otherwise. Clearly, $A_0 = I$ and $A_0 + A_1 + \cdots + A_d = J$, where I denotes the identity matrix and J the all-one matrix.

The real vector space spanned by $\{A_0, A_1, \ldots, A_d\}$ is a commutative $(d+1)$-dimensional algebra of symmetric matrices, known as the *Bose-Mesner algebra*. It can be shown that the Bose-Mesner algebra has a unique basis $\{E_0, E_1, \ldots, E_d\}$ of minimal idempotents for which $E_iE_j = \delta_{ij}E_i$, $\forall i, j \in \{0, 1, \ldots, d\}$, $E_0 + E_1 + \cdots + E_d = I$ and $E_0 = J/|\Omega|$.

The adjacency matrix A_1 of Γ has exactly $d+1$ distinct eigenvalues. There exists a bijective correspondence between these $d+1$ eigenvalues and the $d+1$ minimal idempotents. Indeed, for every minimal idempotent E there exists a unique eigenvalue λ such that $A_1E = \lambda E$, and then the column span of E is precisely the (right) eigenspace of A_1 for λ.

The *dual eigenvalue sequence* of a minimal idempotent E is the unique sequence $\theta_0^*, \theta_1^*, \ldots, \theta_d^*$ of real numbers such that $E = |\Omega|^{-1}(\theta_0^*A_0 + \theta_1^*A_1 + \cdots + \theta_d^*A_d)$. We then say Γ is *Q-polynomial* with respect to E if there is a (necessarily unique) ordering $E_0 = J/|\Omega|, E_1 = E, E_2, \ldots, E_d$ of the minimal idempotents such that every E_j, $j \in \{0, 1, \ldots, d\}$, can be written in the form $|\Omega|^{-1}\sum_{i=0}^{d} q_j(\theta_i^*)A_i$ for some real polynomial q_j of degree j. We point out that the minimal idempotent with respect to which Γ is Q-polynomial need not be unique. If θ is the eigenvalue for A_1 corresponding to E, then we also say that

Γ is *Q-polynomial with respect to θ*. The eigenvalues, their multiplicities, and the existence of Q-polynomial orderings only depend on the parameters of the distance-regular graph. Graphs with classical parameters are Q-polynomial with respect to the eigenvalue $\left[{d-1 \atop 1}\right]_b(\beta-\alpha)-1$. Bannai and Ito [1] proposed the problem of classifying all Q-polynomial distance-regular graphs with sufficiently large diameter.

For more background information on distance-regular graphs and proofs of the above facts, we refer to Brouwer et al. [5]. We now show the validity of Proposition 1.

Proposition 2.1. *Let $\mathcal{S}$ be a regular near hexagon whose parameters (s, t_2, t) satisfy $s \geq 2$ and $t+1 = (s^2-s+1)(t_2+s+1)$. Then the conditions (1), ..., (6) in Proposition 1 hold.*

Proof. These conditions clearly hold if $t_2 = 0$. So, suppose that $t_2 \geq 1$. Then every two points at distance 2 are contained in a unique quad, and so $\mathcal{S}$ has subquadrangles of order (s, t_2). By Higman's inequality ([17, Section 6]; see also Payne and Thas [21, 1.2.3]), we should have $t_2 = 1$ or $\sqrt{s} \leq t_2 \leq s^2$. Moreover, we know that $s + t_2$ is a divisor of $st_2(s+1)(t_2+1)$, see e.g. [21, 1.2.2], or equivalently that $s+t_2$ is a divisor of $s^2(s^2-1)$. So, (1) and (2) of Proposition 1 hold.

Since every two lines that meet in a unique point are contained in a unique quad, the total number of quads through a given line and the total number of quads through a given point are respectively equal to $\frac{t}{t_2}$ and $\frac{t(t+1)}{t_2(t_2+1)}$. Since $\gcd(t_2, t_2+1) = 1$, these numbers are integral if and only if the conditions (3) and (4) of Proposition 1 hold.

We know from Brouwer-Wilbrink [6, p. 158] that $t = t_2^2 + t_2$ or $1 + t \geq (1+t_2)(1+st_2)$. This implies that $t_2 = s^2$ or $s^2 + st_2 \geq t_2^2 + 2t_2$, proving (5).

The collinearity graph Γ of $\mathcal{S}$ has classical parameters $(d, b, \alpha, \beta) = (3, -s, -\frac{s+t_2}{s-1}, s(t_2 + s+1))$ and so the eigenvalues of Γ and their multiplicities can be computed with the aid of [5, 8.4.2 & 8.4.3]. The eigenvalues of Γ are $s(t+1)$, $s^2+t_2s+s-t_2-1$, $-s^2+s-1$ and $-(t+1)$. The multiplicity of the eigenvalue $s(t+1)$ is equal to 1 and the multiplicity of the eigenvalue $-(t+1)$ is equal to $\frac{s^2(s^4+t_2s^3-s^2-t_2s^2+t_2+1)}{s+t_2}$. The latter number is integral since $s+t_2$ is a divisor of $s^2(s^2-1)$. Note that the sum of all multiplicities is equal to $v := (s+1)\left(1+st+\frac{s^2t(t-t_2)}{t_2+1}\right)$, the total number of points of $\mathcal{S}$, and that the condition $\frac{t(t+1)}{t_2(t_2+1)} \in \mathbb{N}$ implies that $v \in \mathbb{N}$. So, all multiplicities are integral if also the multiplicity $\frac{s^2(s+1)(s^2-s+1)^2(s+t_2+1)(s^3+t_2s^2-t_2s+t_2)}{(t_2+1)(2s^2+st_2-t_2)}$ of the eigenvalue $s^2+t_2s+s-t_2-1$ is integral, and thus (6) of Proposition 1 is clear as well. $\square$

The following two results regarding Q-polynomial regular near polygons will be useful.

Proposition 2.2. ([5, pp. 252-253]) *A regular near $2d$-gon $\mathcal{S}$ with parameters $(s, t_2, \ldots, t)$ where $d \geq 3$ and $s \geq 2$ is Q-polynomial if and only if at least one of the following holds:*

(1) *The collinearity graph has classical parameters $(d, t_2, 0, s)$. In this case, $\mathcal{S}$ is a Hamming near $2d$-gon or a dual polar space.*

(2) *The collinearity graph has classical parameters*

$$\left(d, -s, -\frac{t_2+s}{s-1}, s+\frac{s(t_2+s)((-s)^{d-1}-1)}{s^2-1}\right).$$

In this case, the collinearity graph is Q-polynomial with respect to $-(t+1)$.

Proposition 2.3. *A distance-regular graph Γ with classical parameters $\left(d, -s, -\frac{t_2+s}{s-1}, s+s(t_2+s)\cdot\frac{(-s)^{d-1}-1}{s^2-1}\right)$ where $d \geq 3$ and $s \geq 2$ is the collinearity graph of a regular near $2d$-gon.*

Proof. We know from Terwilliger [26, Theorem 2.12] that there are no induced subgraphs of shape $K_{1,1,2}$ (i.e. no so-called kites of length 2). Since $b_i = b_0 - sc_i$ for all $i \in \{0, 1, \ldots, d\}$, it follows from [5, Theorem 6.4.1] that Γ is the collinearity graph of a regular near $2d$-gon. □

3 Maximal regular near hexagons and some of their combinatorial properties

Suppose $\mathcal{S}$ is a finite regular near $2d$-gon, $d \geq 2$, with parameters $(s, t_2, t_3, \ldots, t_{d-1}, t)$ and point set Ω. Then the collinearity graph Γ of $\mathcal{S}$ is distance-regular. Similarly as in Subsection 2.2, let A_i, $i \in \{0, 1, \ldots, d\}$, be the real matrix describing the distance-i-relation on the point set Ω. We define the following matrix:

$$M := \sum_{i=0}^{d} \left(-\frac{1}{s}\right)^i A_i.$$

It is known that, up to a positive scalar, the matrix M is the minimal idempotent for the smallest eigenvalue $-(t+1)$ of Γ (see for instance Neumaier [20, Remark 2.2]).

If X is a set of points in $\mathcal{S}$, then χ_X denotes the *characteristic vector* of X. This is the 0-1 column vector whose rows are indexed by the elements of Ω (in the same way the rows and columns of the matrices A_i have been indexed) such that the entry $(\chi_X)_p$ of χ_X is equal to 1 if and only if $p \in X$. If X is a singleton $\{x\}$, then we will also write χ_x instead of χ_X.

The following proposition is a very particular case of Terwilliger [25, Theorems 3.3, 4.1 and 4.2]. Most of the claims in the proposition are also implied by Theorem 3.2 of De Bruyn and Vanhove [13].

Proposition 3.1. *Let $\mathcal{S}$ be a regular near hexagon with parameters (s, t_2, t), $s \geq 2$. Then $t+1 \leq (s^2-s+1)(t_2+s+1)$, with equality if and only if the collinearity graph is Q-polynomial with respect to the eigenvalue $-(t+1)$.*

If equality holds, then for two any two points x and y of $\mathcal{S}$ at distance 3 from each other, $M\chi = 0$ holds, where

$$\chi = s(t_2 + 1 + s)(\chi_x - \chi_y) + (\chi_X - \chi_Y),$$

with $X := \Gamma_1(x) \cap \Gamma_2(y)$ and $Y := \Gamma_2(x) \cap \Gamma_1(y)$.

For the remainder of this section, $\mathcal{S}$ will be a maximal regular near hexagon. The near hexagon $\mathcal{S}$ has $v = 1 + s(t+1) + \frac{s^2 t(t+1)}{t_2+1} + \frac{s^3 t(t-t_2)}{t_2+1}$ points. If $t_2 \geq 1$, then $\mathcal{S}$ is dense and for every two points x and y at distance 2 from each other, $\widetilde{Q(x,y)}$ is a generalized quadrangle of order (s, t_2). If $t_2 \geq 1$ and Q is a quad of $\mathcal{S}$, then each point belongs to either Q, $\Gamma_1(Q)$ or $\Gamma_2(Q)$. Each point of $Q \cup \Gamma_1(Q)$ is classical with respect to Q and each point of $\Gamma_2(Q)$ is ovoidal with respect to Q. In the following proposition, the notion of a *triad* of a generalized quadrangle occurs. This is a set of three pairwise noncollinear points. A *center* of a triad T is a point that is collinear with every point of T.

Proposition 3.2. *Suppose $\mathcal{S}$ is a maximal regular near hexagon with parameters (s, t_2, t). Let x, y and z be three points of $\mathcal{S}$ such that $\mathrm{d}(x,y) = 2$ and $\mathrm{d}(x,z) = \mathrm{d}(y,z) = 3$. Then:*

(1) $|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| = s^2 + s\Big(t_2 + 1 - |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|\Big) + t_2 + 1,$

(2) *if $t_2 \geq 1$ and $\mathrm{d}(z, Q(x,y)) = 1$, then $|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| \leq s + 1$,*

(3) *if $t_2 \geq 1$ and $\mathrm{d}(z, Q(x,y)) = 2$, then $|\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_2(z)| \leq s$,*

(4) *if $t_2 \geq 1$ and T is a triad of a quad Q having precisely β centers, then there are precisely $\frac{s(t-t_2)(s+t_2)(s+1-\beta)}{t_2+1}$ points in $\Gamma_2(Q)$ subtending an ovoid of $\widetilde{Q}$ containing T.*

Proof. (1) Put $X := \Gamma_1(x) \cap \Gamma_2(z)$, $Z := \Gamma_1(z) \cap \Gamma_2(x)$, $N := |\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)|$ and $\alpha := |\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_2(z)|$. Since $|\Gamma_1(z) \cap \Gamma_2(x)| = t + 1$, we then have $|\Gamma_2(x) \cap \Gamma_3(y) \cap \Gamma_1(z)| = t + 1 - N$.

We show that $|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| = t_2 + 1 - \alpha$. If $u \in \Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)$, then since $\mathrm{d}(u,y) = \mathrm{d}(x,y) = 2$, there exists a unique point on the line xu collinear with y and hence xu is one of the $t_2 + 1$ lines through x containing a point collinear with y. Among the $t_2 + 1$ lines through x containing a point collinear with y, there are α lines for which this point of $\Gamma_1(y)$ lies in $\Gamma_2(z)$. If L is one of the $t_2 + 1 - \alpha$ other lines, then the unique point of $\Gamma_1(x) \cap \Gamma_2(z)$ on L lies at distance 2 from y. It follows that $|\Gamma_1(x) \cap \Gamma_2(y) \cap \Gamma_2(z)| = t_2 + 1 - \alpha$ as we wished to show. We also have $|\Gamma_1(x) \cap \Gamma_3(y) \cap \Gamma_2(z)| = (t+1) - (t_2 + 1 - \alpha) - \alpha = t - t_2$.

By Proposition 3.1, we have

$$\chi_y^T \cdot M \cdot \Big(s(t_2 + 1 + s)(\chi_x - \chi_z) + (\chi_X - \chi_Z)\Big) = 0. \tag{1}$$

Now,

$$\chi_y^T M \chi_x = \left(-\frac{1}{s}\right)^2, \qquad \chi_y^T M \chi_z = \left(-\frac{1}{s}\right)^3,$$

$$\chi_y^T M \chi_X = \alpha \cdot \left(-\frac{1}{s}\right) + (t_2 + 1 - \alpha) \cdot \left(-\frac{1}{s}\right)^2 + (t - t_2) \cdot \left(-\frac{1}{s}\right)^3,$$

$$\chi_y^T M \chi_Z = N \cdot \left(-\frac{1}{s}\right)^2 + (t + 1 - N) \cdot \left(-\frac{1}{s}\right)^3.$$

If we plug these four equalities into equation (1) and solve for N, then we find

$$N = s^2 + s(t_2 + 1 - \alpha) + t_2 + 1.$$

(2) Suppose now that $t_2 \geq 1$ and $z \in \Gamma_1(Q(x,y))$. Let z' denote the unique point of $Q(x,y)$ collinear with z. Since $\Gamma_1(x) \cap \Gamma_1(y) \subseteq Q(x,y)$ and $\mathrm{d}(z,u) = 1 + \mathrm{d}(z',u)$, $\forall u \in Q(x,y)$, we have $\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_2(z) = \Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_1(z')$. Now, put $\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_1(z') = \{u_1, u_2, \ldots, u_\alpha\}$. For every $i \in \{1, 2, \ldots, \alpha\}$, $\Gamma_1(u_i) \cap \Gamma_1(z)$ contains t_2+1 points. One of these t_2+1 points is z'. If w is one of the t_2 other points, then $w \in \Gamma_1(Q(x,y))$, and u_i is the unique point of $Q(x,y)$ collinear with w. So, we see that there are precisely $1 + \alpha t_2$ neighbors of z which are collinear with a point of the set $\{u_1, u_2, \ldots, u_\alpha\}$. All these points are contained in $\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)$. So, by part (1) of this proposition, we have $s^2 + s(t_2 + 1 - \alpha) + t_2 + 1 \geq 1 + \alpha t_2$, i.e. $\alpha \leq s + 1$.

(3) Suppose $t_2 \geq 1$ and $z \in \Gamma_2(Q(x,y))$. As before, put $\Gamma_1(x) \cap \Gamma_1(y) \cap \Gamma_2(z) = \{u_1, u_2, \ldots, u_\alpha\}$. For every $i \in \{1, 2, \ldots, \alpha\}$, the points u_i and z have t_2+1 common neighbors. Suppose that $w \in \Gamma_1(u_i) \cap \Gamma_1(u_j) \cap \Gamma_1(z)$ where i and j are two distinct elements of $\{1, 2, \ldots, \alpha\}$. The points u_i and u_j belong to $Q(x,y)$ and hence also w since $Q(x,y)$ is convex. But that is impossible since z would then be collinear with a point of $Q(x,y)$. So, there are $\alpha(t_2+1)$ points collinear with z and a point of the set $\{u_1, u_2, \ldots, u_\alpha\}$. All these points are contained in $\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)$. So, by part (1) of this proposition, we have

$$s^2 + s(t_2 + 1 - \alpha) + t_2 + 1 \geq (t_2 + 1)\alpha,$$

from which it follows that $\alpha \leq s + \frac{t_2+1}{s+t_2+1}$. So, $\alpha \leq s$.

(4) Now, let $T = \{w_1, w_2, w_3\}$ be a triad of a quad Q having precisely β centers. Put $x' := w_1$ and $y' := w_2$. We count in two different ways the number of pairs (z', z'') where $z' \in \Gamma_1(Q) \cap \Gamma_1(w_3)$, $z'' \in \Gamma_2(Q) \cap \Gamma_1(z')$ and $T \subseteq \Gamma_2(z'') \cap Q$. Observe that if $z'' \in \Gamma_2(Q)$ such that $T \subseteq \Gamma_2(z'') \cap Q$, then there are $t_2 + 1$ possibilities for z', namely the $t_2 + 1$ common neighbors of z'' and w_3. On the other hand, there are $s(t - t_2)$ possibilities for $z' \in \Gamma_1(Q) \cap \Gamma_1(w_3)$. Now, fix $z' \in \Gamma_1(Q) \cap \Gamma_1(w_3)$. Then $\beta = |\Gamma_1(w_1) \cap \Gamma_1(w_2) \cap \Gamma_1(w_3)| = |\Gamma_1(x') \cap \Gamma_1(y') \cap \Gamma_2(z')|$. Hence, by part (1), there are $s^2 + s(t_2 + 1 - \beta) + t_2 + 1$ points in $\Gamma_2(x') \cap \Gamma_2(y') \cap \Gamma_1(z')$. Among these

$s^2+s(t_2+1-\beta)+t_2+1$ points, one (namely w_3) is contained in Q and $t_2\beta$ others are contained in $\Gamma_1(Q)$, see part (2) of this proof. So, for fixed $z' \in \Gamma_1(Q)\cap\Gamma_1(w_3)$, there are $(s^2+s(t_2+1-\beta)+t_2+1)-1-\beta t_2 = (s+t_2)(s+1-\beta)$ points $z'' \in \Gamma_1(z')\cap\Gamma_2(Q)$ for which $T \subseteq \Gamma_2(z'') \cap Q$. We conclude that there are $\frac{s(t-t_2)(s+t_2)(s+1-\beta)}{t_2+1}$ points in $\Gamma_2(Q)$ subtending an ovoid of $\widetilde{Q}$ containing T.

□

The following is an immediate consequence of Proposition 3.2(4).

Corollary 3.3. *If $t_2 \geq 1$ and T is a triad of a quad of a maximal regular near hexagon $\mathcal{S}$ with parameters (s, t_2, t), then T has at most $s+1$ centers. If T has precisely $s+1$ centers, then it cannot be contained in a subtended ovoid.*

Remark 3.4. (1) If Q is a quad of a regular near hexagon with parameters $s \geq 2$, $t_2 \geq 1$ and t, and if $\{x, y\}$ is a pair of noncollinear points of Q, then an easy counting argument yields there are precisely $\frac{s(t-t_2)(t-t_2^2-t_2)}{t_2+1}$ points in $\Gamma_2(Q)$ subtending an ovoid of $\widetilde{Q}$ that contains the points x and y. It is not possible to find a formula in terms of only s, t and t_2 giving the total number of points of $\Gamma_2(Q)$ subtending an ovoid that contains a fixed triad of $\widetilde{Q}$. However, in the case of a maximal regular near hexagon, it is possible by Proposition 3.2(4) to give such a formula if one allows the use of one extra parameter, namely the number of centers of the triad.

(2) If $\mathcal{Q}$ is a generalized quadrangle of order (s, t_2) with $s \neq 1$ and T is a triad of $\mathcal{Q}$, then the number of centers of T is obviously bounded above by t_2+1. By Payne and Thas [21, 1.4.1], the number of centers of T is also bounded above by $\frac{s^2}{2}+1$. So, in case the generalized quadrangle $\mathcal{Q}$ can occur as a quad (or even just as a full subquadrangle) in a maximal regular near hexagon, the upper bound $\frac{s^2}{2}+1$ for the number of centers of T can be improved to $s+1$. (Note that by Shult and Yanushka [24, Proposition 2.5], every full subquadrangle of a dense near polygon must be contained in a unique quad.)

For generalized hexagons, we obtain the following corollary of Proposition 3.2(1).

Corollary 3.5. *Let x, y and z be three points of a generalized hexagon of order (s, s^3), $s \geq 2$, with $d(x, y) = 2$ and $d(x, z) = d(y, z) = 3$. Let p be the unique point collinear with both x and y. Then:*

- $|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| = s^2+1$ *if* $d(p, z) = 2$;
- $|\Gamma_2(x) \cap \Gamma_2(y) \cap \Gamma_1(z)| = s^2+s+1$ *if* $d(p, z) = 3$.

The first property mentioned in Corollary 3.5 generalizes a property of the dual twisted triality hexagons given in Ronan [22, Remark 3.2].

We will now exclude some maximal regular near hexagons. We will need the following result on triads of generalized quadrangles.

Lemma 3.6. (Payne and Thas [21, 1.3.5]) *Suppose x and y are noncollinear points in a generalized quadrangle $\mathcal{Q}$ of order (s, t_2). For every $\beta \in \{0, 1, \ldots, t_2+1\}$, denote by T_β the number of points z noncollinear with both x and y such that the triad $\{x, y, z\}$ has exactly β centers. Then the following equalities hold:*

$$\begin{aligned}
\sum_{0 \le \beta \le t_2+1} T_\beta &= s^2 t_2 - s t_2 - s + t_2, \\
\sum_{0 \le \beta \le t_2+1} \beta \cdot T_\beta &= s t_2^2 - s, \\
\sum_{0 \le \beta \le t_2+1} \beta(\beta-1) \cdot T_\beta &= t_2^3 - t_2.
\end{aligned}$$

Proposition 3.7. *Suppose $\mathcal{S}$ is a maximal regular near hexagon with parameters (s, t_2, t). If β_0 denotes the smallest strictly positive integer β such that $\frac{s(t-t_2)(s+t_2)(s+1-\beta)}{t_2+1}$ is an integer, then $(\beta_0 - 1)s \le t_2$.*

Proof. Note that $\beta_0 = 1$ if $t_2 \in \{0, 1\}$, so we may assume $t_2 \ge 2$. Let Q be any quad in $\mathcal{S}$. Consider two noncollinear points x and y in Q. As in Lemma 3.6, we denote by T_β the number of points $z \in \Gamma_2(x) \cap \Gamma_2(y) \cap Q$ such that the triad $\{x, y, z\}$ has precisely β centers. If $\beta > 0$ and $T_\beta > 0$, then it follows from Proposition 3.2(4) that $\frac{s(t-t_2)(s+t_2)(s+1-\beta)}{t_2+1}$ must be an integer, and hence $\beta \ge \beta_0$. Lemma 3.6 now implies that:

$$(\beta_0 - 1) \cdot s(t_2^2 - 1) = (\beta_0 - 1) \left(\sum_{0 \le \beta \le t_2+1} \beta T_\beta \right) \le \sum_{0 \le \beta \le t_2+1} \beta(\beta-1) T_\beta = t_2(t_2^2 - 1).$$

□

Corollary 3.8. *There exists no maximal regular near hexagon whose parameters (s, t_2, t) are equal to either $(8, 4, 740)$, $(92, 64, 1314560)$, $(95, 19, 1027064)$ or $(105, 147, 2763012)$.*

Proof. Let β_0 be the strictly positive integer as defined in Proposition 3.7. Then one easily checks that $\beta_0 = 4$ if $(s, t_2) = (8, 4)$, $\beta_0 = 3$ if $(s, t_2) = (92, 64)$, $\beta_0 = 2$ if $(s, t_2) = (95, 19)$ and $\beta_0 = 32$ if $(s, t_2) = (105, 147)$. In each of the cases, a contradiction is obtained from Proposition 3.7. □

Remark 3.9. We have verified that the system of three linear equations given in Lemma 3.6 has a non-negative integral solution for all 25 possibilities of (s, t_2) mentioned after Proposition 1, except for those we excluded in Corollary 3.8 (of course, supposing that $T_\beta = 0$ if $\beta > s+1$ or if $t_2 + 1$ is not a divisor of $s(t-t_2)(s+t_2)(s+1-\beta)$).

We can prove an extra restriction on s and t_2 that must hold for a general maximal regular near hexagon with $t_2 \ne s^2$.

Proposition 3.10. *If a maximal regular near hexagon with parameters (s, t_2, t) with $t_2 \ne s^2$ exists, then β_0, the smallest strictly positive integer β such that $\frac{s(t-t_2)(s+t_2)(s+1-\beta)}{t_2+1}$ is an integer, must be 1 or 2, and $t_2 + 1$ must be a divisor of $\gcd(2, s+1) \cdot s(s-1)(s^2 - s + 1)$.*

Proof. Since the proposition clearly holds for $t_2 = 0$, we may assume that $t_2 \geq 1$. Since $t_2 \neq s^2$, it follows from Proposition 1(5) that $(\frac{s}{t_2})^2+\frac{s}{t_2} \geq 1+\frac{2}{t_2} > 1$ and hence $\frac{s}{t_2} > \frac{\sqrt{5}-1}{2}$. It now follows from Proposition 3.7 that $\beta_0-1 \leq \frac{t_2}{s} < \frac{2}{\sqrt{5}-1}$, i.e. $\beta_0 = 1$ or $\beta_0 = 2$. Hence by definition of β_0, $\frac{s^2(t-t_2)(s+t_2)}{t_2+1}$ or $\frac{s(t-t_2)(s+t_2)(s-1)}{t_2+1}$ is an integer. Since $t = s^3+t_2(s^2-s+1)$, this is equivalent to asking that $\frac{s^3(s^2-s+1)(s-1)}{t_2+1}$ or $\frac{s^2(s^2-s+1)(s-1)^2}{t_2+1}$ is an integer. In any case, t_2+1 must divide $s^3(s^2-s+1)(s-1)^2$. We also know from Proposition 1(4) that t_2+1 divides $s(s-1)(s^2+1)(s^2-s+1)$, and hence t_2+1 must divide $\gcd(s^3(s^2-s+1)(s-1)^2, s(s-1)(s^2+1)(s^2-s+1)) = s(s-1)(s^2-s+1)\cdot\gcd(s^2(s-1), s^2+1) = \gcd(2, s+1)\cdot s(s-1)(s^2-s+1)$. $\square$

4 Maximal regular near hexagons with $t_2 = 1$

In this section we study maximal regular near hexagons with parameters (s, t_2, t) where $s \geq 2$ and $t_2 = 1$. Then $t = s^3 + s^2 - s + 1$. The main result is that no such regular near hexagon exists if $s = 3$ (Subsection 4.3). We will also look at the implications of this result for the classification of the dense near polygons with four points per line (Subsection 4.4). The machinery for our investigation will first be developed in Subsections 4.1 and 4.2.

4.1 Nice sets of permutations

Let X be a set of size $s + 1 \geq 1$. In this paper, we call a set $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_{s+1}\}$ of permutations of X a *nice set of permutations* if the following properties hold:

- the trivial permutation of X belongs to Σ;
- for any two elements $x_1, x_2 \in X$, there exists a unique $\sigma \in \Sigma$ for which $x_1^\sigma = x_2$.

We will use the left-to-right convention for compositions of maps. Suppose $\Sigma = \{\sigma_1, \sigma_2, \dots, \sigma_{s+1}\}$ and $\Sigma' = \{\sigma_1', \sigma_2', \dots, \sigma_{s+1}'\}$ are nice sets of permutations of the respective sets X and X'. Then Σ and Σ' are called *isomorphic* if there exists a bijection $\tau : X \to X'$ such that $\Sigma' = \tau^{-1} \cdot \Sigma \cdot \tau := \{\tau^{-1}\sigma_1\tau, \tau^{-1}\sigma_2\tau, \dots, \tau^{-1}\sigma_{s+1}\tau\}$.

Examples. • The set $X = \{1\}$ admits only one nice set of permutations, namely $\Sigma = \{\mathsf{Id}\}$.

• The set $X = \{1, 2\}$ admits only one nice set of permutations, namely $\Sigma = \{\mathsf{Id}, (12)\}$.

• The set $X = \{1, 2, 3\}$ admits only one nice set of permutations, namely $\Sigma = \{\mathsf{Id}, (123), (132)\}$.

• The set $X = \{1, 2, 3, 4\}$ admits four nice sets of permutations, namely $\Sigma_1 = \{\mathsf{Id}, (12)(34), (13)(24), (14)(23)\}$, $\Sigma_2 = \{\mathsf{Id}, (1234), (1432), (13)(24)\}$, $\Sigma_3 = \{\mathsf{Id}, (1243), (1342), (14)(23)\}$ and $\Sigma_4 = \{\mathsf{Id}, (1324), (1423), (12)(34)\}$. Up to isomorphism, there are only two nice sets of permutations of X, namely Σ_1 and Σ_2.

• It is straightforward but tedious to verify that the set $X = \{1,2,3,4,5\}$ admits up to isomorphism three nice sets of permutations, namely

$$\begin{aligned}
\Sigma_1 &= \{\mathsf{Id}, (12345), (13524), (14253), (15432)\},\\
\Sigma_2 &= \{\mathsf{Id}, (12345), (14352), (15324), (13)(254)\},\\
\Sigma_3 &= \{\mathsf{Id}, (12)(345), (13)(254), (14)(235), (15)(243)\}.
\end{aligned}$$

If Σ is a nice set of permutations of X, then any nice set of permutations of X of the form $\Sigma \cdot \sigma^{-1}$ where $\sigma \in \Sigma$ is called a *cousin* of Σ. The relation of being a cousin is an equivalence relation on the set of all nice sets of permutations of X.

The proof of the following lemma is straightforward.

Lemma 4.1. (1) *Let Σ_1 and Σ_2 denote the nice sets of permutations of $\{1,2,3,4\}$ as defined above. Then Σ_i, $i \in \{1,2\}$, has one cousin, namely Σ_i itself.*

(2) *Let Σ_1, Σ_2 and Σ_3 be the nice sets of permutations of $\{1,2,3,4,5\}$ as defined above. Then Σ_1 has only one cousin, namely Σ_1 itself. Σ_2 has five cousins, four of them are isomorphic to Σ_2 while the fifth one is isomorphic to Σ_3. Σ_3 has five cousins, four of them are isomorphic to Σ_2 while the fifth one is equal to Σ_3 itself.*

4.2 Maximal regular near hexagons with grid-quads

Consider a maximal regular near hexagon $\mathcal{S}$ with parameters $(s, t_2, t) = (s, 1, t)$ where $s \geq 2$ and $t = s^3 + s^2 - s + 1$. Note that here the quads are $(s+1) \times (s+1)$-grids. Let G be a quad and L a line of $\mathcal{S}$. We denote by i the smallest distance between a point of L and a point of G. Recall the possible line-quad relations as discussed in Subsection 2.1. We are especially interested here in the cases (3) and (4). The following additional information can be provided for these cases, see Brouwer and Wilbrink [6, Section (b)]. If case (3) occurs, then $i = 1$ and if y denotes the unique point of L contained in $\Gamma_{1,C}(G)$, then $\{O_x \mid x \in L \setminus \{y\}\}$ is a set of ovoids of $\widetilde{G}$ through $\pi_G(y)$ partitioning the set of points of G at distance 2 from $\pi_G(y)$. If case (4) occurs, then $i = 2$ and $\{O_x \mid x \in L\}$ is a partition of G into ovoids of $\widetilde{G}$.

There exist two sets of lines, each of which partitions the point set of G. We denote these sets by $\mathcal{H}$ and $\mathcal{V}$. For every point x of $\mathcal{S}$ which is ovoidal with respect to G and for every $L \in \mathcal{H}$, let $\theta_x(L)$ denote the unique line of $\mathcal{V}$ such that $L \cap \theta_x(L)$ is contained in the ovoid O_x of $\widetilde{G}$. Then θ_x defines a bijection from $\mathcal{H}$ to $\mathcal{V}$. Note that if $x, x' \in \Gamma_{2,O}(G)$ then $\theta_x(L) = \theta_{x'}(L)$ if and only if the ovoids O_x and $O_{x'}$ intersect L in the same point.

Lemma 4.2. (1) *Suppose L is contained in $\Gamma_{2,O}(G)$ and let x_1^* be a fixed point of L. Then $\Sigma(L, x_1^*) := \{\theta_x \theta_{x_1^*}^{-1} \mid x \in L\}$ is a nice set of permutations of $\mathcal{H}$. If x_2^* is another point of L, then $\Sigma(L, x_1^*)$ and $\Sigma(L, x_2^*)$ are cousins.*

(2) *Suppose L is contained in $\Gamma_{1,C}(G) \cup \Gamma_{2,O}(G)$ with $L \cap \Gamma_{1,C}(G) \neq \emptyset$ and $L \cap \Gamma_{2,O}(G) \neq \emptyset$. Let y^* be the unique point in $L \cap \Gamma_{1,C}(G)$ and let M^* denote the unique line of $\mathcal{H}$ containing the point $\pi_G(y^*)$. If x_1^* is a point of $L \setminus \{y^*\}$, then the set $\Sigma(L, x_1^*) := \{\theta_x \theta_{x_1^*}^{-1} \,|\, x \in L \setminus \{y^*\}\}$ induces a nice set $\Sigma'(L, x_1^*)$ of permutations of $\mathcal{H} \setminus \{M^*\}$. If x_2^* is another point of $L \setminus \{y^*\}$, then $\Sigma'(L, x_1^*)$ and $\Sigma'(L, x_2^*)$ are cousins.*

Proof. The fact that $\Sigma(L, x_1^*)$ is a nice set of permutations of $\mathcal{H}$ (in case (1)) is a consequence of the fact that $\{O_x \,|\, x \in L\}$ is a partition of $\widetilde{G}$ into ovoids. The fact that $\Sigma'(L, x_1^*)$ is a nice set of permutations of $\mathcal{H} \setminus \{M^*\}$ (in case (2)) is a consequence of the fact that $\{O_x \,|\, x \in L \setminus \{y^*\}\}$ is a set of ovoids of $\widetilde{G}$ through $\pi_G(y^*)$ partitioning the set $\Gamma_2(\pi_G(y^*)) \cap G$. In both cases, the fact that the nice sets of permutations are cousins follows from the fact that $\theta_x \theta_{x_1^*}^{-1} = \theta_x \theta_{x_2^*}^{-1} (\theta_{x_1^*} \theta_{x_2^*}^{-1})^{-1}$. □

Now consider a quad G in $\mathcal{S}$, and let x be a fixed point in $\Gamma_2(G)$. For every $y \in \Gamma_2(G) \cap \Gamma_1(x)$, let π_y denote the cycle structure of the permutation $\theta_y \theta_x^{-1}$ of $\mathcal{H}$. If the cycle decomposition of $\theta_y \theta_x^{-1}$ contains α_i cycles of length $i \in \mathbb{N} \setminus \{0\}$, then we denote $\pi_y = 1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \cdots$. In the expression π_y, we often omit the terms j^{α_j} for which $\alpha_j = 0$.

Let Π denote the set of all cycle structures of the form

$$1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \cdots$$

where $\alpha_1, \alpha_2, \alpha_3, \ldots$ are nonnegative integers such that $\alpha_1 \in \{0, 1\}$ and $s + 1 = 1 \cdot \alpha_1 + 2 \cdot \alpha_2 + 3 \cdot \alpha_3 + \cdots$. By Lemma 4.2, if $y \in \Gamma_2(G) \cap \Gamma_1(x)$, then the cycle structure π_y belongs to Π.

If π is the cycle structure $1^{\alpha_1} 2^{\alpha_2} 3^{\alpha_3} \cdots$, then we also put $\alpha_1(\pi) := \alpha_1$, $\alpha_2(\pi) := \alpha_2$, $\alpha_3(\pi) := \alpha_3$, etc. For every $\pi \in \Pi$, we put

$$\begin{aligned}
n_1(\pi) &= \alpha_2(\pi), \\
n_2(\pi) &= 3 \cdot \alpha_3(\pi) + 4 \cdot \alpha_4(\pi) + \cdots = s + 1 - \alpha_1(\pi) - 2 \cdot \alpha_2(\pi), \\
n_3(\pi) &= \frac{(s + 1 - \alpha_1(\pi))(s - \alpha_1(\pi))}{2} - n_1(\pi) - n_2(\pi).
\end{aligned}$$

For every $\pi \in \Pi$, let N_π denote the total number of points $y \in \Gamma_2(G) \cap \Gamma_1(x)$ for which $\pi_y = \pi$.

There are $s + 1 = |\Gamma_2(x) \cap G|$ grid-quads through x meeting G (necessarily in a point of $\Gamma_2(x) \cap G$) and these $s + 1$ grid-quads define $2(s + 1)$ lines through x meeting $\Gamma_1(G)$. These $2(s + 1)$ lines through x are all the lines through x meeting $\Gamma_1(G)$.

Since $|\Gamma_1(x)| = s(s^3 + s^2 - s + 2)$ and $|\Gamma_1(x) \cap \Gamma_1(G)| = 2(s + 1)$, we have

$$\sum_{\pi \in \Pi} N_\pi = s^4 + s^3 - s^2 - 2. \tag{2}$$

The points $y \in \Gamma_1(x) \cap \Gamma_2(G)$ for which $\alpha_1(\pi_y) = 1$ are precisely the points $y \in \Gamma_1(x) \cap \Gamma_2(G)$ which lie on a line through x meeting $\Gamma_1(G)$. Since there are $2(s + 1)(s - 1)$ such points, we have

$$\sum_{\pi\in\Pi,\alpha_1(\pi)=1} N_\pi = 2(s+1)(s-1). \tag{3}$$

From equations (2) and (3), we find

$$\sum_{\pi\in\Pi,\alpha_1(\pi)=0} N_\pi = s^4 + s^3 - 3s^2. \tag{4}$$

Lemma 4.3. *Let $y \in \Gamma_1(x)\cap\Gamma_2(G)$.*

(1) *The number of unordered pairs $\{a,b\} \subseteq (\Gamma_2(y)\cap G)\setminus\Gamma_2(x)$ for which $\{a,b\}^\perp \subseteq \Gamma_2(x)\cap G$ is equal to $n_1(\pi_y)$.*

(2) *The number of unordered pairs $\{a,b\} \subseteq (\Gamma_2(y)\cap G)\setminus\Gamma_2(x)$ for which $\{a,b\}^\perp \cap (\Gamma_2(x)\cap G)$ is a singleton is equal to $n_2(\pi_y)$.*

(3) *The number of unordered pairs $\{a,b\} \subseteq (\Gamma_2(y)\cap G)\setminus\Gamma_2(x)$ for which $\{a,b\}^\perp \cap (\Gamma_2(x)\cap G)=\emptyset$ is equal to $n_3(\pi_y)$.*

Proof. The number of unordered pairs $\{a,b\}$ contained in $(\Gamma_2(y)\cap G)\setminus\Gamma_2(x)$ is equal to $\binom{s+1-\alpha_1(\pi_y)}{2}$. Hence, it suffices to prove the claims (1) and (2).

(1) Suppose $\{a,b\} \subseteq (\Gamma_2(y)\cap G)\setminus\Gamma_2(x)$ for which $\{a,b\}^\perp \subseteq \Gamma_2(x)\cap G$. Let L_a and L_b denote the unique lines of $\mathcal{H}$ containing a and b, respectively, and let M_a and M_b denote the unique lines of $\mathcal{V}$ containing a and b, respectively. Then $L_a^{\theta_y\theta_x^{-1}} = M_a^{\theta_x^{-1}} = L_b$ and $L_b^{\theta_y\theta_x^{-1}} = M_b^{\theta_x^{-1}} = L_a$. Hence, (L_aL_b) is a transposition in the cycle decomposition of $\theta_y\theta_x^{-1}$.

Conversely, suppose that (L_1L_2) is a transposition of the cycle decomposition of $\theta_y\theta_x^{-1}$. Then $L_1^{\theta_y} = L_2^{\theta_x}$ and $L_2^{\theta_y} = L_1^{\theta_x}$. So, the unique points a and b in respectively $L_2^{\theta_x}\cap L_1$ and $L_1^{\theta_x}\cap L_2$ belong to $(\Gamma_2(y)\cap G)\setminus\Gamma_2(x)$. Clearly, $\{a,b\}^\perp \subseteq \Gamma_2(x)\cap G$. With the notations of the previous paragraph, we have $L_1 = L_a$ and $L_2 = L_b$.

We conclude that the number of unordered pairs $\{a,b\} \subseteq (\Gamma_2(y)\cap G)\setminus\Gamma_2(x)$ for which $\{a,b\}^\perp \subseteq \Gamma_2(x)$ is equal to the number of transpositions contained in the cycle decomposition of $\theta_y\theta_x^{-1}$, i.e. equal to $n_1(\pi_y)$.

(2) Suppose $\{a,b\} \subseteq (\Gamma_2(y)\cap G)\setminus\Gamma_2(x)$ such that $\{a,b\}^\perp \cap (\Gamma_2(x)\cap G)$ is a singleton $\{u\}$. Without loss of generality, we may suppose that $ua \in \mathcal{H}$. Let L_a and L_b denote the lines of $\mathcal{H}$ containing a and b, respectively, and let M_a and M_b denote the lines of $\mathcal{V}$ containing a and b, respectively. Then $L_b^{\theta_y\theta_x^{-1}} = M_b^{\theta_x^{-1}} = L_a$ and $L_a^{\theta_y\theta_x^{-1}} = M_a^{\theta_x^{-1}} \neq L_b$. So, in the cycle decomposition of $\theta_y\theta_x^{-1}$, the line $L_b \in \mathcal{H}$ does not belong to a cycle of length 1, nor to one of length 2.

Conversely, suppose that L_1 and L_2 are two distinct lines of $\mathcal{H}$ such that $L_2^{\theta_y\theta_x^{-1}} = L_1$ and $L_1^{\theta_y\theta_x^{-1}} \neq L_2$. Then $L_2^{\theta_y} = L_1^{\theta_x}$ and $L_1^{\theta_y} \neq L_2^{\theta_x}$. Let a and b denote the unique points of $L_1 \cap \Gamma_2(y)$ and $L_2 \cap \Gamma_2(y)$, respectively. Then $\{a,b\} \in (\Gamma_2(y) \cap G) \setminus \Gamma_2(x)$ and $\{a,b\}^\perp \cap (\Gamma_2(x) \cap G)$ is a singleton. With the notation of the previous paragraph, we have $L_1 = L_a$ and $L_2 = L_b$.

We conclude that the number of unordered pairs $\{a,b\} \subseteq (\Gamma_2(y) \cap G) \setminus \Gamma_2(x)$ for which $\{a,b\}^\perp \cap (\Gamma_2(x) \cap G)$ is a singleton is equal to the number of elements of $\mathcal{H}$ that do not belong to a cycle of length 1 nor to one of length 2 in the cycle decomposition of $\theta_y \theta_x^{-1}$. This number is equal to $n_2(\pi_y)$. □

Proposition 4.4. *The following hold:*

(1) $\sum_{\pi \in \Pi} n_1(\pi) \cdot N_\pi = \frac{(s+1)s}{2}(s^2-2)$,

(2) $\sum_{\pi \in \Pi} n_2(\pi) \cdot N_\pi = (s+1)s(s-1)(s^2+s)$.

Proof.
- We count in two different ways the number of pairs $(y, \{a,b\})$ where $y \in \Gamma_1(x) \cap \Gamma_2(G)$ and $\{a,b\} \subseteq (\Gamma_2(y) \cap G) \setminus \Gamma_2(x)$ such that $a \neq b$ and $\{a,b\}^\perp \subseteq \Gamma_2(x) \cap G$. The number of such pairs is equal to $\sum_{\pi \in \Pi} n_1(\pi) N_\pi$ by Lemma 4.3(1). On the other hand, there are $\frac{(s+1)s}{2} = \binom{|\Gamma_2(x) \cap G|}{2}$ possibilities for $\{a,b\}$, as one can take two distinct elements u and v of $\Gamma_2(x) \cap G$ and put $\{a,b\} = \{u,v\}^\perp$. For such a fixed choice of $\{a,b\}$, we have $|\Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(x)| = s^2+2$ by Proposition 3.2(1). There are however four points in $\Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(x)$ that are contained in $\Gamma_1(G)$, namely the two common neighbors of x and u and the two common neighbors of x and v. So, for fixed $\{a,b\}$, there are s^2-2 choices for y.

- Next, we count in two different ways the number of pairs $(y, \{a,b\})$ where $y \in \Gamma_1(x) \cap \Gamma_2(G)$ and $\{a,b\} \subseteq (\Gamma_2(y) \cap G) \setminus \Gamma_2(x)$ such that $a \neq b$ and $\{a,b\}^\perp \cap (\Gamma_2(x) \cap G)$ is a singleton $\{u\}$ for which $ua \in \mathcal{H}$ and $ub \in \mathcal{V}$. The number of such pairs is equal to $\sum_{\pi \in \Pi} n_2(\pi) N_\pi$ by Lemma 4.3(2). On the other hand, there are $s+1$ choices for $u \in \Gamma_2(x) \cap G$. For each such u, there are s choices for $a \in \Gamma_1(u)$ such that $ua \in \mathcal{H}$. If u and a have been chosen, then there are $s-1$ choices for $b \in \Gamma_1(u)$ such that $ub \in \mathcal{V}$ and $\{a,b\}^\perp \cap (\Gamma_2(x) \cap G) = \{u\}$. If a, b and u have been chosen, then $|\Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(x)| = s^2+s+2$ by Proposition 3.2(1). However, there are two points in $\Gamma_2(a) \cap \Gamma_2(b) \cap \Gamma_1(x)$ that are contained in $\Gamma_1(G)$, namely the two common neighbors of x and u. This leads to s^2+s possibilities for y.

□

Remark 4.5. Making use of Proposition 3.2(1) and Lemma 4.3(3), we can also prove that

$$\sum_{\pi \in \Pi} n_3(\pi) N_\pi = \frac{(s+1)s(s-1)(s-2)}{2}(s^2+2s+2). \tag{5}$$

However, equation (5) can also be derived from equations (3), (4) and Proposition 4.4.

4.3 On the (non)existence of regular near hexagons with parameters $(s,t_2,t) = (s,1,s^3+s^2-s+1)$

Suppose $\mathcal{S}$ is a maximal regular near hexagon with parameters $s \geq 2$, $t_2 = 1$ and $t = s^3+s^2-s+1$. We first show that there are only a few possible values for s.

Proposition 4.6. *If $\mathcal{S}$ is a maximal regular near hexagon with parameters $s \geq 2$, $t_2 = 1$ and $t = s^3+s^2-s+1$, then $s \in \{2,3,4,5,8,11,18,23,32,53,158\}$.*

Proof. The conditions in Proposition 1 hold if and only if

$$m := \frac{s^2(s+2)(s^3+s^2-s+1)(s^2-s+1)^2}{2(2s-1)} \in \mathbb{Z}.$$

Since the numerator is always even and $2s-1$ is odd, m is integral if and only if $2s-1$ is a divisor of $p(s) = 2^{10}s^2(s+2)(s^3+s^2-s+1)(s^2-s+1)^2$ (which can be regarded as a polynomial in $2s$ with integral coefficients). Since $p(\frac{1}{2}) = 315$, this is equivalent to demanding that $2s-1$ divides 315. This leads to the possibilities for s as stated in the proposition. $\square$

Brouwer [2] showed that there exists up to isomorphism a unique regular near hexagon with parameters $(s,t_2,t) = (2,1,11)$. The next case in Proposition 4.6 is that of a maximal regular near hexagon with parameters $s = 3$, $t_2 = 1$ and $t = 34$. In the next proposition, we show that such a regular near hexagon cannot exist.

Proposition 4.7. *There exists no regular near hexagon with parameters $(s,t_2,t) = (3,1,34)$.*

Proof. We continue with the notation of Subsection 4.2. In this case, we have $\Pi = \{\pi_1,\pi_2,\pi_3\}$ where $\pi_1 = 1^1 3^1$, $\pi_2 = 2^2$ and $\pi_3 = 4^1$. Obviously,

$$n_1(\pi_1) = 0,\ n_1(\pi_2) = 2,\ n_1(\pi_3) = 0,\ n_2(\pi_1) = 3,\ n_2(\pi_2) = 0,\ n_2(\pi_3) = 4.$$

By equations (3), (4) and Proposition 4.4(1), we have $N_{\pi_1} = 16$, $N_{\pi_2} + N_{\pi_3} = 81$ and $2N_{\pi_2} = 42$. So, $N_{\pi_1} = 16$, $N_{\pi_2} = 21$ and $N_{\pi_3} = 60$.

Recall from Subsection 4.1 that there are up to isomorphism only two nice sets of permutations of $\{1,2,3,4\}$, namely $\Sigma_1 = \{\mathsf{Id},(12)(34),(13)(24),(14)(23)\}$ and $\Sigma_2 = \{\mathsf{Id},(1234),(1432),(13)(24)\}$. If $L = \{x,y_1,y_2,y_3\}$ is a line through x contained in $\Gamma_2(G)$, then by Lemma 4.2(1), $\{\mathsf{Id},\theta_{y_1}\theta_x^{-1},\theta_{y_2}\theta_x^{-1},\theta_{y_3}\theta_x^{-1}\}$ is a nice set of permutations of $\mathcal{H}$ isomorphic to either Σ_1 or Σ_2. There are $(t+1)-2(s+1) = 27$ lines through x contained in $\Gamma_2(G)$ and hence there are at most $54 = 2\cdot 27$ points $y \in \Gamma_1(x)\cap\Gamma_2(G)$ for which $\pi_y = \pi_3 = 4^1$. This is in contradiction with the fact that $N_{\pi_3} = 60$. $\square$

We were not able to prove the nonexistence of other maximal regular near hexagons with $t_2 = 1$. For the rest of this subsection, we use the above techniques to discuss a hypothetical maximal regular near hexagon with parameters $s = 4$, $t_2 = 1$ and $t = 77$. Any such near hexagon would have $v = 235625$ points.

In this case, we have $\Pi = \{\pi_1, \pi_2, \pi_3, \pi_4\}$ where $\pi_1 = 1^1 4^1$, $\pi_2 = 1^1 2^2$, $\pi_3 = 5^1$ and $\pi_4 = 2^1 3^1$. From equations (3), (4) and Proposition 4.4(1), we find $N_{\pi_1} + N_{\pi_2} = 30$, $N_{\pi_3} + N_{\pi_4} = 272$ and $2N_{\pi_2} + N_{\pi_4} = 140$. Hence, $N_{\pi_2} = 30 - N_{\pi_1}$, $N_{\pi_3} = 192 - 2N_{\pi_1}$ and $N_{\pi_4} = 2N_{\pi_1} + 80$, where $N_{\pi_1} \in \{0, 1, \ldots, 30\}$.

Recall from Subsection 4.1 that there are up to isomorphism two nice sets of permutations of $\{1, 2, 3, 4\}$, namely $\Sigma_1 = \{\mathsf{Id}, (12)(34), (13)(24), (14)(23)\}$ and $\Sigma_2 = \{\mathsf{Id}, (1234), (1432), (13)(24)\}$. If $L = \{x, y_1, y_2, y_3, z\}$ is a line through x such that $z \in \Gamma_1(G)$ and if M^* denotes the unique line of $\mathcal{H}$ containing a point collinear with z, then $\{\mathsf{Id}, \theta_{y_1}\theta_x^{-1}, \theta_{y_2}\theta_x^{-1}, \theta_{y_3}\theta_x^{-1}\}$ induces by Lemma 4.2(2) a nice set of permutations of $\mathcal{H} \setminus \{M^*\}$ isomorphic to either Σ_1 or Σ_2. Hence N_{π_1} must be even, i.e. $N_{\pi_1} \in \{0, 2, 4, \ldots, 30\}$.

We know from Subsection 4.1 that there are up to isomorphism three nice sets of permutations of the set $\{1, 2, 3, 4, 5\}$, namely $\Sigma_1 = \{\mathsf{Id}, (12345), (13524), (14253), (15432)\}$, $\Sigma_2 = \{\mathsf{Id}, (12345), (14352), (15324), (13)(254)\}$ and $\Sigma_3 = \{\mathsf{Id}, (12)(345), (13)(254), (14)(235), (15)(243)\}$. We also know from Lemma 4.1(2) that all cousins of Σ_1 are equal to Σ_1, that Σ_2 has four cousins isomorphic to Σ_2 and one isomorphic to Σ_3, and that Σ_3 has four cousins isomorphic to Σ_2, and one equal to Σ_3. So, by Lemma 4.2(1), we then see that precisely one of the following two properties holds for a line $L \subseteq \Gamma_2(G)$:

(1) For every point $y \in L$, the nice set $\{\theta_z\theta_y^{-1} \mid z \in L\}$ of permutations of $\mathcal{H}$ is isomorphic to Σ_1.

(2) There are four points y on L for which the nice set $\Sigma_y := \{\theta_z\theta_y^{-1} \mid z \in L\}$ of permutations of $\mathcal{H}$ is isomorphic to Σ_2 and a unique point y^* on L for which Σ_{y^*} is isomorphic to Σ_3.

Let M_i, $i \in \{1, 2\}$, denote the total number of lines $L \subseteq \Gamma_2(G)$ for which case (i) above occurs. If case (1) occurs for a line $L \subseteq \Gamma_2(G)$, then there are $5 \cdot 4 = 20$ ordered pairs (y, z) of distinct points of L for which $\theta_z\theta_y^{-1}$ has cycle structure 5^1. If case (2) occurs, then the number of such pairs is equal to $4 \cdot 3 + 1 \cdot 0 = 12$.

If $i \in \{1, 2, 3, 4\}$, then for each of the $|\Gamma_2(G)| = v - |G| - |\Gamma_1(G)| = 228000$ points $y \in \Gamma_2(G)$, we have a certain value for N_{π_i} which we will denote by $N_{\pi_i}(y)$. We denote by $\overline{N_{\pi_i}}$ the average value of $N_{\pi_i}(y)$ taken over all $y \in \Gamma_2(G)$. Since the total number of lines contained in $\Gamma_2(G)$ is equal to $\frac{|\Gamma_2(G)|\cdot(t+1-2(s+1))}{s+1} = 3100800$, we must have

$$\begin{cases} M_1 + M_2 & = & 3100800, \\ 20 \cdot M_1 + 12 \cdot M_2 & = & |\Gamma_2(G)| \cdot \overline{N_{\pi_3}} = 228000 \cdot (192 - 2 \cdot \overline{N_{\pi_1}}). \end{cases}$$

From $228000 \cdot (192 - 2 \cdot \overline{N_{\pi_1}}) \geq 12(M_1 + M_2) = 12 \cdot 3100800$, we find $\overline{N_{\pi_1}} \leq \frac{72}{5}$.

Hence, we should have the following for every point $y \in \Gamma_2(G)$:

$$\begin{cases} N_{\pi_1}(y) \in \{0, 2, 4, \ldots, 30\} \text{ and } \overline{N_{\pi_1}} \leq \frac{72}{5}, \\ N_{\pi_2}(y) = 30 - N_{\pi_1}(y), \\ N_{\pi_3}(y) = 192 - 2 \cdot N_{\pi_1}(y), \\ N_{\pi_4}(y) = 2 \cdot N_{\pi_1}(y) + 80. \end{cases}$$

We note that by De Bruyn & Vanhove [13, Theorem 3.17], regular near hexagons with parameters $(s, t_2, t) = (4, 1, 77)$ could yield strongly regular graphs with new parameters.

4.4 Applications to the classification of dense near polygons with four points per line

In this subsection, we discuss the implications of Proposition 4.7 to the classification of dense near polygons with four points on each line.

Dixmier and Zara [15] (see also Payne and Thas [21, Section 6.2]) classified all generalized quadrangles of order $(3, t)$. Any such generalized quadrangle is isomorphic to either the (4×4)-grid, $W(3)$, $Q(4, 3)$, $Q(5, 3)$ or $T_2^*(\mathcal{H})$ with $\mathcal{H}$ the unique hyperoval (up to isomorphism) of $\mathrm{PG}(2, 4)$.

There are up to isomorphism ten known examples of finite dense near hexagons with four points on each line, namely the near hexagons $\mathbb{L}_4 \times \mathbb{L}_4 \times \mathbb{L}_4$, $W(3) \times \mathbb{L}_4$, $Q(4, 3) \times \mathbb{L}_4$, $T_2^*(\mathcal{H}) \times \mathbb{L}_4$, $Q(5, 3) \times \mathbb{L}_4$, $DW(5, 3)$, $DQ(6, 3)$, $DH(5, 9)$, $Q(5, 3) \otimes Q(5, 3)$ and $T_2^*(\mathcal{H}) \otimes T_2^*(\mathcal{H})$. In De Bruyn [9, Theorem 4.17], it was shown that any other dense near hexagon with four points per line can only have quads isomorphic to the (4×4)-grid or the generalized quadrangle $Q(4, 3)$. In fact, there exist constants t, a and b such that every point is contained in precisely $t + 1$ lines, a grid-quads and b $Q(4, 3)$-quads. With v denoting the total number of points, it was moreover shown in [9] that (v, t, a, b) is equal to either $(5848, 19, 160, 5)$, $(6736, 21, 171, 10)$, $(8320, 27, 120, 43)$ or $(20608, 34, 595, 0)$. In the last case, the near hexagon must be a regular near hexagon with parameters $s = 3$, $t_2 = 1$ and $t = 34$, which cannot exist by Proposition 4.7. So, in the classification of the finite dense near hexagons with four points per line, three open cases remain (compared to four previously), and in none of these, the near hexagons can be regular.

There are up to isomorphism 28 known examples of finite dense near octagons with four points per line. In De Bruyn [12] (Proposition 1.1, Theorem 1.2 and Section 3.4), it was shown that any other finite dense near octagon with four points per line must be the direct product of a line and a(n unknown) near hexagon or must contain a subhexagon that is a regular near hexagon with parameters $s = 2$, $t_2 = 1$ and $t = 34$. So, also for the near octagons there remain three open cases. Every finite dense near octagon with four points per line is isomorphic to one of the 28 known examples or is the direct product of a line of size four with an "exceptional near hexagon" with parameters $(v, t, a, b) \in \{(5848, 19, 160, 5), (6736, 21, 171, 10), (8320, 27, 120, 43)\}$.

5 A classification of Q-polynomial regular near $2d$-gons

The aim of this section is to prove the classification result for Q-polynomial regular near $2d$-gons stated in Theorem 3. In order to obtain that result we shall also have to prove Theorem 2. The following proposition will be useful to that end.

Proposition 5.1. *Let $\mathcal{S}$ be a finite regular near $2d$-gon with parameters $(s, t_2, t_3, \ldots, t_d)$ and put $t := t_d$. Suppose that $d \geq 3$ and $t_2 = 0$. If $\mathcal{S}$ contains an isometrically embedded generalized subhexagon $\mathcal{S}'$ of order (s, t_3), then*

$$t_3^2\theta^2 + (tt_3 + t_3^2 - 2st_3^2)\theta + (t^2 + s^2t_3^2 - stt_3^2 - stt_3 - st_3^2) \geq 0$$

for any eigenvalue $\theta > -(t+1)$ of the collinearity graph Γ of $\mathcal{S}$.

Proof. Let R_i with $i \in \{0, 1, \ldots, d\}$ denote the distance-i-relation in $\mathcal{S}$. Any point of $\mathcal{S}'$ is at distance 0, 1, 2 or 3 from exactly 1, $s(t_3+1)$, $s^2t_3(t_3+1)$ or $s^3t_3^2$ points of $\mathcal{S}'$, respectively. Hence, the inner distribution $\mathbf{a}$ of the point set T of $\mathcal{S}'$ (whose i-th component is equal to $\frac{|(T\times T)\cap R_i|}{|T|}$) is given by

$$\mathbf{a} = (1, s(t_3+1), s^2t_3(t_3+1), s^3t_3^2, 0, \ldots, 0).$$

We now apply Delsarte's linear programming bound ([14, p. 26]; see also [5, Proposition 2.5.2]) to the set T of vertices. By [5, Proposition 2.2.2 & Section 4.1.B], this bound is equivalent to the $d+1$ inequalities

$$u_0(\theta) + u_1(\theta)\cdot s(t_3+1) + u_2(\theta)\cdot s^2t_3(t_3+1) + u_3(\theta)\cdot s^3t_3^2 \geq 0, \tag{6}$$

where θ is one of the $d+1$ eigenvalues of Γ and the numbers $u_i(\theta)$, $i \in \{0, 1, \ldots, d\}$, are recursively defined by the following equations:

$$u_0(\theta) = 1, \quad u_1(\theta) = \frac{\theta}{s(t+1)},$$

$$(t_i+1)\cdot u_{i-1}(\theta) + (s-1)(t_i+1)\cdot u_i(\theta) + s(t-t_i)\cdot u_{i+1}(\theta) = \theta u_i(\theta),\ i \in \{1, 2, \ldots, d-1\}.$$

We find that

$$\begin{aligned}
u_0(\theta) &= 1,\\
u_1(\theta) &= -\frac{1}{s} + \frac{1}{s(t+1)}\cdot(\theta+t+1),\\
u_2(\theta) &= \frac{1}{s^2} + \frac{1}{s^2t(t+1)}\cdot(\theta+t+1)(\theta-s-t),\\
u_3(\theta) &= -\frac{1}{s^3} + \frac{1}{s^3t^2(t+1)}\cdot(\theta+t+1)(\theta^2-(t+2s-1)\theta+(s^2+t^2-s)).
\end{aligned}$$

Using these equations, inequality (6) reduces to

$$\frac{(\theta+t+1)(t_3^2\theta^2+(tt_3+t_3^2-2st_3^2)\theta+(t^2+s^2t_3^2-stt_3^2-stt_3-st_3^2))}{(t+1)t^2}\geq 0,$$

from which the proposition immediately follows. □

We will use Proposition 5.1 to prove the following nonexistence result for regular near polygons.

Proposition 5.2. *There exists no regular near* 10*-gon with parameters* $(s,t_2,t_3,t_4,t) = (2,0,8,24,120)$.

Proof. Note that the collinearity graph Γ of such a regular near 10-gon has classical parameters $(5,-2,-2,22)$. It follows from Hiraki [18, Corollary 1.2] that this near 10-gon must contain isometrically embedded subpolygons that are generalized hexagons of order $(2,8)$. Proposition 5.1 then implies that $64\theta^2+768\theta-2752\geq 0$ for every eigenvalue θ of Γ distinct from $-(t+1)=-121$. From [5, Corollary 8.4.2] we know that the eigenvalues of Γ are equal to 242, 61, 17, -11, -31 and -121. The inequality $64\theta^2+768\theta-2752\geq 0$ is not valid if $\theta=-11$. □

The following proposition is precisely Theorem 2.

Proposition 5.3. *There are no finite regular near* $2d$*-gons whose parameters* s*,* t *and* t_i*,* $i\in\{0,1,\ldots,d\}$*, satisfy* $d\geq 4$*,* $s\geq 2$ *and* $t_i = s^3\cdot\frac{(s^{i-2}-(-1)^i)(s^{i-1}+(-1)^i)}{(s^2-1)(s+1)}$ *for every* $i\in\{0,1,\ldots,d\}$.

Proof. Note that the collinearity graph Γ of such a regular near $2d$-gon has classical parameters $(d,-s,-\frac{s}{s-1},s+s^2\cdot\frac{(-s)^{d-1}-1}{s^2-1})$.

We first exclude the case $d=4$. If $d=4$, then $t_2=0$, $t_3=s^3$ and $t_4=s^3(s^2-s+1)$. The number of points at distance 3 from any given point is given by

$$\frac{s^3(t+1)t(t-t_2)}{(t_2+1)(t_3+1)}=\frac{s^9(s^3-s^2+1)(s^2-s+1)(s^2+1)}{s+1},$$

and this can only be an integer if $s=2$ or $s=5$. The case $s=2$ is impossible, since nonexistence of regular near octagons with parameters $(s,t_2,t_3,t)=(2,0,8,24)$ was proved in De Bruyn [11]. For the case $s=5$, we consider the eigenvalues. Using for instance [5, 8.4.2 & 8.4.3], we see that here the collinearity graph would have eigenvalues 13130, 529, 104, -121 and -2626, where 529 would have multiplicity $\frac{3453518250}{17}$, which is impossible.

Next, assume that $d=5$. Here, the number $\frac{s^4(t+1)t(t-t_2)(t-t_3)}{(t_2+1)(t_3+1)(t_4+1)}$ of points at distance 4 from any point would be given by:

$$\frac{s^{14}(s^4-s^3+s^2-s+1)(s^3+s+1)(s^3-s^2+2s-1)(s^2-s+1)(s^2+1)}{(s+1)(s^3-s^2+1)}.$$

After Euclidean division of the numerator by the denominator, we obtain the remainder $-156s^3 + 144s^2 + 4s - 146$, which can only be divisible by $(s+1)(s^3 - s^2 + 1)$ if $s = 2$. Hence the regular near 10-gon would have parameters $(s, t_2, t_3, t_4, t) = (2, 0, 8, 24, 120)$, which was proved to be impossible in Proposition 5.2.

Finally, we consider the case $d \geq 6$. Here, it follows from Hiraki [18, Corollary 1.2] that the regular near $2d$-gon must have subgeometries that are regular near octagons with parameters $(s', t'_2, t'_3, t'_4) = (s, 0, s^3, s^3(s^2 - s + 1))$ and we already know from the above that this is impossible. □

As already indicated in Section 1, Propositions 2.3 and 5.3 imply the following.

Corollary 5.4. *There are no distance-regular graphs with classical parameters* $(d, -s, -\frac{s}{s-1}, s + s^2 \cdot \frac{(-s)^{d-1}-1}{s^2-1})$ *if* $d \geq 4$ *and* $s \geq 2$.

We will now prove Theorem 3.

Proof. Suppose $\mathcal{S}$ is a thick Q-polynomial regular near $2d$-gon with $d \geq 3$ which is not a Hamming near $2d$-gon nor a dual polar space of diameter d. Then the collinearity graph Γ of $\mathcal{S}$ has classical parameters $(d, b, \alpha, \beta) = (d, -s, -\frac{t_2+s}{s-1}, s + s(t_2 + s) \cdot \frac{(-s)^{d-1}-1}{s^2-1})$ by Proposition 2.2.

If $d \geq 4$, then Weng [29, Corollary 5.7] implies that $t_2 = 0$. But that is impossible by Corollary 5.4. So, we should have that $d = 3$ and Γ has classical parameters $(3, -s, -\frac{t_2+s}{s-1}, s(t_2 + s + 1))$, or equivalently, that $d = 3$ and the total number $t+1$ of lines through a point is equal to $(s^2 - s + 1)(t_2 + s + 1)$. If $t_2 = 0$, then $\mathcal{S}$ is a generalized hexagon of order (s, s^3).

So, we may suppose that $s \geq 2$ and $t_2 \geq 1$. Then $\mathcal{S}$ has many quads of order (s, t_2). By Higman's inequality we know that $t_2 \leq s^2$. If $t_2 = s^2$, then $t = s^4 + s^2$ and results of Cameron [7] and Brouwer and Wilbrink [6, Lemma 26] (see also [5, Theorem 9.4.4]) imply that $\mathcal{S}$ is a dual polar space, contrary to our assumption. So, we should have $t_2 < s^2$. But then $t_2 \leq s^2 - s$ by Payne and Thas [21, 1.2.5].

If $s = 2$ and $1 \leq t_2 \leq s^2 - s$, then $(s, t_2, t) \in \{(2, 1, 11), (2, 2, 14)\}$. By Brouwer [2, 3], we then know that $\mathcal{S}$ is isomorphic to either the near hexagon related to the extended ternary Golay code or the near hexagon related to the Steiner system $S(5, 8, 24)$. □

Acknowledgements

The research of the second author is supported by the Research Foundation Flanders-Belgium (FWO-Vlaanderen).

References

[1] E. Bannai and T. Ito. *Algebraic combinatorics I. Association schemes.* Benjamin/Cummings Publishing Company, Menlo Park, 1984.

[2] A.E. Brouwer. The uniqueness of the near hexagon on 729 points. *Combinatorica* 2 (1982), 333–340.

[3] A.E. Brouwer. The uniqueness of the near hexagon on 759 points. *Finite geometries* (Pullman, Wash., 1981), pp. 47–60, volume 82 of *Lecture Notes in Pure and Applied Mathematics.* Dekker, New York, 1983.

[4] A.E. Brouwer and A.M. Cohen. Local recognition of Tits geometries of classical type. *Geom. Dedicata* 20 (1986), 181–199.

[5] A.E. Brouwer, A.M. Cohen and A. Neumaier. *Distance-regular graphs.* Results in Mathematics and Related Areas (3) 18. Springer-Verlag, Berlin, 1989.

[6] A.E. Brouwer and H.A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata* 14 (1983), 145–176.

[7] P.J. Cameron. Dual polar spaces. *Geom. Dedicata* 12 (1982), 75–85.

[8] A.M. Cohen and J. Tits. On generalized hexagons and a near octagon whose lines have three points. *European J. Combin.* 6 (1985), 13–27.

[9] B. De Bruyn. Near hexagons with four points on a line. *Adv. Geom.* 1 (2001), 211–228.

[10] B. De Bruyn. The completion of the classification of the regular near octagons with thick quads. *J. Algebraic Combin.* 24 (2006), 23–29.

[11] B. De Bruyn. The nonexistence of regular near octagons with parameters $(s, t, t_2, t_3) = (2, 24, 0, 8)$. *Electron. J. Combin.* 17 (2010), Research Paper 149, 10pp.

[12] B. De Bruyn. Dense near octagons with four points on each line, III. *Ann. Comb.* 15 (2011), 19–35.

[13] B. De Bruyn and F. Vanhove. Inequalities for regular near polygons, with applications to m-ovoids. *European J. Combin.* 34 (2013), 522–538.

[14] P. Delsarte. *An algebraic approach to the association schemes of coding theory.* Philips Res. Rep. Suppl. No. 10 (1973), 97 pp.

[15] S. Dixmier and F. Zara. Etude d'un quadrangle généralisé autour de deux de ses points non liés. Preprint 1976.

[16] W. Haemers and C. Roos. An inequality for generalized hexagons. *Geom. Dedicata* 10 (1981), 219–222.

[17] D.G. Higman. Partial geometries, generalized quadrangles and strongly regular graphs. *Atti del Convegno di Geometria Combinatoria e sue Applicazioni* (Univ. Perugia, Perugia, 1970), pp. 263–293. *Ist. Mat.*, Univ. Perugia, Perugia, 1971.

[18] A. Hiraki. Strongly closed subgraphs in a regular thick near polygon. *European J. Combin.* 20 (1999), 789–796.

[19] R. Mathon. On primitive association schemes with three classes. Unpublished manuscript.

[20] A. Neumaier. Krein conditions and near polygons. *J. Combin. Theory Ser. A* 54 (1990), 201–209.

[21] S.E. Payne and J.A. Thas. *Finite generalized quadrangles.* Second edition. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2009.

[22] M.A. Ronan. A combinatorial characterization of the dual Moufang hexagons. *Geom. Dedicata* 11 (1981), 61–67.

[23] S. Shad and E. Shult. The near n-gon geometries. Unpublished manuscript.

[24] E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata* 9 (1980), 1–72.

[25] P. Terwilliger. A new inequality for distance-regular graphs. *Discrete Math.* 137 (1995), 319–332.

[26] P. Terwilliger. Kite-free distance-regular graphs. *European J. Combin.* 16 (1995), 405–414.

[27] J. Tits. Sur la trialité et certains groupes qui s'en déduisent. *Inst. Hautes Etudes Sci. Publ. Math.* 2 (1959), 13–60.

[28] H. Van Maldeghem. *Generalized polygons.* Monographs in Mathematics 93. Birkhäuser Verlag, Basel, 1998.

[29] C. Weng. D-bounded distance-regular graphs. *European J. Combin.* 18 (1997), 211–229.