# Q- and $\omega_0$-Sensitivities in Positive Gain Second-Order RC Active Filters.

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**Abstract**

An active filter configuration with a general three-terminal second-order RC feedback network and a positive voltage gain is considered. The sensitivities of the Q factor and resonant frequency with respect to the voltage gain are calculated, and a relationship between lower bounds on them is formulated.

Some time ago, Bown deduced a lower bound on the sensitivity $S_A^Q$ of the $Q$ factor with respect to the amplifier gain of a resonant active filter having a biquadratic transfer function [1]. He used the general configuration shown in fig.1. The amplifier is assumed to have infinite input impedance, zero output impedance, and a positive voltage gain $A$. The network $N$ is a second-order transformerless RC network. Assuming certain conditions on the network $N$ (e.g., $Y_{32}$ has a zero in the origin only), he deduced the inequality

$$S_A^Q > 2Q - 1.$$

The conventional definitions for a second-order active filter realizing complex poles $-\alpha \pm j\beta$, where $\alpha \ll \beta$, are being used, namely,

$$Q = \frac{\sqrt{\alpha^2 + \beta^2}}{2\alpha}$$

and

$$S_A^Q = \frac{dQ}{dA}.\frac{A}{Q} \qquad S_A^{\omega_0} = \frac{d\omega_0}{dA}.\frac{A}{\omega_0}$$

In this letter, a new relation between bounds on $S_A^Q$ and $S_A^{\omega_0}$ is presented. The derivation of this relation requires fewer restrictive assumptions on $N$.

$N$ being a linear second-order network, we can write

$$V_3(s) = H_1(s)V_1(s) + H_2(s)V_2(s)$$

where $H_1(s)$ and $H_2(s)$ are voltage transfer functions that can be realized by three-terminal

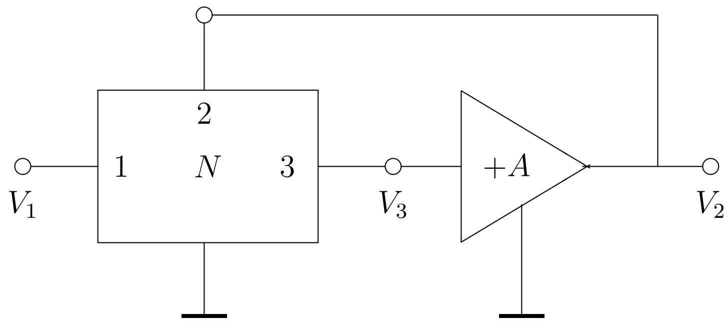


Fig. 1.

transformerless second-order RC networks. We therefore have

$$H_2(s) = \frac{as^2 + bs + c}{(s+\sigma_1)(s+\sigma_2)}$$

where $\sigma_1, \sigma_2, a, b$ and $c$ are real and positive constants satisfying the conditions [2]

$$\begin{aligned} 0 \leq \quad & a \quad \leq 1 \\ 0 \leq \quad & b \quad \leq \sigma_1 + \sigma_2 \\ 0 \leq \quad & c \quad \leq \sigma_1\sigma_2. \end{aligned}$$

The poles $-\alpha \pm j\beta$ of $(V_2(s))/(V_1(s))$ are the roots of

$$(1-aA)s^2 + (\sigma_1 + \sigma_2 - bA)s + (\sigma_1\sigma_2 - cA) = 0$$

Therefore,

$$\begin{aligned} \omega_0 &= (1-aA)^{-1/2}(\sigma_1\sigma_2 - cA)^{+1/2} \\ &= Y(A)/X(A) \qquad (1) \\ Q &= (1-aA)^{1/2}(\sigma_1\sigma_2 - cA)^{1/2} \\ &\quad .(\sigma_1 + \sigma_2 - bA)^{-1} \\ &= X(A).Y(A).Z(A). \qquad (2) \end{aligned}$$

The sensitivities $S_A^Q$ and $S_A^{\omega_0}$ can be calculated as follows:

$$\begin{aligned} S_A^Q &= S_A^X + S_A^Y + S_A^Z \\ &= -\frac{1}{2}\ \frac{aA}{1-aA} - \frac{1}{2}\ \frac{cA}{\sigma_1\sigma_2 - cA} \\ &\quad + \frac{bA}{\sigma_1 + \sigma_2 - bA} \end{aligned}$$

and

$$\begin{aligned} S_A^{\omega_0} &= -S_A^X + S_A^Y \\ &= \frac{1}{2}\ \frac{aA}{1-aA} - \frac{1}{2}\ \frac{cA}{\sigma_1\sigma_2 - cA}. \end{aligned}$$

We can now introduce the $Q$ factor and resonant frequency, given by (1) and (2):

$$S_A^Q = \frac{\sigma_1+\sigma_2}{\omega_0(1-aA)}(Q - \frac{1}{2}\ \frac{\frac{\sigma_1\sigma_2}{\omega_0^2}+1}{\frac{\sigma_1+\sigma_2}{\omega_0}}) \tag{3}$$

$$S_A^{\omega_0} = \frac{1}{2}\ \frac{1-\frac{\sigma_1\sigma_2}{\omega_0^2}}{1-aA} \tag{4}$$

A necessary condition for the absolute stability of the configuration is that

$$\frac{cA}{\sigma_1\sigma_2} < 1.$$

Combining this inequality with (1) we obtain

$$0 < 1 - aA \leq 1 \tag{5}$$

which will be used later on.

Suppose that we want a reasonably small $S_A^Q$sensitivity, say

$$|S_A^Q| \leq k. \tag{6}$$

According to (3), this requires the following inequality to hold as a necessary (but not sufficient) condition:

$$2\ Q\ \frac{\sigma_1+\sigma_2}{\omega_0} \leq \frac{\sigma_1\sigma_2}{\omega_0^2} + m \tag{7}$$

with

$$m = 2k(1-aA) + 1. \tag{8}$$

Because $H_2(s)$ is the voltage transfer function of a passive RC structure, we have

$$\sigma_1 + \sigma_2 \geq 2\sqrt{\sigma_1+\sigma_2} \tag{9}$$

so that condition (7) leads to

$$\frac{\sqrt{\sigma_1\sigma_2}}{\omega_0} + m\frac{\omega_0}{\sqrt{\sigma_1\sigma_2}} \geq 4\ Q \tag{10}$$

If we make the assumption that $m \ll 4Q^2$, a condition that is usually satisfied for high-Q low $S_A^Q$-sensitivity networks, inequality (10) requires

$$\sqrt{\sigma_1\sigma_2} \leq \omega_0 x_1 \quad \text{or} \quad \sqrt{\sigma_1\sigma_2} \geq \omega_0 x_2$$

where $x_1$ and and $x_2$, are given by

$$\begin{aligned} x_1 &= 2Q\ (1 - \sqrt{1 - \frac{m}{4Q^2}}) \\ x_2 &= 2Q\ (1 + \sqrt{1 - \frac{m}{4Q^2}}) \end{aligned}$$

Because, now,

$$\sqrt{1 - \frac{m}{4Q^2}} > 1 - \frac{m}{4Q^2}$$

we have

$$x_1 < \frac{m}{2Q} \qquad x_2 > 4Q - \frac{m}{2Q}.$$

If we, therefore, want $S_A^Q$ to satisfy condition (6), it is necessary that either

$$\sigma_1\sigma_2 < \frac{m^2}{4Q^2}\omega_0^2 \tag{11}$$

or

$$\sigma_1\sigma_2 > (4Q - \frac{m}{2Q})^2\omega_0^2. \tag{12}$$

We now investigate the effect of the foregoing conditions on the sensitivity $S_A^{\omega_0}$. If condition (11) is satisfied, we obtain, using (1) and the fact that $c$ is a positive constant,

$$(1 - aA) < \frac{m^2}{4Q^2} \tag{13}$$

Because, on the other hand,

$$1 - \frac{\sigma_1\sigma_2}{\omega_0^2} > 1 - \frac{m^2}{4Q^2} \geq 0$$

(4) yields

$$S_A^{\omega_0} > 2\frac{Q^2}{m^2} - \frac{1}{2}. \tag{14}$$

If, on the contrary, condition 12 is satisfied, we have

$$1 - \frac{\sigma_1\sigma_2}{\omega_0^2} < -16Q^2 - \frac{m^2}{4Q^2} + 1 + 4m < -16Q^2 + 1 + 4m < 0.$$

Using (4) and inequality (5) we obtain

$$|S_A^{\omega_0}| > 8Q^2 - \frac{1}{2} - 2m. \tag{15}$$

The conclusion is that if the Q-sensitivity $S_A^Q$ is smaller than $k$, the $\omega_0$-sensitivity $S_A^{\omega_0}$ is bounded by either

$$\begin{aligned} |S_A^{\omega_0}| &> 2\frac{Q^2}{m^2} - \frac{1}{2} \quad \text{or} \\ |S_A^{\omega_0}| &> 8Q^2 - \frac{1}{2} - 2m. \end{aligned} \tag{16}$$

A short discussion of these bounds is useful. Let us suppose that we want to realize a circuit with a high Q factor. By letting $a = c = O$ (Sallen-Key filters [3]), we obtain $S_A^{\omega_0} = 0$, but, as shown by Bown, it becomes impossible to obtain a value of $S_A^Q$ (or of k) lower than $2Q - 1$.

If we assume $a$ or $c$ (or both) to be non-zero, we have shown that lower $S_A^Q$values and thus lower $k$-values can be obtained. In that case, however, the inequality (16) states that $S_A^{\omega_0}$ rapidly becomes very high. Often, both $S_A^Q$ and $S_A^{\omega_0}$ are high (as is the case in Inigo's circuits [4]. While not desirable, such a situation is not always to be rejected. In a recent paper, Moschytz [5] argued that even a high $S_A^Q$ can lead to a reasonable Q-variation with temperature, if the passive elements that determine $A$ track very closely and if the loop gain remains large enough. The same argument holds for $S_A^{\omega_0}$. Under those conditions high sensitivities $S_A^{\omega_0}$ and $S_A^Q$ can be tolerated.

## CONCLUSION

If we try to realize a high $Q$-factor with a positive gain second-order RC active filter, it is impossible to obtain simultaneously low sensitivities of the $Q$-factor and the resonant frequency with respect to the amplifier gain.

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## References

[1] G.C. Bown, "Sensitivity in active RC filters," *Electron. Lett. vol.*, 3, pp. 298-299, July 1967.

[2] A. Fialkow and I. Gerst, "The transfer function of general two-terminal-pair RC networks," *Quart. Appl. Math.*, vol. 10, pp, 113-127, July 1952.

[3] R. P. SaJlen and E. L. Key, "A practical method of designing RC active filters," *IRE Trans. Circuit Theory*, vol. CT-2, pp. 74-85, Mar. 1955.

[4] R. M. Inigo, "Active filter realization using finite-gain voltage amplifiers," *IEEE Trans. Circuit Theory (Corresp.)*, vol. CT-17, pp. 445-448. Aug. 1970.

[5] G.S. Moschytz, "Gain-sensitivity product - A figure of merit for hybrid-integrated filters using single operational amplifiers," *IEEE J. Solid-State Circuits, vol. SC-6*, pp. 103-110, June 1971.