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Vekua systems in Hyperbolic Harmonic Analysis

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Vekua systems in Hyperbolic Harmonic Analysis

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Abstract

In this paper we consider the solutions of the equation $\mathcal{M}_\kappa f = 0$, where $\mathcal{M}_\kappa$ is the so called modifier Dirac operator acting on functions f defined in the upper half space and taking values in the Clifford algebra. We look for solutions $f(\underline{x}, x_n)$ where the first variable is invariant under rotations. A special type of solution is generated by the so called spherical monogenic functions. These solutions may be characterize by a vekua-type system and this system may be solved using Bessel functions. We will see that the solution of the equation $\mathcal{M}_\kappa f = 0$ in this case will be a product of Bessel functions.

1 Introduction and History

In the early 1990s, Professor Heinz Leutwiler from University Erlangen Nürnberg began to investigate a function theory related to this model. He looked at the vector valued function $u = (u_0, u_1, ..., u_n)$ satisfying the equation system

$$\begin{aligned} &x_n\Big(\frac{\partial u_0}{\partial x_0} - \frac{\partial u_1}{\partial x_1} - \cdots - \frac{\partial u_n}{\partial x_n}\Big) + (n-1)u_n = 0, \\ &\frac{\partial u_i}{\partial x_k} = \frac{\partial u_k}{\partial x_i},\ i,k = 1, ..., n, \\ &\frac{\partial u_0}{\partial x_k} = -\frac{\partial u_k}{\partial x_0},\ k = 1, ..., n \end{aligned}$$

on the hyperbolic upper half space $\mathbb{R}^{n+1}_+ = \{(x_0, x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$ with the metric $g = x_n^{-2}(dx_0^2 + dx_1^2 + ... + dx_n^2)$ (see [5, 6]). This system is called the system (H_n) (in honor of Hodge). If $u = (u_0, u_1, ..., u_n)$ is a solution of the system then the corresponding 1-form

$$\eta = u_0 dx_0 - u_1 dx_1 - \cdots - u_n dx_n$$

is harmonic on the upper half space i.e. it satisfies the Hodge-de Rham system. After that Leutwiler started to study the solutions of this system using quaternions in $\mathbb{R}^3_+$ and the Clifford algebra. Leutwiler and the first author introduced the so called modified Dirac operator for Clifford algebra valued functions $f : \Omega \subset \mathbb{R}^{n+1} \to \mathcal{C}\ell_n$ in the form

$$Mf = Df + \frac{n-1}{x_n} Q'f,$$

where Ω is an open subset and the function admits the decomposition $f = Pf + Qfe_n$, where Pf and Qf takes values in $\mathcal{C}\ell_{n-1}$, and $Q'f$ is defined such, that it satisfies the equation $Qfe_n = e_n Q'f$. In the above the operator

$$D = \partial_{x_0} + e_1 \partial_{x_1} + \cdots + e_n \partial_{x_n}$$

is the standard Euclidean Dirac-Weyl-Delanghe -operator. A null solution of the equation $Mf = 0$ is called the hypermonogenic functions and if we consider functions with values in $\mathbb{R} \oplus \mathbb{R}^n$ we see, that the components satisfies the system (H_n). Nowadays this theory is called the hyperbolic function theory. See more information e.g. in [3].

Around the same time when Leutwiler started to study the H_n-system, the third author and Pertti Lounesto started, independently, to consider axially monogenic functions in the Euclidean Clifford analysis i.e. functions which are solutions of the system $Df = 0$ invariant under the spin group, see [8] and [7]. The introduction to this topic may be found e.g. in the famous "green book" [1].

In this paper our goal is to consider similar type of axially symmetric solutions on the hyperbolic upper half space. These solutions form a kind of basis for the space of all solutions to the hypermonogenic system. The method of separation of variables leads to solutions that may be expressed as products of Bessel functions.

2 Preliminaries

2.1 Clifford Algebra

Let $\mathcal{C}\ell_n$ be the Clifford algebra for the quadratic form $Q(x) = -|x|^2$ on $\mathbb{R}^n$. If $e_1, ..., e_n$ is a basis of $\mathbb{R}^n$ then we obtain the generating relations

$$e_i e_j + e_j e_i = -2\delta_{ij}$$

for $i, j = 1, ..., n$. We define the subspace $\mathcal{C}\ell_n^k$ as a real vector space spanned by all products of k linearly independent vectors. This may be characterized using the basis as follows. We define the increasing lists $A = \{a_1, ..., a_k\} \subset N = \{1, ..., n\}$ where $0 \leq a_1 < \cdots < a_k \leq n$, and then we define

$$e_A = e_{a_1} \cdots e_{a_k}.$$

The length of a list A is denoted by $|A| = k$. Then $\mathcal{C}\ell_n^0 = \mathbb{R}$ and $\mathcal{C}\ell_n^1 = \mathbb{R}^n$. We may express the Clifford algebra $\mathcal{C}\ell_n$ as a direct sum

$$\mathcal{C}\ell_n = \mathcal{C}\ell_n^0 \oplus \cdots \oplus \mathcal{C}\ell_n^n.$$

The natural projection is denoted by $[\cdot]_k : \mathcal{C}\ell_n \to \mathcal{C}\ell_n^k$.

A (smooth) k-multivector field in an open domain $U \subset \mathbb{R}^n$ is a map

$$F_k : U \to \mathcal{C}\ell_n^k.$$

Any k-multivector field F_k admits the expression

$$F_k = \sum_{|A|=k} F_A e_A,$$

where $F_A : U \to \mathbb{R}$ is the smooth function. The space of k-multivector fields on U is denoted by $\mathcal{E}_k(U)$.

The conjugation is an automorphism $- : \mathcal{E}_k(U) \to \mathcal{E}_k(U)$ defined by

$$\overline{e}_A = (-1)^{\frac{k(k+1)}{2}} e_A.$$

Especially $\overline{e}_A e_A = e_A \overline{e}_A = 1$.

A fundamental differential operator on the space of k-multivector fields is the Dirac operator $\partial_{\mathbf{x}} : \mathcal{E}_k(U) \to \mathcal{E}_{k-1}(U) \oplus \mathcal{E}_{k+1}(U)$ defined by

$$\partial_{\mathbf{x}} : F_k = \sum_{|A|=k} F_A e_A \mapsto \partial_x F_k = \sum_{j=1}^{n} \sum_{|A|=k} \frac{\partial F_A}{\partial x_j} e_j e_A.$$

The product of a 1-vector x and a k-vectorfield F_k may be decomposed into

$$\mathbf{x} F_k = \mathbf{x} \cdot F_k + \mathbf{x} \wedge F_k,$$

where $\mathbf{x} \cdot F_k$ is called the interior product and $x \wedge F_k$ is called the exterior product. They admit the expressions

$$\mathbf{x} \cdot F_k = [\mathbf{x} F_k]_{k-1} = \frac{1}{2}(\mathbf{x} F_k - (-1)^k F_k \mathbf{x})$$

and

$$\mathbf{x} \wedge F_k = [\mathbf{x} F_k]_{k+1} = \frac{1}{2}(\mathbf{x} F_k + (-1)^k F_k \mathbf{x}).$$

For basis elements we make the following observation. For any e_A we get

$$e_A = e_{a_1} \cdots e_{a_k} = e_{a_1} \wedge \cdots \wedge e_{a_k}$$

and if $a_j \in A$ we have

$$e_{a_j} \cdot e_A = (-1)^j e_{A \setminus \{a_j\}}.$$

If $\mathbf{y}$ is a 1-vector we have

$$\mathbf{x} \cdot \mathbf{y} = -\sum_{j=1}^{n} x_j y_j.$$

Using the interior and the exterior product we have

$$\partial_{\mathbf{x}} F_k = \partial_{\mathbf{x}} \cdot F_k + \partial_{\mathbf{x}} \wedge F_k,$$

where $\partial_{\mathbf{x}} \cdot F_k = [\partial_{\mathbf{x}} F_k]_{k-1}$ and $\partial_{\mathbf{x}} \wedge F_k = [\partial_{\mathbf{x}} F_k]_{k+1}$. Then we may define the operators $\partial_{\mathbf{x}} \cdot : \mathcal{E}_k(U) \to \mathcal{E}_{k-1}(U)$ and $\partial_{\mathbf{x}} \wedge : \mathcal{E}_k(U) \to \mathcal{E}_{k+1}(U)$.

2.2 Hodge-de Rham Operators and Modified Dirac operator in Upper-Half Space

In this section we define our fundamental operator in the hyperbolic Upper-Half Space. By hyperbolic upper half space we shall mean the set

$$\mathbb{R}^n_+ = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_n > 0\}$$

with the metric

$$g = \frac{dx_1^2 + ... + dx_n^2}{x_n^2}.$$

We start form the following famous first order operators act on sections of the cotangent bundle. Let

$$\omega^r = \sum_{|A|=r} f_A dx_{a_1} dx_{a_2} \cdots dx_{a_r}$$

be an arbitrary differential form in the upper-half space. The form ω^r is called harmonic, if it is a solution of the system

$$d\omega^r = 0 \quad \text{and} \quad d^* \omega^r = 0,$$

where d is the standard exterior derivative operator and d^* its adjoint (Hodge star). We may make the $1-1$ identification between differential r-forms and Clifford r-multivector fields given by

$$\omega^r = \sum_{|A|=r} f_A dx_{a_1} dx_{a_2} \cdots dx_{a_k} \;\leftrightarrow\; F_r = \sum_{|A|=r} f_A e_{a_1} e_{a_2} \cdots e_{a_k}.$$

Let us define a family of operators

$$\mathcal{D}_r F_r = \partial_x F_r - \frac{n-2r}{x_n} e_n \cdot F_r$$

acting on the multivector field F_r. We have that ω^r is harmonic if and only if the corresponding F_r satisfies the equation $\mathcal{D}_r F_r = 0$, see [4, 9]. So, we may study harmonic differential forms using the operator $\mathcal{D}_r$ and tools from Clifford analysis. Keeping this in our mind, we may make the following generalization.

On differentiable Clifford algebra -valued functions depending on vector variables, the modified Dirac operators defined by

$$\mathcal{M}_\kappa = \partial_{\mathbf{x}} - \frac{\kappa}{x_n} e_n \cdot,$$

where κ is an arbitrary real parameter and

$$\partial_{\mathbf{x}} = \sum_{j=1}^{n} e_j \partial_{x_j}$$

the Euclidean Dirac operator.

If f is a solution of the equation $\mathcal{M}_\kappa f = 0$ it is called κ-hypermonogenic. 0-hypermonogenic functions are called briefly monogenic. The difference to the modified operator in the introduction is that now we consider functions with a vector variable $\mathbf{x}$, but in the original setting authors considered functions with paravector variable $x_0 + \mathbf{x}$. Note that these definition coincide since $Q' f(x) = -e_n \cdot f$. We use the different notation for the operators in order to stress that we consider functions depending on vector variables. See the detailed definition for the operator in [4].

3 A Vekua system related to Rotation Invariant κ-hypermonogenic Solutions

Recall that we are looking for solutions of the modified Dirac operator

$$\mathcal{M}_\kappa f = (\partial_{\mathbf{x}} - \frac{\kappa}{x_n} e_n \cdot) f = 0,$$

where κ is a real parameter. Let $\mathbf{x} \in \mathbb{R}^n_+$ be of the form $\mathbf{x} = (\underline{x}, x_n)$, where $\underline{x} = (x_1, ..., x_{n-1})$. Let us make the similar decomposition for the Dirac operator

$$\partial_{\mathbf{x}} = \partial_{\underline{x}} + e_n \partial_{x_n},$$

where

$$\partial_{\underline{x}} = \sum_{j=1}^{n-1} e_j \partial_{x_j}.$$

We also define

$$r = |\underline{x}| \ \text{ and } \ \underline{\omega} = \frac{\underline{x}}{|\underline{x}|}$$

for every $\underline{x} \neq 0$. Using these concepts we may express

$$\mathbf{x} = \underline{x} + x_n e_n = r\underline{\omega} + x_n e_n.$$

We recall that

Definition 3.1 *A k-homogeneous polynomial $P_k(\underline{x})$ is called k-spherical monogenic if $\partial_{\underline{x}} P_k(\underline{x}) = 0$ in $\mathbb{R}^{n-1}$.*

General homogeneous polynomials and spherical monogenics have the following connection.

Lemma 3.2 ([1]) *Let H_m be a m-homogeneous Clifford algebra valued polynomial in $\mathbb{R}^{n-1}$. Then there exist spherical monogenic polynomials $P_k(\underline{x})$ such that*

$$H_m(\underline{x}) = \sum_{k=0}^{m} \underline{x}^{m-k} P_k(\underline{x}),$$

where $\underline{x} = x_1 e_1 + ... + x_{n-1} e_{n-1}$.

Also it is well known that:

Lemma 3.3 ([2]) *Let Ω be an open subset of the upper half space $\mathbb{R}^n_+$. If $f : \Omega \to \mathcal{C}\ell_n$ is κ-hypermonogenic in Ω then it is real analytic.*

We may prove the following result:

Theorem 3.4 *Let Ω be an open subset of the upper half space $\mathbb{R}^n_+$. If $f : \Omega \to \mathcal{C}\ell_n$ is κ-hypermonogenic on Ω and $\mathbf{a} \in \Omega$ then it has locally the presentation*

$$f(\mathbf{x}) = \sum_{(\alpha,\beta)\in \mathbb{N}_0^{n-1}\times\mathbb{N}_0} \sum_{k=0}^{|\alpha|} \underline{x}^{|\alpha|-k} P_k^{\alpha}(\underline{x})(x_n - a_n)^{\beta} c_{\alpha,\beta}$$

valid in some neighbourhood of $\mathbf{a}$.

Proof. Let $f : \Omega \to \mathcal{C}\ell_n$ be κ-hypermonogenic on Ω. It is enough to prove the result for $\mathbf{a} = a_n e_n$, since κ-hypermonogenic functions are invariant under the translation $\tau_{\mathbf{a}}(\mathbf{x}) = \mathbf{x} - \underline{a}$. Then f is real analytic and therefore it admit the prower series representation

$$f(\mathbf{x}) = \sum_{(\alpha,\beta)\in\mathbb{N}_0^{n-1}\times\mathbb{N}_0} \underline{x}^{\alpha}(x_n - a_n)^{\beta} c_{\alpha,\beta}$$

converging uniformly and absolutely in some neighbood of $\mathbf{a}$. Since $\underline{x}^{\alpha}$ is a homogenous polynomial of order $|\alpha|$ then there exists spherical monogenics $P_k^{\alpha}(\underline{x})$ such that

$$\underline{x}^{\alpha}(x_n - a_n)^{\beta} c_{\alpha,\beta} = \sum_{k=0}^{|\alpha|} \underline{x}^{|\alpha|-k} P_k^{\alpha}(\underline{x})(x_n - a_n)^{\beta} c_{\alpha,\beta},$$

completing the proof. ■

The preceding theorem allow us to look for solutions that depend only on the r and x_n in the form

$$f_k(\mathbf{x}) = \big(A(r^2, x_n)e_n + B(r^2, x_n)\underline{x}\big) P_k(\underline{x}).$$

The functions A and B are real valued and differentiable with respect to both variables, and $P_k(\underline{x})$ is a spherical monogenic. The functions are defined on an open subset $\Omega \subset \mathbb{R}^n_+$ that is rotation symmetric with respect to x_n-axis.

Next we'd like to compute $\mathcal{M}_{\kappa} f_k(\mathbf{x})$, but for that, we need the following lemmata. First we recall that:

Lemma 3.5 *If $P_k(\underline{x})$ is a spherical monogenic then*

$$\frac{\underline{x}}{r^{n+2k-1}} P_k(\underline{x})$$

is monogenic.

Then we may compute the following formula.

Lemma 3.6 *If $P_k(\underline{x})$ is a spherical monogenic and $B(r^2, x_n)$ a real valued differentiable funtion, then*

$$\partial_{\underline{x}}(B(r^2, x_n)\underline{x}P_k(\underline{x})) = -(n+2k-1)B(r^2, x_n)P_k(\underline{x}) - 2\partial_1 B(r^2, x_n) r^2 P_k(\underline{x}),$$

where ∂_1 is the derivative with respect to the first variable r^2.

Proof. Using the above lemma we compute

$$\partial_{\underline{x}}(B(r^2,x_n)\underline{x}P_k(\underline{x})) = \partial_{\underline{x}}(r^{n+2k-1}B(r^2,x_n)\frac{\underline{x}}{r^{n+2k-1}}P_k(\underline{x}))$$
$$= \partial_{\underline{x}}(r^{n+2k-1}B(r^2,x_n))\frac{\underline{x}}{r^{n+2k-1}}P_k(\underline{x}).$$

Now

$$\partial_{\underline{x}} r^{n+2k-1} = (n+2k-1)\underline{x}r^{n-2k-3}$$

and

$$\partial_{\underline{x}} B(r^2,x_n) = 2\underline{x}\partial_1 B(r^2,x_n).$$

Then we have

$$\begin{aligned}
&\partial_{\underline{x}}(B(r^2,x_n)\underline{x}P_k(\underline{x}))\\
&= \partial_{\underline{x}} r^{n+2k-1}B(r^2,x_n)\frac{\underline{x}}{r^{n+2k-1}}P_k(\underline{x}) + r^{n+2k-1}\partial_{\underline{x}}B(r^2,x_n)\frac{\underline{x}}{r^{n+2k-1}}P_k(\underline{x})\\
&= (n+2k-1)\underline{x}r^{n-2k-3}B(r^2,x_n)\frac{\underline{x}}{r^{n+2k-1}}P_k(\underline{x}) + r^{n+2k-1}2\underline{x}\partial_1 B(r^2,x_n)\frac{\underline{x}}{r^{n+2k-1}}P_k(\underline{x})\\
&= (n+2k-1)B(r^2,x_n)\frac{\underline{x}^2}{r^2}P_k(\underline{x}) + 2\partial_1 B(r^2,x_n)\underline{x}^2 P_k(\underline{x})\\
&= -(n+2k-1)B(r^2,x_n)P_k(\underline{x}) - 2\partial_1 B(r^2,x_n)r^2 P_k(\underline{x})
\end{aligned}$$

and the proof ist complete.■

This technical result allow us to prove following theorem.

Theorem 3.7 *A function*

$$f_k(\mathbf{x}) = \big(A(r^2,x_n)e_n + B(r^2,x_n)\underline{x}\big)P_k(\underline{x}).$$

is κ-hypermonigenic if and only if the functions A and B satisfies the Vekua-type system

$$-2\partial_1 B(r^2,x_n)r^2 - \partial_{x_n}A(r^2,x_n) + \frac{\kappa}{x_n}A(r^2,x_n) - (n+2k-1)B(r^2,x_n) = 0,$$
$$\partial_{x_n}B(r^2,x_n) - 2\partial_1 A(r^2,x_n) = 0.$$

Proof. We compute

$$e_n\partial_{x_n}f_k(\mathbf{x}) = (-\partial_{x_n}A(r^2,x_n) + e_n\underline{x}\partial_{x_n}B(r^2,x_n)))P_k(\underline{x})$$

and

$$-\frac{\kappa}{x_n}e_n \cdot f_k(r,x_n) = \frac{\kappa}{x_n}A(r^2,x_n)P_k(\underline{x}).$$

Using the preceding lemma, we have

$$\begin{aligned}
\mathcal{M}_\kappa f_k &= (-2e_n\underline{x}\partial_1 A(r^2,x_n) - (n+2k-1)B(r^2,x_n) - 2\partial_1 B(r^2,x_n)r^2)P_k(\underline{x})\\
&\quad + (-\partial_{x_n}A(r^2,x_n) + e_n\underline{x}\partial_{x_n}B(r^2,x_n)))P_k(\underline{x}) + \frac{\kappa}{x_n}A(r^2,x_n)P_k(\underline{x}),
\end{aligned}$$

completing the proof.■

4 A series solution for the system

Defining $u = r^2$ and $v = x_n$ we get system in the form

$$-2\partial_u B(u,v)u - \partial_v A(u,v) + \frac{\kappa}{v}A(u,v) - (n+2k-1)B(u,v) = 0, \tag{1}$$

$$\partial_v B(u,x_n) - 2\partial_u A(u,v) = 0. \tag{2}$$

We look for solution in the form

$$A = \sum_{i=0}^{\infty} u^i A_i(v)$$

and

$$B = \sum_{j=0}^{\infty} u^j B_j(v).$$

We compute the derivatives

$$\begin{aligned}
\partial_u A &= \sum_{i=1}^{\infty} i u^{i-1} A_i(v) = \sum_{i=0}^{\infty}(i+1)u^i A_{i+1}(v),\\
\partial_v A &= \sum_{i=0}^{\infty} u^i A_i'(v),\\
\partial_u B &= \sum_{j=0}^{\infty} j u^{j-1} B_j(v),\\
\partial_v B &= \sum_{j=0}^{\infty} u^j B_j'(v).
\end{aligned}$$

Using the second equation (2) of the system, we get

$$\begin{aligned}
\partial_v B(u,v) - 2\partial_u A(u,v) &= \sum_{j=0}^{\infty} u^j B_j'(v) - 2\sum_{i=0}^{\infty}(i+1)u^i A_{i+1}(v)\\
&= \sum_{j=0}^{\infty} u^j (B_j'(v) - 2(j+1)A_{j+1}(v)) = 0,
\end{aligned}$$

that is we get the relation

$$B_j'(v) = 2(j+1)A_{j+1}(v),\ j = 0,1,2,.... \tag{3}$$

Similarly, the first equation (1) gives

$$
\begin{aligned}
&-\partial_v A(u,v) - 2u\partial_u B(u,v) + \frac{\kappa}{v}A(u,v) - (n+2k-1)B(u,v) \\
&= -\sum_{i=0}^{\infty} u^i A_i'(v) - 2u\sum_{j=0}^{\infty} j u^{j-1}B_j(v) + \frac{\kappa}{v}\sum_{i=0}^{\infty} u^i A_i(v) - (n+2k-1)\sum_{j=0}^{\infty} u^j B_j(v) \\
&= \sum_{i=0}^{\infty} u^i(-A_i'(v) + \frac{\kappa}{v}A_i(v) - (2j+n+2k-1)B_j(v)) = 0
\end{aligned}
$$

so we get

$$
-A_j'(v) + \frac{\kappa}{v}A_j(v) - (2j+n+2k-1)B_j(v) = 0, \ j = 0,1,2,3,... \quad (4)
$$

Now we get an algorithm to find a solution for the system. If we take an arbitary differentiable function A_0, then we may compute the other unknown functions in the series solution using the above formulae (3) and (4) by

$$
A_0 \xrightarrow{(4)} B_0 \xrightarrow{(3)} A_1 \xrightarrow{(4)} B_1 \xrightarrow{(3)} A_2 \xrightarrow{(4)} \cdots
$$

5 Solutions for the System generated by Bessel fuctions

In this section we look for one special class of solutions. Taking derivatives we get form (4), that

$$
-A_j''(v) + \frac{\kappa}{v}A_j'(v) - \frac{\kappa}{v^2}A_j(v) - (2j+n+2k-1)B_j'(v) = 0
$$

and then we may substitute (3) and we get

$$
-A_j''(v) + \frac{\kappa}{v}A_j'(v) - \frac{\kappa}{v^2}A_j(v) - (2j+n+2k-1)2(j+1)A_{j+1}(v) = 0
$$

that is

$$
A_{j+1}(v) = \frac{1}{(2j+n+2k-1)2(j+1)}\big(-A_j''(v) + \frac{\kappa}{v}A_j'(v) - \frac{\kappa}{v^2}A_j(v)\big). \quad (5)
$$

We look for solution for the equation

$$
-A_j''(v) + \frac{\kappa}{v}A_j'(v) - \frac{\kappa}{v^2}A_j(v) = -A_j(v) \quad (6)
$$

and we have the linear independent solutions

$$
v^{\frac{\kappa+1}{2}} J_{\frac{|\kappa-1|}{2}}(v) \ \text{ or } \ v^{\frac{\kappa+1}{2}} Y_{\frac{|\kappa-1|}{2}}(v).
$$

Combining (5) and (6) we may compute the terms recursively by the formula

$$A_{j+1}(v) = \frac{-1}{(2j+n+2k-1)2(j+1)} A_j(v).$$

Then we have

$$\begin{aligned}
A_1(v) &= \frac{-1}{2(n-2k+1)2} v^{\frac{\kappa+1}{2}} A_0(v) \\
A_2(v) &= \frac{-1}{2(n-2k+3)3} \frac{-1}{2(n-2k+1)2} v^{\frac{\kappa+1}{2}} A_0(v) \\
&= \frac{1}{2^2(n-2k+3)(n-2k+1)2\cdot 3} v^{\frac{\kappa+1}{2}} A_0(v) \\
A_3(v) &= \frac{-1}{2(n-2k+5)4} \frac{1}{2^2(n-2k+3)(n-2k+1)2\cdot 3} v^{\frac{\kappa+1}{2}} A_0(v) \\
&= \frac{-1}{2^3(n-2k+5)(n-2k+3)(n-2k+1)2\cdot 3\cdot 4} v^{\frac{\kappa+1}{2}} A_0(v)
\end{aligned}$$

and in general

$$A_j(v) = -\frac{(-1)^j}{2^j(n-2k+1)(n-2k+3)\cdots(n-2k+(2j-1))j!} v^{\frac{\kappa+1}{2}} A_0(v)$$

and

$$A = -\sum_{j=0}^{\infty} \frac{(-1)^j u^j}{2^j(n-2k+1)(n-2k+3)\cdots(n-2k+(2j-1))j!} v^{\frac{\kappa+1}{2}} A_0(v)$$

And since $u = r^2$ and $v = x_n$ we have

$$A(r^2, x_n) = -\sum_{j=0}^{\infty} \frac{(-1)^j r^{2j}}{2^j(n-2k+1)(n-2k+3)\cdots(n-2k+(2j-1))j!} x_n^{\frac{\kappa+1}{2}} A_0(x_n).$$

Using a standard identity of the gamma function we get

$$\begin{aligned}
&(n-2k+1)(n-2k+3)\cdots(n-2k+(2j-1)) \\
&= 2^j\left(\frac{n-2k-1}{2}+j\right)\cdots\left(\frac{n-2k+1}{2}+1\right)\left(\frac{n-2k+1}{2}\right) \\
&= 2^j \frac{\Gamma(\frac{n-2k+1}{2}+j)}{\Gamma(\frac{n-2k+1}{2})}
\end{aligned}$$

and then

$$A(r^2, x_n) = -\Gamma\left(\frac{n-2k+1}{2}\right) \sum_{j=0}^{\infty} \frac{(-1)^j r^{2j}}{2^{2j}\Gamma(\frac{n-2k+1}{2}+j)j!} x_n^{\frac{\kappa+1}{2}} A_0(x_n).$$

Using the definition of the Bessel function

$$J_\nu(z) = \frac{z^\nu}{2^\nu} \sum_{j=0}^{\infty} \frac{(-1)^j z^{2j}}{j! 2^{2j} \Gamma(\nu + j + 1)}$$

we see that our solution is given by

$$A(r^2, x_n) = -2^{\frac{n-2k+1}{2}} \Gamma(\frac{n-2k-1}{2}) r^{-\frac{n-2k-1}{2}} J_{\frac{n-2k+1}{2}}(r) x_n^{\frac{\kappa+1}{2}} A_0(x_n),$$

where $A_0(v)$ is either

$$v^{\frac{\kappa+1}{2}} J_{\frac{|\kappa-1|}{2}}(v) \quad \text{or} \quad v^{\frac{\kappa+1}{2}} Y_{\frac{|\kappa-1|}{2}}(v).$$

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