On the algebraic variety $\mathcal{V}_{r,t}$

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Abstract

The variety $\mathcal{V}_{r,t}$ is the image under the Grassmannian map of the $(t-1)$–subspaces of $PG(rt-1,q)$ of the elements of a Desarguesian spread. We investigate some properties of this variety, with particular attention to the case $r=2$: in this case we prove that every $t+1$ points of the variety are in general position and we give a new interpretation of linear sets of $PG(1,q^t)$.

Keywords: Desarguesian spread; Grassmann variety; Veronese variety; Segre variety; subgeometry; linear set.

1 Definitions and preliminary results

Let $V(n,q)$ be the vector space of dimension n over $GF(q)$ and $PG(n-1,q)$ be the projective space defined by the lattice of subspaces of $V(n,q)$; we will denote by $(x_0,\ldots,x_{n-1})$ both the vector of homogeneous coordinates of a certain point $P \in \mathrm{PG}(n-1,q)$ and the point P as well. The group $PGL(n,q)$ is the group of all the *projectivities* of $PG(n-1,q)$. A subspace Π of $PG(n-1,q)$ has *dimension* $t-1$ and *rank* t if it is a t–dimensional subspace of $V(n,q)$. A *subgeometry* Σ of $PG(n-1,q)$ is a subset isomorphic to $PG(n-1,q')$, where $GF(q')$ is a subfield of $GF(q)$. Since a frame consisting of $n+1$ points determines a $PG(n-1,q')$ and $PGL(n,q)$ acts transitively on frames, all the subgeometries $PG(n-1,q')$ contained in $PG(n-1,q)$ are projectively equivalent. It is easy to see that a subgeometry $PG(n-1,q')$ is the set of fixed points of a suitable cyclic semilinear (i.e. $GF(q')$–linear) collineation (see [9], Theorem 4.28 and [6], Chapter 1).

A $(t-1)$–*spread* $\mathcal{S}$ of $PG(n-1,q)$ is a partition of the point set of $PG(n-1,q)$ in subspaces of dimension $(t-1)$ and it exists if and only if t divides n ([16]). Let $\mathcal{S}$ be a $(t-1)$–spread of $PG(rt-1,q)$, embed $PG(rt-1,q)$ into $PG(rt,q)$ as a hyperplane and let $A(\mathcal{S})$ be the following incidence structure: the points are the points of $PG(rt,q)\setminus PG(rt-1,q)$, the lines are the t–dimensional subspaces of $PG(rt,q)$ intersecting $PG(rt-1,q)$ in an element of $\mathcal{S}$ and the incidence is the natural one. Then $A(\mathcal{S})$ is a $2-(q^{rt},q^t,1)$ translation design with parallelism (see [1]) and we will say that $\mathcal{S}$ is a *Desarguesian* spread if $A(\mathcal{S})$ is isomorphic to the affine space $AG(r,q^t)$. An easy construction of a Desarguesian spread of $PG(rt-1,q)$ is by the so called *field reduction* of $PG(r-1,q^t)$. The underlying vector space of the projective space $PG(r-1,q^t)$ is $V(r,q^t)$; if we consider $V(r,q^t)$ as a vector space over $GF(q)$, then it has dimension rt and it defines a $PG(rt-1,q)$. Every point $P\in PG(r-1,q^t)$ corresponds in this way to a subspace Π_P of $PG(rt-1,q)$ of dimension $(t-1)$ and the set $\mathcal{S}=\{\Pi_P, P\in PG(r-1,q^t)\}$ is a spread of $PG(rt-1,q)$. Moreover, it is easy to see that any two elements Π_P and $\Pi_{P'}$ of $\mathcal{S}$ span a $(2t-1)$–dimensional subspace completely partitioned by elements of $\mathcal{S}$, and they are precisely the ones corresponding to the points of the line $\langle P, P' \rangle$ of $PG(r-1, q^t)$. For $r > 2$, such a spread is called *normal* in [14] and in [1] it is proven that $\mathcal{S}$ is normal if and only if it is Desarguesian; for $r = 2$, the proof that a spread constructed in such a way is Desarguesian is in [16].

In [14], a *linear set* is defined as a generalization of the concept of subgeometry. More precisely, a $GF(q)$–linear set L of $PG(r-1, q^t)$ of rank s is a set of points of $PG(r-1, q^t)$ defined by a subset U of $V(r, q^t)$ that is an s–dimensional vector space over $GF(q)$. Such a linear set L is equivalent, by field reduction, to the elements of a Desarguesian spread $\mathcal{S}$ of $PG(rt-1, q)$ having non–empty intersection with the subspace of $PG(rt-1, q)$ defined by U. Finally, there is another equivalent way to define a linear set as a (projected) subgeometry of a suitable projective space (for an overview about this topic see [15]). In this paper we present a fourth point of view to describe linear sets of $PG(1, q^t)$.

We now introduce some algebraic varieties that play an important role in finite geometry.

The *Veronese variety* $\mathcal{V}(n, d)$ is an algebraic variety of $PG(\binom{n+d}{d}-1, q)$ image of the injective map $v_{n,d} : PG(n, q) \longrightarrow PG(\binom{n+d}{d}-1, q)$, where $v_{n,d}(x_0, x_1, \ldots, x_n)$ is the vector of all the monomials of degree d in $x_0, \ldots, x_n$ (for $d = 2$, see [10], Chapter 25, and for general d see e.g. [5]) and we recall that $\mathcal{V}(1, d)$ is a *normal rational curve* of $PG(d, q)$. We will use the notation $\mathcal{V}(n, d, q)$ to recall also the field under consideration.

Let $PG(n_1-1, q), PG(n_2-1, q), \ldots, PG(n_k-1, q)$ be k projective spaces, then the *Segre embedding* $\sigma : PG(n_1-1, q) \times PG(n_2-1, q) \times \cdots \times PG(n_k-1, q) \longrightarrow PG(n_1 n_2 \cdots n_k - 1, q)$ is such that $\sigma(\mathbf{x}^1, \ldots, \mathbf{x}^k)$ is the vector of all the products $x^{(1)}_{j_1} x^{(2)}_{j_2} \cdots x^{(k)}_{j_k}$, with $\mathbf{x}^i = (x^{(i)}_0, x^{(i)}_1, \ldots, x^{(i)}_{n_i-1}) \in PG(n_i-1, q)$. The image of σ is called the *Segre variety* $\Sigma_{n_1;n_2;\ldots;n_k}$ and it is in some way the product of projective spaces (see [10], Chapter 25 and [7]): for this reason we will say the image under σ of the subset $S_1 \times S_2 \times \ldots \times S_k$ of $PG(n_1-1, q) \times PG(n_2-1, q) \times \cdots \times PG(n_k-1, q)$ is the *Segre product* of the subsets $S_1, S_2, \ldots, S_k$, $S_i \in PG(n_i-1, q)$. We remark that $\mathcal{V}(n, d)$ is the *diagonal* of the Segre product of d $PG(n, q)$'s.

To introduce the last variety, we give some more details because the way it is defined is useful in the proof of a proposition of the next section. Let Π be an $(r-1)$–dimensional subspace of $PG(n-1, q)$, let $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(r)}$, with $\mathbf{x}^{(i)} \in V(n, q)$ be the coordinate vectors of r linearly independent points of Π and let T_Π be the matrix whose rows are the vectors $\mathbf{x}^{(1)}, \mathbf{x}^{(2)}, \ldots, \mathbf{x}^{(r)}$. After choosing an ordering, we can then construct the vector of length $\binom{n}{r}$ of all possible $r \times r$ minors of T_Π and it is called a *coordinate vector* of Π; by Lemma 24.1.1 of [10], this is unique up to a non–zero scalar factor. So we can define the *Grassmannian map* $g_{n,r} : PG^{(r-1)}(n-1, q) \longrightarrow PG(\binom{n}{r}-1, q)$, where $PG^{(r-1)}(n-1, q)$ is the set of all $(r-1)$–subspaces of $PG(n-1, q)$, such that $g_{n,r}(\Pi)$ is a coordinate vector of Π. This map is injective and its image $\mathcal{G}_{n,r}$ is called the *Grassmannian* or the *Grassmann variety* of the $(r-1)$–subspaces of $PG(n-1, q)$ (for more details we refer to [10], Chapter 24).

The varieties described in this section are the image of injective maps, so every collineation of the projective space where the map is defined induces a collineation fixing the variety setwise and viceversa (for the Grassmann and the Segre variety, see [10] Theorem 24.2.16 and Theorem 25.5.13 respectively; for the Veronese variety, see [5] Theorem 2.15). If σ is a collineation of the projective space, we will denote by σ^* the collineation induced on the variety and we will call it the *lifting* of σ.

2 The algebraic variety $\mathcal{V}_{r,t}$

The algebraic variety $\mathcal{V}_{r,t}$ appeared for the first time in the literature in [16] and it has been described in a more detailed way and with a modern terminology in [14]. This variety is the image under the Grassmannian map $g_{rt,t}$ of the elements of a Desarguesian $(t-1)$–spread $\mathcal{S}$ of $PG(rt-1,q)$: in [14], Lunardon proves that $\mathcal{V}_{r,t}$ is the complete intersection of the Grassmann variety $\mathcal{G}_{rt,t}$ with a suitable (r^t-1)–space. In fact he proves that $\mathcal{V}_{r,t}=\Delta\cap\Sigma_{r;r;\ldots;r}$, where $\Delta=PG(r^t-1,q)$ and $\Sigma_{r;r;\ldots;r}$ is the Segre variety product of t $PG(r-1,q^t)$'s contained in the Grassmannian of the $(r-1)$–subspaces of $PG(n-1,q^t)$. As showed in the previous section, by field reduction, we can get a Desarguesian $(t-1)$–spread $\mathcal{S}$ of $PG(rt-1,q)$ from $PG(r-1,q^t)$: in this way, to every point P of $PG(r-1,q^t)$ corresponds a spread element Π_P and to every line m of $PG(r-1,q^t)$ correspond the spread elements $\Pi_P, P\in m$, hence the incidence structure of the points of $\mathcal{V}_{r,t}$ and $\mathcal{O}_m=\{g_{rt,t}(\Pi_P), P\in m\}$, m a line of $PG(r-1,q^t)$, is isomorphic to $PG(r-1,q^t)$. There are remarkable examples of such varieties: for $r=t=2$, $\mathcal{V}_{2,2}$ is an elliptic quadric contained in the Klein quadric $\mathcal{Q}^+(5,q)$ (see [8], Chapter 16); for $t=2$, we have the so called *Hermitian Veronesean* (see for example [4]); for $t=3, r=2$ and q even, $\mathcal{V}_{3,2}$ is the Desarguesian ovoid of $\mathcal{Q}^+(7,q)$ and for $t=2, r=3$ and $q\equiv 2$ mod 3, a suitable hyperplane section of $\mathcal{V}_{2,3}$ is the Unitary ovoid of $\mathcal{Q}^+(7,q)$, (see [11, 14]).

We start giving an explicit description of $\mathcal{V}_{r,t}$ in terms of coordinates.

Proposition 1. *The algebraic variety $\mathcal{V}_{r,t}$ is isomorphic to the set of points of $PG(r^t-1,q^t)$ with coordinates $(\mathbf{x}^{\alpha_1},\mathbf{x}^{\alpha_2},\ldots,\mathbf{x}^{\alpha_{r^t}})$, where $\mathbf{x}^{\alpha_i}=x_0^{\alpha_0^{(i)}}x_1^{\alpha_1^{(i)}}\cdots x_{r-1}^{\alpha_{r-1}^{(i)}}$, $(\alpha_0^{(i)},\alpha_1^{(i)},\ldots,\alpha_{r-1}^{(i)})$ is such that $\alpha_k^{(i)}$ is a sum of distinct powers of q, $\sum_{k=0}^{r-1}\alpha_k^{(i)}=q^{t-1}+q^{t-2}+\ldots+1$ $\forall i$, $(x_0,x_1,\ldots,x_{r-1})\in PG(r-1,q^t)$ and it is contained in a subgeometry isomorphic to $PG(r^t-1,q)$.*

Proof. In $\Sigma^*=PG(rt-1,q^t)$, consider the subgeometry $\Sigma=\{(x_0,\ldots,x_{r-1},x_0^q,\ldots,x_{r-1}^q,\ldots,x_0^{q^{t-1}},\ldots,x_{r-1}^{q^{t-1}}), x_i\in GF(q^t)\}$: Σ is the set of fixed points of the $GF(q)$–linear collineation

$$\sigma:(\mathbf{x}^{(1)},\mathbf{x}^{(2)},\ldots,\mathbf{x}^{(t)})\longmapsto(\mathbf{x}^{(t)q},\mathbf{x}^{(1)q},\ldots,\mathbf{x}^{(t-1)q}), \mathbf{x}^{(i)}=(x_0^{(i)},\ldots,x_{r-1}^{(i)})\in V(r,q^t)$$

of order t, hence $\Sigma=PG(tr-1,q)$. Let $\Pi=\{(\mathbf{x},\mathbf{0},\ldots,\mathbf{0}),\mathbf{x}\in V(r,q^t)\}\subset\Sigma^*$ and for any $P\in\Pi$ let $\ell(P)=\langle P,P^\sigma,\ldots,P^{\sigma^{q^{t-1}}}\rangle$, then $\mathcal{S}=\{\ell(P),P\in\Pi\}$ is a Desarguesian spread of Σ (see [3]). Let $g^*_{rt,t}$ be the Grassmannian map of subspaces of rank t of Σ^*: by [14], page 250, the image under $g^*_{rt,t}$ of the subspaces of rank t of Σ is the Grassmannian of $(t-1)$–subspaces of Σ. The image under $g^*_{rt,t}$ of $\ell(P)$ is the vector of all minors of order t of the matrix whose rows are the coordinate vectors of $P, P^{\sigma}, \ldots, P^{\sigma^{q^{t-1}}}$, that is the matrix

$T(P) = \begin{pmatrix} \mathbf{x} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{x}^q & \ldots & \mathbf{0} \\ \ldots & \ldots & \ldots & \ldots \\ \mathbf{0} & \ldots & \ldots & \mathbf{x}^{q^{t-1}} \end{pmatrix}$, where $\mathbf{x} = (x_0, \ldots, x_{r-1}) \in V(r, q^t)$ and $P = (\mathbf{x}, 0, \ldots, 0) \in \Pi$. The submatrices of order t of $T(P)$ are such that every column has only one non–zero entry, hence the determinant is 0 or it is in the form $x_0^{\alpha_0^{(i)}} x_1^{\alpha_1^{(i)}} \cdots x_{r-1}^{\alpha_{r-1}^{(i)}}$, $\sum_{k=0}^{r-1} \alpha_k^{(i)} = q^{t-1} + q^{t-2} + \ldots + 1$, $\alpha_k^{(i)}$ is a sum of distinct powers of q. This set of points is contained in a subgeometry isomorphic to $PG(r^t - 1, q)$ by [14], page 250. □

Remark 1 We want to emphasize the analogy of $\mathcal{V}_{r,t}$ with the Veronese variety $\mathcal{V}(r-1, t, q^t)$. We have already mentioned that $\mathcal{V}_{r,t}$ is the intersection of $\Sigma_{r;r;\ldots;r}$ (the Segre variety product of t $PG(r-1, q^t)$'s) with a suitable subgeometry $PG(r^t - 1, q)$, more precisely, it is the Segre embedding of the points of type $(\mathbf{x}, \mathbf{x}^q, \ldots, \mathbf{x}^{q^{t-1}}) \in PG(r-1, q^t) \times PG(r-1, q^t) \times \cdots PG(r-1, q^t)$, whereas $\mathcal{V}(r-1, t, q^t)$ is the diagonal of $\Sigma_{r;r;\ldots;r}$, i.e. is the Segre embedding of the points of type $(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}) \in PG(r-1, q^t) \times PG(r-1, q^t) \times \cdots PG(r-1, q^t)$. Moreover, $\mathcal{V}(r-1, t, q^t)$ is defined by the vectors of all monomial of degree t in $x_0, x_1, \ldots, x_{r-1}$, whereas $\mathcal{V}_{r,t}$ is defined by the vectors of all monomials of degree $1 + q + \ldots q^{t-1}$, but the only powers admitted for x_i are of type $q^{\alpha_1} + \ldots + q^{\alpha_k}, \alpha_i \neq \alpha_j \ \forall i \neq j$.

Example 1 The variety $\mathcal{V}_{3,2}$ is the image of the map $\alpha : (x_0, x_1, x_2) \in PG(2, q^2) \longmapsto (x_0^{q+1}, x_0 x_1^q, x_0^q x_1, x_1^{q+1}, x_1 x_2^q, x_1^q x_2, x_2^{q+1}, x_2 x_0^q, x_2^q x_0) \in PG(8, q^2)$. Let σ be the following $GF(q)$–linear collineation of order two:

$$(y_0, y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8) \in PG(8, q^2) \mapsto (y_0^q, y_2^q, y_1^q, y_3^q, y_5^q, y_4^q, y_6^q, y_8^q, y_7^q) \in PG(8, q^2).$$

The points of $\mathcal{V}_{3,2}$ are fixed by σ and hence $\mathcal{V}_{3,2}$ is contained in the $PG(8, q)$ defined by σ (compare with [4]).

Example 2 The variety $\mathcal{V}_{2,4}$ is the image of the map $\alpha : (x, y) \in PG(1, q^4) \longmapsto$ $(x^{q^3+q^2+q+1}, x^{q^2+q+1} y^{q^3}, x^{q^3+q^2+q} y, x^{1+q^3+q^2} y^q, x^{q+1+q^3} y^{q^2}, x^{q+1} y^{q^3+q^2},$ $x^{q^2+q} y^{1+q^3}, x^{q^3+q^2} y^{q+1}, x^{1+q^3} y^{q^2+q}, x^{q^2+1} y^{q^3+q}, x^{q^3+q} y^{1+q^2}, x y^{q^3+q^2+q},$ $x^q y^{1+q^3+q^2}, x^{q^2} y^{q+1+q^3}, x^{q^3} y^{q^2+q+1}, y^{q^3+q^2+q+1}) \in PG(15, q^4)$. Let τ be the following $GF(q)$–linear collineation of order four: $(z_0, z_1, \ldots, z_{15}) \in PG(15, q^4) \mapsto$ $(z_0^q, z_4^q, z_1^q, z_2^q, z_3^q, z_8^q, z_5^q, z_6^q, z_7^q, z_{10}^{q^3}, z_9^{q^3}, z_{14}^q, z_{11}^q, z_{12}^q, z_{13}^q, z_{15}^q)$. The points of $\mathcal{V}_{2,4}$ are fixed by τ and hence $\mathcal{V}_{2,4}$ is contained in the $PG(15, q)$ defined by τ.

Remark 2 There is a group isomorphic to $PGL(r, q^t)$ acting 2–transitively on $\mathcal{V}_{r,t}$ ([14], Corollary 1).

The following result is a generalization of Theorem 2.6 of [13], where Lunardon proves that a subline $PG(1, q)$ of $PG(1, q^t)$ corresponds in $\mathcal{V}_{2,t}$ to a normal rational curve that is the complete intersection of $\mathcal{V}_{2,t}$ with a suitable t–dimensional space. We keep the notation of the proof of the previous proposition.

Theorem 2. *Let g be the map $P \in PG(r-1, q^t) \longmapsto g_{rt,t}(\ell(P))$. The image under g of a subgeometry $PG(r-1, q^s), s|t$, is the intersection of the Segre product of s Veronese varieties $\mathcal{V}(r-1, \frac{t}{s}, q^s)$ with a $PG(\binom{r-1+\frac{t}{s}}{\frac{t}{s}}^s - 1, q)$ and* *it is the complete intersection of $\mathcal{V}_{r,t}$ with a suitable space of rank $\binom{r-1+\frac{t}{s}}{\frac{t}{s}}^s$. In particular, the image of a subgeometry $PG(r-1,q)$ is a Veronese variety $\mathcal{V}(r-1,t,q)$ and it is the intersection of $\mathcal{V}_{r,t}$ with a suitable space of rank $\binom{r-1+t}{t}$.*

Proof. Since all the subgeometries are projectively equivalent and by Remark 2, we can assume that the points of $PG(r-1,q^s)$ are the ones with coordinates in $GF(q^s)$. If $P \in PG(r-1,q^s)$, then the image under the Grassmannian map of $\ell(P)$ is the vector of all minors of order t of the matrix

$$T(P) = \begin{pmatrix}
\mathbf{x} & \mathbf{0} & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{x}^q & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\mathbf{0} & \mathbf{0} & \dots & \mathbf{x}^{q^{s-1}} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{x} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{x}^q & \dots & \mathbf{0} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{x}^{q^{s-1}}
\end{pmatrix}$$

where $\mathbf{x} = (x_0, \dots, x_{r-1}) \in V(r,q^s)$. Next, consider the following matrix:

$$T(P)^* = \begin{pmatrix}
\mathbf{x}_1 & \mathbf{0} & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{x}_2 & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\mathbf{0} & \mathbf{0} & \dots & \mathbf{x}_s & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{x}_1 & \mathbf{0} & \dots & \mathbf{0} \\
\mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{x}_2 & \dots & \mathbf{0} \\
\dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\
\mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & \dots & \dots & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{x}_s
\end{pmatrix}$$

where $\mathbf{x} = (x_0, \dots, x_{r-1}) \in V(r,q^s)$; the vectors of all the minors of $T(P)^*$ is the Segre product of s Veronese varieties $\mathcal{V}(r-1,\frac{t}{s},q^s)$ and the minors of $T(P)$ are the points of this variety fixed by the $GF(q)$–linear collineation $\sigma^{\frac{t}{s}}$. Hence, as in [14] page 250, this variety is $\mathcal{V}(r-1,\frac{t}{s},q^s) \cap \Delta$, where $\Delta = PG(\binom{r-1+\frac{t}{s}}{\frac{t}{s}}^s - 1, q)$. □

2.1 The case $r = 2$

In this section, we focus on the case $r = 2$. In [14], Theorem 1, Lunardon proves that the algebraic variety $\mathcal{V}_{r,t}$ is a cap of $PG(r^t-1,q)$, i.e. any three points of $\mathcal{V}_{r,t}$ are not collinear. In the case $r = 2$, we can prove a stronger result, but we first need a technical lemma.

Lemma 1. *Let $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a set of n distinct non–negative integers, with $n \leq t$ and $\alpha_i < t$ $\forall i$. Let M be the $(n+1) \times 2^n$ matrix over $GF(q^t)$, such that the columns of M are in bijective correspondence with the elements of the power set of S, namely $\mathcal{P}(S)$, and $M_{i,j} = x_i^{v(j)}$, where $v(j) = q^{\alpha_{i_1}} + \ldots + q^{\alpha_{i_k}}$ and $\{i_1, \ldots, i_k\}$ is the j–th element of $\mathcal{P}(S)$ (by convention, if the j–th element is the empty set, then $x_i^{v(j)} = 1$). If $x_h \neq x_k$ $\forall h \neq k$, then the $GF(q^t)$–rank of M is $n+1$.*

Proof. We prove the statement by induction on n. For $n = 1$, $M = \begin{pmatrix} 1 & x^{q^\alpha} \\ 1 & y^{q^\alpha} \end{pmatrix}$ and the statement is obviously true. Let now $n > 1$ and suppose it is true for $n-1$. We assume that the first column is the all–one column. After adding to every column a suitable linear combination of the other ones, we can get a matrix M' such that the first row is the vector $(1, 0, \ldots, 0)$ and $M'_{i,j} = (x_i - x_1)^{v(j)}$, $\forall i = 2, \ldots, n+1$ and $\forall j = 1, \ldots, 2^n$. Consider the submatrix of components $M'_{i,j}$ with $i \geq 2$ and j such that the j–th element of $\mathcal{P}(S)$ contains α_1; under the hypothesis that $x_i \neq x_1$ $\forall i \geq 2$, we can divide each row by $(x_i - x_1)^{\alpha_1}$ and in this way we get a $n \times 2^{n-1}$ matrix over $GF(q^t)$ determined by the set $S' = S \setminus \{\alpha_1\}$: by the induction hypothesis the rank of this matrix is n and so the rank of M is $n+1$. □

Theorem 3. *Any $t+1$ points of $\mathcal{V}_{2,t}$ are in general position, i.e. any $t+1$ points of $\mathcal{V}_{2,t}$ span a t–dimensional space.*

Proof. The points of $\mathcal{V}_{2,t}$ are $\{(x^{\alpha_1}, x^{\alpha_2}, \ldots, x^{\alpha_{2^t}}), \alpha_i$ are all the possible sums of distinct powers $q^i, 0 \leq i \leq t-1\} \cup \{P = (0, 0, \ldots, 0, 1)\}$. Since by Remark 2 there is a transitive group fixing $\mathcal{V}_{2,t}$, we can assume that the $t+1$ points we consider are distinct from P. Let M be the matrix the rows of which are the coordinate vectors of $t+1$ points of $\mathcal{V}_{2,t} \setminus \{P\}$. We can apply the Lemma 1 to M with $n = t$, hence the $t+1$ rows vectors of M are $GF(q^t)$–linearly independent and so they are also $GF(q)$–linearly independent. □

Remark3 This is another analogy with the Veronese variety: $\mathcal{V}(1,t)$ is a normal rational curve and it has the property that any $t+1$ points span a t–dimensional space.

The next theorem is about linear sets of $PG(1, q^t)$. In Section 1 we have recalled the three different ways to define a linear set of a projective geomerty, but for our proof we shall use the following: a linear set of $PG(1, q^t)$ of rank r is the set of the elements of $\mathcal{S}$, where $\mathcal{S}$ is a Desarguesian $(t-1)$–spread of $PG(2t-1, q)$, with non–empty intersection with a subspace of $PG(2t-1, q)$ of dimension $r-1$; in this case, a linear set is a proper one when $r \leq t$.

We need to recall the following property of the Grassmannian. Let $\mathcal{G}$ be the Grassmannian of the $(t-1)$–subspaces of $PG(2t-1, q)$: $\mathcal{G}$ is in $PG(N-1, q)$, where $N = \binom{2t}{t}$. By [10], page 109, in $PG(N-1, q)$ there exists a polarity $\perp$, called the *fundamental polarity* of $\mathcal{G}$, such that for every $(t-1)$–space Π, the $(t-1)$–spaces with non–empty intersection with Π correspond to the points of $\mathcal{G} \cap g(\Pi)^\perp$, where g is the Grassmannian map.

Theorem 4. *A linear set L of rank $r \leq t$ of $PG(1, q^t)$ corresponds to the points of $\Pi \cap \mathcal{V}_{2,t}$, where Π is a suitable subspace of the $PG(2^t - 1, q)$ containing $\mathcal{V}_{2,t}$. Moreover, if $r = t$, then Π is a hyperplane of $PG(2^t - 1, q)$; if $r = t-1$, then Π is a subspace of codimension $t+1$ of $PG(2^t - 1, q)$.*

Proof. The points of L correspond to the elements of $\mathcal{S}$ intersecting an $(r-1)$–dimensional subspaces Ω of $PG(2t-1,q)$. An element $\pi \in \mathcal{S}$ intersects Ω if and only if π intersects all the $(t-1)$–spaces through Ω. In $PG(N-1,q)$, let Λ be the (2^t-1)–dimensional subspace containing $\mathcal{V}_{2,t}$ and let $\mathcal{G}' = \{g(\pi), \Omega \subseteq \pi\}$: by [10],Corollary 1 page 117, $\mathcal{G}'$ is projectively equivalent to the Grassmannian of the $(t-r-1)$–spaces of $PG(2t-r-1,q)$, hence $\langle \mathcal{G}' \rangle = \Sigma$ is a $(\binom{2t-r}{t-r}-1)$–space. Hence, the points of L correspond to the points of $\mathcal{V}_{2,t} \cap \Sigma^{\perp}$. If $r=t$, then Σ is a point and $\mathcal{V}_{2,t} \cap \Sigma^{\perp}$ is a hyperplane section of $\mathcal{V}_{2,t}$ ($\mathcal{V}_{2,t}$ can not be contained in the hyperplane because not all the elements of $\mathcal{S}$ can intersect a given $(t-1)$–space). If $r=t-1$, then $\mathcal{G}'$ is a maximal subspace of $\mathcal{G}$ and it has dimension t. The space $\Lambda^{\perp}$ has empty intersection with $\mathcal{G}$, since no $(t-1)$–space can intersect all the spread elements, hence $\Lambda^{\perp} \cap \mathcal{G}' = \emptyset$, and so $\Lambda \cap \mathcal{G}'^{\perp}$ is the minimum possible, i.e. it is a subspace of codimension $t+1$ of Λ. □

The following result is a generalization of the main result of Section 3 of [12], where Lavrauw and Van de Voorde show how a $GF(q)$–linear set of $PG(1,q^t)$ can intersect a subline $PG(1,q)$.

Proposition 5. *A $GF(q)$–linear set L of $PG(1,q^t)$ either contains a fixed subline $PG(1,q^s)$, $s|t$, or it intersects it in at most $\frac{t}{s}(q^{s-1}+q^{s-2}+\ldots+1)$ points.*

Proof. The points of L correspond to the points of the intersection of $\mathcal{V}_{2,t}$ with a suitable subspace. The variety $\mathcal{V}_{2,t}$ consists of the points $(\mathbf{x}^{\alpha_1}, \mathbf{x}^{\alpha_2}, \ldots, \mathbf{x}^{\alpha_m}) \in PG((1+\frac{t}{s})^s-1,q)$, where $\mathbf{x}^{\alpha_i} = x_0^{\alpha_0^{(i)}} x_1^{\alpha_1^{(i)}}$, $(\alpha_0^{(i)}, \alpha_1^{(i)})$ is such that $\alpha_k^{(i)}$ is a sum of distinct powers of q, $\alpha_0^{(i)}+\alpha_1^{(i)} = \frac{t}{s}(q^{s-1}+q^{s-2}+\ldots+1)$ $\forall i$, $\mathbf{x}^{\alpha_i} \neq (\mathbf{x}^{\alpha_i})^{q^h}$ $\forall i \neq j$, $\forall h = 0,\ldots,t-1$, and $(x_0,x_1) \in PG(1,q^s)$. Hence, if a hyperplane section of $\mathcal{V}_{2,t}$ does not contain the image of $PG(1,q^s)$, then it consists of the points corresponding to the points of $PG(1,q^s)$ that satisfy a homogeneous equation of degree $\frac{t}{s}(q^{s-1}+q^{s-2}+\ldots+1)$ and so they are at most $\frac{t}{s}(q^{s-1}+q^{s-2}+\ldots+1)$. □

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