

Characterisations of elementary pseudo-caps and good eggs

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Abstract

In this note, we use the theory of Desarguesian spreads to investigate good eggs. Thas showed that an egg in $\text{PG}(4n - 1, q)$, q odd, with two good elements is elementary. By a short combinatorial argument, we show that a similar statement holds for large pseudo-caps, in odd and even characteristic. As a corollary, this improves and extends the result of Thas, Thas and Van Maldeghem (2006) where one needs at least 4 good elements of an egg in even characteristic to obtain the same conclusion. We rephrase this corollary to obtain a characterisation of the generalised quadrangle $T_3(\mathcal{O})$ of Tits.

Lavrauw (2005) characterises elementary eggs in odd characteristic as those good eggs containing a space that contains at least 5 elements of the egg, but not the good element. We provide an adaptation of this characterisation for weak eggs in odd and even characteristic. As a corollary, we obtain a direct geometric proof for the theorem of Lavrauw.

Keywords: Pseudo-caps; Eggs; Translation generalised quadrangles

1 Preliminaries

In this note, we study *eggs* and *pseudo-caps* in the projective space $\text{PG}(n, q)$, where $\text{PG}(n, q)$ denotes the n -dimensional projective space over the finite field \mathbb{F}_q with q elements, $q = p^h$, p prime. Many previous proofs and characterisations of eggs rely on the connection with eggs and *translation generalised quadrangles* [16]. It is our aim to study eggs from a purely geometric perspective, without using this connection or coordinates.

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In Section 2 we obtain a connection between good eggs and Desarguesian spreads. This link will enable us to reprove, improve or extend known results in Sections 3 and 4. We begin by repeating some well-known definitions.

Definition 1. A *cap* in $\text{PG}(n, q)$ is a set of points such that every three points span a plane. A cap of size k is denoted as a *k-cap*.

A k -cap of $\text{PG}(2, q)$ is often called a *k-arc*. A k -arc in $\text{PG}(2, q)$ satisfies $k \leq q + 1$ for q odd and $k \leq q + 2$ for q even. A $(q + 1)$ -arc is called an *oval*, a $(q + 2)$ -arc a *hyperoval*. A k -cap of $\text{PG}(3, q)$, $q > 2$ satisfies $k \leq q^2 + 1$, moreover, a $(q^2 + 1)$ -cap of $\text{PG}(3, q)$ is often called an *ovoid*. We will consider the higher dimensional equivalent of these structures.

Definition 2. A *pseudo-cap* is a set \mathcal{C} of $(n - 1)$ -spaces in $\text{PG}(2n + m - 1, q)$ such that any three elements of \mathcal{C} span a $(3n - 1)$ -space.

If $m = n$, a pseudo-cap is often called a *pseudo-arc*. By [15], a pseudo-arc \mathcal{A} in $\text{PG}(3n - 1, q)$ satisfies $|\mathcal{A}| \leq q^n + 1$ for q odd and $|\mathcal{A}| \leq q^n + 2$ for q even. If a pseudo-arc \mathcal{A} has $q^n + 1$ or $q^n + 2$ elements, \mathcal{A} is a *pseudo-oval* or *pseudo-hyperoval* respectively. If $m = 2n$, a pseudo-cap with $q^{2n} + 1$ elements is called a *pseudo-ovoid*.

Examples of pseudo-caps in $\text{PG}(kn - 1, q)$ arise by applying field reduction to caps in $\text{PG}(k - 1, q^n)$ and if a pseudo-cap is obtained by field reduction, we call it *elementary*. *Field reduction* is the concept where a point in $\text{PG}(k - 1, q^n)$ corresponds in a natural way to an $(n - 1)$ -space of $\text{PG}(kn - 1, q)$. The set of all points of $\text{PG}(k - 1, q^n)$ then correspond to a set of disjoint $(n - 1)$ -spaces partitioning $\text{PG}(kn - 1, q)$, forming a *Desarguesian spread*. Every Desarguesian spread \mathcal{D} has the property that the space spanned by 2 elements of \mathcal{D} is partitioned by elements of \mathcal{D} , i.e. \mathcal{D} is *normal*. Moreover, a normal $(n - 1)$ -spread of $\text{PG}(kn - 1, q)$, $k > 2$, is Desarguesian, see [2]. For more information on field reduction and Desarguesian spreads we refer to [9].

A *partial spread* in $\text{PG}(n + m - 1, q)$ is a set of mutually disjoint $(n - 1)$ -spaces. Every element E_i of a pseudo-cap \mathcal{E} of $\text{PG}(2n + m - 1, q)$ defines a partial spread

$$\mathcal{S}_i := \{E_0, \dots, E_{i-1}, E_{i+1}, \dots, E_{|\mathcal{E}|}\} / E_i$$

in $\text{PG}(n + m - 1, q) \cong \text{PG}(2n + m - 1, q) / E_i$ and we say that the element E_i *induces* the partial spread \mathcal{S}_i .

A partial spread of $\text{PG}(2n - 1, q)$ of size q^n is said to have *deficiency 1*. From [3], we know that a partial spread of $\text{PG}(2n - 1, q)$ with deficiency 1 can be extended to a spread in a unique way, i.e. the set of points in $\text{PG}(2n - 1, q)$ not lying on an element of the partial spread, form an $(n - 1)$ -space.

Definition 3. A *weak egg* in $\text{PG}(2n + m - 1, q)$ is a pseudo-cap of size $q^m + 1$.

Clearly, pseudo-ovals and pseudo-ovoids are examples of weak eggs. A weak egg \mathcal{E} in $\text{PG}(2n + m - 1, q)$ is called an *egg* if each element $E \in \mathcal{E}$ is contained in a $(n + m - 1)$ -space, T_E , which is skew from every element of \mathcal{E} different from E . The space T_E is called the *tangent space* of \mathcal{E} at E . It is not hard to show that if $n = m$, then every weak egg is an

egg. Eggs are studied mostly because of their one-to-one correspondance with *translation generalised quadrangles* of order (q^n, q^{2n}) , see Subsection 3.2.

The only known examples of eggs in $\text{PG}(2n+m-1, q)$ have either $m = n$ or $m = 2n$ and we have the following theorem restricting the number of possibilities for the parameters n and m .

Theorem 4. [11, Theorem 8.7.2] *If \mathcal{E} is an egg of $\text{PG}(2n + m - 1, q)$ then $m = n$ or $ma = n(a + 1)$ with a odd. Moreover, if q even, then $m = n$ or $m = 2n$.*

This explains why the study of eggs is mainly focused on pseudo-ovals and pseudo-ovals.

In the case of pseudo-ovals, all known examples are elementary. The classical example of an oval in $\text{PG}(2, q^n)$ is a conic. It is a well-known theorem of Segre that an oval of $\text{PG}(2, q^n)$, q odd, is always a conic. A *pseudo-conic* in $\text{PG}(3n - 1, q)$ is an elementary pseudo-oval, arising from applying field reduction to a conic in $\text{PG}(2, q^n)$. We have the following theorems characterising elementary pseudo-ovals using the induced Desarguesian spreads.

Theorem 5. [6] *If \mathcal{O} is a pseudo-oval in $\text{PG}(3n - 1, q)$, q odd, such that for at least one element the induced spread is Desarguesian, then \mathcal{O} is a pseudo-conic.*

Theorem 6. [13] *If \mathcal{O} is a pseudo-oval in $\text{PG}(3n - 1, q)$, n prime, $q > 2$ even, such that all elements induce a Desarguesian spread, then \mathcal{O} is elementary.*

In the case that q is odd, we have the following theorem which extends Theorem 5 from pseudo-ovals to large pseudo-arcs in $\text{PG}(3n - 1, q)$.

Theorem 7. [12] *Consider $\mathcal{K} = \{K_1, \dots, K_s\}$ a pseudo-arc in $\text{PG}(3n - 1, q)$, q odd, of size greater than the size of the second largest complete arc in $\text{PG}(2, q^n)$. If for at least one element K_i of \mathcal{K} , the partial spread $\mathcal{S} = \{K_1, \dots, K_{i-1}, K_{i+1}, \dots, K_s\}/K_i$ extends to a Desarguesian spread of $\text{PG}(2n - 1, q) = \text{PG}(3n - 1, q)/K_i$, then \mathcal{K} is contained in a pseudo-conic.*

In Theorem 15, we will prove a similar statement for pseudo-caps in $\text{PG}(4n - 1, q)$.

All known examples of pseudo-ovals in $\text{PG}(4n - 1, q)$ are elementary when q is even, but in contrast to the situation for pseudo-ovals, when q is odd, there are non-elementary examples of pseudo-ovals. The standard example of an ovoid in $\text{PG}(3, q^n)$ is an elliptic quadric $Q^-(3, q^n)$. By the famous result of Barlotti and Panella [1, 10], every ovoid of $\text{PG}(3, q^n)$, q odd, is an elliptic quadric $Q^-(3, q^n)$, however, there is no classification of ovoids in $\text{PG}(3, q^n)$ for q even. For both even and odd order q , the classification of pseudo-ovals is an open problem.

2 Good eggs and Desarguesian spreads

A (weak) egg \mathcal{E} in $\text{PG}(2n + m - 1, q)$, $m > n$, is *good* at an element $E \in \mathcal{E}$ if every $(3n - 1)$ -space containing E and at least two other elements of \mathcal{E} , contains exactly $q^n + 1$

elements of \mathcal{E} . A (weak) egg that has at least one good element is called a *good (weak) egg*. If \mathcal{E} is good at E , then for any two elements $E_1, E_2 \in \mathcal{E} \setminus \{E\}$ the $(3n - 1)$ -space $\langle E, E_1, E_2 \rangle$ intersects \mathcal{E} in a pseudo-oval.

Lemma 8. *Good weak eggs in $\text{PG}(2n + m - 1, q)$ can only exist if n is a divisor of m . Good eggs only exist in $\text{PG}(4n - 1, q)$.*

Proof. Consider a weak egg \mathcal{E} of $\text{PG}(2n + m - 1, q)$, $m > n$, good at an element $E_1 \in \mathcal{E}$. Consider a second element $E_2 \in \mathcal{E} \setminus \{E_1\}$. For every element $E \in \mathcal{E} \setminus \{E_1, E_2\}$, the $(3n - 1)$ -space $\langle E, E_1, E_2 \rangle$ intersects \mathcal{E} in a pseudo-oval. By considering the elements of $\mathcal{E} \setminus \{E_1, E_2\}$, we find a set \mathcal{T} of $(3n - 1)$ -spaces containing $\langle E_1, E_2 \rangle$, such that each space of \mathcal{T} intersects \mathcal{E} in a pseudo-oval. Every two spaces in \mathcal{T} meet exactly in $\langle E_1, E_2 \rangle$ and \mathcal{E} is the union of the pseudo-ovals $\{T \cap \mathcal{E} \mid T \in \mathcal{T}\}$. The set \mathcal{T} contains $\frac{q^m - 1}{q^n - 1}$ $(3n - 1)$ -spaces; as $q^n - 1$ has to be a divisor of $q^m - 1$, it follows that n is a divisor of m .

Suppose \mathcal{E} is an egg. For q even, by Theorem 4, eggs only exist in $\text{PG}(4n - 1, q)$ (or $\text{PG}(3n - 1, q)$). Consider now a good egg of $\text{PG}(2n + m - 1, q)$, q odd, where m is a multiple of n . By Theorem 4, $m = \frac{a+1}{a}n$, for some odd integer a , so we conclude that $m = 2n$. \square

We will show that the good elements of an egg are exactly those inducing a partial spread which is extendable to a Desarguesian spread. Part (i) of the following theorem, for \mathcal{E} an egg, is mentioned in [16, Remark 5.1.7].

Theorem 9.

- (i) *If a weak egg \mathcal{E} in $\text{PG}(2n + m - 1, q)$ is good at an element E , then E induces a partial spread which extends to a Desarguesian spread.*
- (ii) *Let \mathcal{E} be a weak egg in $\text{PG}(2n+m-1, q)$ for q odd and an egg in $\text{PG}(2n+m-1, q)$ for q even. If an element $E \in \mathcal{E}$ induces a partial spread which extends to a Desarguesian spread, then \mathcal{E} is good at E .*

Proof. (i) Suppose \mathcal{E} is a weak egg which is good at E . Consider the partial spread \mathcal{S} of $\text{PG}(n + m - 1, q)$ of size q^m induced by E . Because \mathcal{E} is good at E , any two elements of \mathcal{S} span a $(2n - 1)$ -space which contains a partial spread of q^n elements of \mathcal{S} . This partial spread has deficiency 1, so extends uniquely to a spread by one $(n - 1)$ -space (by [3]).

Consider three elements $S_1, S_2, S_3 \in \mathcal{S}$ not lying in the same $(2n - 1)$ -space, hence spanning a $(3n - 1)$ -space π . There are q^n elements of \mathcal{S} contained in $\langle S_2, S_3 \rangle$. For every element R of $\mathcal{S} \cap \langle S_2, S_3 \rangle$, the $(2n - 1)$ -space $\langle S_1, R \rangle$ contains q^n elements of \mathcal{S} . Hence, there are q^n $(2n - 1)$ -spaces of π containing S_1 and $q^n - 1$ other elements of \mathcal{S} . Similarly, there are q^n $(2n - 1)$ -spaces of π containing S_2 and $q^n - 1$ other elements of \mathcal{S} . Since π has dimension $3n - 1$, two such distinct $(2n - 1)$ -spaces, one containing S_1 and the other containing S_2 , intersect in at least an $(n - 1)$ -space, hence, in exactly an $(n - 1)$ -space. This space is either an element of \mathcal{S} or the $(n - 1)$ -space which extends both of them to a spread. It follows that there are q^{2n} elements of \mathcal{S} contained in π and if an element of \mathcal{S}

intersects π , then it is contained in π . Hence, if $\langle S_2, S_3 \rangle$ meets a $(2n - 1)$ -space spanned by S_1 and another element of \mathcal{S} , then they meet in an $(n - 1)$ -space.

As S_1, S_2, S_3 were chosen randomly, it follows in general that if two distinct $(2n - 1)$ -spaces spanned by elements of \mathcal{S} intersect, then they meet in an $(n - 1)$ -space. They meet either in an $(n - 1)$ -space of \mathcal{S} or in the $(n - 1)$ -space which extends the partial spreads of both $(2n - 1)$ -spaces to a spread. We see that \mathcal{S} can be uniquely extended to a spread which is normal, thus Desarguesian.

(ii) Now let \mathcal{E} be an egg if q is even and a weak egg if q is odd. Suppose E induces a partial spread \mathcal{S} of size q^m which extends to a Desarguesian $(n - 1)$ -spread \mathcal{D} of $\text{PG}(n + m - 1, q)$, hence $m = kn$ for some $k > 1$. There are $\frac{q^m - 1}{q^n - 1}$ elements of \mathcal{D} not contained in \mathcal{S} .

When \mathcal{E} is an egg, the elements of $\mathcal{D} \setminus \mathcal{S}$ span a $(m - 1)$ -space, corresponding to T_E . Hence, any $(2n - 1)$ -space spanned by two elements of \mathcal{S} contains q^n elements of \mathcal{S} and one element $\mathcal{D} \setminus \mathcal{S}$. So, \mathcal{E} is good at E .

Suppose \mathcal{E} is a weak egg, with q odd. As q is odd, no $(3n - 1)$ -space intersects \mathcal{E} in a pseudo-hyperoval. Hence, any $(3n - 1)$ -space containing E intersects \mathcal{E} in at most $q^n + 1$ elements, so any $(2n - 1)$ -space spanned by two elements of \mathcal{S} can contain at most q^n elements of \mathcal{S} . Hence, any such space must contain at least one element of $\mathcal{D} \setminus \mathcal{S}$. By field reduction, the elements of the Desarguesian spread \mathcal{D} of $\text{PG}(n + m - 1, q)$ are in one-to-one correspondance with the points of $\text{PG}(\frac{m}{n}, q^n)$. Any $(2n - 1)$ -space spanned by two elements of \mathcal{D} must contain at least one element of $\mathcal{D} \setminus \mathcal{S}$. Hence, the points corresponding to $\mathcal{D} \setminus \mathcal{S}$ form a line-blocking set of $\text{PG}(\frac{m}{n}, q^n)$. Since $|\mathcal{D} \setminus \mathcal{S}| = \frac{q^m - 1}{q^n - 1}$, from [4] it follows that the points corresponding to $\mathcal{D} \setminus \mathcal{S}$ are the points of a $(\frac{m}{n} - 1)$ -space, hence the elements of $\mathcal{D} \setminus \mathcal{S}$ span a $(m - 1)$ -space. As before, it follows that \mathcal{E} is good at E . \square

The following corollary, for \mathcal{E} an egg, was also mentioned in [14, Theorem 4.3.4] in terms of translation generalised quadrangles.

Corollary 10. *If a weak egg \mathcal{E} , q odd, is good at an element E , then every pseudo-oval of \mathcal{E} containing E is a pseudo-conic.*

Proof. Let Π be a $(n + m - 1)$ -space disjoint from E . By Theorem 9, the partial spread \mathcal{E}/E in Π extends to a Desarguesian spread. Consider a pseudo-oval \mathcal{O} of \mathcal{E} containing E . The q^n elements of \mathcal{O}/E are contained in \mathcal{E}/E and thus extend to a Desarguesian spread of the $(2n - 1)$ -space $\langle \mathcal{O} \rangle \cap \Pi$.

The element E of the pseudo-oval \mathcal{O} induces a partial spread \mathcal{O}/E which extends to a Desarguesian spread, hence, by Theorem 5, the statement follows. \square

3 A characterisation of good eggs

3.1 Eggs with two good elements

An elementary pseudo-ovoid that arises from applying field reduction to an elliptic quadric is called *classical*. We recall the following theorem from [16].

Theorem 11. [16, Theorem 5.1.12]

If q is odd and an egg \mathcal{E} in $\text{PG}(4n - 1, q)$ has at least two good elements, then \mathcal{E} is classical. If q is even and an egg \mathcal{E} in $\text{PG}(4n - 1, q)$ has at least four good elements, not contained in a common pseudo-oval on \mathcal{E} , then \mathcal{E} is elementary.

It was open problem whether, for q even, being good at two elements is sufficient to be elementary, this was posed as Problem A.5.6 in [16]. We will give an affirmative answer to this question in a more general setting, namely in terms of pseudo-caps. We first need two lemma's concerning Desarguesian spreads.

Lemma 12. [13, Corollary 1.8] Consider two Desarguesian $(n - 1)$ -spreads \mathcal{S}_1 and \mathcal{S}_2 in $\text{PG}(2n - 1, q)$, $q > 2$. If \mathcal{S}_1 and \mathcal{S}_2 have at least 3 elements in common, then they share exactly $q^t + 1$ elements for some $t|n$.

The following lemma is a generalisation of [13, Lemma 1.4] and the proof is analogous. We introduce some necessary definitions and notations.

A *regulus* \mathcal{R} in $\text{PG}(2n - 1, q)$ is a set of $q + 1$ mutually disjoint $(n - 1)$ -spaces having the property that if a line meets 3 elements of \mathcal{R} , then it meets all elements of \mathcal{R} . Let us denote the unique regulus through 3 mutually disjoint $(n - 1)$ -spaces A, B and C in $\text{PG}(2n - 1, q)$ by $\mathcal{R}(A, B, C)$. Every Desarguesian spread \mathcal{D} has the property that for 3 elements A, B, C in \mathcal{D} , the elements of $\mathcal{R}(A, B, C)$ are also contained in \mathcal{D} , i.e. \mathcal{D} is regular (see also [5]).

We will use the following notation for points of a projective space $\text{PG}(r - 1, q^n)$. A point P of $\text{PG}(r - 1, q^n)$ defined by a vector $(x_1, x_2, \dots, x_r) \in (\mathbb{F}_{q^n})^r$ is denoted by $\mathbb{F}_{q^n}(x_1, x_2, \dots, x_r)$, reflecting the fact that every \mathbb{F}_{q^n} -multiple of (x_1, x_2, \dots, x_r) gives rise to the point P . We can identify the vector space $\mathbb{F}_{q^{nr}}$ with $(\mathbb{F}_{q^n})^r$, and hence write every point of $\text{PG}(rn - 1, q)$ as $\mathbb{F}_q(x_1, \dots, x_r)$ with $x_i \in \mathbb{F}_{q^n}$. In this way, when applying field reduction, a point $\mathbb{F}_{q^n}(x_1, \dots, x_r)$ in $\text{PG}(r - 1, q^n)$ corresponds to the $(n - 1)$ -space $\mathbb{F}_{q^n}(x_1, \dots, x_r) = \{\mathbb{F}_q(\alpha x_1, \dots, \alpha x_r) | \alpha \in \mathbb{F}_{q^n}\}$ of $\text{PG}(rn - 1, q)$.

Lemma 13. Let \mathcal{D}_1 be a Desarguesian $(n - 1)$ -spread in a $(kn - 1)$ -dimensional subspace Π of $\text{PG}((k + 1)n - 1, q)$, let μ be an element of \mathcal{D}_1 and let E_1 and E_2 be mutually disjoint $(n - 1)$ -spaces such that $\langle E_1, E_2 \rangle$ meets Π exactly in the space μ . Then there exists a unique Desarguesian $(n - 1)$ -spread of $\text{PG}((k + 1)n - 1, q)$ containing E_1 , E_2 and all elements of \mathcal{D}_1 .

Proof. Since \mathcal{D}_1 is a Desarguesian spread in Π , we can choose coordinates for Π such that $\mathcal{D}_1 = \{\mathbb{F}_{q^n}(x_1, x_2, \dots, x_k) | x_i \in \mathbb{F}_{q^n}\}$ and $\mu = \mathbb{F}_{q^n}(0, \dots, 0, 1)$. We embed Π in $\text{PG}((k + 1)n - 1, q)$ by mapping a point $\mathbb{F}_q(x_1, \dots, x_k)$, $x_i \in \mathbb{F}_{q^n}$, of Π onto $\mathbb{F}_q(x_1, \dots, x_k, 0)$. Let ℓ_P denote the unique transversal line through a point P of μ to the regulus $\mathcal{R}(\mu, E_1, E_2)$.

We can still choose coordinates for $n + 1$ points in general position in $\text{PG}((k + 1)n - 1, q) \setminus \Pi$. We will choose these $n + 1$ points such that n of them belong to E_1 and one of them belongs to E_2 . Consider a set $\{y_i | i = 1, \dots, n\}$ forming a basis of \mathbb{F}_{q^n} over \mathbb{F}_q . We may assume that the line ℓ_{P_i} through $P_i = \mathbb{F}_q(0, \dots, 0, y_i, 0) \in \mu$ meets E_1 in the point $\mathbb{F}_q(0, \dots, 0, 0, y_i)$. It follows that $E_1 = \mathbb{F}_{q^n}(0, \dots, 0, 0, 1)$. Moreover, we may assume

that ℓ_Q with $Q = \mathbb{F}_q(0, \dots, 0, 0, \sum_{i=1}^n y_i, 0) \in \mu$ meets E_2 in $\mathbb{F}_q(0, \dots, 0, \sum_{i=1}^n y_i, \sum_{i=1}^n y_i)$. Since $\mathbb{F}_q(0, \dots, 0, \sum_{i=1}^n y_i, \sum_{i=1}^n y_i)$ has to be in the space spanned by the intersection points $R_i = \ell_{P_i} \cap E_2$, it follows that $R_i = \mathbb{F}_q(0, \dots, 0, y_i, y_i)$ and consequently, that $E_2 = \mathbb{F}_{q^n}(0, \dots, 0, 1, 1)$.

It is clear that the Desarguesian spread $\mathcal{D} = \{\mathbb{F}_{q^n}(x_1, \dots, x_{k+1}) \mid x_i \in \mathbb{F}_{q^n}\}$ contains the spread \mathcal{D}_1 and the $(n-1)$ -spaces E_1 and E_2 . Moreover, since every element of \mathcal{D} , not in $\langle E_1, E_2 \rangle$, is obtained as the intersection of $\langle E_1, X \rangle \cap \langle E_2, Y \rangle$, where $X, Y \in \mathcal{D}_1$, it is clear that \mathcal{D} is the unique Desarguesian spread satisfying our hypothesis. \square

Lemma 14. *Consider a pseudo-cap \mathcal{E} of $\text{PG}(4n-1, q)$ containing an element E that induces a partial spread which extends to a Desarguesian spread. If Π is a $(3n-1)$ -space spanned by E and two other elements of \mathcal{E} , then every element of \mathcal{E} is either disjoint from Π or contained in Π .*

Proof. Let Σ be a $(3n-1)$ -space skew from E and consider the induced partial spread \mathcal{E}/E in Σ . If F is an element of \mathcal{E} which meets Π , then the projection F/E of F from E onto Σ is an element of \mathcal{E}/E which meets the space Π/E . By assumption, the space Π/E is spanned by spread elements of a partial spread extending to a Desarguesian spread. Hence, since a Desarguesian spread is normal, F/E is contained in Π/E . It follows that since Π contains E , the element F is contained in Π . \square

Theorem 15. *Consider a pseudo-cap \mathcal{E} in $\text{PG}(4n-1, q)$, $q > 2$, with $|\mathcal{E}| > q^{n+k} + q^n - q^k + 1$, q odd, and $|\mathcal{E}| > q^{n+k} + q^n + 2$, q even, where k is the largest divisor of n with $k \neq n$. The pseudo-cap \mathcal{E} is elementary if and only if two of its elements induce a partial spread which extends to a Desarguesian spread.*

Proof. If \mathcal{E} is elementary, then the elements of \mathcal{E} are contained in a Desarguesian spread of $\text{PG}(4n-1, q)$, so every element of \mathcal{E} induces a partial spread which extends to a Desarguesian spread.

Now suppose that \mathcal{E} contains two distinct elements E_1, E_2 that induce a partial spread which extends to a Desarguesian spread. Since $|\mathcal{E}| > q^n + 2$, using Lemma 14, we can find two elements $E_3, E_4 \in \mathcal{E}$ such that $\langle E_1, E_2, E_3, E_4 \rangle$ spans $\text{PG}(4n-1, q)$.

The partial spread induced by E_1 in the space $\langle E_2, E_3, E_4 \rangle$ can be extended to a Desarguesian spread \mathcal{D}_1 . Analogously, the partial spread induced by E_2 in the space $\langle E_1, E_3, E_4 \rangle$ can be extended to a Desarguesian spread \mathcal{D}_2 . Since E_3 and E_4 are elements of the spreads \mathcal{D}_1 and \mathcal{D}_2 , the Desarguesian spreads \mathcal{D}_1 and \mathcal{D}_2 intersect the $(2n-1)$ -space $\langle E_3, E_4 \rangle$ each in a Desarguesian spread, say \mathcal{S}_1 and \mathcal{S}_2 respectively.

Take an element $E \in \mathcal{E} \setminus \{E_1, E_2\}$ and consider the $(3n-1)$ -subspace $\langle E_1, E_2, E \rangle$. From Lemma 14 it follows that any element of \mathcal{E} is either contained in or disjoint from $\langle E_1, E_2, E \rangle$. By considering the elements of $\mathcal{E} \setminus \{E_1, E_2\}$, we find a set \mathcal{T} of $(3n-1)$ -spaces containing $\langle E_1, E_2 \rangle$, such that each space of \mathcal{T} intersects \mathcal{E} in a pseudo-arc. Every two spaces in \mathcal{T} meet exactly in $\langle E_1, E_2 \rangle$ and \mathcal{E} is the union of the pseudo-arcs $\{T \cap \mathcal{E} \mid T \in \mathcal{T}\}$. The set \mathcal{T} intersects $\langle E_3, E_4 \rangle$ in a partial $(n-1)$ -spread \mathcal{P} .

Let P be an element of \mathcal{P} , then $\langle P, E_1, E_2 \rangle$ is a $(3n-1)$ -space containing at least one element E of $\mathcal{E} \setminus \{E_1, E_2\}$. The projection E' of E from E_1 onto $\langle E_2, E_3, E_4 \rangle$ is contained

in \mathcal{D}_1 . We obtain that $P = \langle E_2, E' \rangle \cap \langle E_3, E_4 \rangle$, and since the elements E_2, E', E_3, E_4 are contained in \mathcal{D}_1 , this implies that P is contained in \mathcal{D}_1 . Moreover, since $P \subset \langle E_3, E_4 \rangle$, the element P is contained in \mathcal{S}_1 . Similarly, we obtain that P is contained in \mathcal{S}_2 and we conclude that every element of \mathcal{P} must be contained in both \mathcal{S}_1 and \mathcal{S}_2 .

Suppose that k is the largest divisor of n with $k \neq n$. The pseudo-cap \mathcal{E} has size $|\mathcal{E}| > (q^n - \epsilon)(q^k + 1) + 2$ and every $(3n - 1)$ -space of \mathcal{T} contains at most $q^n - \epsilon$ elements different from E_1, E_2 , where $\epsilon = 1$ for q odd and $\epsilon = 0$ for q even. By the pigeonhole principle, it follows that $|\mathcal{P}| \geq q^k + 2$. Hence, the Desarguesian spreads \mathcal{S}_1 and \mathcal{S}_2 have at least $q^k + 2$ elements in common, where k is the largest divisor of n with $k \neq n$. As $q > 2$, by Lemma 12, we find that $\mathcal{S}_1 = \mathcal{S}_2$.

By Theorem 13, consider the unique Desarguesian spread \mathcal{D} of $\text{PG}(4n - 1, q)$ containing all elements of \mathcal{D}_1 and two distinct elements of $\mathcal{D}_2 \setminus \mathcal{D}_1$. It is clear that, since $\mathcal{S}_1 = \mathcal{S}_2$, the spread \mathcal{D} contains all elements of \mathcal{D}_2 .

Every element of \mathcal{E} , not in $\mathcal{D}_1 \cup \mathcal{D}_2$, arises as the intersection $\langle E_1, X \rangle \cap \langle E_2, Y \rangle$ for some $X \in \mathcal{D}_1 \subset \mathcal{D}$ and $Y \in \mathcal{D}_2 \subset \mathcal{D}$, hence, since a Desarguesian spread is normal, every element of \mathcal{E} belongs to \mathcal{D} . It follows that \mathcal{E} is elementary. \square

We obtain the following corollary which improves [16, Theorem 5.1.12].

Corollary 16. *A weak egg in $\text{PG}(4n - 1, q)$ which is good at two distinct elements is elementary.*

Proof. A weak egg is a pseudo-cap of size $q^{2n} + 1$ in $\text{PG}(4n - 1, q)$. By Theorem 9, if the weak egg is good at two elements, these elements induce a partial spread which extends to a Desarguesian spread. We can repeat the proof of Theorem 15. Now the partial spread \mathcal{P} has size $q^n + 1$, so the conclusion $\mathcal{S}_1 = \mathcal{S}_2$ follows immediately. We do not require Lemma 12, hence the restriction $q > 2$ can be dropped. \square

3.2 A corollary in terms of translation generalised quadrangles

A *generalised quadrangle* of order (s, t) , $s, t > 1$, is an incidence structure of points and lines satisfying the following axioms:

- every line has exactly $s + 1$ points,
- through every point, there are exactly $t + 1$ lines,
- if P is a point, not on the line L , then there is exactly one line through P which meets L non-trivially.

From every egg \mathcal{E} in $\Sigma_\infty = \text{PG}(2n + m - 1, q)$ we can construct a generalised quadrangle $(\mathcal{P}, \mathcal{L})$ as follows. Embed Σ_∞ as a hyperplane at infinity of $\text{PG}(2n + m, q)$.

- \mathcal{P} :
- (i) affine points of $\text{PG}(2n + m, q)$, i.e. the points not lying in Σ_∞ ,
 - (ii) the $(n + m)$ -spaces meeting Σ_∞ in T_E for some $E \in \mathcal{E}$,
 - (iii) the symbol (∞) .

- \mathcal{L} : (a) the n -spaces meeting Σ_∞ in an element of \mathcal{E} ,
 (b) the elements of \mathcal{E} .

Incidence is defined as follows.

- A point of type (i) is incident with the lines of type (a) through it.
- A point of type (ii) is incident with the lines of type (a) it contains and the line of type (b) it contains.
- The point (∞) is incident with all lines of type (b).

The obtained generalised quadrangle is denoted as $T(\mathcal{E})$ and is called a *translation generalised quadrangle* (TGQ) with base point (∞) . In [11, Theorem 8.7.1] it is proven that every TGQ of order (q^n, q^m) , where \mathbb{F}_q is a subfield of its kernel, is isomorphic to a $T(\mathcal{E})$ for some egg \mathcal{E} of $\text{PG}(2n + m - 1, q)$.

When $n = m = 1$, then \mathcal{O} is an oval of $\text{PG}(2, q)$ and the construction above gives the well-known construction of $T_2(\mathcal{O})$. When $n = 1$ and $m = 2$, then \mathcal{O} is an ovoid of $\text{PG}(3, q)$ and the construction above is the construction of Tits of $T_3(\mathcal{O})$ (see [16]).

Lemma 17. *Let $T = T(\mathcal{E})$ be a TGQ of order (q^n, q^{2n}) with base point (∞) . Let m_1, m_2, m_3 be three distinct lines through (∞) , and let E_1, E_2, E_3 denote the elements of \mathcal{E} corresponding to m_1, m_2, m_3 respectively. Then there is a subquadrangle of order q^n through m_1, m_2, m_3 if and only if the $(3n - 1)$ -dimensional space $\langle E_1, E_2, E_3 \rangle$ contains exactly $q^n + 1$ elements of \mathcal{E} .*

Proof. Suppose that the $(3n - 1)$ -space $\Sigma = \langle E_1, E_2, E_3 \rangle$ contains a set \mathcal{O} of exactly $q^n + 1$ elements of \mathcal{E} , then it is clear that $T(\mathcal{E})$ defines the incidence structure $T(\mathcal{O})$ in a $3n$ -space through Σ . The structure $T(\mathcal{O})$ is a generalised quadrangle of order q^n , forming a subquadrangle of $T(\mathcal{E})$ and containing the lines m_1, m_2, m_3 .

On the other hand, suppose that there is a subquadrangle T' of order q^n containing m_1, m_2, m_3 , where the lines m_1, m_2, m_3 are incident with (∞) . This implies that the point (∞) is in T' , and since (∞) lies only on lines of type (b) (i.e. the lines corresponding to elements of \mathcal{E}), we know that T' contains exactly $q^n + 1$ lines of type (b), among which the lines m_1, m_2 and m_3 . Let $\{E_1, \dots, E_{q^n+1}\}$ be the egg elements corresponding to these lines. This means that there are $(q^n + 1)q^{2n}$ lines in T' of type (a), containing in total $(q^n + 1)q^{2n}(q^n)/(q^n + 1) = q^{3n}$ points of type (i) (i.e. affine points).

Each $(n - 1)$ -space E_j is contained in q^{2n} n -spaces corresponding to a line of type (a) of T' and every affine point is contained in exactly one n -space containing E_j . Let P_j be a point of the space E_j , then we see that the q^{3n} affine points of T' lie on q^{2n} lines through P_j . As this holds for every $j \in \{1, \dots, q^n + 1\}$, it is clear that the q^{3n} affine points of T' are contained in a $3n$ -space. This in turn implies that the elements $\{E_1, \dots, E_{q^n+1}\}$ are contained in a $(3n - 1)$ -space, namely $\langle E_1, E_2, E_3 \rangle$. Hence, this space contains at least $q^n + 1$ elements of \mathcal{E} . Since \mathcal{E} is an egg, it is not possible that a $(3n - 1)$ -space contains more than $q^n + 1$ elements of \mathcal{E} , which concludes the proof. \square

Lemma 18. *Let $T = T(\mathcal{E})$ be a TGQ of order (q^n, q^{2n}) with base point (∞) . Let ℓ be a line through (∞) and E_ℓ the element of \mathcal{E} corresponding to ℓ . The egg \mathcal{E} is good at E_ℓ if and only if for every two distinct lines m_1, m_2 through (∞) , where $m_1, m_2 \neq \ell$, there is a subquadrangle of order q^n through m_1, m_2, ℓ .*

Proof. This follows immediately from Lemma 17 and the definition of a being good at an element. \square

We are now ready to state the promised characterisation of the translation generalised quadrangle $T_3(\mathcal{O})$.

Theorem 19. *Let T be a TGQ of order (q^n, q^{2n}) with base point (∞) . Suppose that T contains two distinct lines $\ell_i, i = 1, 2$ such that for every two distinct lines m_1, m_2 through (∞) , where $m_1, m_2 \neq \ell_i, i = 1, 2$ there is a subquadrangle through $m_1, m_2, \ell_i, i = 1, 2$, then T is isomorphic to $T_3(\mathcal{O})$, where \mathcal{O} is an ovoid of $\text{PG}(3, q^n)$.*

4 A geometric proof of a Theorem of Lavrauw

In this section we obtain a second characterisation of good weak eggs. We need the following lemma stating that every good element of a weak egg has a tangent space.

Lemma 20. *If a weak egg \mathcal{E} in $\text{PG}(2n + m - 1, q)$ is good at an element E , then there exists a unique $(n + m - 1)$ -space T , such that $T \cap \mathcal{E} = \{E\}$.*

Proof. Consider a $(n + m - 1)$ -space Σ disjoint from E . If \mathcal{E} is good at E , the element E induces a partial spread $\mathcal{S} = \mathcal{E}/E$ which extends to a Desarguesian spread \mathcal{D} of Σ . By following the proof of Theorem 9, part (ii), for both q odd and q even, the elements of $\mathcal{D} \setminus \mathcal{S}$ span a $(m - 1)$ -space. It is clear that the $(n + m - 1)$ -space $T = \langle E, \mathcal{D} \setminus \mathcal{S} \rangle$ satisfies $T \cap \mathcal{E} = E$. \square

In [8] the authors proved that every egg of $\text{PG}(7, 2)$ arises from an elliptic quadric $Q^-(3, 4)$ by field reduction. Hence, in the following characterisation, when \mathcal{E} is an egg in $\text{PG}(4n - 1, q)$, the condition $q^n > 4$ is essentially not a restriction.

Theorem 21. *Suppose $n > 1, q^n > 4$, consider \mathcal{E} a weak egg in $\text{PG}(4n - 1, q)$. Then \mathcal{E} is elementary if and only if the following three properties hold:*

- \mathcal{E} is good at an element E ,
- there exists a $(3n - 1)$ -space, disjoint from E , with at least 5 elements E_1, E_2, E_3, E_4, E_5 of \mathcal{E} ,
- all pseudo-ovals of \mathcal{E} containing $\{E, E_1\}, \{E, E_2\}$ or $\{E, E_3\}$ are elementary.

Proof. Clearly, if an egg is elementary, the statement is valid.

For the converse, consider the $(3n-1)$ -space Π containing 5 elements E_1, E_2, E_3, E_4, E_5 of \mathcal{E} , but not the element E . As \mathcal{E} is good at E , the element E induces a partial spread which extends to a Desarguesian $(n-1)$ -spread \mathcal{D}_0 in Π , which contains $E_i, i = 1, \dots, 5$.

By Lemma 20, there exists a unique $(3n-1)$ -space T , such that $T \cap \mathcal{E} = \{E\}$. When \mathcal{E} is an egg, this space corresponds to the tangent space T_E .

Consider the two $(n-1)$ -spaces $F = \langle E_1, E_5 \rangle \cap \langle E_2, E_4 \rangle$ and $F' = \langle E_1, E_5 \rangle \cap \langle E_3, E_4 \rangle$. Both F and F' are contained in \mathcal{D}_0 , but at most one of them can be contained in the $(2n-1)$ -space $\Pi \cap T$. Suppose F is not contained in T (note that this choice has no further impact as E_2 and E_3 play the same role). This implies that the $(2n-1)$ -space $\langle E, F \rangle$ contains an element $E_6 \in \mathcal{E} \setminus \{E\}$. By Theorem 13, there exists a unique Desarguesian spread \mathcal{D} containing E, E_6 and all elements of \mathcal{D}_0 . We will prove that \mathcal{E} is contained in \mathcal{D} .

The $(3n-1)$ -space $\langle E, E_1, E_5 \rangle$ intersect \mathcal{E} in a pseudo-oval \mathcal{O}_1 , and the $(3n-1)$ -space $\langle E, E_2, E_4 \rangle$ intersect \mathcal{E} in a pseudo-oval \mathcal{O}_2 . Clearly, \mathcal{O}_1 and \mathcal{O}_2 both contain E_6 .

By assumption, \mathcal{O}_1 and \mathcal{O}_2 are elementary pseudo-ovals. The Desarguesian $(n-1)$ -spread in $\langle E, E_1, E_5 \rangle$ containing \mathcal{O}_1 contains E, E_6 and the q^n+1 elements of $\mathcal{D}_0 \cap \langle E_1, E_5 \rangle$. It follows that this Desarguesian spread is contained in \mathcal{D} , hence \mathcal{O}_1 is contained in \mathcal{D} . Analogously, the pseudo-oval \mathcal{O}_2 is also contained in \mathcal{D} .

There are $q^n - 2$ pseudo-ovals \mathcal{O} of \mathcal{E} , containing $\{E, E_3\}$, but not E_6 , such that the $(3n-1)$ -space $\langle \mathcal{O} \rangle$ does not contain the $(n-1)$ -space $T \cap \langle \mathcal{O}_1 \rangle$, nor the $(n-1)$ -space $T \cap \langle \mathcal{O}_2 \rangle$. Take such an oval \mathcal{O} , then there is an element E_7 of $\mathcal{E} \setminus \{E\}$ contained in $\langle \mathcal{O} \rangle \cap \langle \mathcal{O}_1 \rangle$, hence, $E_7 \in \mathcal{O} \cap \mathcal{O}_1$. Likewise, there is an element E_8 of $\mathcal{E} \setminus \{E\}$ contained in $\mathcal{O} \cap \mathcal{O}_2$.

By assumption, \mathcal{O} is elementary; let $\mathcal{S}_{\mathcal{O}}$ be the Desarguesian $(n-1)$ -spread containing \mathcal{O} . As E_7 and E_8 are contained in \mathcal{D} , the Desarguesian spread \mathcal{D} intersects $\langle E_7, E_8 \rangle$ in a Desarguesian spread. Let P be an element of $\mathcal{D} \cap \langle E_7, E_8 \rangle$, not contained in T , then $\langle E, P \rangle$ meets Π in an element of \mathcal{D} , and hence, $\langle E, P \rangle$ contains an element P' of $\mathcal{E} \setminus E$. As $\langle E, P \rangle$ is contained in $\langle \mathcal{O} \rangle$, P' is an element of \mathcal{O} , and hence also of $\mathcal{S}_{\mathcal{O}}$. Since P', E, E_7, E_8 are contained in $\mathcal{S}_{\mathcal{O}}$, the element $P = \langle E, P' \rangle \cap \langle E_7, E_8 \rangle$ is an element of $\mathcal{S}_{\mathcal{O}}$. This implies that $\mathcal{D} \cap \langle E_7, E_8 \rangle$ and $\mathcal{S}_{\mathcal{O}}$ have at least q^n elements in common, which implies in turn that they have all their elements in common. We conclude that $\mathcal{S}_{\mathcal{O}}$ contains E, E_3 and the q^n+1 elements of $\mathcal{D} \cap \langle E_7, E_8 \rangle$, hence $\mathcal{S}_{\mathcal{O}}$ and thus all elements of \mathcal{O} are contained in \mathcal{D} .

Now, consider an element $E_9 \in \mathcal{E}$, not contained in $\mathcal{O}_1, \mathcal{O}_2$ or any of the previously considered $q^n - 2$ pseudo-ovals \mathcal{O} . Look at the pseudo-oval $\mathcal{O}' = \langle E, E_1, E_9 \rangle \cap \mathcal{E}$ and the pseudo-oval $\mathcal{O}'' = \langle E, E_2, E_9 \rangle \cap \mathcal{E}$. At least one of them does not contain E_3 . Suppose \mathcal{O}' does not contain E_3 (the proof goes analogously if \mathcal{O}'' does not contain E_3). For at most one of the $q^n - 2$ pseudo-ovals \mathcal{O} containing $\{E, E_3\}$ we have $\langle \mathcal{O} \rangle \cap \langle \mathcal{O}' \rangle \in T$. Hence, since $q^n - 2 \geq 3$, we can find two distinct elementary pseudo-ovals containing $\{E, E_3\}$ that are contained in \mathcal{D} and have an element E_{10} and E_{11} respectively in common with \mathcal{O}' .

Let $\mathcal{S}_{\mathcal{O}'}$ be the Desarguesian $(n-1)$ -spread containing \mathcal{O}' . As E_{10} and E_{11} are elements of \mathcal{D} the same argument as before shows that all but one element of the Desarguesian

spread $\mathcal{D} \cap \langle E_{10}, E_{11} \rangle$ can be written as the intersection of $\langle E, P'' \rangle$ with $\langle E_{10}, E_{11} \rangle$ for some P'' in \mathcal{O}' . It follows that $\mathcal{S}_{\mathcal{O}'}$ contains E, E_1 and the $q^n + 1$ elements of $\mathcal{D} \cap \langle E_{10}, E_{11} \rangle$, hence, that $\mathcal{S}_{\mathcal{O}'}$ is contained in \mathcal{D} . In particular, the element E_9 is contained in \mathcal{D} , which implies that $\mathcal{E} \subset \mathcal{D}$ and so that \mathcal{E} is elementary and more specifically, a field reduced ovoid. \square

When \mathcal{E} is good at E and q is odd, by Corollary 10 all pseudo-ovals of \mathcal{E} containing E are pseudo-conics; we use this to obtain the following corollary. The same statement, where \mathcal{E} is an egg, was proven in [7, Theorem 3.2] using coordinates. For \mathcal{E} an egg, this was also shown in [16, Theorem 5.2.3] where a different proof was obtained independently, relying on a technical theorem concerning the \mathbb{F}_{q^n} -extension of the egg elements. We have now obtained a direct geometric proof.

Corollary 22. *A weak egg \mathcal{E} of $\text{PG}(4n - 1, q)$, q odd, $n > 1$, is classical if and only if it is good at an element E and there exists a $(3n - 1)$ -space, not containing E , with at least 5 elements of \mathcal{E} .*

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