A decomposition of the universal embedding space for the near polygon $\mathbb{H}_n$

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Abstract

Let $\mathbb{H}_n$, $n \geq 1$, be the near $2n$-gon defined on the 1-factors of the complete graph on $2n+2$ vertices, and let e denote the absolutely universal embedding of $\mathbb{H}_n$ into $\mathrm{PG}(W)$, where W is a $\frac{1}{n+2}\binom{2n+2}{n+1}$-dimensional vector space over the field $\mathbb{F}_2$ with two elements. For every point z of $\mathbb{H}_n$ and every $i \in \mathbb{N}$, let $\Delta_i(z)$ denote the set of points of $\mathbb{H}_n$ at distance i from z. We show that for every pair $\{x, y\}$ of mutually opposite points of $\mathbb{H}_n$, W can be written as a direct sum $W_0 \oplus W_1 \oplus \cdots \oplus W_n$ such that the following four properties hold for every $i \in \{0, \ldots, n\}$: (1) $\langle e(\Delta_i(x) \cap \Delta_{n-i}(y)) \rangle = \mathrm{PG}(W_i)$; (2) $\langle e\Big(\bigcup_{j \leq i} \Delta_j(x)\Big) \rangle = \mathrm{PG}(W_0 \oplus W_1 \oplus \cdots \oplus W_i)$; (3) $\langle e\Big(\bigcup_{j \leq i} \Delta_j(y)\Big) \rangle = \mathrm{PG}(W_{n-i} \oplus W_{n-i+1} \oplus \cdots \oplus W_n)$; (4) $\dim(W_i) = |\Delta_i(x) \cap \Delta_{n-i}(y)| = \binom{n}{i}^2 - \binom{n}{i-1} \cdot \binom{n}{i+1}$.

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1 Introduction

Let $\mathbb{H}_n$, $n \geq 1$, be the following point-line geometry:

• The points of $\mathbb{H}_n$ are the partitions of $\{1, 2, \ldots, 2n+2\}$ in $n+1$ subsets of size 2.

• The lines of $\mathbb{H}_n$ are the partitions of $\{1, 2, \ldots, 2n+2\}$ in $n-1$ subsets of size 2 and 1 subset of size 4.

• A point is incident with a line if and only if the partition corresponding to the point is a refinement of the partition corresponding to the line.

The point-line geometry $\mathbb{H}_n$, $n \geq 1$, is a so-called dense near polygon with three points per line. An alternative description of $\mathbb{H}_n$ can be given where the points are the 1-factors of a complete graph on $2n+2$ vertices. Indeed, there exists a natural bijective correspondence between the partitions of $\{1, 2, \ldots, 2n+2\}$ in $n+1$ subsets of size 2 and the 1-factors of the complete graph with vertex set $\{1, 2, \ldots, 2n+2\}$.

The near polygon $\mathbb{H}_n$, $n \geq 1$, is embeddable into a projective space and hence admits the so-called absolutely universal embedding.

For every two points x and y of $\mathbb{H}_n$ (i.e. partitions x and y of $\{1, \ldots, 2n+2\}$ in $n+1$ subsets of size 2), let $\Gamma_{x,y}$ denote the graph with vertex set $\{1, 2, \ldots, 2n+2\}$ and edge set $x \cup y$. Then the distance $\mathrm{d}(x, y)$ between x and y in the collinearity graph of $\mathbb{H}_n$ is equal to $n+1-N$, where N denotes the number of connected components of $\Gamma_{x,y}$. If x is a point of $\mathbb{H}_n$ and $i \in \mathbb{N}$, then $\Delta_i(x)$ denotes the set of points at distance i from x and $\Delta_i^*(x)$ the set of points at distance at most i from x.

A set S of points of $\mathbb{H}_n$ is called a *subspace* if every line of $\mathbb{H}_n$ which has at least two points in S has all its points in S. If the smallest subspace of $\mathbb{H}_n$ which contains a given set X of points coincides with the whole set of points of $\mathbb{H}_n$, then X is called a *generating set* of $\mathbb{H}_n$. In Blokhuis and Brouwer [1], it was mentioned that if x and y are two opposite points of $\mathbb{H}_n$ and if $C(x, y)$ denotes the union of all geodesics from x to y, then $C(x, y)$ is a generating set of $\mathbb{H}_n$ whose size is equal to the Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$. In the present paper, we refine this result in the following way.

Theorem 1.1 *Let x and y be two opposite points of the near polygon $\mathbb{H}_n$, $n \geq 1$, and put $X_i := \Delta_i(x) \cap \Delta_{n-i}(y)$, $i \in \{0, \ldots, n\}$. Then*

(1) $|X_i| = \binom{n}{i}^2 - \binom{n}{i-1} \cdot \binom{n}{i+1}$;

(2) $X_0 \cup X_1 \cup \cdots \cup X_n$ *is a generating set of* $\mathbb{H}_n$.

In the previous theorem and elsewhere in the paper, we have adopted the convention that $\binom{n}{j} = 0$ for every $n \in \mathbb{N}$ and every $j \in \mathbb{Z} \setminus \{0, \ldots, n\}$. Using Theorem 1.1, we are able to prove a decomposition theorem for the absolutely universal embedding of $\mathbb{H}_n$.

Theorem 1.2 *Let e denote the absolutely universal embedding of $\mathbb{H}_n$, $n \geq 1$, into $\mathrm{PG}(W)$, where W is a $\frac{1}{n+2}\binom{2n+2}{n+1}$-dimensional vector space over the field $\mathbb{F}_2$ with two elements. Then for every pair $\{x, y\}$ of mutually opposite points of $\mathbb{H}_n$, W can be written as a direct sum $W_0 \oplus W_1 \oplus \cdots \oplus W_n$ such that the following four properties hold for every $i \in \{0, \ldots, n\}$:*

(1) $\langle e(\Delta_i(x) \cap \Delta_{n-i}(y)) \rangle = \mathrm{PG}(W_i)$;

(2) $\langle e(\Delta_i^*(x)) \rangle = \mathrm{PG}(W_0 \oplus W_1 \oplus \cdots \oplus W_i)$;

(3) $\langle e(\Delta_i^*(y))\rangle = \mathrm{PG}(W_{n-i} \oplus W_{n-i+1} \oplus \cdots \oplus W_n)$;
(4) $\dim(W_i) = \binom{n}{i}^2 - \binom{n}{i-1} \cdot \binom{n}{i+1}$.

Remark A. In the literature, decomposition theorems for other projective embeddings of dense near polygons have been proved:

- the Grassmann embeddings of symplectic dual polar spaces [5, Theorem 1.1], see also [8] for another approach;
- the spin-embeddings of some dual polar spaces and the near polygons $\mathbb{H}_n$ and $\mathbb{I}_n$ [6, Theorem 1.7];
- the Grassmann embeddings of Hermitian dual polar spaces [7, Theorem 1.4].

Remark B. Theorems 1.1 and 1.2 can also be deduced from the decomposition theorem for the Grassmann embedding of the symplectic dual polar space $DW(2n-1,2)$. Such an approach would be highly artificial and not very elegant. Indeed, a proof of Theorem 1.1 which needs the introduction of a $2n$-dimensional vector space V equipped with a nondegenerate alternating bilinear form and technical computations in the exterior algebra of V is quite a detour. The approach discussed in the present paper avoids all this machinery. Notice that in the proof of Theorem 1.2, we introduce a symplectic dual polar space $DW(2n-1,2)$ but we only need to invoke some elementary properties of this dual polar space.

2 Preliminaries

2.1 A recursively defined series of numbers

In this section, we define in a recursive way numbers $f_n(k,l)$, $n \in \mathbb{N} \setminus \{0,1\}$ and $k,l \in \{0,\ldots,n\}$, and give a closed expression for these numbers.

The numbers $f_2(k,l)$, $k,l \in \{0,1,2\}$, are defined in the following table.

$f_2(k,l)$	$l=0$	$l=1$	$l=2$
$k=0$	1	1	0
$k=1$	0	1	1
$k=2$	0	1	0

For every $n \geq 3$ and $k,l \in \{0,\ldots,n\}$, we define

- If k is even and $l=n$, then we define $f_n(k,l) := 0$.
- If k is odd and $l=0$, then we define $f_n(k,l) := 0$.

- If $k = 0$ and $l \neq n$, then we define $f_n(k,l) := \sum_{i=0}^{n-1} f_{n-1}(i,l)$.
- If $k \neq 0$ is even and $l \neq n$, then we define $f_n(k,l) := \sum_{i=k-1}^{n-1} f_{n-1}(i,l)$.
- If k is odd and $l \neq 0$, then we define $f_n(k,l) := \sum_{i=k-1}^{n-1} f_{n-1}(i,l-1)$.

It was show in De Bruyn [5, Section 2] that for every $n \in \mathbb{N} \setminus \{0,1\}$ and all $k,l \in \{0,1,\dots,n\}$, we have

$$f_n(k,l) = \binom{n-1-\lfloor\frac{k}{2}\rfloor}{l-\lfloor\frac{k+1}{2}\rfloor} \cdot \binom{n-\lfloor\frac{k+1}{2}\rfloor}{l+\frac{(-1)^k-1}{2}} - \binom{n-1-\lfloor\frac{k}{2}\rfloor}{l-1-\lfloor\frac{k+1}{2}\rfloor} \cdot \binom{n-\lfloor\frac{k+1}{2}\rfloor}{l+\frac{(-1)^k+1}{2}}.$$

2.2 The big maxes of $\mathbb{H}_n$

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point x and every line L, there exists a unique point on L nearest to x. Here, distances are measured in the collinearity graph Γ of $\mathcal{S}$. If n is the diameter of Γ, then the near polygon is called a *near* $2n$*-gon*. A near 0-gon is just a point and a near 2-gon is a line. Near quadrangles are usually called generalized quadrangles (Payne and Thas [10]).

A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 from each other have at least two common neighbors. By Theorem 4 of Brouwer and Wilbrink [3], every two points of a dense near $2n$-gon at distance $\delta \in \{0,\dots,n\}$ from each other are contained in a unique convex sub-2δ-gon. These sub-2δ-gons are called *quads* if $\delta = 2$ and *maxes* if $\delta = n-1$. The existence of quads in a dense near polygon was already shown by Shult and Yanushka [12, Proposition 2.5].

A max M of a dense near polygon $\mathcal{S}$ is called *big* in $\mathcal{S}$ if every point x of $\mathcal{S}$ not contained in M is collinear with a necessarily unique point $\pi_M(x)$ of M. If M is big in $\mathcal{S}$ and x is a point of $\mathcal{S}$ not contained in M, then $\mathrm{d}(x,y) = 1 + \mathrm{d}(\pi_M(x), y)$ for every point y of M. If M is big in $\mathcal{S}$ and every line of $\mathcal{S}$ is incident with precisely three points, then a reflection $\mathcal{R}_M$ about M can be defined which is an automorphism of $\mathcal{S}$ (see [4, Theorem 1.11]). If $x \in M$, then we define $\mathcal{R}_M(x) := x$. If $x \notin M$, then $\mathcal{R}_M(x)$ denotes the unique point of the line $x\pi_M(x)$ different from x and $\pi_M(x)$. More information on dense near polygons can be found in the book [4].

Let $\mathbb{H}_n$, $n \geq 2$, be the dense near $2n$-gon defined on the partitions of $\{1,2,\dots,2n+2\}$ in $n+1$ subsets of size 2 (see Section 1). There exists a bijective correspondence between the big maxes of $\mathbb{H}_n$ and the subsets of size 2 of $\{1,2,\dots,2n+2\}$. If $\{i,j\}$ is a subset of size 2 of $\{1,2,\dots,2n+2\}$, then the set of all partitions P of $\{1,2,\dots,2n+2\}$ for which $\{i,j\} \in P$ is a big max $M[i,j]$ of $\mathbb{H}_n$. Conversely, every big max of $\mathbb{H}_n$ is obtained in this way. The point-line geometry $\widetilde{M}$ induced on a big max M (by the points and lines which are contained in it) is isomorphic to $\mathbb{H}_{n-1}$. Suppose now that $\{i_1, j_1\}$ and $\{i_2, j_2\}$ are two distinct subsets of size 2 of $\{1, 2, \ldots, 2n+2\}$. If $\{i_1, j_1\} \cap \{i_2, j_2\} = \emptyset$, then the big maxes $M[i_1, j_1]$ and $M[i_2, j_2]$ meet. If $\{i_1, j_1\} \cap \{i_2, j_2\}$ is a singleton, say $\{i_1\} = \{i_2\}$, then the reflection of $M[i_1, j_1]$ about $M[i_2, j_2] = M[i_1, j_2]$ is equal to the big max $M[j_1, j_2]$. More information about the near polygon $\mathbb{H}_n$ can be found in [4, Section 6.2].

2.3 The absolutely universal embedding of $\mathbb{H}_n$

By Ronan [11], every point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ with three points per line which is fully embeddable into a projective space admits the absolutely universal embedding which is obtained in the following way. Let V be a vector space over the field $\mathbb{F}_2$ with a basis B whose vectors are indexed by the elements of $\mathcal{P}$, say $B = \{\bar{e}_p \,|\, p \in \mathcal{P}\}$. Let W denote the subspace of V generated by all vectors $\bar{e}_{p_1} + \bar{e}_{p_2} + \bar{e}_{p_3}$, where $\{p_1, p_2, p_3\}$ is a line of $\mathcal{S}$. Then the map $p \in \mathcal{P} \mapsto \{\bar{e}_p + W, W\}$ defines a full embedding of $\mathcal{S}$ into the projective space $\mathrm{PG}(V/W)$. This full embedding is isomorphic to the so-called absolutely universal embedding of $\mathcal{S}$.

The absolutely universal embedding of the near polygon $\mathbb{H}_n$, $n \geq 1$, is described in Blokhuis and Brouwer [1, Section 3]. Let V be a $(2n+2)$-dimensional vector space over $\mathbb{F}_2$ with basis $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n+2}\}$. For every point $P = \{\{i_1, i_2\}, \{i_3, i_4\}, \ldots, \{i_{2n+1}, i_{2n+2}\}\}$ of $\mathbb{H}_n$, put $e(P)$ equal to the point $\langle (\bar{e}_{i_1} + \bar{e}_{i_2}) \wedge (\bar{e}_{i_3} + \bar{e}_{i_4}) \wedge \cdots \wedge (\bar{e}_{i_{2n+1}} + \bar{e}_{i_{2n+2}}) \rangle$ of $\mathrm{PG}(\bigwedge^{n+1} V)$. Then e defines a full embedding of $\mathbb{H}_n$ into a subspace of $\mathrm{PG}(\bigwedge^{n+1} V)$ of dimension $\frac{1}{n+2}\binom{2n+2}{n+1}$. This projective embedding is isomorphic to the absolutely universal embedding of $\mathbb{H}_n$.

3 Proof of Theorem 1.1

3.1 A generating set of points of $\mathbb{H}_n$

Suppose the points of $\mathbb{H}_n$, $n \geq 1$, are the 1-factors of the complete graph K_{2n+2}, and suppose the $2n+2$ vertices of K_{2n+2} are drawn as the vertices of a convex $(2n+2)$-gon $\mathbb{P}$ in the plane. Blokhuis and Brouwer [1] proved that the set Y^* of all 1-factors of K_{2n+2} without crossing edges is a generating set of $\mathbb{H}_n$. They also mentioned that the cardinality of Y^* is equal to the Catalan number $\frac{1}{n+2}\binom{2n+2}{n+1}$, and referred to van Lint [9, Section 3.1] for a proof of this fact.

In [1] another more geometric description of the generating set Y^* was given. Suppose x and y are two opposite vertices of the near polygon $\mathbb{H}_n$ such that x and y are alternating edges of $\mathbb{P}$. Then the above generating set Y^* of vertices coincides with the union $C(x, y)$ of all geodesics between x and y. The proof given in [1] seems however not to be valid. (The claims $c(x \cup z) + c(y \cup z) = n + 1 - cr(z)$ and $\mathrm{d}(x, z) + \mathrm{d}(y, z) = n - 1 + cr(z)$ on lines -15, -14 and -13 of page 300 do have counter examples; in fact, there might exist 1-factors z for which $cr(z) > n + 1$.) We will now give a proof of this claim since we will need it in the present paper.

Proposition 3.1 *The generating set Y^* is equal to $C(x, y)$.*

Proof. We start with the proof of two similar properties.

Claim I. *If $z \in C(x, y)$, then there exists an edge in z which is also an edge of x or y.*
PROOF. Since $\mathrm{d}(x, z) + \mathrm{d}(z, y) = n$, we have $\mathrm{d}(x, z) \leq \frac{n}{2}$ or $\mathrm{d}(y, z) \leq \frac{n}{2}$.

Suppose $k := \mathrm{d}(x, z) \leq \frac{n}{2}$ and let $x = z_0, z_1, \dots, z_k = z$ be a shortest path between x and z. The 1-factor x has $n + 1$ edges. Let N_i, $i \in \{0, \dots, k\}$, denote the number of edges of x which are also edges of z_i. Then $N_0 = n+1$ and $|N_i - N_{i+1}| \leq 2$ for every $i \in \{0, \dots, k - 1\}$. Hence, $N_k \geq N_0 - 2k \geq (n + 1) - 2 \cdot \frac{n}{2} = 1$. So, there is an edge in z which is also an edge of x.

In a similar way, one proves that if $\mathrm{d}(y, z) \leq \frac{n}{2}$, then there is an edge in z which is also an edge of y. (qed)

Claim II. *If $z \in Y^*$, then there is an edge of z which is also an edge of either x or y.*
PROOF. We define a distance function $dist(\cdot, \cdot)$ on the set of vertices of $\mathbb{P}$. If i_1 and i_2 are two vertices of $\mathbb{P}$, then $dist(i_1, i_2)$ is the smallest nonnegative integer k for which there exist $k + 1$ vertices $j_0, j_1, \dots, j_k$ of $\mathbb{P}$ satisfying (a) $j_0 = i_1$, (b) $j_k = i_2$, (c) j_{i-1}, j_i are neighboring vertices of $\mathbb{P}$ for every $i \in \{1, \dots, k\}$.

Now, let $\{i_1, i_2\}$ be an edge of z for which $dist(i_1, i_2)$ is as small as possible and suppose that $dist(i_1, i_2) > 1$. Let i_3 be a vertex of $\mathbb{P}$ which lies on a shortest path γ from i_1 to i_2 and let i_4 be the unique vertex of $\mathbb{P}$ such that $\{i_3, i_4\}$ is an edge of z. Since there are no crossing edges of z, also i_4 is contained on the path γ. It follows that $dist(i_3, i_4) < dist(i_1, i_2)$, contradicting the minimality of $dist(i_1, i_2)$. Hence, $dist(i_1, i_2) = 1$ and the edge $\{i_1, i_2\}$ of z is also an edge of either x or y. (qed)

We will now prove the proposition by induction on $n \geq 1$. Suppose first that $n = 1$. Label the vertices of $\mathbb{P}$ with the numbers 1, 2, 3 and 4 such that $x = \{\{1, 2\}, \{3, 4\}\}$ and $y = \{\{2, 3\}, \{1, 4\}\}$. One has $C(x, y) = Y^* = \{x, y\}$.

We will now suppose that $n \geq 2$. By Claims I and II, it suffices to prove that $z \in Y^* \Leftrightarrow z \in C(x,y)$ for 1-factors z which contain a given edge $\{i_1, i_2\}$ of either x or y. Without loss of generality, we may suppose that $\{i_1, i_2\}$ is an edge of x. Let i_0, i_3 denote the unique vertex of $\mathbb{P}$ such that $\{i_0, i_1\}$ and $\{i_2, i_3\}$ are edges of Y. Let K_{2n} denote the complete graph on the set of vertices of $\mathbb{P}$ distinct from i_1 and i_2 and let $\mathbb{H}_{n-1}$ denote the near polygon defined on the 1-factors of K_{2n}. Let x' denote the 1-factor of K_{2n} obtained from x by removing the edge $\{i_1, i_2\}$ and let y' denote the 1-factor of K_{2n} obtained from y by removing the edges $\{i_0, i_1\}, \{i_2, i_3\}$ and adding the edge $\{i_0, i_3\}$. Then x' and y' are opposite vertices of $\mathbb{H}_{n-1}$. For every 1-factor w of K_{2n}, let $\theta(w)$ denote the 1-factor of K_{2n+2} obtained from w by adding the edge $\{i_1, i_2\}$. Then θ defines an isomorphism between $\mathbb{H}_{n-1}$ and a big max M of $\mathbb{H}_n$. We have $\theta(x') = x$ and $\theta(y')$ is the unique point of M collinear with y. Moreover, $\mathrm{d}(y,u) = 1 + \mathrm{d}(\theta(y'), u)$ for every point u of M. The following should now be obvious:

(a) A point u of $\mathbb{H}_{n-1}$ lies on a shortest path between x' and y' if and only if $\theta(u')$ lies on a shortest path between x and y.

(b) By the induction hypothesis, a point u of $\mathbb{H}_{n-1}$ lies on a shortest path between x' and y' if and only if u, regarded as 1-factor of K_{2n}, has no crossing edges.

(c) u, regarded as a 1-factor of K_{2n} has no crossing edges if and only if the 1-factor $\theta(u)$ of K_{2n+2} has no crossing edges.

By (a), (b), (c) above, the statement $z \in Y^* \Leftrightarrow z \in C(x,y)$ holds for all 1-factors z which contain the edge $\{i_1, i_2\}$. This was precisely what we needed to prove. □

As mentioned above, Blokhuis and Brouwer [1] proved that the set Y^* is a generating set of $\mathbb{H}_n$. In view of Proposition 3.1, it is then clear that also the set $C(x,y)$ is a generating set of points of $\mathbb{H}_n$. This fact can also be shown in a direct way.

Proposition 3.2 *$C(x,y)$ is a generating set of $\mathbb{H}_n$.*

Proof. We will prove the proposition by induction on n. Obviously, the proposition holds if $n = 1$. So, we will suppose that $n \geq 2$. We will regard the points of $\mathbb{H}_n$ as partitions of $\{1, 2, \ldots, 2n+2\}$ in $n+1$ subsets of size 2. Without loss of generality, we may suppose that $x = \{\{1,2\},\{3,4\},\ldots,\{2n+1, 2n+2\}\}$ and $y = \{\{2,3\},\{4,5\},\ldots,\{2n+2, 1\}\}$. Let S denote the smallest subspace of $\mathbb{H}_n$ containing $C(x,y)$.

We will prove that all big maxes $M[i, i+1]$, $i \in \{1, \ldots, 2n+1\}$, are contained in S. If i is odd, then $x \in M[i, i+1]$ and $y \notin M[i, i+1]$. In this case, we define $x' := x$ and y' denotes the unique point of $M[i, i+1]$ collinear with y. If i is even, then $x \notin M[i, i+1]$ and $y \in M[i, i+1]$. In this case, we define $y' := y$ and x' denotes the unique point of $M[i, i+1]$ collinear with x. Then $C(x', y') \subseteq C(x, y)$. Since x' and y' are opposite points of $M[i, i+1] \cong \mathbb{H}_{n-1}$, the smallest subspace of $\mathbb{H}_n$ containing $C(x', y')$ coincides with $M[i, i+1]$ by the induction hypothesis. Hence, $M[i, i+1] \subseteq S$.

Notice that if i_1, i_2 and i_3 are three distinct elements of $\{1, 2, \ldots, 2n+2\}$ such that $M[i_1, i_2] \subseteq S$ and $M[i_1, i_3] \subseteq S$, then also $M[i_2, i_3] \subseteq S$ since $M[i_2, i_3]$ is the reflection of $M[i_1, i_3]$ about $M[i_1, i_2]$. By the previous paragraph it then follows that all big maxes $M[i, j]$, $i, j \in \{1, \ldots, 2n+2\}$ and $i \neq j$, are contained in S. Since every point of $\mathbb{H}_n$ is contained in a big max, $C(x, y)$ is a generating set of $\mathbb{H}_n$. □

The following proposition improves Claim I of Proposition 3.1.

Proposition 3.3 *Let $z \in C(x, y)$ and let E denote a set of $n+1$ consecutive edges of the polygon $\mathbb{P}$. Then there is an edge in z which is contained in E.*

Proof. We label the points of $\mathbb{P}$ by the numbers $1, 2, \ldots, 2n+2$, either clockwise or counterclockwise. Without loss of generality, we may suppose that $E = \{\{1, 2\}, \{2, 3\}, \ldots, \{n+1, n+2\}\}$. Let $\{j, i_j\}$ denote the edge of z containing the vertex with label $j \in \{1, \ldots, n+1\}$. If $i_j = j+1$ for a certain $j \in \{1, \ldots, n+1\}$, then we are done.

In the sequel, we suppose that $i_j \neq j+1$ for every $j \in \{1, \ldots, n+1\}$ and derive a contradiction. We prove by induction on $j \in \{1, \ldots, n+1\}$ that (a) $i_j > j+1$ and (b) $i_j < i_{j-1}$ if $j \neq 1$. Since $i_1 \neq 2$, these claims hold if $j = 1$. So, suppose $j \in \{2, \ldots, n+1\}$. Since $\{j-1, i_{j-1}\}$ and $\{j, i_j\}$ are non-crossing edges and $i_{j-1} > j$, we have that $i_j \in \{j+1, \ldots, i_{j-1}-1\}$. Since $i_j \neq j+1$, we have $i_j > j+1$ and $i_j < i_{j-1}$.

In particular, we have $i_{n+1} \leq i_1 - n \leq 2n+2-n = n+2$ and $i_{n+1} > n+2$, clearly a contradiction. □

3.2 The sizes of the sets $\Delta_i(x) \cap \Delta_{n-i}(y)$

Consider the near polygon $\mathbb{H}_n$, $n \geq 2$, whose points are the partitions of $\{1, 2, \ldots, 2n+2\}$ in $n+1$ subsets of size 2. Let x and y be two points of $\mathbb{H}_n$ at maximal distance from each other. Since the automorphism group of $\mathbb{H}_n$ acts transitively on the ordered pairs of opposite points of $\mathbb{H}_n$, we may without loss of generality suppose that $x = \{\{1, 2\}, \{3, 4\}, \cdots, \{2n+1, 2n+2\}\}$ and $y = \{\{2, 3\}, \{4, 5\}, \ldots, \{2n, 2n+1\}, \{2n+2, 1\}\}$. For every $i \in \{1, 2, \ldots, 2n+1\}$, we define $M_i := M[i, i+1]$. We also define $M_{2n+2} := M[1, 2n+2]$. We call $\{M_1, M_2, \ldots, M_{2n+2}\}$ the *nice set of big maxes of $\mathbb{H}_n$ induced by (x, y)*.

The following corollary is an immediate consequence of Proposition 3.3.

Corollary 3.4 *We have* $C(x,y) \subseteq M_1 \cup M_2 \cup \cdots \cup M_{n+1}$.

For all $k,l \in \{0,\ldots,n\}$, let $Y_n(k,l)$ denote the set of all points of $\Delta_l(x) \cap \Delta_{n-l}(y)$ which are contained in $M_{k+1} \setminus (M_1 \cup \cdots \cup M_k)$ and put $g_n(k,l) := |Y_n(k,l)|$. Notice that the sets $Y_n(k,l)$, $k,l \in \{0,\ldots,n\}$, are mutually disjoint. Notice also that since $M_k \cap M_{k+1} = \emptyset$, we also have $M_{k+1} \setminus (M_1 \cup \cdots \cup M_k) = M_{k+1} \setminus (M_1 \cup \cdots \cup M_{k-1})$. (Here, $M_1 \cup \cdots \cup M_k = \emptyset$ if $k = 0$ and $M_1 \cup \cdots \cup M_{k-1} = \emptyset$ if $k \in \{0,1\}$.) The following is an immediate consequence of Corollary 3.4.

Corollary 3.5 (1) *For every* $l \in \{0,\ldots,n\}$, *we have* $\Delta_l(x) \cap \Delta_{n-l}(y) = \bigcup_{0 \leq k \leq n} Y_n(k,l)$.

(2) $C(x,y) = \bigcup_{0 \leq k,l \leq n} Y_n(k,l)$.

Lemma 3.6 *We have* $g_2(0,0) = 1$, $g_2(0,1) = 1$, $g_2(0,2) = 0$, $g_2(1,0) = 0$, $g_2(1,1) = 1$, $g_2(1,2) = 1$, $g_2(2,0) = 0$, $g_2(2,1) = 1$ *and* $g_2(2,2) = 0$.

Proof. We have $x = \{\{1,2\},\{3,4\},\{5,6\}\}$, $y = \{\{2,3\},\{4,5\},\{1,6\}\}$, $M_1 = M[1,2]$, $M_2 = M[2,3]$ and $M_3 = M[3,4]$. It is straightforward to verify that

$$Y_2(0,0) = \{\{\{1,2\},\{3,4\},\{5,6\}\}\} = \{x\},$$

$$Y_2(0,1) = \{\{\{1,2\},\{4,5\},\{3,6\}\}\},$$

$$Y_2(1,1) = \{\{\{2,3\},\{5,6\},\{1,4\}\}\},$$

$$Y_2(1,2) = \{\{\{2,3\},\{4,5\},\{1,6\}\}\} = \{y\},$$

$$Y_2(2,1) = \{\{\{3,4\},\{1,6\},\{2,5\}\}\},$$

$$Y_2(0,2) = Y_2(1,0) = Y_2(2,0) = Y_2(2,2) = \emptyset.$$

The values of $g_2(k,l)$, $k,l \in \{0,1,2\}$, follow. □

In Lemmas 3.7, 3.8, 3.9 and 3.10 below, we suppose that $n \geq 3$ and that k and l are elements of the set $\{0,\ldots,n\}$.

Lemma 3.7 (1) *If* k *is even and* $l = n$, *then* $g_n(k,l) = 0$.

(2) *If* k *is odd and* $l = 0$, *then* $g_n(k,l) = 0$.

Proof. (1) If k is even, then the max M_{k+1} does not contain y. Hence, $g_n(k,l) = 0$ if k is even and $l = n$.

(2) If k is odd, then the max M_{k+1} does not contain x. Hence, $g_n(k,l) = 0$ if k is odd and $l = 0$. □

Lemma 3.8 *If $k = 0$ and $l \neq n$, then $g_n(k,l) = \sum_{i=0}^{n-1} g_{n-1}(i,l)$.*

Proof. The max M_1 contains the point x, but not the point y. Let y' denote the unique point of M_1 collinear with y. Then x and y' are opposite points of $\widetilde{M_1}$ and $\Delta_l(x) \cap \Delta_{n-l}(y) \cap M_1 = \Delta_l(x) \cap \Delta_{n-1-l}(y') \cap M_1$. By Corollary 3.5(1), the size of $\Delta_l(x) \cap \Delta_{n-1-l}(y') \cap M_1$ is equal to $\sum_{i=0}^{n-1} g_{n-1}(i,l)$. This equals the size $g_n(0,l)$ of $\Delta_l(x) \cap \Delta_{n-l}(y) \cap M_1$. $\square$

Lemma 3.9 *If $k \neq 0$ is even and $l \neq n$, then $g_n(k,l) = \sum_{i=k-1}^{n-1} g_{n-1}(i,l)$.*

Proof. The max M_{k+1} contains the point x, but not the point y. Let y' denote the unique point of M_{k+1} collinear with y. Then x and y' are opposite points of $\widetilde{M_{k+1}}$ and $\Delta_l(x) \cap \Delta_{n-l}(y) \cap M_{k+1} = \Delta_l(x) \cap \Delta_{n-1-l}(y') \cap M_{k+1}$. Suppose the points of the near polygon $\mathbb{H}_{n-1}$ are the partitions of $\{1,2,\ldots,2n+2\} \setminus \{k+1,k+2\}$ in n subsets of size 2. For every point u of $\mathbb{H}_{n-1}$, let $\theta(u)$ denote the partition of $\{1,2,\ldots,2n+2\}$ obtained from u by adding the subset $\{k+1,k+2\}$. Then θ defines an isomorphism between $\mathbb{H}_{n-1}$ and $\widetilde{M_{k+1}}$. We have

$$\theta^{-1}(x) = \{\{1,2\},\ldots,\{k-1,k\},\{k+3,k+4\},\ldots,\{2n+1,2n+2\}\},$$

$$\theta^{-1}(y') = \{\{2,3\},\ldots,\{k-2,k-1\},\{k,k+3\},\{k+4,k+5\},\ldots,\{2n+2,1\}\}.$$

Now, define the following maxes:

$$\begin{aligned} M'_i &:= M[i,i+1] \cap M_{k+1}, \quad && i \in \{1,\ldots,k-1\}; \\ M'_k &:= M[k,k+3] \cap M_{k+1}; && \\ M'_i &:= M[i+2,i+3] \cap M_{k+1}, \quad && i \in \{k+1,\ldots,2n-1\}; \\ M'_{2n} &:= M[1,2n+2] \cap M_{k+1}. && \end{aligned}$$

Then it is clear that $\{M'_1, M'_2, \ldots, M'_{2n}\}$ is the nice set of maxes of $\widetilde{M_{k+1}} \cong \mathbb{H}_{n-1}$ induced by (x,y'). The $g_n(k,l)$ points of $\Delta_l(x) \cap \Delta_{n-l}(y) \cap M_{k+1}$ which are not contained in $M_1 \cup M_2 \cup \cdots \cup M_k$ are precisely the points of $\Delta_l(x) \cap \Delta_{n-1-l}(y') \cap M_{k+1}$, which are not contained in $M'_1 \cup M'_2 \cup \cdots \cup M'_{k-1}$. By Corollary 3.5(1), there are $\sum_{i=k-1}^{n-1} g_{n-1}(i,l)$ such points. $\square$

Lemma 3.10 *If k is odd and $l \neq 0$, then $g_n(k,l) = \sum_{i=k-1}^{n-1} g_{n-1}(i,l-1)$.*

Proof. The max M_{k+1} contains the point y, but not the point x. Let x' denote the unique point of M_{k+1} collinear with x. Then x' and y are opposite points of $\widetilde{M_{k+1}}$ and $\Delta_l(x) \cap \Delta_{n-l}(y) \cap M_{k+1} = \Delta_{l-1}(x') \cap \Delta_{n-l}(y) \cap M_{k+1}$. Suppose the points of the near polygon $\mathbb{H}_{n-1}$ are the partitions of $\{1,2,\ldots,2n+2\} \setminus \{k+1,k+2\}$ in n subsets of size 2. For every point u

of $\mathbb{H}_{n-1}$, let $\theta(u)$ denote the partition of $\{1, 2, \ldots, 2n+2\}$ obtained from u by adding the subset $\{k+1, k+2\}$. Then θ defines an isomorphism between $\mathbb{H}_{n-1}$ and $\widetilde{M_{k+1}}$. We have

$$\theta^{-1}(x') = \{\{1,2\}, \ldots, \{k-2, k-1\}, \{k, k+3\}, \{k+4, k+5\}, \ldots, \{2n+1, 2n+2\}\},$$

$$\theta^{-1}(y) = \{\{2,3\}, \ldots, \{k-1, k\}, \{k+3, k+4\}, \ldots, \{2n+2, 1\}\}.$$

Now, define the following maxes:

$$\begin{array}{rcll} M'_i & := & M[i, i+1] \cap M_{k+1}, & i \in \{1, \ldots, k-1\}; \\ M'_k & := & M[k, k+3] \cap M_{k+1}; & \\ M'_i & := & M[i+2, i+3] \cap M_{k+1}, & i \in \{k+1, \ldots, 2n-1\}; \\ M'_{2n} & := & M[1, 2n+2] \cap M_{k+1}. & \end{array}$$

Then it is clear that $\{M'_1, M'_2, \ldots, M'_{2n}\}$ is the nice set of maxes of $\widetilde{M_{k+1}} \cong \mathbb{H}_{n-1}$ induced by (x', y). The $g_n(k, l)$ points of $\Delta_l(x) \cap \Delta_{n-l}(y) \cap M_{k+1}$ which are not contained in $M_1 \cup M_2 \cup \cdots \cup M_k$ are precisely the points of $\Delta_{l-1}(x') \cap \Delta_{n-l}(y) \cap M_{k+1}$ which are not contained in $M'_1 \cup M'_2 \cup \cdots \cup M'_{k-1}$. By Corollary 3.5(1), there are $g_n(k, l) = \sum_{i=k-1}^{n-1} g_{n-1}(i, l-1)$ such points. □

By Lemmas 3.6, 3.7, 3.8, 3.9, 3.10 and Section 2.1, we have:

Corollary 3.11 *For all $k, l \in \{0, 1, \ldots, n\}$, we have*

$$g_n(k,l) = \binom{n-1-\lfloor \frac{k}{2} \rfloor}{l - \lfloor \frac{k+1}{2} \rfloor} \cdot \binom{n - \lfloor \frac{k+1}{2} \rfloor}{l + \frac{(-1)^k - 1}{2}} - \binom{n-1-\lfloor \frac{k}{2} \rfloor}{l-1-\lfloor \frac{k+1}{2} \rfloor} \cdot \binom{n - \lfloor \frac{k+1}{2} \rfloor}{l + \frac{(-1)^k + 1}{2}}.$$

Proposition 3.12 *For every $l \in \{0, \ldots, n\}$, we have $|\Delta_l(x) \cap \Delta_{n-l}(y)| = \binom{n}{l}^2 - \binom{n}{l-1} \cdot \binom{n}{l+1}$.*

Proof. By Corollary 3.5(1), Lemma 3.8 and Corollary 3.11, we have

$$\begin{aligned} |\Delta_l(x) \cap \Delta_{n-l}(y)| &= \sum_{i=0}^{n} g_n(i, l) = g_{n+1}(0, l) \\ &= \binom{n}{l} \cdot \binom{n+1}{l} - \binom{n}{l-1} \cdot \binom{n+1}{l+1} \\ &= \binom{n}{l} \cdot \left[\binom{n+1}{l} - \binom{n}{l-1}\right] - \binom{n}{l-1} \cdot \left[\binom{n+1}{l+1} - \binom{n}{l}\right] \\ &= \binom{n}{l}^2 - \binom{n}{l-1} \cdot \binom{n}{l+1}. \end{aligned}$$

$\square$

Remarks. (1) We have $|C(x,y)| = \sum_{l=0}^{n} \binom{n}{l}^2 - \binom{n}{l-1} \cdot \binom{n}{l+1} = \sum_{l=0}^{n} \binom{n}{l} \cdot \binom{n}{n-l} - \sum_{l=0}^{n} \binom{n}{l-1} \cdot \binom{n}{n-l-1} = \binom{2n}{n} - \binom{2n}{n-2} = \frac{1}{n+2}\binom{2n+2}{n+1}$. It was already mentioned above that $Y^* = C(x,y)$ and $|Y^*| = \frac{1}{n+2}\binom{2n+2}{n+1}$.

(2) Notice that the conclusion of Proposition 3.12 is also valid for the near polygon $\mathbb{H}_1$ (which is a line with three points).

4 Proof of Theorem 1.2

Theorem 1.2 trivially holds if $n = 1$. So, we will suppose that $n \geq 2$.

Lemma 4.1 *Suppose e is the absolutely universal embedding of $\mathbb{H}_n$ into $\mathrm{PG}(W)$ and that x and y are two opposite points of $\mathbb{H}_n$. If W_i, $i \in \{0, \ldots, n\}$, is the subspace of W for which $\mathrm{PG}(W_i) = \langle e(\Delta_i(x) \cap \Delta_{n-i}(y))\rangle$, then*

(1) $W = W_0 \oplus W_1 \oplus \cdots \oplus W_n$,

(2) $\dim(W_i) = \binom{n}{i}^2 - \binom{n}{i-1} \cdot \binom{n}{i+1}$ *for every* $i \in \{0, \ldots, n\}$.

Proof. In Blokhuis and Brouwer [1], it as shown that $e(C(x,y))$ is an independent generating set of points of the projective space $\mathrm{PG}(W)$. The lemma then follows from Proposition 3.12. $\square$

Let V be a $(2n+2)$-dimensional vector space over $\mathbb{F}_2$ with basis $\{\bar{e}_1, \bar{e}_2, \ldots, \bar{e}_{2n+2}\}$. Put $\bar{e} = \bar{e}_1 + \bar{e}_2 + \cdots + \bar{e}_{2n+2}$. Let V' denote the subspace of V consisting of all vectors of the form $\sum_{i=1}^{2n+2} \lambda_i \bar{e}_i$ where $\lambda_1 + \lambda_2 + \cdots + \lambda_{2n+2} = 0$. If $\bar{x} = \sum_{i=1}^{2n+2} \lambda_i \bar{e}_i$ and $\bar{y} = \sum_{i=1}^{2n+2} \mu_i \bar{e}_i$ are two vectors of V', then we define $(\bar{x}, \bar{y}) = \sum_{i=1}^{2n+2} \lambda_i \mu_i$. Then $(\cdot,\cdot)$ is an alternating bilinear form on V' whose radical is equal to $\langle \bar{e} \rangle$. Let $DW(2n-1,2)$ denote the point-line geometry whose points, respectively lines, are the $(n+1)$-dimensional, respectively n-dimensional, subspaces of V' which are totally isotropic with respect to $(\cdot,\cdot)$, with incidence being reverse containment. Then $DW(2n-1,2)$ is a symplectic dual polar space of rank n. Let A and B denote the following mutually opposite points of $DW(2n-1,2)$:

$$A = \langle \bar{e}, \bar{e}_1 + \bar{e}_2, \bar{e}_3 + \bar{e}_4, \ldots, \bar{e}_{2n-1} + \bar{e}_{2n} \rangle,$$
$$B = \langle \bar{e}, \bar{e}_2 + \bar{e}_3, \bar{e}_4 + \bar{e}_5, \ldots, \bar{e}_{2n} + \bar{e}_{2n+1} \rangle.$$

We can write

$$\bar{e} \wedge \bigwedge^{n} V' = U_0 \oplus U_1 \oplus \cdots \oplus U_n, \qquad (1)$$

where U_i, $i \in \{0, \dots, n\}$, is the subspace of $\bigwedge^{n+1} V'$ generated by all vectors of the form $\bar{e} \wedge \bar{a}_1 \wedge \cdots \bar{a}_{n-i} \wedge \bar{b}_{n-i+1} \wedge \cdots \wedge \bar{b}_n$, where $\bar{a}_1, \dots, \bar{a}_{n-i} \in A$ and $\bar{b}_{n-i+1}, \dots, \bar{b}_n \in B$. For every point $p = \langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n+1} \rangle$ of $DW(2n-1, 2)$, let $f(p)$ denote the vector $\bar{v}_1 \wedge \bar{v}_2 \wedge \cdots \wedge \bar{v}_{n+1}$ of $\bigwedge^{n+1} V'$. The point $f(p)$ is independent from the generating set $\{\bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n+1}\}$ of p.

Suppose $p_1 = \langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n+1} \rangle$ is a point of $DW(2n-1, 2)$ at distance i from A and distance $n-i$ from B. Then $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n+1} \rangle$ intersects A in an $(n+1-i)$-dimensional subspace containing $\bar{e}$ and B in an $(i+1)$-dimensional subspace containing $\bar{e}$. So, without loss of generality, we may suppose that $\bar{v}_1 = \bar{e}$, $\langle \bar{v}_2, \dots, \bar{v}_{n+1-i} \rangle \subseteq A$ and $\langle \bar{v}_{n+2-i}, \dots, \bar{v}_{n+1} \rangle \subseteq B$. It is then clear that

$$f(p_1) \in U_i. \tag{2}$$

Suppose $p_2 = \langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n+1} \rangle$ is a point of $DW(2n-1, 2)$ at distance at most i from A. Then $\langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n+1} \rangle$ intersects A in a subspace of dimension at least $n+1-i$ which contains $\bar{e}$. So, without loss of generality, we may suppose that $\bar{v}_1 = \bar{e}$ and $\langle \bar{v}_2, \dots, \bar{v}_{n+1-i} \rangle \subseteq A$. It is then clear that

$$f(p_2) \in U_0 \oplus U_1 \oplus \cdots \oplus U_i. \tag{3}$$

Similarly, as in the previous paragraph, one shows that if $p_3 = \langle \bar{v}_1, \bar{v}_2, \dots, \bar{v}_{n+1} \rangle$ is a point of $DW(2n-1, 2)$ at distance at most i from B, then

$$f(p_3) \in U_{n-i} \oplus U_{n+1-i} \oplus \cdots \oplus U_n. \tag{4}$$

Now, suppose the points of $\mathbb{H}_n$ are the partitions of $\{1, 2, \dots, 2n+2\}$ in $n+1$ subsets of size 2. Let x and y be two opposite points of $\mathbb{H}_n$. Without loss of generality, we may suppose that $x = \{\{1,2\}, \{3,4\}, \dots, \{2n+1, 2n+2\}\}$ and $y = \{\{2,3\}, \{4,5\}, \dots, \{2n+2, 1\}\}$. For every point $\{\{i_1, i_2\}, \{i_3, i_4\}, \dots, \{i_{2n+1}, i_{2n+2}\}\}$ of $\mathbb{H}_n$, we define $\epsilon(p) = \langle \bar{e}_{i_1} + \bar{e}_{i_2}, \bar{e}_{i_3} + \bar{e}_{i_4}, \dots, \bar{e}_{i_{2n+1}} + \bar{e}_{i_{2n+2}} \rangle$. Then by Brouwer et al. [2, p. 356], ϵ is a full isometric embedding of $\mathbb{H}_n$ into $DW(2n-1, 2)$, i.e. ϵ is a map from the point set of $\mathbb{H}_n$ to the point set of $DW(2n-1, 2)$ which maps lines to lines and preserves the distances between points. Notice that $f \circ \epsilon$ is isomorphic to the absolutely universal embedding of $\mathbb{H}_n$. Also, $\epsilon(x) = A$ and $\epsilon(y) = B$. Theorem 1.2 is now a consequence of Lemma 4.1 and equations (1), (2), (3), (4).

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References

[1] A. Blokhuis and A. E. Brouwer. The universal embedding dimension of the near polygon on the 1-factors of a complete graph. *Des. Codes Cryptogr.* 17 (1999), 299–303.

[2] A. E. Brouwer, A. M. Cohen, J. I. Hall and H. A. Wilbrink. Near polygons and Fischer spaces. *Geom. Dedicata* 49 (1994), 349–368.

[3] A. E. Brouwer and H. A. Wilbrink. The structure of near polygons with quads. *Geom. Dedicata* 14 (1983), 145–176.

[4] B. De Bruyn. *Near polygons.* Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[5] B. De Bruyn. A decomposition of the natural embedding spaces for the symplectic dual polar spaces. *Linear Algebra Appl.* 426 (2007), 462–477.

[6] B. De Bruyn. The structure of the spin-embeddings of dual polar spaces and related geometries. *European J. Combin.* 29 (2008), 1242–1256.

[7] B. De Bruyn. On the Grassmann-embeddings of the hermitian dual polar spaces. *Linear Multilinear Algebra* 56 (2008), 665–677.

[8] B. De Bruyn. Some subspaces of the k-th exterior power of a symplectic vector space. *Linear Algebra Appl.* 430 (2009), 3095–3104.

[9] J. H. van Lint. *Combinatorial Theory Seminar, Eindhoven University of Technology.* Lecture Notes in Mathematics 382. Springer-Verlag, Berlin-New York, 1974.

[10] S. E. Payne and J. A. Thas. *Finite generalized quadrangles.* Second edition. EMS Series of Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2009.

[11] M. A. Ronan. Embeddings and hyperplanes of discrete geometries. *European J. Combin.* 8 (1987), 179–185.

[12] E. E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata* 9 (1980), 1–72.