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Convex Order Approximations in the case of Cash Flows of Mixed Signs

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Abstract

In Van Weert et al. (2010), results are obtained showing that, when allowing some of the cash flows to be negative, convex order lower bound approximations can still be used to solve general investment problems in a context of provisioning or terminal wealth. In this paper, a correction and further clarification of the reasoning of Van Weert et al. (2010) are given, thereby significantly expanding the scope of problems and cash flow patterns for which the terminal wealth or initial provision can be accurately approximated. Also an interval for the probability level is derived in which the quantiles of the lower bound approximation can be computed. Finally, it is shown how one can move from a context of provisioning of future obligations to a saving and terminal wealth problem by inverting the time axis.
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1 Introduction

In Van Weert et al. (2010), optimal portfolio selection problems for arbitrary cash flow patterns have been discussed. They allow for liabilities that can be both positive or negative, as opposed to Dhaene et al. (2005), where all liabilities have to be of the same sign. They generalize portfolio selection problems to the case where a minimal return requirement is imposed. The results that they

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propose are an extension of the solution of Vanduffel et al. (2005) to the more general context of provisioning and saving as described in Dhaene et al. (2005). However, the proof of the presented results contains an error. For example Lemma 1 can only be applied to partial sums of the form as in Vanduffel et al. (2005) constituting the function $f(p)$ in formula (16) of Van Weert et al. (2010). In this paper we will show how to split $f(p)$ in these building blocks and to derive the interval for p where $f(p) > 0$ implies $f'(p) > 0$. In a next step, we will enlarge this interval for p using the theory of zeros of general polynomials. In addition, we will formulate a sufficient condition for the main result to hold. This will be a slightly stronger condition on the signs and amounts of the cash flows but will be satisfied in many practical cases. We will illustrate our results with numerical examples. The framework of optimal portfolio selection in which we work is the same as in Dhaene et al. (2005) and Van Weert et al. (2010) and we refer to those papers for more details, notations and terminology.

2 Problem Description

To a series of future payments α_i at time i, $i = 0, 1, \ldots, n$, we attach the random variable S defined by

$$S = \sum_{i=0}^{n} \alpha_i e^{Z_i} \tag{1}$$

where the cash flows α_i of mixed signs are deterministic constants and the Z_i's are linear combinations of the components of the multivariate normal random vector $(Y_1, Y_2, \ldots, Y_n)$:

$$Z_i = \sum_{j=1}^{n} \lambda_{ij} Y_j.$$

It is well-known that the random variables Z_i are normally distributed with mean $\mathrm{E}[Z_i]$ and variance $\sigma^2_{Z_i}$. Depending on the choice of the Z_i's, the random variable S in (1) can be interpreted as a stochastic present value or stochastic accumulated value of the cash flows, in a model with multivariate normal logreturns. However, it is impossible to determine the distribution function of S analytically in closed form, because S is a sum of non-independent lognormal variables. We will use the convex upper and lower bounds for S satisfying $S^\ell \leq_{cx} S \leq_{cx} S^c$ as introduced in Dhaene et al. (2002b):

$$S^c = \sum_{i=0}^{n} \alpha_i e^{\mathrm{E}[Z_i] + \sigma_{Z_i} \Phi^{-1}(U)}, \tag{2}$$

$$S^\ell = \mathrm{E}\left[S \mid \Lambda\right] = \sum_{i=0}^{n} \alpha_i e^{\mathrm{E}[Z_i] + \frac{1}{2}(1 - r_i^2)\sigma^2_{Z_i} + r_i \sigma_{Z_i} \Phi^{-1}(U)}, \tag{3}$$

with U uniformly distributed on the unit interval, Φ the standard normal cumulative distribution function (cdf), r_i the correlations between the random variables Z_i and the conditioning random variable

$$\Lambda = \sum_{j=1}^{n} \beta_j Y_j. \tag{4}$$

If all the amounts α_i have the same sign, the upper bound (2) is a comonotonic sum, which implies that distortion risk measures related to these bounds can be obtained by simply summing the individual terms in the sum. For the lower bound S^ℓ (3) this is not a sufficient condition. In view of the factor r_i in the exponent, all r_i's should also have the same sign (not necessarily the same as that of the α_i's).

In case of payments α_i with changing signs, S^c is not a comonotonic sum. However, the upper bound approximation (2) can be adapted easily as follows:

$$S^c = \sum_{i=0}^{n} \alpha_i e^{\mathrm{E}[Z_i] + \mathrm{sign}(\alpha_i)\sigma_{Z_i}\Phi^{-1}(U)}, \tag{5}$$

with $\mathrm{sign}(x) = 1$ for $x > 0$ and $\mathrm{sign}(x) = -1$ for $x < 0$ (see, e.g., Dhaene et al. (2002a,b)). It holds that (5) is a comonotonic sum. However, the upper bound does in general not give a very accurate approximation of the distribution function of S; the accuracy of the lower bound (3) is usually much higher. For this lower bound, the problem is that there is no general rule to find a conditioning random variable Λ, leading to an accurate approximation of S, such that S^ℓ is a sum of non-decreasing (or non-increasing) functions of Λ and, hence, such that S^ℓ is a comonotonic sum in case the α_i's have changing signs. One needs to find a Λ such that the products $\alpha_i r_i$ have the same sign for all i. S^ℓ not being a comonotonic sum would imply that the additivity property would no longer hold and, hence, that distortion risk measures related to S^ℓ cannot be obtained by simply summing the individual terms in the sum, which would make the lower bound approximations useless in practice.

In this paper, however, we show that it is possible, under some mild conditions, to allow for more arbitrary cash flow patterns. We show that, although the lower bound approximations are not comonotonic sums anymore, allowing some of the cash flows to be negative does not necessarily imply that the convex order lower bound approximations cannot be used. As a result we significantly expand the scope of problems and cash flow patterns for which the quantiles can be accurately approximated.

3 Savings and Terminal Wealth

3.1 General results

In this section we consider a terminal wealth problem: we determine how periodic amounts should be invested in order to reach some target capital at a predetermined future time n. We consider a set of deterministic amounts

$\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ with $n \geq 1$. The conditions under which our main result holds, require α_0 to be positive. However, we do not impose a priori a sign condition on the other amounts α_l with $l \in \{1, \ldots, n-1\}$ which can take positive or negative values. This series of payments will be interpreted in a first approach as a combination of series of positive payments (savings) followed by negative ones (consumptions). Hence this is a generalization of the so-called "saving-consumption" problem described in Vanduffel et al. (2005) where only one of such series is taken into account.

We recall some notations and terminology based on Vanduffel et al. (2005) and Van Weert et al. (2010). We assume that the return on the account is generated by a Brownian motion process. Let V_k denote the *surplus* at time k. By convention, the surplus at time k has to be understood as the surplus just after saving or withdrawal. Starting from the initial value $V_0 = \alpha_0$, the surplus V_k available at time k is given by the following recursive relation:

$$V_k = V_{k-1} e^{Y_k} + \alpha_k, \qquad k = 1, \ldots, n-1. \tag{6}$$

The surplus at time n is then equal to $V_n = V_{n-1} e^{Y_n}$. Solving recursion (6), we can rewrite V_k in the form of (1) as

$$V_k = \sum_{l=0}^{k} \alpha_l e^{Z_{l,k}}, \qquad k = 0, \ldots, n-1, \tag{7}$$

with $Z_{l,k} = \sum_{j=l+1}^{k} Y_j$, for $l = 0, \ldots, k$. By convention $\sum_{j=k+1}^{k} Y_j = 0$. The surplus at time n equals

$$V_n = \sum_{l=0}^{n-1} \alpha_l e^{Z_{l,n}}, \qquad \text{with } Z_{l,n} = \sum_{j=l+1}^{n} Y_j.$$

Our goal is to determine the distribution of the final surplus V_n. However this surplus can become negative, which would imply shortselling of units of the investment portfolio. To avoid this, we limit our study to the distribution of the *terminal wealth* W_n, which is defined as:

$$W_n = \max[V_n, 0]. \tag{8}$$

As explained in the previous section, we focus on the convex order lower bound (3), which we denote here as V_n^ℓ. We approximate the distribution of the terminal wealth W_n by $W_n^\ell = \max[V_n^\ell, 0]$.

Choosing Λ such that the variance of V_n^ℓ is maximized and hence as close as possible to $\mathrm{Var}(V_n)$, results in the optimal conditioning random variable Λ of the form (4), with coefficients β_j equal to, see Dhaene et al. (2005):

$$\beta_j = \sum_{l=0}^{j-1} \alpha_l e^{(n-l)\mu}, \tag{9}$$

for $j = 1, \ldots, n$, with μ the drift of the yearly logreturns Y_j.

From (3) we find that the random variable V_n^ℓ with Λ chosen as (4) with coefficients (9) equals in distribution

$$V_n^\ell \stackrel{d}{=} \sum_{l=0}^{n-1} \alpha_l e^{(n-l)\mu - \frac{1}{2} r_l^2 (n-l)\sigma^2 + r_l \sigma \sqrt{n-l}\Phi^{-1}(U)}, \tag{10}$$

with $U \sim \mathcal{U}(0,1)$, Φ the standard normal cdf and σ the standard deviation of the normally distributed random variables Y_j. The correlation coefficients r_l are given by

$$r_l = \frac{\mathrm{Cov}\left(Z_{l,n}, \Lambda\right)}{\sigma_{Z_{l,n}} \sigma_\Lambda} = \frac{\sum_{j=l+1}^n \beta_j}{\sqrt{n-l}\sqrt{\sum_{j=1}^n \beta_j^2}}, \qquad l = 0, \ldots, n-1. \tag{11}$$

In the remainder of this section we use the notation f for the following function:

$$f(p) = \sum_{l=0}^{n-1} \alpha_l e^{(n-l)(\mu - \frac{1}{2} r_l^2 \sigma^2) + r_l \sigma \sqrt{n-l}\Phi^{-1}(p)}, \ \ p \in (0,1). \tag{12}$$

Combining (8) and (10) leads to $V_n^\ell \stackrel{d}{=} f(U)$ and $W_n^\ell \stackrel{d}{=} \max[f(U), 0]$. Further, we assume that there are $m > 0$ series of negative cash flows:

$$\begin{cases} \alpha_{k_1} < 0, \ldots, \alpha_{k_1+j_1} < 0 & k_1 > 0, \, j_1 \geq 0; \\ \vdots & \\ \alpha_{k_m} < 0, \ldots, \alpha_{k_m+j_m} < 0 & k_m > k_{m-1} + j_{m-1}, \, j_m \geq 0, \end{cases} \tag{13}$$

with $k_m + j_m \leq n-1$. All other cash flows are assumed to be positive. Then, we can rewrite (12) as the following combination:

$$f(p) = \sum_{i=1}^m f_i(p) + f_{m+1}(p), \tag{14}$$

where

$$\begin{aligned} f_i(p) = & \sum_{l=k_{i-1}+j_{i-1}+1}^{k_i-1} \alpha_l e^{(n-l)(\mu - \frac{1}{2} r_l^2 \sigma^2) + r_l \sigma \sqrt{n-l}\Phi^{-1}(p)} \\ & - \sum_{l=k_i}^{k_i+j_i} |\alpha_l| e^{(n-l)(\mu - \frac{1}{2} r_l^2 \sigma^2) + r_l \sigma \sqrt{n-l}\Phi^{-1}(p)} \end{aligned} \tag{15}$$

for $i = 1, \ldots, m$, with $k_0 + j_0 + 1 = 0$, and

$$f_{m+1}(p) = \sum_{l=k_m+j_m+1}^{n-1} \alpha_l e^{(n-l)(\mu - \frac{1}{2} r_l^2 \sigma^2) + r_l \sigma \sqrt{n-l}\Phi^{-1}(p)} \geq 0, \tag{16}$$

being zero when $k_m + j_m = n - 1$. In other words, we divide (12) into separate sums f_i, where each f_i for $i = 1, \ldots, m$ represents a series of positive cash flows, followed by a series of negative cash flows to which the result of Vanduffel et al. (2005) can be applied. We will state this result applied to the functions f_i and combine it to arrive at a result for the function f as given by (12) or (14).

Lemma 1 *1. Let f_i be defined by* (15) *and β_j by* (9). *If $\beta_j > 0$ for $j = k_{i-1} + j_{i-1} + 2, \ldots, n$, then there exists a probability level $p_i \in (0, 1)$ such that*

$$f_i(p) > 0 \implies f_i'(p) > 0 \quad \text{for } p \in (p_i, 1). \tag{17}$$

When $k_m + j_m < n-1$, we also have that $f_{m+1}(p) > 0$ implies $f_{m+1}'(p) > 0$ for all $p \in (0, 1)$.

2. Let f be defined by (14) *and β_j by* (9). *If $\beta_j > 0$ for $j = 1, \ldots, n$, then $f(p) > 0$ implies $f'(p) > 0$ for $p \in (\max_{i \in \{1,\ldots,m\}} p_i, 1)$, with the p_i determined in* (17).

Proof.

1. Since $\beta_j > 0$ for $j = k_{i-1} + j_{i-1} + 2, \ldots, n$ it follows from (11) that $r_l > 0$ for $l = k_{i-1} + j_{i-1} + 1, \ldots, k_i + j_i$ and, hence, that

$$\lim_{p \to 0} f_i(p) = 0 \quad \text{and} \quad \lim_{p \to 1} f_i(p) = +\infty.$$

   Furthermore, we may apply Lemma 1 of Vanduffel et al. (2005) to f_i which implies that for those $p \in (0, 1)$ for which $f_i(p) \geq 0$, also $f_i'(p) > 0$. It is clear that once the continuous function f_i becomes positive in a $p \in (0, 1)$ it is increasing and will not drop below zero again. We may therefore conclude that there exists a value $p_i \in (0, 1)$ such that the implication (17) holds.

2. From (14) it is clear that when all terms $f_i(p)$ and $f_{m+1}(p)$ are positive and increasing then also the sum $f(p)$ will be. In view of (17) this will be satisfied for p in the stated interval.

■

In what follows we will enlarge the interval for p by lowering the lower bound $\max_{i \in \{1,\ldots,m\}} p_i$. Hereto we make the change of variables:

$$x = e^{\sigma \Phi^{-1}(p)} \text{ with } x \in (0, +\infty) \tag{18}$$

in the function $f(p)$ (14):

$$f(p) = h(x) = \sum_{i=1}^{m} h_i(x) + h_{m+1}(x) \tag{19}$$

with according to (15) and (16)

$$h_i(x) = \sum_{l=\ell_i}^{k_i-1} \alpha_{i,l} x^{r_l\sqrt{n-l}} - \sum_{l=k_i}^{u_i} |\alpha_{i,l}| x^{r_l\sqrt{n-l}} \tag{20}$$

$$h_{m+1}(x) = \sum_{l=\ell_m}^{n-1} \alpha_{m+1,l} x^{r_l\sqrt{n-l}}, \tag{21}$$

where we introduced the short hand notations:

$$\ell_i = k_{i-1} + j_{i-1} + 1,\ u_i = k_i + j_i,\ \alpha_{i,l} = \alpha_l e^{(n-l)(\mu - \frac{1}{2} r_l^2 \sigma^2)}.$$

Lemma 2 *1. Let h_i be defined by (20) and β_j by (9). If $\beta_j > 0$ for $j = k_{i-1}+j_{i-1}+2, \ldots, n$, then there exist an $x_i \in [0, +\infty)$ and an $x_i' \in [0, +\infty)$ with $x_i' \leq x_i$ such that*

$$h_i(x) > 0 \implies h_i'(x) > 0 \quad \text{for } x \in (x_i, +\infty) \tag{22}$$

$$h_i'(x) > 0 \implies h_i''(x) > 0 \quad \text{for } x \in (x_i', +\infty). \tag{23}$$

When $k_m+j_m < n-1$, we also have that $h_{m+1}(x) > 0$ implies $h_{m+1}'(x) > 0$ and $h_{m+1}''(x) > 0$ for all $x \in (0, +\infty)$.

2. Let h be defined by (19) and β_j by (9). If $\beta_j > 0$ for $j = 1, \ldots n$, then $h(x) > 0$ and $h'(x) > 0$ for $x \in (x^\star, +\infty)$ with $x^\star = \max(x_{\max}, x'_{\max}) \leq \max_{i=1,\ldots,m} x_i$ where $x_{\max}$ corresponds to the largest zero of h, and $x'_{\max}$ to the largest zero of h', and where the x_i are determined in (22).

3. Let f be defined by (12) and β_j by (9). If $\beta_j > 0$ for $j = 1, \ldots, n$, then there exists a $p^\star = \Phi\left(\frac{1}{\sigma}\log x^\star\right)$, with $x^\star$ determined in assertion 2, such that $f(p) > 0$ and $f'(p) > 0$ for $p \in (p^\star, +\infty)$.

Proof.

1. We note that the coefficients $\alpha_{i,l}$ in (20) have the same sign pattern as the original coefficients α_l. Also, we note that the first order derivative functions h_i' are of the same form as h_i:

$$h_i'(x) = \sum_{l=\ell_i}^{k_i-1} \alpha_{i,l} r_l \sqrt{n-l}\, x^{r_l\sqrt{n-l}-1} - \sum_{l=k_i}^{u_i} |\alpha_{i,l}| r_l \sqrt{n-l}\, x^{r_l\sqrt{n-l}-1}.$$

The coefficients $\alpha_{i,l} r_l \sqrt{n-l}$ still have the same sign pattern when all r_l are positive which is satisfied by the assumption that $\beta_j > 0$ for all $j \in \{1, \ldots, n\}$. Hence, a reasoning as in the proof of Lemma 1 of Vanduffel et al. (2005) and of Lemma 1 above applied to h_i as well as to h_i' leads

to (22) and (23). Thus the functions h_i are strictly increasing and convex from x_i' on, but positive only from $x_i \geq x_i'$ on.
The result for h_{m+1} is clear from the relation of (16) and (21) through (18).

2. From (19), it is clear that $h(x) > 0$ implies $h'(x) > 0$ for $x \in [\max_{i=1,\dots,m} x_i, +\infty)$. To determine $x^\star \leq \max_{i=1,\dots,m} x_i$ we note that $h(x)$ and $h'(x)$ are generalized polynomials to which Descartes' rule of signs applies, see Jameson (2006) and references therein, when the exponents $r_l\sqrt{n-l}$ are listed in descending order. This is satisfied when the sequence $(r_l\sqrt{n-l})_l$ is decreasing in l or equivalently, in view of (11), when $\beta_j > 0$ for all $j \in \{1,\dots,n\}$ (in particular, $\beta_1 = \alpha_0 e^{n\mu} > 0$ requires $\alpha_0 > 0$ as mentioned at the beginning of this section). Further we find that $r_0\sqrt{n} > 1$ since $\left(\sum_{j=1}^n \beta_j\right)^2 > \sum_{j=1}^n \beta_j^2$ under the assumption for the β_j's. Hence we obtain $\lim_{x\to+\infty} h(x) = \lim_{x\to+\infty} h'(x) = \text{sign}(\alpha_0)\infty = +\infty$. Both h and h' cannot have more zeros than the number of sign changes of the sequence $\alpha_0,\dots,\alpha_{n-1}$, and the number of zeros of h in the interval $(0,+\infty)$ is less than or equal to the number of zeros of h' on $(0,+\infty)$ plus one (taking the order of the zeros into account), see Jameson (2006). Therefore, it is clear that beyond the largest zero $x_{\max}$ of h, the function h is strictly positive but can still decrease and increase again before going to infinity. Similarly for h', beyond its largest zero $x'_{\max}$, h' is strictly positive. Thus, both h and h' are strictly positive for $x \in (\max(x_{\max}, x'_{\max}), +\infty)$. Is $x^\star = \max(x_{\max}, x'_{\max}) \leq \max_{i=1,\dots,m} x_i$? Adding h_1 and h_2 produces a zero in the interval $(\min(x_1,x_2), \max(x_1,x_2))$ and by induction we arrive at $x_{\max} \leq \max_{i=1,\dots,m} x_i$. Analogously, we obtain for the zeros of h' and h_i' that $x'_{\max} \leq \max_{i=1,\dots,m} x_i'$ and $x'_{\max} \leq \max_{i=1,\dots,m} x_i$ since $x_i' \leq x_i$ for all i.

3. Applying the chain rule when taking the first order derivative of f in (19), making use of (18), we find
$$f'(p) = h'(x)x\frac{\sigma}{\varphi(p)},$$
with φ the density function of a standard normal random variable, which implies that also $f'(p)$ and $h'(x)$ have the same sign for x and p related by (18).

∎

Remark The functions h_i (20) are also generalized polynomials with one sign change in the coefficients and thus by Descartes' sign rule have at most one zero $x_i > 0$ (besides zero itself). Indeed, since the functions h_i are strictly convex increasing, once they crossed the x-axis, they cannot return to zero again.

We recall that V_n^ℓ is not a comonotonic sum: Lemma 2 only states that the total sum $V_n^\ell \overset{d}{=} f(U)$ is a non-decreasing function of one random variable U (for realizations of U at least equal to $p^\star$); the separate terms in the sum are *not* all non-decreasing functions of U. However, Lemma 2 implies that the lower bound approximation can still be used, as the quantiles of V_n^ℓ can easily be determined. This result, which is the main result of this section, is stated in the following theorem:

Theorem 1 *If the conditioning random variable* Λ *is chosen as in* (4) *with coefficients* β_j *given by* (9), *and if the surplus* V_l *in* (6) *satisfies*

$$E[V_l] > 0, \quad l = 0, \dots, n-1, \tag{24}$$

then the quantiles of W_n^ℓ *are given by*

$$Q_p[W_n^\ell] = \max[f(p), 0] = f(p), \quad p^\star < p < 1, \tag{25}$$

where $p^\star$ *is determined in Lemma 2 and* $f(p)$ *is defined by* (12). *The distribution function of* W_n^ℓ *follows from*

$$f(F_{W_n^\ell}(x)) = x, \quad x \geq Q_{p^\star}[W_n^\ell]. \tag{26}$$

Proof. Define the (left-continuous) function g on the interval $(0, 1)$ as

$$g(p) = \begin{cases} f(p) & p^\star \leq p < 1, \\ f(p^\star) & 0 < p < p^\star. \end{cases}$$

We recall from Van Weert et al. (2010) that for $l = 0, \dots, n-1$

$$\mathrm{E}[V_l] = e^{-(n-l)\mu}\beta_{l+1}. \tag{27}$$

Hence, by condition (24) we can apply Lemma 2 which implies that the function g is non-decreasing. As stated in Vanduffel et al. (2005), the quantiles of W_n^ℓ, for $p \geq p^\star$, can easily be determined analytically by (25) and the distribution function of W_n^ℓ from (26), where, in the present case, x has to be at least equal to $Q_{p^\star}[W_n^\ell]$ according to Lemma 2. ∎

Remarks Combining (27) and (9), we can rewrite the average surplus as a polynomial in x with $x = e^\mu > 0$:

$$\mathrm{E}[V_l] = \sum_{k=0}^{l} \alpha_k e^{(l-k)\mu} = \sum_{k=0}^{l} \alpha_k x^{l-k}. \tag{28}$$

From Descartes' rule of signs it is known that the number of positive roots of this polynomial in x is either equal to the number of sign variations in the coefficients or is less than this number by an even integer. Thus for given cash flows α_l, $l = 0, \dots, n-1$ it is not possible to rewrite the conditions (24) as the single condition (23) on μ in Van Weert et al. (2010).
However, it is possible to find a lower bound $\mu^\star$ for μ such that for $\mu > \mu^\star$ we are sure that conditions (24) are satisfied. Since $\alpha_0 > 0$ the polynomial (28)

in x will go to $+\infty$ when x tends to infinity. Denoting the largest zero of the polynomial (28) by $x_{l,\max}$, this polynomial will be increasing for $x > x_{l,\max}$. Since $x = e^{\mu}$ we obtain that for μ larger than

$$\mu^\star = \max_{l=0,\dots,n-1}\left(\max\left\{\mu \mid \mu > 0 \text{ and } \sum_{k=0}^{l}\alpha_k e^{(l-k)\mu} = 0\right\}\right) \tag{29}$$

conditions (24) will be satisfied. This is a sufficient condition.
Further, we note that from the recursion

$$\mathrm{E}[V_l] = x\mathrm{E}[V_{l-1}] + \alpha_l, l = 1,\dots,n-1, \text{ with } \mathrm{E}[V_0] = \alpha_0$$

it follows that we only have to impose conditions (24) when adding a negative coefficient α_l, thus from the set (13) or for $l \in \{0, k_1, \dots, k_1 + j_1, \dots, k_m, \dots, k_m + j_m\}$ as was also noted in section 2.2.2 from Van Weert et al. (2010).

3.2 Sufficient condition

We derive a sufficient condition imposed on the cash flows so that the $\beta_j > 0$ for all j under the realistic assumption that $\mu > 0$ and hence Theorem 1 holds. It is not a necessary condition, since there exist sequences of cash flows that will not satisfy that condition while all the β_j's are strictly positive, as we will see in the numerical examples. In a first lemma we prove the following general result.

Lemma 3 *Suppose we have deterministic cash flows $\alpha_0,\dots,\ \alpha_{n-1}$ such that*

$$\sum_{k=0}^{l}\alpha_k \geq 0$$

for $l = 0,\dots,n-1$. Suppose $\{\gamma_l\}_{l=0,\dots,n-1}$ is a sequence of positive numbers, which is strictly decreasing in l. Then

$$\sum_{k=0}^{l}\alpha_k\gamma_k > 0 \tag{30}$$

for $l = 0,\dots,n-1$.

Proof. Consider the series of negative cash flows (13). All other cash flows are assumed to be positive. For $l = 0,\dots,k_1 - 1$ it is straightforward that $\sum_{k=0}^{l}\alpha_k\gamma_k > 0$. Next, we look at the first series of negative cash flows, $\alpha_{k_1},\dots,\alpha_{k_1+j_1}$. We know that

$$\sum_{k=0}^{k_1-1}\alpha_k \geq -\sum_{k=k_1}^{k_1+j_1}\alpha_k.$$

This implies we can choose $\alpha_k^{(1)}$ such that $0 \le \alpha_k^{(1)} \le \alpha_k$ for $k = 0, \dots, k_1 - 1$ and

$$\sum_{k=0}^{k_1-1} \alpha_k^{(1)} = -\sum_{k=k_1}^{k_1+j_1} \alpha_k. \tag{31}$$

Since the terms γ_k are strictly decreasing in k, we find that

$$\begin{aligned}\sum_{k=0}^{k_1-1} \alpha_k^{(1)} \gamma_k &> \left(\sum_{k=0}^{k_1-1} \alpha_k^{(1)}\right) \gamma_{k_1} \\ &= \left(-\sum_{k=k_1}^{k_1+j_1} \alpha_k\right) \gamma_{k_1} > -\sum_{k=k_1}^{k_1+j_1} \alpha_k \gamma_k,\end{aligned}$$

which means, taking into account that $\gamma_k > 0$ for all k:

$$\sum_{k=0}^{k_1+j_1} \alpha_k \gamma_k \ge \sum_{k=0}^{k_1-1} \alpha_k^{(1)} \gamma_k + \sum_{k=k_1}^{k_1+j_1} \alpha_k \gamma_k > 0. \tag{32}$$

Note that, following a similar reasoning, we obtain that $\sum_{k=0}^{l} \alpha_k \gamma_k > 0$ for all $l \in \{k_1, \dots, k_1 + j_1\}$.

Now consider the second series of negative cash flows, $\alpha_{k_2}, \dots, \alpha_{k_2+j_2}$. Using (31) and the fact that $\sum_{k=0}^{k_2+j_2} \alpha_k \ge 0$, we know that we can choose $\alpha_k^{(2)}$ such that $0 \le \alpha_k^{(2)} \le \alpha_k - \alpha_k^{(1)}$ for $k = 0, \dots, k_1 - 1$, $0 \le \alpha_k^{(2)} \le \alpha_k$ for $k = k_1 + j_1 + 1, \dots, k_2 - 1$ and

$$\sum_{k=0}^{k_1-1} \alpha_k^{(2)} + \sum_{k=k_1+j_1+1}^{k_2-1} \alpha_k^{(2)} = -\sum_{k=k_2}^{k_2+j_2} \alpha_k.$$

Since the terms γ_k are decreasing in k, we find that

$$\begin{aligned}&\sum_{k=0}^{k_1-1} \alpha_k^{(2)} \gamma_k + \sum_{k=k_1+j_1+1}^{k_2-1} \alpha_k^{(2)} \gamma_k \\ &> \left(\sum_{k=0}^{k_1-1} \alpha_k^{(2)} + \sum_{k=k_1+j_1+1}^{k_2-1} \alpha_k^{(2)}\right) \gamma_{k_2} \\ &= \left(-\sum_{k=k_2}^{k_2+j_2} \alpha_k\right) \gamma_{k_2} > -\sum_{k=k_2}^{k_2+j_2} \alpha_k \gamma_k,\end{aligned}$$

which means

$$\sum_{k=0}^{k_1-1} \alpha_k^{(2)} \gamma_k + \sum_{k=k_1+j_1+1}^{k_2-1} \alpha_k^{(2)} \gamma_k + \sum_{k=k_2}^{k_2+j_2} \alpha_k \gamma_k > 0. \tag{33}$$

Adding (32) and (33) and taking the ranges for $\alpha_l^{(1)}$ and $\alpha_l^{(2)}$ into account, we have

$$\sum_{k=0}^{k_2+j_2} \alpha_k \gamma_k \geq \sum_{k=0}^{k_1-1} (\alpha_k^{(1)} + \alpha_k^{(2)}) \gamma_k + \sum_{k=k_1}^{k_1+j_1} \alpha_k \gamma_k$$
$$+ \sum_{k=k_1+j_1+1}^{k_2-1} \alpha_k^{(2)} \gamma_k + \sum_{k=k_2}^{k_2+j_2} \alpha_k \gamma_k > 0.$$

Following a similar reasoning, it is clear that $\sum_{k=0}^{l} \alpha_k \gamma_k > 0$ for all $l \in \{k_2, \ldots, k_2 + j_2\}$.

Repeating this reasoning for the remaining negative cash flows, we find the stated result (30). ∎

In the following theorem we use Lemma 3 to show that requiring the total amount of savings to be positive at any time is a sufficient condition for the coefficients β_j to be strictly positive when the drift μ of the yearly logreturns is positive (which is a realistic assumption):

Theorem 2 *Suppose we have deterministic cash flows $\alpha_0, \ldots, \alpha_{n-1}$ such that*

$$\sum_{k=0}^{l} \alpha_k \geq 0 \tag{34}$$

for $l = 0, \ldots, n-1$. Then $\beta_j > 0$ or equivalently $E[V_{j-1}] > 0$ for $j = 1, \ldots, n$, with β_j defined by (9) *with positive μ and $E[V_{j-1}]$ related to β_j according to* (27)*.*

Proof. Recall that

$$\beta_j = \sum_{l=0}^{j-1} \alpha_l e^{(n-l)\mu}$$

for $j = 1, \ldots, n$. The exponential terms $e^{(n-l)\mu}$ are clearly strictly decreasing in l for positive μ. Therefore, applying Lemma 3, we immediately find that $\beta_j > 0$ for $j = 1, \ldots, n$. The result for $\mathrm{E}[V_{j-1}]$ then follows from (27). ∎

Next, we show that when requiring the total amount of savings to be positive at any time, it is possible to construct an analytical expression for the probability level $p_{\min}$ such that the function f (12) is positive and increasing in p for $p \in (p_{\min}, 1)$.

Theorem 3 *Suppose we have deterministic cash flows $\alpha_0, \ldots, \alpha_{n-1}$ such that*

$$\sum_{k=0}^{l} \alpha_k \geq 0, \tag{35}$$

for $l = 0, \ldots, n-1$ and let μ be positive. Then it follows that $f(p) > 0$ and $f'(p) > 0$ for $p_{min} < p < 1$, with $f(p)$ defined by (12) *and*

$$
\begin{aligned}
&p_{\min} \\
&= \max_{l=0,\ldots,n-1} \left\{ \Phi\left(\frac{\frac{1}{2}\sigma^2\left((n-l)r_l^2 - (n-l-1)r_{l+1}^2\right) - \mu}{\sigma\left(r_l\sqrt{n-l} - r_{l+1}\sqrt{n-l-1}\right)} \right) \right\}.
\end{aligned}
\tag{36}
$$

Proof. As seen in Vanduffel et al. (2005), by application of the chain rule, we find for $p \in (0,1)$ that

$$
\begin{aligned}
f'(p) = \frac{1}{\varphi(\Phi^{-1}(p))} \sum_{l=0}^{n-1} &\alpha_l r_l \sigma \sqrt{n-l} \\
&\times e^{(n-l)\mu - \frac{1}{2} r_l^2 (n-l)\sigma^2 + r_l \sigma \sqrt{n-l}\ \Phi^{-1}(p)},
\end{aligned}
\tag{37}
$$

with r_l given by (11). First of all note that $\frac{1}{\varphi(\Phi^{-1}(p))}$ is a positive constant. Theorem 2 implies that $r_l > 0$ for all l, and that the sequence $\{r_l\sigma\sqrt{n-l}\}$ is strictly decreasing in l. Now look at the exponential terms in (12) and (37). Two consecutive terms are strictly decreasing if:

$$
\begin{aligned}
&e^{(n-l)\mu - \frac{1}{2} r_l^2 (n-l)\sigma^2 + r_l \sigma \sqrt{n-l}\Phi^{-1}(p)} \\
&\quad > e^{(n-l-1)\mu - \frac{1}{2} r_{l+1}^2 (n-l-1)\sigma^2 + r_{l+1} \sigma \sqrt{n-l-1}\Phi^{-1}(p)} \\
\Leftrightarrow &(n-l)\mu - \frac{1}{2} r_l^2 (n-l)\sigma^2 + r_l \sigma \sqrt{n-l}\Phi^{-1}(p) \\
&\quad > (n-l-1)\mu - \frac{1}{2} r_{l+1}^2 (n-l-1)\sigma^2 \\
&\qquad + r_{l+1}\sigma\sqrt{n-l-1}\Phi^{-1}(p) \\
\Leftrightarrow &\mu - \frac{1}{2}\sigma^2\left((n-l)r_l^2 - (n-l-1)r_{l+1}^2\right) \\
&\quad + \sigma\left(r_l\sqrt{n-l} - r_{l+1}\sqrt{n-l-1}\right)\Phi^{-1}(p) > 0 \\
\Leftrightarrow &p > \Phi\left(\frac{\frac{1}{2}\sigma^2\left((n-l)r_l^2 - (n-l-1)r_{l+1}^2\right) - \mu}{\sigma\left(r_l\sqrt{n-l} - r_{l+1}\sqrt{n-l-1}\right)} \right).
\end{aligned}
$$

From this last expression we see that if $p > p_{\min}$, the exponential factors in (12) and (37) are strictly decreasing in l, for all l. As a consequence, using Lemma 3, we see that, if (35) is satisfied, both $f(p)$ and its derivative $f'(p)$ are strictly positive for all $p > p_{\min}$. ∎

Although the minimal probability level $p_{\min}$ given by (36) does not have an interpretation, its value can easily be determined numerically. Following a similar reasoning as in the proof of Theorem 3, it can be seen that $f(\frac{1}{2}) \geq 0$ and $f'(p) > 0$ for all $\frac{1}{2} \leq p < 1$, which implies that $p_{\min}$ is in general lower than $\frac{1}{2}$. Indeed, for $p \geq \frac{1}{2}$, we have that $\Phi^{-1}(p) > 0$. As a consequence, the exponential terms in (12) and (37) can in this case easily be seen to be strictly decreasing in l for all reasonable combinations of μ and σ[1].

In practical situations, $p_{\min}$ turns out to be significantly lower than $\frac{1}{2}$, and often even close to zero. In general, we can conclude that the fewer negative cash flows there are, and, moreover, the later they occur in time, the lower $p_{\min}$ will be. Also, higher values of μ, and lower values of σ lead to lower values of $p_{\min}$, see (36). We refer to the following section for some numerical illustrations where we compare $p^\star$, $p_{\min}$ and $\max_{i=1,\dots,m} p_i$.

It is clear that in practical situations, when working in a saving environment, conditions (34), which state that the total amount of cash saved to the account should be non-negative at any time, will often be satisfied. Practical situations where sporadic negative payments occur, exist. For instance, when determining the liabilities of a pension fund, outgoing and incoming cash flows are typically compared. It may happen that in some years the incoming cash flows are larger than the outgoing ones, leading to negative liabilities in those years.

Theorem 1 and 3 are generalizations of the main result of Vanduffel et al. (2005), as in our case the sign of the cash flows is allowed to change several times, and an addition to the results in Van Weert et al. (2010). Note that in the case of a "saving-consumption" plan, Theorem 1 reduces to the result of Vanduffel et al. (2005), since only the *average final* surplus has to be non-negative for (25) to hold.

3.3 Numerical illustration

The accuracy and speed of the lower bound approximation was confirmed by numerical illustrations in section 2.2.2 of Van Weert et al. (2010), by comparing with results obtained through simulation. We recall that compared to simulation, the analytical approximations are significantly less time-consuming. As a consequence, the analytical approach allows us for example to optimize over the whole spectrum of investment portfolios, whereas when using simulation the analysis is typically restricted to a subset of the admissible portfolios. Also, the analytical approach allows us to consider a high number of assets or asset classes without significantly increasing the computational complexity. Here, we will focus on the interval of the probability level in which the approximation is valid. We will compare the lower bounds $p^\star$, $p_{\min}$ and $\max_{i=1,\dots,m} p_i$.

We consider the constant savings and consumptions setting of section 2.2.2 and example 1 in Van Weert et al. (2010). Suppose we have a fixed yearly income $\alpha(>0)$ but also a fixed liability of one every five years over a period of

[1] For the terms $(n-l)\mu - \frac{1}{2}r_l^2(n-l)\sigma^2$ to be positive for all l, it is sufficient to require that $\mu - \frac{1}{2}\sigma^2 \geq 0$. For these terms to be decreasing in l, a further restriction has to be made, which unfortunately is hard to quantify. However, it can be seen numerically that $\mu - \frac{1}{2}\sigma^2\varepsilon \geq 0$ must hold, for an ε sufficiently small such that the condition is satisfied for realistic choices of μ and σ.

25 years. Then, the deterministic cash flow stream equals

$$\alpha_l = \begin{cases} \alpha - 1 & \text{if } l = 5k, k = 1, \ldots, 5 \\ \alpha & \text{otherwise}, 0 \leq l < 26. \end{cases}$$

with the cash flows α_{5k} being negative, when $\alpha < 1$. As shown in Van Weert et al. (2010) conditions (24) will be satisfied when the yearly income α satisfies

$$\alpha > \alpha^\star = \frac{1 - e^{25\mu}}{1 - e^{26\mu}} \frac{1 - e^{\mu}}{1 - e^{5\mu}}. \tag{38}$$

As for conditions (24), conditions (34) will be fulfilled when $\sum_{l=0}^{5k} \alpha_l \geq 0$ for $k = 1, \ldots, 5$ as these are the only years in which negative cash flows are involved. These latter conditions are equivalent to

$$\begin{aligned} &(5k+1)\alpha - k \geq 0, \ k = 1, \ldots, 5 \\ &\Leftrightarrow \alpha \geq \alpha_{\max} = \max_{k=1,\ldots,5} \frac{k}{5k+1} = \frac{5}{26} \approx 0.1923. \end{aligned}$$

A simple calculation shows that $\alpha_{\max} > \alpha^\star$ for $\mu > 0$.
For different combinations of μ and σ, and for different values of α we will compare $p^\star$, $\max_{i=1,\ldots,5} p_i$ and $p_{\min}$. We will report also the values of $p'_{\max}$, $p_{\max}$ and $\max_{i=1,\ldots,5} p'_i$. Note that it has only sense to report the value of $p_{\min}$ when $\alpha > \frac{5}{26} \approx 0.1923$. Further, the value of $\alpha^\star$ for $\mu = 0.07$ and $\mu = 0.10$ is according to (38) 0.1591 and 0.1455 respectively. For values of α less than $\alpha^\star$ the entries in the table are also empty. From the numerical results in Table 1 we can conclude that when α increases $p^\star$ decreases to become nearly zero for values of α larger then 0.30. This means that in such case the convex lower bound can be used for all relevant probability levels. We observe that $p_{\min}$ decreases when μ increases or σ decreases, as could be seen from the expression (36). In this example, $p'_{\max}$ is smaller than $p_{\max}$ such that $p^\star$ equals $p_{\max}$. As proven in Lemma 2 we find that $p^\star \leq \max_{i=1,\ldots,5} p_i$. We also observe that $p_{\min}$ is too high compared to $p^\star$.

In view of our comment that a lower bound $\mu^\star$ given by (29) on the drift μ is a sufficient condition for the conditions (24) to hold, relation (35) in the application to optimal portfolio selection in Van Weert et al. (2010) should be replaced by

$$\{\underline{\pi} \mid \mu(\underline{\pi}) > \mu^\star\} \subset \Theta = \{\underline{\pi} \mid \mathrm{E}[V_i(\underline{\pi})] > 0; \ \ i = 0, \ldots, n-1\}.$$

Similarly, relation (45) in Van Weert et al. (2010) should be adapted in the numerical illustration, however without any consequences for the validity of the reported results. The value 0.0242 of $\mu^\star$ in this example coincides precisely with the one obtained by (29).

In the next section we will have a closer look at the reserving problem discussed in section 2.3 of Van Weert et al. (2010).

α	(μ, σ)	$p'_{\max}$	$\max_{i=1,\ldots,5} p'_i$	$\mathbf{p}^\star = \mathbf{p}_{\max}$	$\max_{i=1,\ldots,5} p_i$	$p_{\min}$
0.15	(0.07, 0.15)	–	–	–	–	–
	(0.10, 0.15)	9.9189E-4	0.0077	**0.5487**	0.9844	–
	(0.10, 0.20)	0.0188	0.0491	**0.6235**	0.9875	–
0.16	(0.07, 0.15)	3.2287E-4	0.0090	**0.6199**	0.9976	–
	(0.10, 0.15)	6.3914E-4	0.0165	**0.3341**	0.7040	–
	(0.10, 0.20)	0.0164	0.0840	**0.4743**	0.7332	–
0.18	(0.07, 0.15)	1.0018E-5	0.0107	**0.1896**	0.4802	–
	(0.10, 0.15)	3.7136E-7	0.0012	**0.0361**	0.1846	–
	(0.10, 0.20)	3.2536E-4	0.0231	**0.1384**	0.3454	–
0.20	(0.07, 0.15)	5.0928E-15	2.2757E-4	**0.0113**	0.0635	0.1721
	(0.10, 0.15)	3.5659E-18	4.6831E-6	**5.2586E-4**	0.0067	0.0477
	(0.10, 0.20)	2.7000E-10	0.0012	**0.0129**	0.0599	0.1883
0.25	(0.07, 0.15)	1.3768E-64	3.2973E-14	**1.9944E-8**	8.2588E-4	0.0642
	(0.10, 0.15)	2.4859E-76	2.0219E-17	**2.4694E-11**	1.3823E-10	0.0093
	(0.10, 0.20)	1.5218E-43	1.0606E-9	**8.6283E-7**	2.8758E-6	0.0835
0.30	(0.07, 0.15)	3.0374E-95	3.4301E-29	**1.2646E-16**	1.1496E-15	0.0516
	(0.10, 0.15)	1.0147E-112	2.2990E-34	**2.6483E-21**	1.4794E-20	0.0052
	(0.10, 0.20)	0.0	3.2296E-19	**1.8026E-12**	3.2804E-12	0.0510

Table 1: Minimal probability level

4 Provisions for future obligations

In this section we will correct the reasoning of Van Weert et al. (2010) leading to the main result. Further we show how this provisioning problem can be transformed into a saving and consumption problem by inverting the time axis.

4.1 General results

We shortly recall the problem description and some notations from Dhaene et al. (2005) and Van Weert et al. (2010). For the provisioning problem we consider a sequence of deterministic obligations $\alpha_1, \ldots, \alpha_n$ due at time $1, \ldots, n$ respectively. In order to be able to meet these obligations a provision has to be set up at time zero. The obligations can take positive or negative values, except for α_n which has to be positive in view of the imposed conditions in the main result.

We consider the stochastically discounted value R_k at time k of all future obligations from time k on:

$$R_l = \sum_{k=l+1}^{n} \alpha_k e^{Z_{l,k}}, \quad l = 0, \ldots, n-1, \tag{39}$$

with $Z_{l,k} = -\sum_{j=l+1}^{k} Y_j$, for $k = l+1, \ldots, n$. The goal is to approximate the distribution function of R_0 which, however, can become negative. Therefore, we consider the stochastic provision S_0 available at time zero defined as:

$$S_0 = \max[R_0, 0]. \tag{40}$$

We note that (39) is of the general form (1). Again, we will focus on the lower bound approximation, denoted as R_0^ℓ and S_0^ℓ respectively, and obtained by conditioning on a random variable Λ of the form (4) with coefficients β_j equal to, see Dhaene et al. (2005):

$$\beta_j = -\sum_{k=j}^{n} \alpha_k e^{k(-\mu+\sigma^2)}, \tag{41}$$

for $j = 1, \ldots, n$ with μ the drift and σ the standard deviation of the yearly logreturns Y_j. This leads to the lower bound approximation R_0^ℓ:

$$R_0^\ell \stackrel{d}{=} \sum_{j=1}^{n} \alpha_j e^{-j\mu+(1-\frac{1}{2}r_j^2)j\sigma^2+r_j\sqrt{j}\sigma\Phi^{-1}(U)}, \tag{42}$$

with $U \sim \mathcal{U}(0,1)$ and Φ the standard normal cdf. The correlation coefficients r_j are given by

$$r_j = \frac{-\sum_{k=1}^{j} \beta_k}{\sqrt{j}\sqrt{\sum_{k=1}^{n} \beta_k^2}}, \quad j = 1, \ldots, n. \tag{43}$$

In view of (40) and (42) we study the function

$$f(p) = \sum_{j=1}^{n} \alpha_j e^{-j\mu+(1-\frac{1}{2}r_j^2)l\sigma^2+r_j\sqrt{j}\sigma\Phi^{-1}(p)}, \; p \in (0,1). \tag{44}$$

so that $S_0^\ell \stackrel{d}{=} \max[f(U), 0]$. Therefore, we change from the variable p to x, (18), and introduce the function

$$h(x) = \sum_{l=0}^{n-1} a_{n-l} x^{r_{n-l}\sqrt{n-l}}, \quad x \in (0, +\infty), \tag{45}$$

with $a_{n-l} = \alpha_{n-l} e^{-(n-l)\mu+(1-\frac{1}{2}r_{n-l}^2)(n-l)\sigma^2}$ having the same sign pattern as α_{n-l} for $l = 0, \ldots, n-1$. Note that we changed the running variable j to $n-l$ going from f to h such that the function h takes the form of a generalized polynomial with exponents $r_{n-l}\sqrt{n-l}$ listed in descending order when the sequence $(r_{n-l}\sqrt{n-l})_l$ is decreasing. We state the analogue of Lemma 2.

Lemma 4 *1. Let h be defined by (45) and β_j by (41). If $\beta_j < 0$ for $j = 1, \ldots, n$, then $h(x) > 0$ and $h'(x) > 0$ for $x \in (x^\star, +\infty)$ with $x^\star = \max(x_{\max}, x'_{\max})$, where $x_{\max}$ stands for the largest zero of h and $x'_{\max}$ for the largest zero of h'.*

2. Let f be defined by (44) and β_j by (41). If $\beta_j < 0$ for $j = 1, \ldots, n$, then there exists a $p^\star = \Phi\left(\frac{1}{\sigma}\log x^\star\right)$, with $x^\star$ determined in assertion 1, such that $f(p) > 0$ and $f'(p) > 0$ for $p \in (p^\star, +\infty)$.

Proof. The reasoning is similar to the proof of Lemma 2 and based on an application of Descartes' rule of sign to the generalized polynomials $h(x)$ and $h'(x)$ with

$$h'(x) = \sum_{l=0}^{n-1} a_{n-l} x^{r_{n-l}\sqrt{n-l}-1} r_{n-l}\sqrt{n-l}, \quad x \in (0, +\infty),$$

where in view of (43) the sequence $(r_{n-l}\sqrt{n-l})_l$ is decreasing in l when $\beta_j < 0$ for all $j \in \{1, \ldots, n\}$, which in turn implies that $r_{n-l} > 0$ for all $l \in \{0, \ldots, n-1\}$. In particular $\beta_n = -\alpha_n e^{n(-\mu+\sigma^2)} < 0$ requires $\alpha_n > 0$. Further, since $\left(-\sum_{j=1}^n \beta_j\right)^2 > \sum_{j=1}^n \beta_j^2$, it holds that $r_n\sqrt{n} > 1$ under the assumption for the β_j's. ■

The main result of this section is stated in the following theorem.

Theorem 4 *If the conditioning random variable* Λ *is chosen as in* (4) *with coefficients* β_j *given by* (41), *and if the functions* R_l (39) *satisfy*

$$E[R_l] > 0, \quad l = 0, \ldots, n-1, \tag{46}$$

then the quantiles of S_0^ℓ *are given by*

$$Q_p[S_0^\ell] = \max[f(p), 0] = f(p), \quad p^\star < p < 1, \tag{47}$$

where $p^\star$ *is determined in Lemma 4 and* $f(p)$ *is defined by* (44). *The distribution function of* S_0^ℓ *follows from*

$$f(F_{S_0^\ell}(x)) = x, \quad x \geq Q_{p^\star}[S_0^\ell]. \tag{48}$$

Proof. Recalling from Van Weert et al. (2010) that for $l = 0, \ldots, n-1$

$$E[R_l] = -e^{l(-\mu+\sigma^2)}\beta_{l+1}, \tag{49}$$

requiring (46) is equivalent to the condition on the β_l's in Lemma 4 which can be applied along the similar lines as in the proof of Theorem 1, leading to the stated result. ■

To conclude, we show how the reserving problem can be translated to a terminal wealth setting by inverting the time axis such that the results of section 3.1 (all symbols of that section are here denoted by a tilde) can be applied instead of those stated here. We carry out the following substitutions for $l = 0, \ldots, n-1$:

$$\begin{aligned}
\tilde{\alpha}_l &= \alpha_{n-l} \\
\tilde{\mu} &= -\mu + \sigma^2 \\
\tilde{Y}_l &= -Y_{n-l+1} \sim \mathcal{N}(\tilde{\mu} - \frac{1}{2}\sigma^2, \sigma^2).
\end{aligned}$$

Next, we can derive the following relations for $l = 0, \ldots, n-1$:

$$\begin{aligned}
\tilde{\beta}_{l+1} &= -\beta_{n-l} \\
\tilde{r}_l &= r_{n-l} \\
\tilde{Z}_{l,k} &= Z_{n-k,n-l}, \;\; l \le k \\
\tilde{V}_n &= R_0 \\
\mathrm{E}[\tilde{V}_l] &= \alpha_{n-l} + \mathrm{E}[R_{n-l}] \\
\tilde{f}(U) &\stackrel{d}{=} \tilde{V}_n^\ell = R_0^\ell \stackrel{d}{=} f(U).
\end{aligned}$$

Since in general $\tilde{\mu}$ will be negative, Theorem 2 will not hold. Thus $\sum_{k=0}^{l} \tilde{\alpha}_k = \sum_{i=n-l}^{n} \alpha_i \ge 0$ for all $l = 0, \ldots, n-1$ or, equivalently, $\sum_{k=j}^{n} \alpha_k \ge 0$ for all $j = 1, \ldots, n$ does not imply that $\mathrm{E}[\tilde{V}_{j-1}] > 0$. In fact, it is the converse. Conditions (46) will imply a condition on the obligations.

Theorem 5 *Suppose we have deterministic obligations $\alpha_1, \ldots, \alpha_n$ such that conditions (46) hold. If $\mu - \sigma^2 > 0$, it holds that*

$$\sum_{k=j}^{n} \alpha_k \ge 0 \tag{50}$$

for $j = 1, \ldots, n$.

Proof. Combining (49) with (41) and changing the running variable we obtain:

$$\mathrm{E}[R_l] = \sum_{t=0}^{n-l-1} \alpha_{n-t} e^{-(n-l-t)(\mu-\sigma^2)}.$$

Denote $\alpha_{n-t} e^{-(n-l-t)(\mu-\sigma^2)} = a_t$. Since $\sum_{t=0}^{n-l-1} a_t \ge 0$ we can apply Theorem 2 with $\gamma_t = e^{(n-l-t)(\mu-\sigma^2)}$ which is positive and strictly decreasing in t for positive $\mu - \sigma^2$. This leads for all $l = 0, \ldots, n-1$ to

$$\sum_{t=0}^{n-l-1} \alpha_{n-t} \ge 0 \;\; \Leftrightarrow \;\; \sum_{k=l+1}^{n} \alpha_k \ge 0.$$

■

Note, however, that when conditions (46) hold the reasoning of the proof of Theorem 3 can be applied when $\mu - \sigma^2$ is positive. Thus we can state the following result.

Theorem 6 *Suppose we have deterministic obligations $\alpha_1, \ldots, \alpha_n$ such that conditions (46) hold and let $\mu - \sigma^2$ be positive. Then, $f(p) > 0$ and $f'(p) > 0$ for $p_{min} < p < 1$, with $f(p)$ defined by (44) and*

$$\begin{aligned}
&p_{\min} \\
&= \max_{l=0,\ldots,n-1} \left\{ \Phi\left(\frac{\frac{1}{2}\sigma^2 \left((n-l) r_{n-l}^2 - (n-l-1) r_{n-l-1}^2 \right) + \mu - \sigma^2}{\sigma \left(r_{n-l}\sqrt{n-l} - r_{n-l-1}\sqrt{n-l-1} \right)} \right) \right\}.
\end{aligned}$$

Proof. Since the conditions (46) hold, relation (49) implies that all $\beta_j < 0$, which is equivalent to $\tilde{\beta}_j > 0$ for all $j = 1, \ldots, n$. Hence $\tilde{r}_l\sqrt{n-l} > 0$ is strictly decreasing in l. On the other hand by Theorem 5 we have that $\sum_{t=0}^{n-l-1} \tilde{\alpha}_t \geq 0$ holds for all $l = 0, \ldots, n-1$. In this way all elements of the proof of Theorem 3 are available so that a similar reasoning leads to

$$p_{\min} = \max_{l=0,\ldots,n-1}\left\{\Phi\left(\frac{\frac{1}{2}\sigma^2\left((n-l)\tilde{r}_l^2 - (n-l-1)\tilde{r}_{l+1}^2\right) - \tilde{\mu}}{\sigma\left(\tilde{r}_l\sqrt{n-l} - \tilde{r}_{l+1}\sqrt{n-l-1}\right)}\right)\right\}.$$

Carrying out the substitution mentioned above gives the stated expression for $p_{\min}$. ■

5 Conclusion

We corrected the reasoning that led to the results in Van Weert et al. (2010) which show that when allowing some of the cash flows to be negative, convex order lower bound approximations can still be used. In particular we showed these results for the choice (4) of the conditioning random variable Λ when the cash flows are such that all expected surpluses after saving or withdrawal are strictly positive. Further we proved that imposing the stronger condition (34) on the cash flows is a sufficient condition for these expected surpluses to be strictly positive. These results significantly expand the scope of problems and cash flow patterns for which the quantiles of the terminal wealth can be accurately approximated. In addition, we derived an interval for the probability level in which the quantiles of the lower bound approximation can be computed. Further, we showed how by an inversion of the time axis the provisioning of future obligations can be transformed into the savings and terminal wealth problem.

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