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Authors: Hans Vernaeve and Jasson Vindas

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CHARACTERIZATION OF DISTRIBUTIONS HAVING A VALUE AT A POINT IN THE SENSE OF ROBINSON

HANS VERNAEVE AND JASSON VINDAS

ABSTRACT. We characterize Schwartz distributions having a value at a single point in the sense introduced by means of nonstandard analysis by A. Robinson. They appear to be continuous functions in a neighborhood of the point. This characterization improves a result by P. Loeb which assumes the everywhere existence of point values.

1. Introduction

In [10, §5.3], A. Robinson initiated the use of nonstandard analysis in the theory of Schwartz distributions. Among other things, he introduces nonstandard representatives of a Schwartz distribution and, by means of an infinitesimal property of the representatives, he introduces a notion of point value of a distribution.

A natural question to raise is how Robinson's notion of point value is related to the classical definition of point value in the sense of Łojasiewicz [2, 5, 6]. Through investigation of this question, we arrived at a characterization which is the main result of this paper: a distribution has a value at $x_0 \in \Omega$ in the sense of Robinson iff it is a continuous function in a neighborhood of x_0. Our characterization substantially improves an earlier result of P. Loeb [4] which has to assume the everywhere existence of Robinson point values (cf. section 3).

Nonstandard analysis has proved useful to study algebras of generalized functions [7, 8]. Recently, tools inspired by nonstandard analysis have systematically been introduced in the nonlinear theory of generalized functions [9, 12]. We hope that the current paper may provide new insights into regularity theory in these algebras, which is of much relevance to the analysis of nonlinear PDE within the generalized function approach.

2. Notations

By Ω, we always denote an open subset of $\mathbb{R}^n$. We denote $B(a,r) := \{x \in \mathbb{R}^n : |x-a| < r\}$.

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2.1. **Schwartz distributions.** We denote by $\mathcal{D}(\Omega)$ the space of functions in $\mathcal{C}^\infty(\Omega)$ with compact support contained in Ω. Its dual space $\mathcal{D}'(\Omega)$ is the space of Schwartz distributions on Ω. We denote the action of a linear map $T\colon \mathcal{D}(\Omega) \to \mathbb{C}$ on an element $\phi \in \mathcal{D}(\Omega)$ by means of the pairing $\langle T, \phi\rangle$. Sometimes it is useful to denote functions and distributions by means of their action on a dummy variable (e.g., as in [1]); we then denote the action of T on ϕ as $\langle T(x), \phi(x)\rangle$. This allows us to write changes of variables $y = F(x)$ simply as $\langle T(F(x)), \phi(x)\rangle = \big\langle T(y), \phi(F^{-1}(y)) \cdot \big|DF^{-1}(y)\big|\big\rangle$. We refer to [1, 11] for further information about Schwartz distributions.

2.2. **Nonstandard analysis.** We work in the framework of nonstandard analysis as introduced by Robinson [10]. We refer to [3] for a more accessible introduction to nonstandard analysis. As usual, if $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is a function, we keep the notation f for its canonical extension ${}^*f$ (and similarly for relations on $\mathbb{R}^n$). Also for the integral $\int$ (considered as a map which takes an integration domain and a function as input), the canonical extension ${}^*\!\int$ will still be denoted by $\int$ (it is thus a map which takes an internal integration domain and an internal function as input). We denote the set of all finite numbers in ${}^*\mathbb{C}$ by $\mathrm{Fin}({}^*\mathbb{C})$ and write $x \approx y$ if $|x - y|$ is infinitesimal ($x, y \in {}^*\mathbb{R}^n$). We write $x \lessapprox y$ if $x \le y$ or $x \approx y$ ($x, y \in {}^*\mathbb{R}$). We denote the standard part (a.k.a. shadow) by st.

3. Known results

Robinson works with real valued distributions on the real line, but the generalization to complex valued distributions on an open subset Ω of $\mathbb{R}^n$ is in most cases straightforward. We say that a function $f \in {}^*\mathcal{C}^\infty(\Omega)$ represents (in the sense of Robinson) a (not necessarily continuous) linear map $T\colon \mathcal{D}(\Omega) \to \mathbb{C}$ if $\int_\Omega f\phi \approx \langle T, \phi\rangle$ for each $\phi \in \mathcal{D}(\Omega)$. In fact, more general functions than ${}^*\mathcal{C}^\infty(\Omega)$-functions are allowed as representatives, but ${}^*\mathcal{C}^\infty(\Omega)$-functions suffice to develop distribution theory by infinitesimal means. Robinson calls equivalence classes of functions representing the same map T predistributions. We will identify the predistribution with the map T. Thus, in general, a predistribution is not necessarily a distribution.

Robinson calls a predistribution standard at $x_0 \in \Omega$ if it has a representative f that is S-continuous at x_0, i.e., such that $f(x) \approx f(x_0)$ for each $x \approx x_0$ [10, p. 140]. He shows:

Theorem 3.1 (Robinson)**.** *Let T be a linear map $\mathcal{D}(\Omega) \to \mathbb{C}$. If T has a representative f that is S-continuous at $x_0 \in \Omega$, then $f(x_0) \in \mathrm{Fin}({}^*\mathbb{C})$. Moreover, the value $\operatorname{st} f(x_0)$ does not depend on the chosen S-continuous representative.*

The number $\operatorname{st} f(x_0)$ is called the value (in the sense of Robinson) of the predistribution at x_0.

P. Loeb [4] proves that if T admits a value $g(x)$ at each $x \in \Omega$, then the resulting map $g\colon \Omega \to \mathbb{C}$ is continuous. In that case, ${}^*g$ represents T

[10, 5.3.15], and hence T is a continuous function (as a regular Schwartz distribution). Actually, Loeb's result is a particular case of theorem 4.2, shown below.

Of crucial importance for our result is the following theorem [6, §6.2]:

Theorem 3.2 (Łojasiewicz). *Let $T \in \mathcal{D}'(\Omega)$ and $x_0 \in \Omega$. If*

$$\lim_{\varepsilon\to 0,\lambda\to 0} T(x_0 + \varepsilon x + \lambda)$$

exists in the distributional sense, i.e.,

$$\lim_{\varepsilon\to 0,\lambda\to 0} \langle T(x_0 + \varepsilon x + \lambda), \phi(x)\rangle \quad \textit{exists} \quad \forall \phi \in \mathcal{D}(\Omega),$$

then T is a continuous function in a neighborhood of x_0.

4. Main result

We first need to show that Robinson's notion of point values is a local property.

Lemma 4.1. *Let ω be an open subset of Ω and let $x_0 \in \omega$. Then a linear map $T\colon \mathcal{D}(\Omega) \to \mathbb{C}$ admits the value c at x_0 (in the sense of Robinson) iff $T_{|\omega}\colon \mathcal{D}(\omega) \to \mathbb{C}$ admits the value c at x_0 (in the sense of Robinson).*

Proof. $\Rightarrow$: immediate.
$\Leftarrow$: Let $f \in {}^*\mathcal{C}^\infty(\omega)$ be a representative of $T_{|\omega}$, i.e., $\int_{^*\omega} f\phi \approx \langle T, \phi\rangle$, $\forall \phi \in \mathcal{D}(\omega)$, and suppose that f is S-continuous in x_0. Let $g \in {}^*\mathcal{C}^\infty(\Omega)$ be any representative of T. Then $f - g_{|\omega}$ is a representative of the 0-distribution on ω. Let $\chi \in \mathcal{D}(\omega)$ with $\chi = 1$ on some (standard) neighborhood V of x_0. Then $\int_{^*\Omega}(f - g)\chi\phi \approx 0$, $\forall \phi \in \mathcal{D}(\Omega)$, since $\chi\phi \in \mathcal{D}(\omega)$. So $(f - g)\chi$ is a representative of 0 on Ω, and $f\chi + g(1 - \chi)$ is a representative of T on Ω which is equal to f in a (standard) neighborhood of x_0. □

We are now in the position to state and prove our main result.

Theorem 4.2. *Let T be a predistribution and $x_0 \in \Omega$. Then T is standard at x_0 iff T is a continuous function in a neighborhood of x_0.*

Proof. $\Rightarrow$: Let $f \in {}^*\mathcal{C}^\infty(\Omega)$ be a representative of T that is S-continuous at x_0. Let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Since f is internal and S-continuous, we find by overspill (see e.g. [3, 11.9.1]) on the set

$$\{r \in {}^*\mathbb{R}, r > 0 : (\forall x \in {}^*\Omega)\, |x - x_0| \le r \implies |f(x) - f(x_0)| \le \varepsilon\}$$

that there exists $r \in \mathbb{R}$, $r > 0$ such that $|f(x) - f(x_0)| \le \varepsilon$ for each $x \in {}^*B(x_0, r) \subseteq {}^*\Omega$. Now let $\phi \in \mathcal{D}(B(x_0, r))$. As f represents T,

$$\left|\langle T, \phi\rangle - f(x_0)\int_\Omega \phi\right| \approx \left|\int_{^*B(x_0,r)} (f(x) - f(x_0))\phi(x)\,dx\right| \le \varepsilon \int_\Omega |\phi|\,.$$

By Robinson's theorem 3.1, $c := \operatorname{st} f(x_0) \in \mathbb{C}$. Taking standard parts,

$$\left|\langle T, \phi\rangle - c\int_\Omega \phi\right| \le \varepsilon \int_\Omega |\phi|\,, \quad \forall \phi \in \mathcal{D}(B(x_0, r)). \tag{1}$$

Hence $|\langle T, \phi \rangle| \le C \sup_\Omega |\phi|$ for some $C \in \mathbb{R}$, and $T_{|B(x_0,r)}$ is a distribution. By transfer on (1), we obtain

$$\left| \langle {}^*T, \phi \rangle - c \int_{{}^*\Omega} \phi \right| \le \varepsilon \int_{{}^*\Omega} |\phi| \,, \quad \forall \phi \in {}^*\mathcal{D}(B(x_0, r)).$$

Now let $\phi \in \mathcal{D}(\Omega)$, let $\varepsilon \in {}^*\mathbb{R}$, $\varepsilon \approx 0$ and $\lambda \in {}^*\mathbb{R}^n$, $\lambda \approx 0$. Then

$$\psi(x) := \frac{1}{\varepsilon^n} \phi\Big(\frac{x - x_0 - \lambda}{\varepsilon}\Big) \in {}^*\mathcal{D}(B(x_0, r)), \quad \forall r \in \mathbb{R},\ r > 0$$

and $\int_{{}^*\Omega} |\psi| = \int_\Omega |\phi| \in \mathbb{R}$. By (1),

$$\big\langle {}^*T(x_0 + \varepsilon x + \lambda), \phi(x) \big\rangle = \langle {}^*T, \psi \rangle \approx c \int_\Omega \phi.$$

As ε and λ are arbitrary, by a nonstandard characterization of limits (see e.g. [3, §7.3])

$$\lim_{\lambda \to 0, \varepsilon \to 0} \big\langle T(x_0 + \varepsilon x + \lambda), \phi(x) \big\rangle = c \int_\Omega \phi.$$

By Łojasiewicz's theorem 3.2, T is a continuous function in a neighborhood of x_0.

$\Leftarrow$: Let ω be an open neighborhood of x_0 and $f \in \mathcal{C}(\omega)$ such that $T = f$ on ω. Let $\phi_0 \in \mathcal{D}(\mathbb{R}^n)$ with $\int \phi_0 = 1$. Let $\varepsilon \in {}^*\mathbb{R}$, $\varepsilon \approx 0$ and $\psi(x) := \varepsilon^{-n} \phi_0(x/\varepsilon)$. Let $g := f \star \psi$ (where $\star$ denotes convolution). Let ω' be a neighborhood of x_0 whose closure is contained in ω. Then $g \in {}^*\mathcal{C}^\infty(\omega')$. Let $x \in {}^*\omega'$. Since f is S-continuous at x (see e.g. [3, §7.1]),

$$\begin{aligned} |f(x) - g(x)| &= \left| \int_{\operatorname{supp} \psi} \big(f(x) - f(x - y)\big) \psi(y)\, dy \right| \\ &\le \sup_{x \in {}^*\omega'} |f(x) - f(x - y)| \int |\phi_0| \approx 0. \end{aligned}$$

Then for any $\phi \in \mathcal{D}(\omega')$,

$$\left| \int_{{}^*\omega'} g\phi - \langle T, \phi \rangle \right| = \left| \int_{{}^*\omega'} g\phi - \int_{{}^*\omega'} f\phi \right| \le \sup_{{}^*\omega'} |g - f| \int_{\omega'} |\phi| \approx 0,$$

so g represents T on ω'. By lemma 4.1, T admits the value $\operatorname{st} g(x_0) = f(x_0)$ at x_0. □

Remark. Theorem 4.2 implies that Łojasiewicz's notion of point value is much more general than that of Robinson: the existence of the point value of a distribution T at a point x_0 in the classical sense of Łojasiewicz [5, 6] does not imply that T is continuous in a neighborhood of x_0, as shown by the function $T(x) = |x|^{-\frac{1}{2}} e^{i/x}$, which is unbounded at the origin but it admits the Łojasiewicz value 0 at $x_0 = 0$. More generally, any function of the form $|x|^{-\beta} e^{i/|x|^\alpha}$, where $\alpha, \beta > 0$, uniquely determines a distribution that has Łojasiewicz value 0 at the origin [2, 5].

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Department of Mathematics, Ghent University, Krijgslaan 281, B-9000 Gent, Belgium.
E-mail address: hvernaev@cage.ugent.be

Department of Mathematics, Ghent University, Krijgslaan 281, B-9000 Gent, Belgium.
E-mail address: jvindas@cage.ugent.be