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Isospin-breaking effects in the two-pion contribution to hadronic vacuum polarization

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Abstract: Isospin-breaking (IB) effects in the two-pion contribution to hadronic vacuum polarization (HVP) can be resonantly enhanced, if related to the interference of the $\rho(770)$ and $\omega(782)$ resonances. This particular IB contribution to the pion vector form factor and thus the line shape in $e^+e^- \to \pi^+\pi^-$ can be described by the residue at the ω pole — the ρ-ω mixing parameter ϵ_ω. Here, we argue that while in general analyticity requires this parameter to be real, the radiative channels $\pi^0\gamma$, $\pi\pi\gamma$, $\eta\gamma$ can induce a small phase, whose size we estimate as $\delta_\epsilon = 3.5(1.0)^\circ$ by using a narrow-width approximation for the intermediate-state vector mesons. We then perform fits to the $e^+e^- \to \pi^+\pi^-$ data base and study the consequences for the two-pion HVP contribution to the anomalous magnetic moment of the muon, its IB part due to ρ-ω mixing, and the mass of the ω resonance. We find that the global fit does prefer a non-vanishing value of $\delta_\epsilon = 4.5(1.2)^\circ$, close to the narrow-resonance expectation, but with a large spread among the data sets, indicating systematic differences in the ρ-ω region.

Keywords: Chiral Lagrangian, Precision QED

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1 Introduction

The two-pion channel gives the dominant contribution to hadronic vacuum polarization (HVP) in the low-energy region most relevant for the anomalous magnetic moment of the muon [1], adding about 70% of the total leading-order (LO) effect [2–7]

$$a_\mu^{\text{HVP, LO}}\big|_{e^+e^-} = 693.1(4.0) \times 10^{-10}. \tag{1.1}$$

Its contribution needs to be understood at a level of at least 0.3% to match the final precision expected from the Fermilab E989 experiment [8]. The nominal combined sensitivity of the 2π data sets entering eq. (1.1) — from SND [9, 10], CMD-2 [11–14], BESIII [15], CLEO [16], and dominated by the precision data sets from BaBar [17, 18] and KLOE [19–22] — does reach 0.4%, but becomes diluted due to a tension between BaBar and KLOE, inflating the 2π uncertainty included in eq. (1.1) to 0.7%. In dispersive approaches [4–6, 23, 24] also space-like data [25, 26] can be used, and while stabilizing the extrapolation to the space-like region, their impact on the time-like HVP integral is minor. More recently, new data from SND [27] have become available, lying in between BaBar and KLOE, but not at a comparable level of precision that would allow one to resolve the tension. Such new precision measurements are expected from CMD-3 [28], BaBar [29], BESIII [30], and Belle II [31] in the future.

Improved understanding of the 2π channel has further become critical to address the emerging tension between lattice QCD [32–35] and e^+e^- data at least for the intermediate window quantity [36], with immediate consequences for the current 4.2σ discrepancy for the anomalous magnetic moment of the muon between experiment [37–41] and the prediction in the Standard Model [1–7, 42–59] when the HVP contribution is derived from

$e^+e^- \to$ hadrons cross-section data. While the detailed comparison to lattice QCD as well as related observables defines an important path forward [60–67], so does renewed scrutiny of the data-driven approach.

For the 2π channel, new precision data sets constitute the clear first priority, but another aspect concerns the role of radiative corrections [29, 68], in particular, the question in which cases the use of a point-like approximation [69–73] for the pion might miss relevant effects, as recently observed in the forward-backward asymmetry [74, 75], and currently under study for the C-even contributions [76]. In this work, we study a complementary point, i.e., not isospin-breaking (IB) effects that manifest themselves as final- or initial-state radiation, but corrections that are typically absorbed into the pion vector form factor (VFF) itself.[1] The most prominent such correction arises from ρ-ω mixing. From a dispersive point of view the fact that the $\omega(782)$ resonance is so narrow allows one to describe this interference in terms of a single real parameter: the ρ-ω mixing parameter ϵ_ω. But given the extraordinary precision requirements for the 2π channel together with the resonance enhancement in the ρ-ω region, even higher-order effects may affect the value of this parameter and generate non-negligible effects in $a_\mu^{\text{HVP, LO}}$. Most notably, the radiative channels $\pi^0\gamma$, $\pi\pi\gamma$, $\eta\gamma$, all of which couple to both ρ and ω, can induce imaginary parts in the mixing and thereby an effective small phase δ_ϵ in the parameter ϵ_ω to which e^+e^- data might be sensitive.

To derive the phenomenological consequences of this phase we first generalize the dispersive representation of the pion VFF from ref. [4] and estimate its size based on a narrow-resonance approach, see section 2. We then perform fits, to individual data sets and globally, allowing for a free phase δ_ϵ, to assess consistency both among the data sets and with the narrow-resonance expectation, see section 3. Consequences for the IB contribution to a_μ due to ρ-ω mixing and the ω mass are discussed in sections 4 and 5, respectively, before concluding in section 6.

2 Radiative channels and phase in the ρ–ω mixing parameter

2.1 Dispersive representation

Dispersive representations for the pion VFF, $F_\pi^V(s)$, that link the matrix element of the electromagnetic current $j_{\text{em}}^\mu = (2\bar{u}\gamma^\mu u - \bar{d}\gamma^\mu d - \bar{s}\gamma^\mu s)/3$,

$$\langle \pi^\pm(p')|j_{\text{em}}^\mu(0)|\pi^\pm(p)\rangle = \pm(p'+p)^\mu F_\pi^V((p'-p)^2), \tag{2.1}$$

to $\pi\pi$ scattering have been used for a long time in the literature [4, 6, 23, 82–91], not only for the HVP application, but also for hadronic light-by-light scattering (HLbL), where the extrapolation into the space-like region enters [50, 51, 92–95].

Here, we build upon the representation from ref. [4] (in turn based on refs. [83, 84]), whose main features can be summarized as follows: the VFF is decomposed into three

[1] Such corrections were studied before in the context of relating VFF measurements in $e^+e^- \to \pi^+\pi^-$ to $\tau^\pm \to \pi^\pm\pi^0\nu_\tau$ data [77–81]. Here, we aim instead at a rigorous implementation of ρ-ω mixing in a dispersive framework, both to quantify its impact on $a_\mu^{\text{HVP, LO}}$ and as another powerful consistency check on the e^+e^- data base.

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factors

$$F_\pi^V(s) = \Omega_1^1(s) G_\omega(s) G_{\text{in}}^N(s), \tag{2.2}$$

corresponding to 2π, 3π, and higher intermediate states, respectively. The Omnès function [96]

$$\Omega_1^1(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\} \tag{2.3}$$

implements 2π singularities in terms of the isospin $I = 1$ elastic $\pi\pi$ phase shift $\delta_1^1(s)$ in the isospin limit. The phase shift is further constrained by $\pi\pi$ Roy equations [97], which are solved with the phase shifts at $s_0 = (0.8\,\text{GeV})^2$ and $s_1 = (1.15\,\text{GeV})^2$ as free parameters [98, 99]. Systematic errors from the asymptotic continuation of δ_1^1 beyond s_1 are treated as described in ref. [4].

The focus of this work is the second factor, G_ω, which takes into account the effect of 3π intermediate states. Here, the parameterization from ref. [4] reads

$$G_\omega(s) = 1 + \frac{s}{\pi}\int_{9M_\pi^2}^\infty ds' \frac{\operatorname{Im} g_\omega(s')}{s'(s'-s)}\left(\frac{1-\frac{9M_\pi^2}{s'}}{1-\frac{9M_\pi^2}{M_\omega^2}}\right)^4, \tag{2.4}$$

with

$$g_\omega(s) = 1 + \epsilon_\omega \frac{s}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s}. \tag{2.5}$$

The dispersive reconstruction in eq. (2.4) ensures both the absence of unphysical imaginary parts below $s = 9M_\pi^2$ (for real ϵ_ω) and the correct threshold behavior above [83]. In this formulation, ϵ_ω is an effective parameter tightly related to the residue at the ω pole. The latter, however, is complex in general, but its phase is expected to be tiny: $\delta_\epsilon \simeq \arctan \Gamma_\omega/M_\omega \simeq 0.6^\circ$ arising from the analytic continuation from the real axis to the pole position in the complex plane. Such a small difference is of no concern and if we take it as a measure of the systematic uncertainty in the phase of ϵ_ω, allows us to view the latter as the residue at the ω pole. With the threshold behavior and the pole parameters determined, the resulting $G_\omega(s)$ is then largely insensitive to the parameterization of $g_\omega(s)$, e.g., the numerator could be taken to a constant without any relevant changes to the fit outcome. The main observation in this paper is that the assumption of a real ϵ_ω no longer holds if further IB effects due to radiative channels, $X = \pi^0\gamma, \pi\pi\gamma, \eta\gamma, \ldots$, are considered, and these imaginary parts, despite being small, can alter the fit parameters in a significant way, as only the modulus $|F_\pi^V(s)|^2$ is probed by the fit to the cross-section data.

Finally, for the inelastic channels we continue to use a conformal polynomial, whose phase is constrained by the Eidelman–Łukaszuk bound [100, 101]. Its threshold is chosen as the $\omega\pi^0$ threshold, below which inelasticities are negligibly small. After removing the S-wave cusp, $G_{\text{in}}^N(s)$ involves $N-1$ free parameters, which together with ϵ_ω and $\delta_1^1(s_0)$, $\delta_1^1(s_1)$ are to be constrained in the fit. The fit range is restricted to $s \leq 1\,\text{GeV}^2$, going beyond would require including the effects of ρ', ρ'' resonances along the lines of refs. [86, 102], an extension left for future work.

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2.2 Radiative channels

Both ρ and ω possess non-negligible branching fractions into radiative channels, of which $\pi^0\gamma$ yields the largest contribution. The corresponding imaginary part in the pion VFF can be expressed as

$$\operatorname{Im} F_\pi^V(s)\big|_{\pi^0\gamma} = \frac{\alpha(s-M_{\pi^0}^2)^3}{48s} F_{\pi^0\gamma^*\gamma^*}(s,0)\big(f_1(s)\big)^*, \tag{2.6}$$

where $F_{\pi^0\gamma^*\gamma^*}$ is the pion transition form factor normalized according to

$$F_{\pi^0\gamma^*\gamma^*}(0,0) = F_{\pi\gamma\gamma} = \sqrt{\frac{4\Gamma[\pi^0\to\gamma\gamma]}{\pi\alpha^2 M_{\pi^0}^3}} \tag{2.7}$$

and f_1 denotes the P-wave projection of the $\gamma\pi\to\pi\pi$ amplitude, see refs. [103–107] for detailed discussions of these amplitudes. To map this imaginary part onto $\operatorname{Im}\epsilon_\omega$, we first write the full pion VFF in the approximation

$$F_\pi^V(s) = \bigg(1+\epsilon_\omega\frac{s}{M_\omega^2-s-i\epsilon}\bigg)\Omega_1^1(s), \tag{2.8}$$

where we have neglected inelastic corrections for the time being to focus on the interplay of ρ and ω resonances, with the ω approximated in the narrow-width limit for simplicity. As a first step, we show that an imaginary part in ϵ_ω in this form is actually compatible with unitarity. Applying Cutkosky rules to eq. (2.8), we have

$$\begin{aligned}
\frac{1}{2i}\operatorname{disc} F_\pi^V(s)\big|_{2\pi} &= \bigg(1+\epsilon_\omega\frac{s}{M_\omega^2-s-i\epsilon}\bigg)\operatorname{Im}\Omega_1^1(s),\\
\frac{1}{2i}\operatorname{disc} F_\pi^V(s)\big|_{3\pi} &= \epsilon_\omega^* s\pi\delta(s-M_\omega^2)\big(\Omega_1^1(s)\big)^*,\\
\frac{1}{2i}\operatorname{disc} F_\pi^V(s)\big|_{\pi^0\gamma} &= \operatorname{Im}\epsilon_\omega\frac{s}{M_\omega^2-s-i\epsilon}\big(\Omega_1^1(s)\big)^*.
\end{aligned} \tag{2.9}$$

These equations are only consistent as long as the sum of these discontinuities is purely imaginary. Collecting all terms, this consistency check is indeed satisfied,

$$\begin{aligned}
&\operatorname{Im}\bigg[\frac{1}{2i}\operatorname{disc} F_\pi^V(s)\big|_{2\pi}+\frac{1}{2i}\operatorname{disc} F_\pi^V(s)\big|_{3\pi}+\frac{1}{2i}\operatorname{disc} F_\pi^V(s)\big|_{\pi^0\gamma}\bigg]\\
&=\operatorname{Im}\Omega_1^1(s)\bigg(\operatorname{Im}\epsilon_\omega\frac{s}{M_\omega^2-s}+\operatorname{Re}\epsilon_\omega s\pi\delta(s-M_\omega^2)\bigg)\\
&\quad+s\pi\delta(s-M_\omega^2)\bigg(-\operatorname{Re}\epsilon_\omega\operatorname{Im}\Omega_1^1(s)-\operatorname{Re}\Omega_1^1(s)\operatorname{Im}\epsilon_\omega\bigg)\\
&\quad+\operatorname{Im}\epsilon_\omega\bigg(-\frac{s}{M_\omega^2-s}\operatorname{Im}\Omega_1^1(s)+\operatorname{Re}\Omega_1^1(s)s\pi\delta(s-M_\omega^2)\bigg)\\
&=0,
\end{aligned} \tag{2.10}$$

so that as long as imaginary parts are avoided below the respective thresholds a phase in ϵ_ω is indeed possible.

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Next, the comparison of eqs. (2.6) and (2.9) points towards a strategy for a practical implementation, with $\Omega_1^1(s)$ corresponding to $f_1(s)$, and the ω propagator to the ω contribution in $F_{\pi^0\gamma^*\gamma^*}(s,0)$. The latter is given by

$$F_{\pi^0\gamma^*\gamma^*}(s,0) \simeq \frac{g_{\omega\pi\gamma}}{g_{\omega\gamma}} \frac{M_\omega^2}{M_\omega^2 - s - i\epsilon} \tag{2.11}$$

(see, e.g., ref. [108]), while the former can be approximated by [106]

$$f_1(s) \simeq \frac{2g_{\rho\pi\gamma}g_{\rho\pi\pi}}{M_\rho^2 - s - iM_\rho\Gamma_\rho}, \qquad \Omega_1^1(s) \simeq \frac{M_\rho^2}{M_\rho^2 - s - iM_\rho\Gamma_\rho}, \tag{2.12}$$

leading to

$$\operatorname{Im}\epsilon_\omega \simeq \frac{\alpha(s-M_{\pi^0}^2)^3}{24s} \frac{g_{\omega\pi\gamma}g_{\rho\pi\gamma}g_{\rho\pi\pi}}{g_{\omega\gamma}M_\rho^2}. \tag{2.13}$$

Inserting the expressions for the radiative decay widths,

$$\Gamma[V \to \pi^0\gamma] = \frac{\alpha(M_V^2 - M_{\pi^0}^2)^3}{24M_V^3}|g_{V\pi\gamma}|^2, \qquad V = \omega, \rho, \tag{2.14}$$

as well as the VMD predictions $g_{\rho\pi\pi} = g_{\rho\gamma} = g_{\omega\gamma}/3$, and evaluating eq. (2.13) at $s = M_V^2 \simeq M_\rho^2 \simeq M_\omega^2$, we find

$$\operatorname{Im}\epsilon_\omega \simeq \frac{\sqrt{\Gamma[\omega \to \pi^0\gamma]\Gamma[\rho \to \pi^0\gamma]}}{3M_V}. \tag{2.15}$$

In fact, in the narrow-width limit the same relation can be established for an arbitrary intermediate state X, leading to estimates for the phases around 2.8° ($\pi^0\gamma$), 0.2° ($\eta\gamma$), and 0.02° ($\pi^0\pi^0\gamma$) when using the averages from ref. [109] for branching fractions and masses. For the charged channel $\pi^+\pi^-\gamma$ one needs to take into account the fact that the presence of the Born-term contribution leads to an infrared divergence, in such a way that branching fractions are typically quoted with a cut $E_\gamma = 50\,\text{MeV}$ in the photon energy [110, 111]. However, combined with virtual corrections calculated in a scalar-QED approximation one can define an infrared-safe decay width as

$$\Gamma[V \to \pi^+\pi^-\gamma] = \Gamma[V \to \pi^+\pi^-]\frac{\alpha}{\pi}\eta(M_V^2), \tag{2.16}$$

where explicit expressions for the function η can be found in refs. [69–72]. This procedure gives an estimate of 0.4° for the $\pi^+\pi^-\gamma$ channel, subject to minor corrections from non-Born contributions [111]. As for the relative signs, VMD arguments show that the $\pi^0\gamma$ and $\eta\gamma$ channels enter with the same sign (in the standard phase conventions, both intermediate states couple with the same sign to ρ and ω), and Born-term dominance for $\rho, \omega \to \pi^+\pi^-\gamma$, as well as the positive sign of $\operatorname{Re}\epsilon_\omega$, suggest that its contribution should also add to the other two. Taking further potential corrections due to the analytic continuation to the ω pole as the uncertainty, we conclude that the range $\delta_\epsilon = 3.5(1.0)°$ should give a realistic estimate of the phase in ϵ_ω that can be expected.

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Finally, the discussion of the $\pi^0\gamma$ channel suggests that the dominant threshold can be reproduced by implementing the imaginary part in ϵ_ω via

$$\begin{aligned} G_\omega(s) = 1 &+ \frac{s}{\pi}\int_{9M_\pi^2}^\infty ds' \frac{\mathrm{Re}\,\epsilon_\omega}{s'(s'-s)} \mathrm{Im}\left[\frac{s'}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s'}\right]\left(\frac{1-\frac{9M_\pi^2}{s'}}{1-\frac{9M_\pi^2}{M_\omega^2}}\right)^4 \\ &+ \frac{s}{\pi}\int_{M_{\pi^0}^2}^\infty ds' \frac{\mathrm{Im}\,\epsilon_\omega}{s'(s'-s)} \mathrm{Re}\left[\frac{s'}{(M_\omega - \frac{i}{2}\Gamma_\omega)^2 - s'}\right]\left(\frac{1-\frac{M_{\pi^0}^2}{s'}}{1-\frac{M_{\pi^0}^2}{M_\omega^2}}\right)^3, \end{aligned} \tag{2.17}$$

but we also checked that the fit results are largely insensitive to the details of the implementation, such as the inclusion of the explicit $\pi^0\gamma$ threshold in the unphysical region of the pion VFF. In fact, even replacing $G_\omega(s)$ with $g_\omega(s)$ only leads to small changes as long as the imaginary part in ϵ_ω is kept, in line with the expectation that it is solely the resonance enhancement that makes these higher-order effects relevant.

3 Fits to $e^+e^- \to 2\pi$ data

To gauge the impact of a possible phase in ϵ_ω on the HVP contribution to a_μ, we generalize the global fits from ref. [4], including a free imaginary part via the prescription (2.17), and express our results in terms of $\mathrm{Re}\,\epsilon_\omega$ and δ_ϵ. In particular, we now include the BESIII data [15] and the SND measurement [27], which became available after ref. [4].[2] The results for the fits are shown in table 1 (single experiments) and table 2 (combinations), in terms of the most relevant parameters: goodness of fit, the ω mass, real part and phase of ϵ_ω, and the contribution to a_μ. In table 3, we also provide the decomposition into the Euclidean windows from ref. [36].

In most cases, we observe a moderate improvement when a non-vanishing phase is admitted, the main exception being the SND20 data, which we cannot describe with our dispersive representation otherwise. Accordingly, in this case the resulting phase comes out around 10° and thus much larger than can be justified via radiative intermediate states. A similarly large phase is also found for the previous energy-scan experiments SND06 and CMD-2, but in these cases good fits can still be found when imposing a realistic size of δ_ϵ. Even if a large phase is admitted in the fit to the SND20 data, the fit quality remains rather poor.[3] As long as the reason for this behavior, which might point towards underestimated systematic effects, is not understood, we will therefore take the global fit to all experiments apart from SND20 as our new central result, i.e.

$$\begin{aligned} \mathrm{Re}\,\epsilon_\omega &= 1.97(3)\times 10^{-3}, & \delta_\epsilon &= 4.5(1.2)^\circ, \\ a_\mu^{\pi\pi}|_{\leq 1\,\mathrm{GeV}} &= 494.6(2.3)\times 10^{-10}, & M_\omega &= 782.09(12)\,\mathrm{MeV}. \end{aligned} \tag{3.1}$$

[2] In the case of ref. [15] the corrected covariance matrix was critical for the inclusion of this data set in a statistically meaningful way. For the uncertainty of the energy calibration at the ρ peak we use $\Delta E = 0.6\,\mathrm{MeV}$ [112] and $\Delta E = 0.26\,\mathrm{MeV}$ [113], respectively.

[3] The fit presented in ref. [27] in terms of a sum of Breit-Wigner functions for $V = \rho, \omega, \rho'$ displays a slightly better fit quality, $\chi^2/\mathrm{dof} = 47/30 = 1.57$, with p-value of 2.5%, but such a representation cannot be reconciled with the analytic properties of the pion VFF.

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	χ^2/dof	p-value	M_ω [MeV]	$10^3 \times \mathrm{Re}\,\epsilon_\omega$	δ_ϵ [°]	$10^{10} \times a_\mu^{\pi\pi}\|_{\leq 1\,\mathrm{GeV}}$
SND06	1.40	5.3%	781.49(32)(2)	2.03(5)(2)		499.7(6.9)(4.1)
	1.08	35%	782.11(32)(2)	1.98(4)(2)	8.5(2.3)(0.3)	497.8(6.1)(4.9)
CMD-2	1.18	14%	781.98(29)(1)	1.88(6)(2)		496.9(4.0)(2.3)
	1.01	45%	782.64(33)(4)	1.85(6)(4)	11.4(3.1)(1.0)	495.8(3.7)(4.2)
BaBar	1.14	5.7%	781.86(14)(1)	2.04(3)(2)		501.9(3.3)(2.0)
	1.14	5.5%	781.93(18)(4)	2.03(4)(1)	1.3(1.9)(0.7)	501.9(3.3)(1.8)
KLOE	1.36	7.4×10^{-4}	781.82(17)(4)	1.97(4)(2)		492.0(2.2)(1.8)
	1.27	6.7×10^{-3}	782.50(25)(6)	1.94(5)(2)	6.8(1.8)(0.5)	491.0(2.2)(2.0)
KLOE″	1.20	3.1%	781.81(16)(3)	1.98(4)(1)		491.8(2.1)(1.8)
	1.13	10%	782.42(23)(5)	1.95(4)(2)	6.1(1.7)(0.6)	490.8(2.0)(1.7)
BESIII	1.12	25%	782.18(51)(7)	2.01(19)(9)		490.8(4.8)(3.9)
	1.02	44%	783.05(60)(2)	1.99(19)(7)	17.6(6.9)(1.2)	490.3(4.5)(3.1)
SND20	2.93	3.3×10^{-7}	781.79(30)(6)	2.04(6)(3)		494.2(6.7)(9.0)
	1.87	4.1×10^{-3}	782.37(28)(6)	2.02(5)(2)	10.1(2.4)(1.4)	494.9(5.3)(3.1)

Table 1. Comparison of fits to single experiments with and without a phase δ_ϵ in ϵ_ω. Note that BESIII has only a few data points in the interference region and hence is not able to put a strong constraint on δ_ϵ (the corresponding line is indicated in gray). The first error is the fit uncertainty, inflated by $\sqrt{\chi^2/\mathrm{dof}}$, the second error is the combination of all systematic uncertainties.

	χ^2/dof	p-value	M_ω [MeV]	$10^3 \times \mathrm{Re}\,\epsilon_\omega$	δ_ϵ [°]	$10^{10} \times a_\mu^{\pi\pi}\|_{\leq 1\,\mathrm{GeV}}$
Energy scan w/o SND20	1.28	2.1%	781.75(22)(1)	1.97(4)(2)		498.5(3.4)(2.6)
	1.05	33%	782.39(23)(2)	1.93(4)(3)	9.9(1.8)(0.4)	497.3(3.1)(3.9)
Energy scan	1.65	6.3×10^{-7}	781.74(17)(2)	2.01(3)(3)		497.4(3.0)(4.4)
	1.19	5.2%	782.37(16)(3)	1.97(3)(3)	10.1(1.3)(0.7)	496.0(2.6)(5.5)
All e^+e^- w/o SND20	1.25	1.8×10^{-5}	781.70(9)(4)	2.02(2)(3)		494.5(1.5)(2.3)
	1.20	3.3×10^{-4}	782.10(12)(4)	1.96(2)(2)	4.5(9)(8)	494.2(1.4)(2.1)
NA7 + all e^+e^- w/o SND20	1.23	3.0×10^{-5}	781.69(9)(3)	2.02(2)(3)		494.8(1.4)(2.1)
	1.19	4.8×10^{-4}	782.09(12)(4)	1.97(2)(2)	4.5(9)(8)	494.6(1.5)(1.7)
All e^+e^-	1.36	1.0×10^{-9}	781.71(8)(3)	2.02(2)(3)		495.0(1.4)(2.4)
	1.30	2.3×10^{-7}	782.09(10)(4)	1.97(2)(2)	4.5(8)(8)	494.6(1.4)(2.1)
NA7 + all e^+e^-	1.34	2.5×10^{-9}	781.71(8)(3)	2.02(2)(3)		495.2(1.4)(2.2)
	1.28	4.5×10^{-7}	782.09(10)(4)	1.97(2)(2)	4.5(8)(8)	494.9(1.4)(1.8)

Table 2. Comparison of fits to combinations of experiments with and without a phase δ_ϵ in ϵ_ω. The first error is the fit uncertainty, inflated by $\sqrt{\chi^2/\mathrm{dof}}$, the second error is the combination of all systematic uncertainties.

For BESIII, the preferred central value for δ_ϵ comes out even larger, yet with a very large uncertainty that reflects the limited sensitivity to δ_ϵ, resulting from a relatively small number of data points in the ρ-ω region (accordingly, this line is indicated in light gray in tables 1 and 3). Finally, the KLOE fits produce a phase slightly larger than expected, while the BaBar data are even consistent with $\delta_\epsilon = 0$. We thus observe a large spread in the results for the phase of δ_ϵ, pointing towards systematic differences among the data sets in the ρ-ω region.

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	δ_ϵ [°]	SD window	int window	LD window	$10^{10} \times a_\mu^{\pi\pi}\|_{\leq 1\,\text{GeV}}$
SND06		13.9(2)(1)	140.0(2.0)(1.0)	345.8(4.7)(3.0)	499.7(6.9)(4.1)
	8.5(2.3)(0.3)	13.9(2)(1)	139.6(1.8)(1.2)	344.3(4.1)(3.6)	497.8(6.1)(4.9)
CMD-2		13.9(1)(0)	139.5(1.1)(0.4)	343.6(2.7)(1.8)	496.9(4.0)(2.3)
	11.4(3.1)(1.0)	13.9(1)(1)	139.4(1.0)(0.9)	342.6(2.5)(3.2)	495.8(3.7)(4.2)
BaBar		14.0(1)(0)	140.6(1.0)(0.5)	347.3(2.2)(1.5)	501.9(3.3)(2.0)
	1.3(1.9)(0.7)	14.0(1)(0)	140.6(1.0)(0.5)	347.3(2.3)(1.3)	501.9(3.3)(1.8)
KLOE″		13.6(1)(1)	137.3(6)(6)	340.9(1.4)(1.2)	491.8(2.1)(1.8)
	6.1(1.7)(0.6)	13.6(1)(0)	137.1(6)(4)	340.2(1.4)(1.3)	490.8(2.0)(1.7)
BESIII		13.7(1)(0)	138.0(1.4)(0.5)	339.0(3.3)(3.4)	490.8(4.8)(3.9)
	17.6(6.9)(1.2)	13.7(1)(0)	137.8(1.3)(0.4)	338.8(3.1)(2.6)	490.3(4.5)(3.1)
SND20		13.9(2)(1)	139.4(1.9)(1.5)	340.9(4.6)(7.4)	494.2(6.7)(9.0)
	10.1(2.4)(1.4)	13.8(2)(0)	139.2(1.5)(0.5)	341.9(3.7)(2.6)	494.9(5.3)(3.1)
NA7 + all e^+e^- w/o SND20		13.7(0)(0)	138.3(4)(5)	342.7(1.0)(1.6)	494.8(1.4)(2.1)
	4.5(9)(8)	13.7(0)(0)	138.3(4)(4)	342.5(1.0)(1.3)	494.6(1.5)(1.7)

Table 3. Decomposition of $10^{10} \times a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$ into the Euclidean windows from ref. [36]. The first error is the fit uncertainty, inflated by $\sqrt{\chi^2/\text{dof}}$, the second error is the combination of all systematic uncertainties.

In addition, we confirm a correlation between δ_ϵ and M_ω, as already observed in ref. [18]: the larger the phase, the larger the extracted value of M_ω. However, as discussed in more detail in section 5, the size of the phase permitted by radiative intermediate states, roughly in line with the result of the global fit shown in table 2, does not suffice to remove the tension with ω-mass determinations from $e^+e^- \to 3\pi$ and $e^+e^- \to \pi^0\gamma$.

In this regard, we also observe that the BESIII data suggest larger values of M_ω than all other data sets, with the result for $\delta_\epsilon = 0$ close to the global fit with non-vanishing phase. Within uncertainties there is still consistency, but it is noteworthy that the size and direction of the effect echo a similar tension in $e^+e^- \to 3\pi$ [12, 114–116] and $\eta' \to \pi^+\pi^-\gamma$ [117, 118].

4 Isospin-breaking contribution to a_μ from ρ–ω mixing

Based on the dispersive representation (2.2) we can quantify $a_\mu^{\rho-\omega}$ — the IB contribution to a_μ due to ϵ_ω — by contrasting the full result to the HVP integral evaluated with $\epsilon_\omega = 0$. In principle, there is some ambiguity due to final-state radiation (FSR), but in practice this effect comes out well below 0.1×10^{-10}. For definiteness, in table 4 we show the variant without FSR, to isolate the pure $\mathcal{O}(\epsilon_\omega)$ terms.

In general, $a_\mu^{\rho-\omega}$ is sensitive to the assumed line shape [119]. However, we find that the dispersive representations (2.4) or (2.17) are quite robust in that regard, i.e., with the threshold behavior and the properties close to the ω pole determined, the remaining interpolation only has a marginal effect, e.g., changing $s \to M_\omega^2$ in the numerator of eq. (2.5) changes the outcome for $a_\mu^{\rho-\omega}$ again by less than 0.1×10^{-10}. In contrast, whether or not a phase in ϵ_ω is permitted does change the resulting value for $a_\mu^{\rho-\omega}$ in a significant way, and we show results for both scenarios (the difference in the FSR contribution is of $\mathcal{O}(e^2\epsilon_\omega)$ and negligible). Since ϵ_ω in the global fit comes out close to the narrow-resonance

	SD window	int window	LD window	total
$10^{10} \times a_\mu^{\rho\text{-}\omega}$, $\delta_\epsilon = 0$	0.08(0)(0)	1.06(1)(2)	3.23(3)(5)	4.37(4)(7)
$10^{10} \times a_\mu^{\pi\pi,\text{FSR}}$, $\delta_\epsilon = 0$	0.11(0)(0)	1.12(0)(0)	3.00(1)(1)	4.23(1)(2)
$10^{10} \times a_\mu^{\rho\text{-}\omega}$, $\delta_\epsilon = 4.5(1.2)^\circ$	0.05(0)(0)	0.83(5)(4)	2.79(9)(6)	3.68(14)(10)
$10^{10} \times a_\mu^{\pi\pi,\text{FSR}}$, $\delta_\epsilon = 4.5(1.2)^\circ$	0.11(0)(0)	1.12(0)(0)	3.00(1)(1)	4.24(1)(2)

Table 4. IB contribution to $a_\mu^{\pi\pi}|_{\leq 1\,\text{GeV}}$ due to ρ-ω mixing, compared to the effect of FSR and split into the different Euclidean windows from ref. [36]. We only include the linear effects, i.e., $\mathcal{O}(\epsilon_\omega)$ for the ρ-ω-mixing contribution and the $\mathcal{O}(e^2)$ effect for FSR — $\mathcal{O}(e^2\epsilon_\omega)$ effects give very small corrections. The results correspond to the combined fit to all experiments apart from SND20. The first error is the fit uncertainty, inflated by $\sqrt{\chi^2/\text{dof}}$, the second error is the combination of all systematic uncertainties.

expectation, we quote the variant with non-vanishing δ_ϵ as our preferred result, which has already been used as input in estimating the three-flavor quark-disconnected contribution to a_μ in ref. [120].[4] Finally, we also provide the breakdown of $a_\mu^{\rho\text{-}\omega}$ onto the Euclidean windows from ref. [36].

Further, for the comparison to lattice QCD it is also of interest to study the decomposition of ϵ_ω into its $\mathcal{O}(e^2)$ and $\mathcal{O}(m_u - m_d)$ pieces, as was discussed in the context of resonance chiral perturbation theory in ref. [121]. Translated to our normalization one has the prediction [121]

$$\tilde{\epsilon}_\omega = \frac{2}{3R}\frac{M_{K^*} - M_V}{M_V} - \frac{e^2}{|g_{\omega\gamma}|^2}, \qquad R = \frac{m_s - \hat{m}}{m_d - m_u}, \qquad \hat{m} = \frac{m_u + m_d}{2}, \tag{4.1}$$

where we have written the electromagnetic component in terms of the ω-γ coupling, as this is the quantity that enters directly in the corresponding diagram. However, this latter diagram produces a one-particle-reducible correction, and would thus be subtracted when vacuum polarization is removed from the $e^+e^- \to \pi^+\pi^-$ cross sections. Accordingly, we have that our conventions are related to the ones of ref. [121] by

$$\tilde{\epsilon}_\omega = \epsilon_\omega + \tilde{\epsilon}_\omega|_{e^2}, \qquad \tilde{\epsilon}_\omega|_{e^2} = -\frac{e^2}{|g_{\omega\gamma}|^2} = -0.34(1) \times 10^{-3}, \tag{4.2}$$

using $\Gamma[\omega \to e^+e^-] = 4\pi\alpha^2 M_\omega/(3|g_{\omega\gamma}|^2)$ and the average of ref. [109] for the $\omega \to e^+e^-$ branching fraction.

The prediction for the $\mathcal{O}(m_u - m_d)$ part of $\tilde{\epsilon}_\omega$, which coincides with our ϵ_ω, is far less robust, as already from higher-order quark-mass and SU(3)-breaking corrections one would expect an accuracy around 30%. Using the $N_f = 2+1$ and $N_f = 2+1+1$ averages from ref. [122], $R = 38.1(1.5)$ [123–128] and $R = 35.9(1.7)$ [129–133], respectively, and identifying the vector mesons with the neutral ρ and K^* resonances, the predictions for the strong IB contribution become $\tilde{\epsilon}_\omega|^{2+1}_{m_u-m_d} = 2.71(11) \times 10^{-3}$, $\tilde{\epsilon}_\omega|^{2+1+1}_{m_u-m_d} = 2.88(14) \times 10^{-3}$,

[4] The tiny difference to the number for $a_\mu^{\rho\text{-}\omega}$ quoted therein as private communication originates from the improved implementation of the $\pi^0\gamma$ threshold (2.17).

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Reference	$e^+e^- \to 3\pi$	$e^+e^- \to \pi^0\gamma$	$e^+e^- \to 2\pi$	PDG average
Ref. [5]	782.631(28)			
Ref. [46]		782.584(28)		
Ref. [4]			781.68(10)	
This work, $\delta_\epsilon = 0$			781.69(9)	
This work, $\delta_\epsilon = 4.5(1.2)^\circ$			782.09(12)	
Ref. [109]				782.53(13)

Table 5. Dispersive determinations of M_ω in MeV from e^+e^- reactions, compared to the global average from ref. [109]. In all cases, vacuum-polarization corrections are not included, and the average from ref. [109] has been adjusted accordingly using $\Delta M_\omega = 0.13\,\text{MeV}$ [117].

about 40% larger than results from our fit to the $e^+e^- \to \pi^+\pi^-$ data. This is in line with subsequent work on vector mesons in chiral perturbation theory [134, 135], which concluded that higher-order corrections can be substantial. This includes photon loops, short-distance corrections, and meson loops, parts of which scale with e^2 and thus lead to electromagnetic effects not subtracted when removing vacuum polarization in the definition of the bare cross section.

However, from the LO expression (4.1) it still follows that the ρ-ω-mixing contribution to a_μ should be considered primarily a quark-mass effect,

$$a_\mu^{\rho\text{-}\omega}[e^2, \text{LO}] = 0, \qquad a_\mu^{\rho\text{-}\omega}[m_u - m_d, \text{LO}] = 3.68(17) \times 10^{-10}, \tag{4.3}$$

which is expected to yield the dominant strong IB contribution to a_μ.[5] This number agrees well with a recent estimate from SU(3) chiral perturbation theory, $a_\mu[m_u - m_d]|_{[136]} = 3.32(89) \times 10^{-10}$, where the required low-energy constant is determined from hadronic τ decays. Both indicate a somewhat larger central value than the lattice-QCD result $a_\mu[m_u - m_d]|_{[32]} = 1.9(1.2) \times 10^{-10}$.

5 Consequences for the ω mass

The correlation between δ_ϵ and M_ω discussed in section 3 affects the resulting determination of M_ω from $e^+e^- \to 2\pi$. In table 5 we compare our updated extraction from the 2π data to analogous ones from $e^+e^- \to 3\pi$, $e^+e^- \to \pi^0\gamma$, as well as the average from ref. [109]. As discussed in more detail in refs. [4, 5, 46], the PDG average involves a cancellation between determinations from $e^+e^- \to \pi^0\gamma$ [137] and $\bar{p}p \to \omega\pi^0\pi^0$ [138], while dominated by Breit-Wigner-based extractions from $e^+e^- \to 3\pi$ [12, 116] that are in agreement with the dispersive result given in table 5 (further confirmed by the recent BaBar measurement [115], while BESIII suggests a larger value [114]). Our updated value for $\delta_\epsilon = 0$ changes only marginally compared to ref. [4], leading to the same 5σ tension with the PDG value observed therein. Allowing a finite value for δ_ϵ instead removes about half the discrepancy, but we

[5]Resonance-enhanced threshold effects in the $\bar{K}K$ channels largely cancel between K^+K^- and K_SK_L.

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emphasize that this effect cannot explain the entire tension as it would require a size of δ_ϵ that cannot be reconciled with the strength of the radiative channels giving rise to a phase in ϵ_ω in the first place.

6 Conclusions

In this work we performed a detailed study of ρ-ω mixing in $e^+e^- \to \pi^+\pi^-$, based on a dispersive representation of the pion vector form factor. In particular, we investigated the role of imaginary parts that can be generated by radiative intermediate states coupling ω and ρ resonances, estimated their size by narrow-width arguments, and devised a strategy to include their effect in fits to the $e^+e^- \to \pi^+\pi^-$ data base. We found that while the size of the phase of the ρ-ω mixing parameter in a global fit does come out in agreement with narrow-resonance expectations, see eq. (3.1) for the central results, there is a substantial spread among the different data sets, ranging from a vanishing phase to values as large as $10°$. As applications, we derived the isospin-breaking part of the HVP contribution to a_μ originating from ρ-ω mixing and quantified the changes in the extracted value of the ω mass when a non-vanishing phase is permitted.

Our work reveals systematic differences in the low-energy hadronic cross sections that go beyond the well-known BaBar-KLOE tension in the $e^+e^- \to \pi^+\pi^-$ total cross section, including the spread in the phase of the ρ-ω mixing parameter and discrepancies in the ω mass extracted from different decay channels, both of which can be unambiguously defined in terms of pole parameters and residues. While of course resolving the tension in the HVP integral itself carries the highest priority, forthcoming high-precision data on $e^+e^- \to \pi^+\pi^-$ should also allow one to address the tensions pointed out here, and thus increase confidence that the hadronic cross sections are understood at the level required for robust data-driven evaluations of the HVP contribution to the anomalous magnetic moment of the muon.

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