A note on near hexagons with lines of size 3

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Abstract

We classify all finite near hexagons which satisfy the following properties for a certain $t_2 \in \{1, 2, 4\}$: (i) every line is incident with precisely three points; (ii) for every point x, there exists a point y at distance 3 from x; (iii) every two points at distance 2 from each other have either 1 or $t_2 + 1$ common neighbours; (iv) every quad is big. As a corollary, we obtain a classification of all finite near hexagons satisfying (i), (ii) and (iii) with t_2 equal to 4.

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1 Introduction

1.1 Basic definitions

A *near polygon* is a connected partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$, $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$, with the property that for every point x and every line L, there exists a unique point on L nearest to x. Here distances are measured in the point graph or collinearity graph of the geometry. If the maximal distance between two points of $\mathcal{S}$ is equal to d, then the near polygon is called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. In the sequel, we will denote the unique line with $s + 1$ points by $\mathbb{L}_{s+1}$.

We denote the distance between two points x and y of a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ by $\mathrm{d}(x, y)$. For every point x of $\mathcal{S}$, for every nonempty subset X of $\mathcal{P}$ and every $i \in \mathbb{N}$, we define $\Gamma_i(x) = \{y \in \mathcal{P} \,|\, \mathrm{d}(x, y) = i\}$, $\mathrm{d}(x, X) =$

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$\min\{d(x,y) \mid y \in X\}$ and $\Gamma_i(X) = \{y \in \mathcal{P} \mid d(y,X) = i\}$. A point x of a near hexagon $\mathcal{S}$ is called *special* if $\Gamma_3(x) = \emptyset$.

A near $2d$-gon, $d \geq 2$, is called a *generalized* $2d$*-gon* if $|\Gamma_{i-1}(x) \cap \Gamma_1(y)| = 1$ for every $i \in \{1, \ldots, d-1\}$ and for all points x and y at distance i from each other. A generalized $2d$-gon is called *nondegenerate* if it contains an ordinary $2d$-gon as subgeometry. The class of the generalized quadrangles coincides with the class of the near quadrangles. A degenerate generalized quadrangle consists of a number of lines through a given point. We refer to Payne and Thas [8] and Van Maldeghem [10] for more background information on generalized quadrangles and generalized polygons.

If $\mathcal{S}_1$ and $\mathcal{S}_2$ are two near polygons, then there exists up to isomorphism a unique near polygon $\mathcal{S}$ whose point graph is isomorphic to the cartesian product of the point graphs of $\mathcal{S}_1$ and $\mathcal{S}_2$. We denote $\mathcal{S}$ also by $\mathcal{S}_1 \times \mathcal{S}_2$ and we call it the *direct product* of $\mathcal{S}_1$ and $\mathcal{S}_2$.

If A is a nonempty convex subspace of a near polygon $\mathcal{S}$, then the points and lines of $\mathcal{S}$ contained in A determine a sub(-near-)polygon of $\mathcal{S}$. A convex subspace inducing a nondegenerate generalized quadrangle is called a *quad*. A quad Q of a near hexagon $\mathcal{S}$ is called *big* if every point of $\mathcal{S}$ has distance at most 1 from Q. If x and y are two points of a near polygon, then we denote by $\langle x, y \rangle$ the smallest convex subspace containing x and y, i.e., $\langle x, y \rangle$ is the intersection of all convex subspaces containing x and y.

A near polygon is said to have *order* (s,t) if every line is incident with precisely $s+1$ points and if every point is incident with precisely $t+1$ lines. If $s = t$, then the near polygon is said to have order s. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbours. A near polygon is called *slim* if every line is incident with precisely three points. There are up to isomorphism three slim nondegenerate generalized quadrangles. The (3×3)-grid $\mathbb{L}_3 \times \mathbb{L}_3$ is the unique generalized quadrangle of order $(2,1)$. The unique generalized quadrangle of order 2 is the generalized quadrangle $W(2)$ of the points and lines of $\mathrm{PG}(3,2)$ which are totally isotropic with respect to a given symplectic polarity. The points and lines lying on a given nonsingular elliptic quadric of $\mathrm{PG}(5,2)$ define the generalized quadrangle $Q(5,2)$ which is the unique generalized quadrangle of order $(2,4)$. If $\mathcal{S}$ is a slim near polygon and if x and y are two points of $\mathcal{S}$ at distance 2 from each other having at least two common neighbours, then the convex closure $\langle x, y \rangle$ is a quad by Proposition 2.5 of Shult and Yanushka [9]. This quad necessarily is isomorphic to either $\mathbb{L}_3 \times \mathbb{L}_3$, $W(2)$ or $Q(5,2)$. Hence, every two points of a slim near polygon at distance 2 from each other have either 1, 2, 3 or 5 common neighbours.

The present paper deals with finite slim near hexagons. We will meet the following slim near hexagons in the main result of this paper: (i) $\mathrm{GH}(2,1)$, the unique generalized hexagon of order $(2,1)$ which is the point-line dual of the double of the Fano plane; (ii) the split Cayley hexagon $H(2)$ (see Van Maldeghem [10]); (iii) the point-line dual $H(2)^D$ of $H(2)$; (iv) $\mathrm{GH}(2,8)$, the unique generalized hexagon of order (2,8) (see [10]); (v) the Hamming near hexagon $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$; (vi) the dual polar space $DW(5,2)$ whose points and lines are the totally isotropic planes and lines, respectively, of a symplectic polarity of $\mathrm{PG}(5,2)$; (vii) the dual polar space $DH(5,4)$ whose points and lines are the planes and lines, respectively, lying on a given nonsingular hermitian variety of $\mathrm{PG}(5,4)$.

1.2 Overview

All slim dense near hexagons were classified by Brouwer, Cohen, Hall and Wilbrink in [1]. In recent years several publications occurred dealing with classifications of slim dense near $2d$-gons with $d \geq 4$. We refer to Chapters 6 till 9 of De Bruyn [5] for an overview of these results. In two recent papers (De Bruyn [3] and [4]), also nondense slim near hexagons were investigated. In [3] all slim near hexagons with special points were determined and in [3] + [4] it was show that there are only finitely many finite slim near hexagons without special points. Examples of slim near hexagons without special points can be found in [4] and in the list of references of that paper. The following two classification results which we take from De Bruyn [3] are of special interest for the present paper.

Proposition 1.1 ([3, Theorem 3]) *Let $\mathcal{S}$ be a near hexagon of order $(2,t)$. If $\mathcal{S}$ contains a $Q(5,2)$-quad, then $\mathcal{S}$ is isomorphic to either $Q(5,2) \times \mathbb{L}_3$, $Q(5,2) \otimes Q(5,2)$, $\mathbb{G}_3$, $\mathbb{E}_3$ or $DH(5,4)$.* ∎

Proposition 1.2 ([3, Theorem 5]) *Let $\mathcal{S}$ be a near hexagon of order $(2,t)$. If $\mathcal{S}$ contains a big $W(2)$-quad, then $\mathcal{S}$ is isomorphic to either $W(2) \times \mathbb{L}_3$, $DW(5,2)$ or $\mathbb{H}_3$.* ∎

For the definitions of the near hexagons mentioned in Propositions 1.1 and 1.2, we refer to [5, Chapter 6]. In Section 2, we will also prove the missing classification result which we will need for our main result.

Proposition 1.3 (Section 2) *Let $\mathcal{S}$ be a near hexagon of order $(2,t)$. If $\mathcal{S}$ contains a big grid-quad, then $\mathcal{S}$ is isomorphic to either $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $\mathbb{L}_3 \times W(2)$ or $\mathbb{L}_3 \times Q(5,2)$.*

It can be shown, see [3, Lemma 23], that no near hexagon of order $(2, t)$ has a special point. In the present paper, we continue the classification of the slim near hexagons without special points and with big quads. We will show the following.

Theorem 1.4 (Section 3) *Let $\mathcal{S}$ be a finite slim near hexagon without special points. Suppose that the following property holds for a certain $t_2 \in \{1, 2, 4\}$: if x and y are two points at distance 2 having at least two common neighbours, then x and y are contained in a big quad of order $(2, t_2)$. Then $\mathcal{S}$ is isomorphic to either* $\mathrm{GH}(2, 1)$*,* $H(2)$*,* $H(2)^D$*,* $\mathrm{GH}(2, 8)$*,* $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$*,* $DW(5, 2)$ *or* $DH(5, 4)$*.*

Theorem 1.4 has the following corollary.

Corollary 1.5 *Let $\mathcal{S}$ be a finite slim near hexagon without special points satisfying the property that every two points at distance 2 have either 1 or 5 common neighbours. Then $\mathcal{S}$ is isomorphic to either* $\mathrm{GH}(2, 1)$*,* $H(2)$*,* $H(2)^D$*,* $\mathrm{GH}(2, 8)$ *or* $DH(5, 4)$*.*

Proof. Suppose that there exists a quad Q which is not big. Then there exists a point $x \in \Gamma_2(Q)$ and $\Gamma_2(x) \cap Q$ is an ovoid of Q, i.e. a set of points of Q meeting each line of Q in a unique point. Now, $Q \cong Q(5, 2)$ and $Q(5, 2)$ has no ovoids, see e.g. Payne and Thas [8, 3.4.1]. This leads to a contradiction. Hence, every quad of $\mathcal{S}$ is big. The corollary now follows from Theorem 1.4. ∎

2 Proof of Proposition 1.3

Let $\mathcal{S}$ be a near hexagon of order $(2, t)$ containing a big grid-quad Q.

Lemma 2.1 *The total number of points of $\mathcal{S}$ is equal to $v = 9 + 18(t - 1)$. Every grid-quad is big.*

Proof. If R is a grid-quad, then $|R| + |\Gamma_1(R)| = 9 + 18(t - 1)$. The lemma now readily follows. ∎

Lemma 2.2 *Let R be a (big) grid-quad, let x be a point of $\mathcal{S}$ not contained in R and let x' denote the unique point of R collinear with x. Then the quads through the line xx' partition the set of lines through x (or x') different from xx'*

Proof. Let L denote a line through x different from xx', let y be a point of $L \setminus \{x\}$ and let y' denote the unique point of R collinear with y. Then x' and y are two common neighbours of x and y'. Hence, x and y' are contained in a quad which contains the line L. This proves that the quads through xx' partition the set of lines through x different from xx'. Since the point x' is incident with the same number of lines as x, the quads through xx' also partition the set of lines through x' different from xx'. ■

Lemma 2.3 *Every point x is contained in a big grid-quad.*

Proof. We may suppose that x is not contained in the big grid-quad Q. Let x' denote the unique point of Q collinear with x and let L_1 and L_2 denote the two lines of Q through x'. By Lemma 2.2, there exists a quad R_i, $i \in \{1, 2\}$, containing x and the line L_i. Let R_i, $i \in \{1, 2\}$, have order $(2, \alpha_i)$. Then $t + 1 \geq 1 + \alpha_1 + \alpha_2$. Suppose none of R_1, R_2 is a grid-quad. Then $\alpha_1, \alpha_2 \geq 2$ and $|\Gamma_1(R_1)| + |R_1| = 3(1 + 2\alpha_1)(1 + 2(t - \alpha_1)) = 3(1 + 2t + 4\alpha_1(t - \alpha_1))$. This number is at most v. By Lemma 2.1,

$$\begin{aligned} 3(1 + 2t) + 12\alpha_1(t - \alpha_1) &\leq 18t - 9 \\ 12\alpha_1(t - \alpha_1) &\leq 12t - 12 \\ \alpha_1(t - \alpha_1) &\leq t - 1. \end{aligned}$$

This is impossible, since $\alpha_1 \geq 2$ and $t \geq \alpha_1 + \alpha_2 > \alpha_1 + 1$. Hence, at least one of R_1, R_2 is a (necessarily big) grid-quad. This proves the lemma. ■

Lemma 2.4 *The near hexagon $\mathcal{S}$ is dense.*

Proof. We must prove that every two points x and y at distance 2 are contained in a unique quad. Let R denote a big quad through x and let z denote a common neighbour of x and y. If $x, y \in R$, then we are done. If $z \notin R$, then the quads through xz partition the set of lines through z different from xz and it follows that x and y are contained in a quad. If $z \in R$ and $y \notin R$, then the quads through yz partition the set of lines through z different from yz and again it follows that x and y are contained in a quad. ■

By the classification of the slim dense near hexagons ([1]), it now follows that $\mathcal{S}$ is isomorphic to either $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $\mathbb{L}_3 \times W(2)$ or $\mathbb{L}_3 \times Q(5, 2)$.

3 Proof of Theorem 1.4

Let $\mathcal{S}$ be a finite slim near hexagon satisfying the conditions of Theorem 1.4. Let v denote the total number of points of $\mathcal{S}$. For every point x of $\mathcal{S}$, let

$t_x + 1$ denote the total number of lines through x and let α_x denote the total number of quads of order $(2, t_2)$ through x.

Lemma 3.1 *If there exists no quad, then $\mathcal{S}$ is isomorphic to either* $\mathrm{GH}(2,1)$*, $H(2)$, $H(2)^D$ or* $\mathrm{GH}(2,8)$.

Proof. Since there are no quads, every two points at distance 2 have precisely one common neighbour. If x is a point which is incident with at least two lines, then there exists an ordinary hexagon through x and any point at distance 3 from x. This proves that $\mathcal{S}$ is a nondegenerate generalized hexagon. $\mathcal{S}$ must have an order $(2, t)$ (see e.g. Van Maldeghem [10]) and by results of Feit - Higman [6] and Haemers - Roos [7], we know that $t \in \{1, 2, 8\}$. It is easily seen that there exists a unique generalized hexagon of order $(2, 1)$. The generalized hexagons of order 2 and order (2,8) have been classified by Cohen and Tits [2]. ∎

Lemma 3.2 *If $\mathcal{S}$ has an order $(2, t)$ and if $\mathcal{S}$ contains a (big) quad of order $(2, t_2)$, then $\mathcal{S}$ is isomorphic to either $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$, $DW(5, 2)$ and $DH(5, 4)$.*

Proof. This is an immediate corollary of Propositions 1.1, 1.2 and 1.3. ∎

In the sequel, we will suppose that $\mathcal{S}$ contains a big quad and that t_x is not constant. We will derive a contradiction.

Lemma 3.3 *If x and y are two points at distance 3 from each other, then $t_x + 1 - t_2 \cdot \alpha_x = t_y + 1 - t_2 \cdot \alpha_y$.*

Proof. Every line through x contains a unique point at distance 2 from y and each of the α_y quads through y contains a unique point collinear with x (since each such quad is big). Hence, the total number of paths of length 3 connecting x and y is equal to $(t_x + 1 - \alpha_y) \cdot 1 + \alpha_y \cdot (t_2 + 1) = t_x + 1 + t_2 \cdot \alpha_y$. By symmetry, this number is also equal to $t_y + 1 + t_2 \cdot \alpha_x$. The lemma now follows. ∎

Lemma 3.4 *Let Γ be the graph whose vertices are the points of $\mathcal{S}$ with two vertices adjacent whenever they are at distance 3 from each other. Then Γ is connected.*

Proof. Let Q denote a (big) quad of order $(2, t_2)$. The lemma follows from the following two observations: (i) if x and y are two points of $\mathcal{S}$ not contained in Q, then there exists a point z in Q at distance 3 from x and y; (ii) if u is a point of Q, then every point at distance 3 from u is not contained in Q. (Property (i) follows from the fact that there exists a point in Q at distance 2 from the unique points of Q collinear with x and y.) ∎

Lemmas 3.3 and 3.4 have the following corollary.

Corollary 3.5 *There exists a constant a such that $a = t_x - t_2 \cdot \alpha_x$ for every point x of $\mathcal{S}$.* ∎

Let N denote the total number of quads of order $(2, t_2)$. For every point x of $\mathcal{S}$, let $n_2(x)$ (respectively $n_3(x)$) denote the total number of points at distance 2 (respectively distance 3) from x.

Lemma 3.6 *For every point x of $\mathcal{S}$, we have $n_2(x) = (4a+2)t_x + 2a + 2t_2N$.*

Proof. Counting the number of pairs (u, v) with $u \in \Gamma_1(x)$, $v \in \Gamma_2(x)$ and $u \sim v$, yields $(n_2(x) - 4t_2\alpha_x) \cdot 1 + (4t_2\alpha_x) \cdot (t_2 + 1) = \sum_{y \in \Gamma_1(x)} (2t_y)$, or

$$n_2(x) = \sum_{y \in \Gamma_1(x)} (2t_y) - 4t_2^2\alpha_x.$$

Since every quad either contains x or has a unique point collinear with x,

$$\sum_{y \in \Gamma_1(x)} \alpha_y = N - \alpha_x + 2(t_2 + 1)\alpha_x.$$

So, with the aid of Corollary 3.5, we obtain

$$\begin{aligned}
n_2(x) &= \sum_{y \in \Gamma_1(x)} (2t_y) - 4t_2^2\alpha_x \\
&= 2 \sum_{y \in \Gamma_1(x)} (a + t_2\alpha_y) - 4t_2^2\alpha_x \\
&= 4a(t_x + 1) + 2t_2 \sum_{y \in \Gamma_1(x)} \alpha_y - 4t_2^2\alpha_x \\
&= 4a(t_x + 1) + 2t_2(N - \alpha_x + 2(t_2 + 1)\alpha_x) - 4t_2^2\alpha_x \\
&= 4a(t_x + 1) + 2t_2N + 2t_2\alpha_x \\
&= 4a(t_x + 1) + 2t_2N + 2(t_x - a) \\
&= (4a + 2)t_x + 2a + 2t_2N.
\end{aligned}$$

∎

Lemma 3.7 *For every line $L = \{x, y, z\}$ of $\mathcal{S}$, $v = 3 + 4a(t_x + t_y + t_z) + 6a + 6t_2N$.*

Proof. Since every point of $\Gamma_1(L)$ has distance 2 to precisely two points of L, $v = 3 + n_2(x) + n_2(y) + n_2(z) - 2(t_x + t_y + t_z)$. Hence, $v = 3 + 4a(t_x + t_y + t_z) + 6a + 6t_2N$ by Lemma 3.6. ∎

We will now distinguish between the cases $a \neq 0$ and $a = 0$.

(I) The case $a \neq 0$

By Lemma 3.7, there exists a constant b such that $t_x + t_y + t_z = b$ for every line $L = \{x, y, z\}$ of $\mathcal{S}$.

Lemma 3.8 $v = (2t_2 + 1)[2b - 3(2t_2 - 1)]$.

Proof. Let Q be a quad of order $(2, t_2)$. Then $(t_2 + 1) \cdot \sum_{x \in Q} t_x = \sum_{L \subseteq Q} \sum_{x \in L} t_x = \sum_{L \subseteq Q} b = (t_2 + 1)(1 + 2t_2)b$ (summation over all lines L of Q) or $\sum_{x \in Q} t_x = (1 + 2t_2)b$. Now, $v = |Q| + |\Gamma_1(Q)| = 3(1 + 2t_2) + \sum_{x \in Q} 2(t_x - t_2) = 2(1 + 2t_2)b - 3(2t_2 - 1)(2t_2 + 1)$. ■

Lemma 3.9 $n_2(x) = (2b - 2 - 4t_2)t_x - 2t_x^2 + 2b + 4t_2 a$.

Proof. Taking the expression for $n_2(x)$ derived in the proof of Lemma 3.6, we obtain

$$\begin{aligned} n_2(x) &= \sum_{y \in \Gamma_1(x)} (2t_y) - 4t_2^2 \alpha_x \\ &= \sum_{LIx} (2b - 2t_x) - 4t_2^2 \alpha_x \\ &= 2b(t_x + 1) - 2t_x(t_x + 1) - 4t_2(t_x - a) \\ &= (2b - 2 - 4t_2)t_x - 2t_x^2 + 2b + 4t_2 a. \end{aligned}$$

■

Let r_1 and r_2 denote the two roots of the equation $(4a + 2)X + 2a + 2t_2 N = (2b - 2 - 4t_2)X - 2X^2 + 2b + 4t_2 a$. By Lemmas 3.6 and 3.9, $t_x \in \{r_1, r_2\}$ for every point x of $\mathcal{S}$. Since t_x is not constant, $r_1 \neq r_2$. So, there exists a δ such that $b = \delta r_1 + (3 - \delta) r_2$ and it follows that every line is incident with precisely δ points x with $t_x = r_1$ and $3 - \delta$ points y with $t_y = r_2$. Since t_x is not constant, $\delta \in \{1, 2\}$. Without loss of generality, we may suppose that every line is incident with a unique point x with $t_x = r_1$ and two points y with $t_y = r_2$.

Lemma 3.10 *It holds* $t_2 \neq 4$.

Proof. Suppose that $t_2 = 4$ and let Q denote a quad of order $(2, 4)$, i.e. a $Q(5, 2)$-quad. Then the points x in Q with $t_x = r_1$ form an ovoid of Q, a contradiction, since $Q(5, 2)$ has no ovoids. ■

Lemma 3.11 *It holds* $t_2 \neq 2$.

Proof. Suppose $t_2 = 2$, let Q denote a $W(2)$-quad, let x denote a point not contained in Q and let x' be the unique point of Q nearest to x. Let α'_x denote the total number of $W(2)$-quads through xx'. Then $\alpha_x \geq \alpha'_x$. Now, the quads through xx' partition the set of lines through x different from xx' (similar proof as in Lemma 2.2). Hence, $t_x = 2\alpha'_x$. Now, $0 \neq a = t_x - 2\alpha_x = 2(\alpha'_x - \alpha_x)$. Hence, $\alpha_x > \alpha'_x$ and there exists a $W(2)$-quad Q' through x not containing xx'. Let Q'' denote the set of points not contained in $Q \cup Q'$ which are contained in a unique line connecting a point of Q with a point of Q'. Then Q'' is a $W(2)$-quad and $H = Q \cup Q' \cup Q''$ is a subspace inducing a subhexagon isomorphic to $W(2) \times \mathbb{L}_3$. The points z in H with $t_z = r_1$ form an ovoid of H, a contradiction, since $W(2) \times \mathbb{L}_3$ has no ovoids ($W(2)$ has no partition in ovoids). ■

Lemma 3.12 *It holds $t_2 \neq 1$.*

Proof. Suppose $t_2 = 1$ and let Q denote a grid-quad, let x denote a point not contained in Q and let x' denote the unique point of Q nearest to x. Let α'_x denote the total number of grid-quads through xx'. Then $\alpha_x \geq \alpha'_x$. Since the quads through xx' partition the set of lines through x different from xx', $t_x = \alpha'_x$. Now, $0 \neq a = t_x - \alpha_x = \alpha'_x - \alpha_x$. Hence, $\alpha_x > \alpha'_x$ and there exists a grid-quad Q' through x not containing xx'. Let Q'' denote the set of points not contained in $Q \cup Q'$ which are contained in a line connecting a point of Q with a point of Q'. Then $H := Q \cup Q' \cup Q''$ is a subspace which induces a subhexagon isomorphic to $\mathbb{L}_3 \times \mathbb{L}_3 \times \mathbb{L}_3$. Since $\mathcal{S} \neq H$ (t_x is not constant), there exists a point z in $\mathcal{S}$ not contained in H. Let z_1 denote the unique point of Q collinear with z and let z_2 denote the unique point of Q' collinear with z. Then the points z_1 and z_2 have distance 2 from each other and have at least three common neighbours (the point z, a point of Q and a point of Q'), a contradiction. ■

Lemmas 3.10, 3.11 and 3.12 give the desired contradiction.

(II) The case $a = 0$

Lemma 3.13 *Every two distinct quads intersect in a line.*

Proof. Suppose the contrary and let Q and Q' denote two disjoint quads. (Since every quad is big, every two distinct quads either are disjoint or intersect in a line, see e.g. Theorem 1.7 of [5].) Let x and x' be two collinear points in respectively Q and Q'. Let β denote the total number of quads through xx'. Then $\alpha_x \geq \beta + 1$ and $t_x = t_2\beta$ (the quads through xx' partition the set of lines through x). Now, $0 = a = t_x - t_2\alpha_x = t_2(\beta - \alpha_x)$ or $\beta = \alpha_x$, a contradiction. ■

Let I be the intersection of all quads.

Lemma 3.14 *There are three possibilities: (i) I is a quad; (ii) I is a line; (iii) I is a point.*

Proof. Let Q denote a quad of $\mathcal{S}$. If Q is the only quad, then we have possibility (i). Suppose therefore that there are at least two quads. If R_1 and R_2 are two quads different from Q such that the lines $R_1 \cap Q$ and $R_2 \cap Q$ are disjoint, then R_1 and R_2 themselves are also disjoint, a contradiction. Hence, the lines $R \cap Q$, R a quad different from Q, do mutually intersect. So, we have possibility (ii) or (iii). ■

Lemma 3.15 *I cannot be a point or a line.*

Proof. Suppose the contrary. Let x denote an arbitrary point of I and let y denote an arbitrary point at distance 3 from x. Since y is not contained in a quad, $t_y = 0$ by Corollary 3.5. Now, there are two different quads Q_1 and Q_2 through x. Through y, there is a line L_i, $i \in \{1, 2\}$, meeting Q_i. Since $L_1 \neq L_2$, we have $t_y + 1 \geq 2$, a contradiction. ■

Lemma 3.16 *I cannot be a quad.*

Proof. Suppose the contrary. Then by Corollary 3.5, $t_x = t_2$ for every point x of Q. So, there are no lines intersecting Q in a point, a contradiction. ■

A contradiction follows from Lemmas 3.14, 3.15 and 3.16.

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