# The valuations of the near octagon $\mathbb{G}_{4}$ 

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#### Abstract

In [4] it was shown that the dual polar space $D H(2 n-1,4), n \geq 2$, has a sub near- $2 n$-gon $\mathbb{G}_{n}$ with a large automorphism group. In this paper, we classify the valuations of the near octagon $\mathbb{G}_{4}$. We show that each such valuation is either classical, the extension of a non-classical valuation of a $\mathbb{G}_{3}$-hex or is associated with a valuation of Fano-type of an $\mathbb{H}_{3}$-hex. In order to describe the latter type of valuation we must study the structure of $\mathbb{G}_{4}$ with respect to an $\mathbb{H}_{3}$-hex. This study also allows us to construct new hyperplanes of $\mathbb{G}_{4}$. We also show that each valuation of $\mathbb{G}_{4}$ is induced by a (classical) valuation of the dual polar space $D H(7,4)$.


Keywords: near polygon, generalized quadrangle, dual polar space, valuation, hyperplane.
MSC2000: 51A50, 51E12, 05B25

## 1 Introduction

### 1.1 Basic definitions

Let $\mathcal{S}$ be a dense near $2 n$-gon, i.e. $\mathcal{S}$ is a partial linear space which satisfies the following properties:
(i) For every point $p$ and every line $L$, there exists a unique point $\pi_{L}(p)$ on $L$ nearest to $p$. Here, distances $\mathrm{d}(\cdot, \cdot)$ are measured in the collinearity graph of $\mathcal{S}$.
(ii) Every line of $\mathcal{S}$ is incident with at least three points.
(iii) Every two points of $\mathcal{S}$ at distance 2 from each other have at least two common neighbours.
(iv) The maximal distance between two points of $\mathcal{S}$ is equal to $n$.

A dense near 0 -gon is a point, a dense near 2 -gon is a line and a dense near quadrangle is a generalized quadrangle (Payne and Thas [17]).

For every point $y$ of $\mathcal{S}$ and every non-empty set $X$ of points, we define $\mathrm{d}(y, X):=\min \{\mathrm{d}(y, x) \mid x \in X\}$. If $X$ is a non-empty set of points of $\mathcal{S}$, then for every $i \in \mathbb{N}, \Gamma_{i}(X)$ denotes the set of points $y$ of $\mathcal{S}$ at distance $i$ from $X$. If $X$ is a singleton $\{x\}$, we will also write $\Gamma_{i}(x)$ instead of $\Gamma_{i}(X)$.

One of the following two cases occurs for two lines $K$ and $L$ of $\mathcal{S}$ (see e.g. [5, Theorem 1.3]): (i) there exist unique points $k^{*} \in K$ and $l^{*} \in L$ such that $\mathrm{d}(k, l)=\mathrm{d}\left(k, k^{*}\right)+\mathrm{d}\left(k^{*}, l^{*}\right)+\mathrm{d}\left(l^{*}, l\right)$ for all $k \in K$ and $l \in L$; (ii) the map $K \rightarrow L ; x \mapsto \pi_{L}(x)$ is a bijection and its inverse is equal to the map $L \rightarrow K ; y \mapsto \pi_{K}(y)$. If the latter case occurs, then $K$ and $L$ are called parallel.

By Theorem 4 of Brouwer and Wilbrink [2], every two points $x$ and $y$ of $\mathcal{S}$ at distance $\delta \in\{0, \ldots, n\}$ from each other are contained in a unique convex subspace $\langle x, y\rangle$ of diameter $\delta$. These convex subspaces are called quads, respectively hexes, if $\delta=2$, respectively $\delta=3$. The lines and quads through a given point $x$ of $\mathcal{S}$ define a linear space which is called the local space at $x$. If $X_{1}, X_{2}, \ldots, X_{k}$ are non-empty sets of points, then $\left\langle X_{1}, X_{2}, \ldots, X_{k}\right\rangle$ denotes the smallest convex subspace containing $X_{1} \cup X_{2} \cup \cdots \cup X_{k}$. Clearly, $\left\langle X_{1}, X_{2}, \ldots, X_{k}\right\rangle$ is the intersection of all convex subspaces containing $X_{1} \cup$ $X_{2} \cup \cdots \cup X_{k}$.

A point $x$ of $\mathcal{S}$ is called classical with respect to a convex subspace $F$ of $\mathcal{S}$ if there exists a (necessarily unique) point $\pi_{F}(x)$ in $F$ such that $\mathrm{d}(x, y)=$ $\mathrm{d}\left(x, \pi_{F}(x)\right)+\mathrm{d}\left(\pi_{F}(x), y\right)$ for every point $y$ of $F$. Every point of $\Gamma_{1}(F)$ is classical with respect to $F$. A convex subspace $F$ of $\mathcal{S}$ is called classical (in $\mathcal{S}$ ) if every point of $\mathcal{S}$ is classical with respect to $F$. Every line of $\mathcal{S}$ is classical. If every quad of $\mathcal{S}$ is classical, then $\mathcal{S}$ is a so-called dual polar space (Cameron [3]). The near polygon $\mathcal{S}$ is then isomorphic to a geometry $\Delta$ whose points and lines are the maximal and next-to-maximal singular subspaces of a given polar space $\Pi$ (natural incidence). A proper convex subspace $F$ of $\mathcal{S}$ is called big (in $\mathcal{S}$ ) if every point of $\mathcal{S}$ has distance at most 1 from $F$. If $F$ is big, then $F$ is also classical. If $F$ is big and if every line of $\mathcal{S}$ is incident with precisely 3 points, then we can define a reflection $\mathcal{R}_{F}$ about $F$ which is an automorphism of $\mathcal{S}$ : if $x \in F$, then we define $\mathcal{R}_{F}(x):=x$; if $x \notin F$, then $\mathcal{R}_{F}(x)$ is the unique point on the line $x \pi_{F}(x)$ different from $x$ and $\pi_{F}(x)$.

Near polygons were introduced by Shult and Yanushka [18]. We refer to (Chapter 2 of) De Bruyn [5] for more background information on (dense) near polygons.

A function $f$ from the point-set of $\mathcal{S}$ to $\mathbb{N}$ is called a valuation of $\mathcal{S}$ if it satisfies the following properties:
(V1) $f^{-1}(0) \neq \emptyset$;
(V2) every line $L$ of $\mathcal{S}$ contains a unique point $x_{L}$ such that $f(x)=f\left(x_{L}\right)+1$ for every point $x$ of $L$ different from $x_{L}$;
(V3) every point $x$ of $\mathcal{S}$ is contained in a (necessarily unique) convex subspace $F_{x}$ such that the following properties are satisfied for every $y \in F_{x}$ :
(i) $f(y) \leq f(x)$;
(ii) if $z$ is a point collinear with $y$ such that $f(z)=f(y)-1$, then $z \in F_{x}$.

Valuations of dense near polygons were introduced in De Bruyn and Vandecasteele [11]. For many classes of dense near polygons, see [10], it can be shown that property (V3) is a consequence of property (V2).

If $f$ is a valuation of $\mathcal{S}$, then we denote by $O_{f}$ the set of points with value 0 . A quad $Q$ of $\mathcal{S}$ is called special (with respect to $f$ ) if it contains two distinct points of $O_{f}$, or equivalently (see [11]), if it intersects $O_{f}$ in an ovoid of $Q$. We denote by $G_{f}$ the partial linear space with points the elements of $O_{f}$ and with lines the special quads (natural incidence).

Proposition 1.1 (Proposition 2.12 of [11]) Let $\mathcal{S}$ be a dense near polygon and let $F=\left(\mathcal{P}^{\prime}, \mathcal{L}^{\prime}, \mathrm{I}^{\prime}\right)$ be a (not necessarily convex) subpolygon of $\mathcal{S}$ for which the following holds: (1) $F$ is a dense near polygon; (2) $F$ is a subspace of $\mathcal{S}$; (3) if $x$ and $y$ are two points of $F$, then $d_{F}(x, y)=d_{\mathcal{S}}(x, y)$. Let $f$ denote a valuation of $\mathcal{S}$ and put $m:=\min \left\{f(x) \mid x \in \mathcal{P}^{\prime}\right\}$. Then the map $f_{F}: \mathcal{P}^{\prime} \rightarrow \mathbb{N} ; x \mapsto f(x)-m$ is a valuation of $F$.

Definition. The valuation $f_{F}$ of $F$ defined in Proposition 1.1 is called the valuation of $F$ induced by $f$.

Examples. Let $\mathcal{S}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a dense near $2 n$-gon, $n \geq 2$.
(1) For every point $x$ of $\mathcal{S}$, the map $\mathcal{P} \rightarrow \mathbb{N} ; y \mapsto \mathrm{~d}(x, y)$ is a valuation of $\mathcal{S}$ which we call a classical valuation.
(2) Suppose $O$ is an ovoid of $\mathcal{S}$, i.e. a set of points meeting each line in a unique point. For every point $x$ of $\mathcal{S}$, we define $f_{O}(x)=0$ if $x \in O$ and $f_{O}(x)=1$ otherwise. Then $f_{O}$ is a valuation of $\mathcal{S}$ which we call an ovoidal valuation.
(3) Let $x$ be a point of $\mathcal{S}$ and let $O$ be a set of points at distance $n$ from $x$ having a unique point in common with every line at distance $n-1$ from $x$. For every point $y$ of $\mathcal{S}$, we define $f(y)=\mathrm{d}(x, y)$ if $\mathrm{d}(x, y) \leq n-1$,
$f(y)=n-2$ if $y \in O$ and $f(y)=n-1$ otherwise. Then $f$ is a valuation of $\mathcal{S}$ which we call a semi-classical valuation.
(4) Suppose $F=\left(\mathcal{P}^{\prime}, \mathcal{L}^{\prime}, I^{\prime}\right)$ is a convex subspace of $\mathcal{S}$ which is classical in $\mathcal{S}$. Suppose that $f^{\prime}: \mathcal{P}^{\prime} \rightarrow \mathbb{N}$ is a valuation of $F$. Then the map $f: \mathcal{P} \rightarrow$ $\mathbb{N} ; x \mapsto f(x):=\mathrm{d}\left(x, \pi_{F}(x)\right)+f^{\prime}\left(\pi_{F}(x)\right)$ is a valuation of $\mathcal{S}$. We call $f$ the extension of $f^{\prime}$.

In the literature, valuations have been used for the following important applications: (i) classification of dense near polygons ([9], [16]); (ii) constructions of new hyperplanes of dense near polygons, in particular of dual polar spaces ([8], [12]); (iii) classification of certain hyperplanes of dense near polygons, in particular of dual polar spaces ([6]); (iv) study of isometric full embeddings between dense near polygons, in particular between dual polar spaces ([7], [14], [15]).

We will now define two classes of dense near polygons which will be important throughout this paper.
(I) Let $X$ be a set of size $2 n+2, n \geq 2$, and let $\mathbb{H}_{n}=(\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the following point-line geometry:
(i) $\mathcal{P}$ is the set of all partitions of $X$ in $n+1$ subsets of size 2 ;
(ii) $\mathcal{L}$ is the set of all partitions of $X$ in $n-1$ subsets of size 2 and one subset of size 4;
(iii) a point $p \in \mathcal{P}$ is incident with a line $L \in \mathcal{L}$ if and only if the partition determined by the point $p$ is a refinement of the partition determined by $L$.
By Brouwer, Cohen, Hall and Wilbrink [1], see also De Bruyn [5, Section 6.2], $\mathbb{H}_{n}$ is a dense near $2 n$-gon with three points per line. The near polygon $\mathbb{H}_{n}$ has $\frac{(2 n+2)!}{2^{n+1} \cdot(n+1)!}$ points and each point is incident with $\binom{n+1}{2}$ lines. Every quad of $\mathbb{H}_{n}$ is isomorphic to either the $(3 \times 3)$-grid or the generalized quadrangle $W(2)$. Every $W(2)$-quad is classical in $\mathbb{H}_{n}$. By De Bruyn [10, Corollary 1.4], a map $f: \mathcal{P} \rightarrow \mathbb{N}$ is a valuation of $\mathbb{H}_{n}$ if and only if it satisfies properties (V1) and (V2).

The near hexagon $\mathbb{H}_{3}$ will be of interest in this paper. Every $W(2)$-quad of $\mathbb{H}_{3}$ is big. Every local space of $\mathbb{H}_{3}$ is isomorphic to the Fano-plane in which a point has been removed. Hence, every point of $\mathbb{H}_{3}$ is contained in three grid-quads and these grid-quads partition the set of lines through $x$. If $x$ is a point of $\mathbb{H}_{3}$ at distance 2 from a grid-quad $Q$, then $\Gamma_{2}(x) \cap Q$ is an ovoid of $Q$. Moreover, the three quads through $x$ which meet $Q$ are grids.
(II) Let $H(2 n-1,4), n \geq 2$, denote the Hermitian variety $X_{0}^{3}+X_{1}^{3}+\cdots+$ $X_{2 n-1}^{3}=0$ of $\operatorname{PG}(2 n-1,4)$ (with respect to a given reference system). The
number of nonzero coordinates (with respect to the same reference system) of a point $p$ of $\mathrm{PG}(2 n-1,4)$ is called the weight of $p$. With the Hermitian variety $H(2 n-1,4)$, there is associated a dual polar space which is denoted by $D H(2 n-1,4)$. The points and lines of $D H(2 n-1,4)$ are the maximal and next-to-maximal subspaces of $H(2 n-1,4)$ (natural incidence). Let $\mathbb{G}_{n}=(\mathcal{P}, \mathcal{L}$, I $)$ be the following subgeometry of $D H(2 n-1,4)$ :
(i) $\mathcal{P}$ is the set of all maximal subspaces of $H(2 n-1,4)$ containing $n$ points with weight 2 ;
(ii) $\mathcal{L}$ is the set of all $(n-2)$-dimensional subspaces of $H(2 n-1,4)$ containing at least $n-2$ points of weight 2 ;
(iii) incidence is reverse containment.

By De Bruyn [4], see also De Bruyn [5, 6.3], $\mathbb{G}_{n}$ is a dense near $2 n$-gon with three points on each line and its above-defined embedding in $D H(2 n-1,4)$ is isometric, i.e. preserves distances. The near polygon $\mathbb{G}_{n}$ has $\frac{3^{n} \cdot(2 n)!}{2^{n} \cdot n!}$ points and each point of $\mathbb{G}_{n}$ is contained in precisely $\frac{n(3 n-1)}{2}$ lines. Every quad of $\mathbb{G}_{n}$ is isomorphic to either the $(3 \times 3)$-grid, $W(2)$ or the generalized quadrangle $Q(5,2)$. Every $Q(5,2)$-quad is classical in $\mathbb{G}_{n}$. By De Bruyn [10, Corollary 1.4], a map $f: \mathcal{P} \rightarrow \mathbb{N}$ is a valuation of $\mathbb{G}_{n}$ if and only if it satisfies properties (V1) and (V2).

### 1.2 The main result

The near octagon $\mathbb{G}_{4}$ has hexes isomorphic to $\mathbb{G}_{3}$ and $\mathbb{H}_{3}$. Every $\mathbb{G}_{3}$-hex $F$ is big in $\mathbb{G}_{4}$ and hence every valuation $f$ of $F$ will give rise to a valuation of $\mathbb{G}_{4}$, namely the extension of $f$. No $\mathbb{H}_{3}$-hex is big in $F$. We will later show (Propositions 5.1 and 6.10 ) that if $f$ is a valuation of an $\mathbb{H}_{3}$-hex $F$ such that $G_{f}$ is a Fano-plane, then there exists a unique valuation $\bar{f}$ of $\mathbb{G}_{4}$ such that $O_{\bar{f}}=O_{f}$. We will call $\bar{f}$ a valuation of Fano-type of $\mathbb{G}_{4}$. In this paper, we classify all valuations of $\mathbb{G}_{4}$. We will show the following.

Theorem 1.2 (Section 6) If $f$ is a valuation of $\mathbb{G}_{4}$, then $f$ is one of the following:
(1) $f$ is a classical valuation of $\mathbb{G}_{4}$;
(2) $f$ is the extension of a non-classical valuation of $a \mathbb{G}_{3}$-hex of $\mathbb{G}_{4}$;
(3) $f$ is a valuation of Fano-type of $\mathbb{G}_{4}$.

Each of these valuations is induced by a unique (classical) valuation of $D H(7,4)$.
Notice that all valuations of $\operatorname{DH}(7,4)$ are classical by Theorem 6.8 of De Bruyn [5]. In order to describe the valuations of Fano-type of $\mathbb{G}_{4}$ (see Section 5), we must study the structure of $\mathbb{G}_{4}$ with respect to an $\mathbb{H}_{3}$-hex (Section 4).

This study allows us to construct a class of hyperplanes of $\mathbb{G}_{4}$ (Proposition 4.14).

## 2 The valuations of the near hexagons $\mathbb{G}_{3}, \mathbb{H}_{3}$, $Q(5,2) \times \mathbb{L}_{3}$ and $W(2) \times \mathbb{L}_{3}$

The valuations of the near hexagons $\mathbb{G}_{3}, \mathbb{H}_{3}, Q(5,2) \times \mathbb{L}_{3}$ and $W(2) \times \mathbb{L}_{3}$ were determined in De Bruyn and Vandecasteele [13].

There are two types of valuations in $\mathbb{G}_{3}$ : the classical valuations and the non-classical valuations. In the following lemma, we collect some known facts about non-classical valuations of $\mathbb{G}_{3}$.

Lemma 2.1 ([13]) Suppose $f$ is a non-classical valuation of $\mathbb{G}_{3}$. Then:
(i) $G_{f}$ is isomorphic to $\overline{W(2)}$, the linear space obtained from the generalized quadrangle $W(2)$ by adding its ovoids as extra lines.
(ii) $\left|O_{f}\right|=15$ and every two distinct points of $O_{f}$ lie at distance 2 from each other.
(iii) Every point with value 1 is contained in a unique special quad.
(iv) Every $Q(5,2)$-quad $Q$ of $\mathbb{G}_{3}$ contains a unique point with value 0. Moreover, $f(y)=d\left(y, Q \cap O_{f}\right)$ for every point $y$ of $Q$.
(v) Every point $x$ of $O_{f}$ is contained in three special grid-quads and two special $W(2)$-quads. These five quads determine a partition of the set of lines through $x$.

If $f$ is a valuation of $\mathbb{H}_{3}$, then any two distinct points of $O_{f}$ lie at distance 2 from each other. There are four types of valuations in the near hexagon $\mathbb{H}_{3}$ : the classical valuations, the extensions of the ovoidal valuations of the $W(2)$-quads (valuations of extended type), the valuations $f$ for which $G_{f}$ is a line of size 3 (valuations of grid-type) and the valuations $f$ for which $G_{f}$ is a Fano-plane (valuations of Fano-type). In the following two lemmas, we collect some known facts about valuations of grid-type and Fano-type.

Lemma 2.2 ([13]) Let $f$ be a valuation of grid-type of $\mathbb{H}_{3}$. Then $O_{f}$ is an ovoid of a grid-quad $Q$ of $\mathbb{H}_{3}$. If $d\left(x, O_{f}\right) \leq 2$, then $f(x)=d\left(x, O_{f}\right)$. If $d\left(x, O_{f}\right)=3$, then $f(x)=1$.

Lemma 2.3 ([13]) Let $f$ be a valuation of Fano-type of $\mathbb{H}_{3}$. Then:
(i) Every $W(2)$-quad $R$ contains a unique point of $O_{f}$ and $f(y)=d\left(y, O_{f} \cap\right.$ R) for every $y \in R$.
(ii) Every grid-quad intersects $O_{f}$ in either the empty set or an ovoid of the grid-quad. If a grid-quad $Q$ is disjoint from $O_{f}$, then $Q$ intersects the set of points with value 1 in an ovoid of $Q$.
(iii) For every $x \in O_{f}$, the three grid-quads through $x$ are special.
(iv) Every point with value 1 is contained in a unique special quad.

Lemma 2.4 Let $f$ be a valuation of Fano-type of $\mathbb{H}_{3}$. Let $Q$ be a $W(2)$-quad of $\mathbb{H}_{3}$ and let $G_{2}$ and $G_{3}$ be two grid-quads of $\mathbb{H}_{3}$ such that $(i) Q, G_{2}$ and $G_{3}$ are mutually disjoint, and (ii) $\mathcal{R}_{Q}\left(G_{2}\right)=G_{3}$. Put $G_{1}:=\pi_{Q}\left(G_{2}\right)=\pi_{Q}\left(G_{3}\right)$. Then one of the following cases occurs:
(1) There exists precisely one $i \in\{2,3\}$ such that $\left|G_{i} \cap O_{f}\right|=3$ and $\left|G_{5-i} \cap O_{f}\right|=0$. Moreover, the unique point in $O_{f} \cap Q$ is not contained in $G_{1}$.
(2) $\left|G_{2} \cap O_{f}\right|=\left|G_{3} \cap O_{f}\right|=0$ and the unique point in $O_{f} \cap Q$ is contained in $G_{1}$.

Proof. Let $x^{*}$ denote the unique point of $O_{f} \cap Q$. Recall that $f(y)=\mathrm{d}\left(y, x^{*}\right)$ for every $y \in Q$. We distinguish two cases.
(1) Suppose $x^{*}$ is not contained in $G_{1}$. Put $\Gamma_{1}\left(x^{*}\right) \cap G_{1}=\left\{x_{1}, x_{2}, x_{3}\right\}$ and let $L_{i}, i \in\{1,2,3\}$, denote the unique line through $x_{i}$ meeting $G_{2}$ and $G_{3}$. Since $x^{*} \notin G_{1}$, we have $\mathrm{d}\left(x^{*}, G_{2}\right)=\mathrm{d}\left(x^{*}, G_{3}\right)=2$. Hence, each of the three quads through $x^{*}$ meeting $G_{2}\left(G_{3}\right)$ is a grid. So, $\left\langle x^{*} x_{1}, L_{1}\right\rangle,\left\langle x^{*} x_{2}, L_{2}\right\rangle$ and $\left\langle x^{*} x_{3}, L_{3}\right\rangle$ are the three grid-quads through $x^{*}$ meeting $G_{2}\left(G_{3}\right)$ in a point. By Lemma 2.3(iii) these three grid-quads are special with respect to the valuation $f$ (recall $x^{*} \in O_{f}$ ). Hence, $\left|L_{1} \cap O_{f}\right|=1$. Choose $i \in\{2,3\}$ such that $G_{i} \cap L_{1} \cap O_{f} \neq \emptyset$. Then again by Lemma 2.3(iii), $\left|G_{i} \cap O_{f}\right|=3$. Since every point of $G_{1} \backslash\left\{x_{1}, x_{2}, x_{3}\right\}$ has value 2 , no point of $\left(G_{2} \cup G_{3}\right) \backslash\left(L_{1} \cup L_{2} \cup L_{3}\right)$ belongs to $O_{f}$ by property (V2) in the definition of valuation. It follows that $G_{i} \cap O_{f}=\left(G_{i} \cap L_{1}\right) \cup\left(G_{i} \cap L_{2}\right) \cup\left(G_{i} \cap L_{3}\right)$. For every $j \in\{1,2,3\}, L_{j} \cap G_{i}$ has value 0 and $L_{j} \cap Q$ has value 1 . Hence, $L_{j} \cap G_{5-i}$ has value 1 by property (V2). Together with $\left(G_{5-i} \backslash\left(L_{1} \cup L_{2} \cup L_{3}\right)\right) \cap O_{f}=\emptyset$, this implies that $G_{5-i} \cap O_{f}=\emptyset$.
(2) Suppose that the unique point $x^{*}$ in $O_{f} \cap Q$ is contained in $G_{1}$. Suppose $y^{*}$ is a point of $O_{f} \cap G_{2}$. Then $\mathrm{d}\left(x^{*}, y^{*}\right)=2$. Hence, the unique point $z^{*}$ of $G_{2}$ collinear with $x^{*}$ is also collinear with $y^{*}$. It follows that $\left\langle x^{*}, y^{*}\right\rangle$ and $G_{2}$ are two special grid-quads meeting in the line $y^{*} z^{*}$, a contradiction. Hence, $G_{2} \cap O_{f}=\emptyset$. In a similar way, one shows that $G_{3} \cap O_{f}=\emptyset$.

The near hexagon $Q(5,2) \times \mathbb{L}_{3}$ is obtained by taking three isomorphic copies of the generalized quadrangle $Q(5,2)$ and joining the corresponding points to form lines of size 3 . There are two types of valuations in $Q(5,2) \times \mathbb{L}_{3}$ :
the classical valuations and the extensions of the ovoidal valuations of the grid-quads.

The near hexagon $W(2) \times \mathbb{L}_{3}$ is obtained by taking three isomorphic copies of the generalized quadrangle $W(2)$ and joining the corresponding points to form lines of size 3 . There are four types of valuations in $W(2) \times \mathbb{L}_{3}$ : the classical valuations, the extensions of the ovoidal valuations of the gridquads, the extensions of the ovoidal valuations of the $W(2)$-quads and the semi-classical valuations.

## 3 Properties of the near octagon $\mathbb{G}_{4}$

We start with some properties of the near $2 n$-gon $\mathbb{G}_{n}, n \geq 3$, whose proofs can be found in the book [5]. Let $U$ denote the set of points of weight 1 and 2 of $\mathrm{PG}(n-1,4)$ (with respect to a certain reference system) and let $\mathcal{L}_{U}$ denote the linear space induced on the set $U$ by the lines of $\operatorname{PG}(n-1,4)$. Then every local space of $\mathbb{G}_{n}$ is isomorphic to $\mathcal{L}_{U}$. Every quad of $\mathbb{G}_{n}, n \geq 3$, is isomorphic to either the $(3 \times 3)$-grid, $W(2)$ or $Q(5,2)$. The near polygon $\mathbb{G}_{n}, n \geq 3$, has two types of lines:
(i) SPECIAL LINES: these are lines which are not contained in a $W(2)$ quad.
(ii) ORDINARY LINES: these are lines which are contained in at least one $W$ (2)-quad.
There are two possible grid-quads in $\mathbb{G}_{n}, n \geq 3$.
(i) GRID-QUADS OF TYPE I: these grid-quads contain three ordinary and three special lines; the lines of each type partition the point set of the grid.
(ii) GRID-QUADS OF TYPE II: these grid-quads contain six ordinary lines. If $n=3$, then every grid-quad is of type I. If $n \geq 4$, then both types of grid-quads occur.

The automorphism group of $\mathbb{G}_{n}, n \geq 3$, acts transitively on the set of special lines, the set of ordinary lines, the set of $Q(5,2)$-quads, the set of $W(2)$-quads, the set of grid-quads of type $I$ and the set of grid-quads of type $I I$.

In the following lemma, we collect some properties of the near octagon $\mathbb{G}_{4}$.

Lemma 3.1 (1) The near octagon $\mathbb{G}_{4}$ has 8505 points, each line of $\mathbb{G}_{4}$ contains 3 points and each point of $\mathbb{G}_{4}$ is contained in 22 lines.
(2) Every quad of $\mathbb{G}_{4}$ is isomorphic to either the $(3 \times 3)$-grid, $W(2)$ or $Q(5,2)$. Every $Q(5,2)$-quad is classical in $\mathbb{G}_{4}$.
(3) Every hex of $\mathbb{G}_{4}$ is isomorphic to either $\mathbb{G}_{3}, \mathbb{H}_{3}, W(2) \times \mathbb{L}_{3}$ or $Q(5,2) \times$ $\mathbb{L}_{3}$. Every $\mathbb{G}_{3}$-hex is big in $\mathbb{G}_{4}$.
(4) If $x$ is a point of $\mathbb{G}_{4}$, then every $Q(5,2)$-quad through $x$ contains precisely two special lines through $x$. Conversely, every two distinct special lines through $x$ are contained in a unique $Q(5,2)$-quad.
(5) If $x$ is a point of $\mathbb{G}_{4}$, then every $\mathbb{G}_{3}$-hex through $x$ contains precisely three special lines through $x$. Conversely, every three distinct special lines through $x$ are contained in a unique $\mathbb{G}_{3}$-hex.
(6) Every point is contained in 4 special lines, 18 ordinary lines, 36 gridquads of type I, 27 grid-quads of type II, $36 \mathrm{~W}(2)$-quads, 6 Q(5, 2)-quads, 4 $\mathbb{G}_{3}$-hexes, $18 Q(5,2) \times \mathbb{L}_{3}$-hexes, $36 W(2) \times \mathbb{L}_{3}$-hexes and $27 \mathbb{H}_{3}$-hexes.
(7) Every special line is contained in 9 grid-quads of type I, 0 grid-quads of type II, $0 \mathrm{~W}(2)$-quads, $3 Q(5,2)$-quads, $0 \mathbb{H}_{3}$-hexes, $3 \mathbb{G}_{3}$-hexes, $9 Q(5,2) \times$ $\mathbb{L}_{3}$-hexes and $9 W(2) \times \mathbb{L}_{3}$-hexes.
(8) Every ordinary line is contained in 2 grid-quads of type I, 3 gridquads of type II, $6 \mathrm{~W}(2)$-quads, $1 Q(5,2)$-quad, $9 \mathbb{H}_{3}$-hexes, $2 \mathbb{G}_{3}$-hexes, 4 $Q(5,2) \times \mathbb{L}_{3}$-hexes and $6 \mathrm{~W}(2) \times \mathbb{L}_{3}$-hexes.
(9) Every $W(2)$-quad is contained in precisely $1 \mathbb{G}_{3}$-hex, $1 W(2) \times \mathbb{L}_{3}$-hex, $0 Q(5,2) \times \mathbb{L}_{3}$-hexes and $3 \mathbb{H}_{3}$-hexes.
(10) Every $Q(5,2)$-quad is contained in precisely $2 \mathbb{G}_{3}$-hexes, $3 Q(5,2) \times$ $\mathbb{L}_{3}$-hexes, $0 W(2) \times \mathbb{L}_{3}$-hexes and $0 \mathbb{H}_{3}$-hexes.
(11) Every grid-quad of type $I$ is contained in $1 \mathbb{G}_{3}$-hex, $0 \mathbb{H}_{3}$-hexes, 1 $Q(5,2) \times \mathbb{L}_{3}$-hex and $3 W(2) \times \mathbb{L}_{3}$-hexes.
(12) Every grid-quad of type $I I$ is contained in $0 \mathbb{G}_{3}$-hexes, $3 \mathbb{H}_{3}$-hexes, $2 Q(5,2) \times \mathbb{L}_{3}$-hexes and $0 W(2) \times \mathbb{L}_{3}$-hexes.
(13) Suppose the point $x$ of $\mathbb{G}_{4}$ is contained in a $Q(5,2)$-quad $Q$ and a hex $H$, then $Q \cap H$ is either $Q$ or a line of $Q$.
(14) Suppose the point $x$ of $\mathbb{G}_{4}$ is contained in a $\mathbb{G}_{3}$-hex $H$ and an $\mathbb{H}_{3}$-hex $H^{\prime}$. Then $H \cap H^{\prime}$ is a $W(2)$-quad.

Proof. Claims (1), (2), (3) (as well as parts of Claims (4), (5), (6), (7) and (8)) were proved in De Bruyn [5, Section 6.3] in a more general context, namely that of the near $2 n$-gon $\mathbb{G}_{n}, n \geq 3$. Claims (4)-(14) readily follow from information on the local spaces which we will now provide.

Let $x$ be an arbitrary point of $\mathbb{G}_{4}$. Then the local space of $\mathbb{G}_{4}$ at the point $x$ is isomorphic to $\mathcal{L}_{U}$ where $U$ is the set of all points of weight 1 or 2 of $\mathrm{PG}(3,4)$ with respect to a certain reference system $\left(\bar{e}_{1}, \bar{e}_{2}, \bar{e}_{3}, \bar{e}_{4}\right)$ of $V(4,4)$. A convex subspace $F$ through $x$ corresponds to a certain subspace of $\mathcal{L}_{U}$ and hence to a certain set $X_{F}$ of points of $\mathrm{PG}(3,4)$. If $F_{1}$ and $F_{2}$ are two convex subspaces through $x$, then $F_{1} \subset F_{2}$ if and only if $X_{F_{1}} \subset X_{F_{2}}$. We discuss all the possibilities for the lines, quads and hexes.
(i) If $F$ is a special line, then $X_{F}=\left\{\left\langle\bar{e}_{i}\right\rangle\right\}$ for some $i \in\{1,2,3,4\}$.
(ii) If $F$ is an ordinary line, then $X_{F}=\left\{\left\langle\bar{e}_{i}+\lambda \bar{e}_{j}\right\rangle\right\}$ for two distinct $i, j \in\{1,2,3,4\}$ and a $\lambda \in \mathbb{F}_{4}^{*}:=\mathbb{F}_{4} \backslash\{0\}$.
(iii) If $F$ is a $Q(5,2)$-quad, then $X_{F}=\left\{\left\langle\bar{e}_{j}\right\rangle,\left\langle\bar{e}_{i}+\lambda \bar{e}_{j}\right\rangle \mid \lambda \in \mathbb{F}_{4}\right\}$ for two distinct $i, j \in\{1,2,3,4\}$.
(iv) If $F$ is a $W(2)$-quad, then $X_{F}=\left\{\left\langle\bar{e}_{i}+\lambda \bar{e}_{j}\right\rangle,\left\langle\bar{e}_{i}+\mu \bar{e}_{k}\right\rangle,\left\langle\lambda \bar{e}_{j}+\mu \bar{e}_{k}\right\rangle\right\}$ for three mutually distinct $i, j, k \in\{1,2,3,4\}$ and some $\lambda, \mu \in \mathbb{F}_{4}^{*}$.
(v) If $F$ is a grid-quad of type I, then $X_{F}=\left\{\left\langle\bar{e}_{i}\right\rangle,\left\langle\bar{e}_{j}+\lambda \bar{e}_{k}\right\rangle\right\}$ for three mutually distinct $i, j, k \in\{1,2,3,4\}$ and some $\lambda \in \mathbb{F}_{4}^{*}$.
(vi) If $F$ is a grid-quad of type II, then $X_{F}=\left\{\left\langle\bar{e}_{i}+\lambda \bar{e}_{j}\right\rangle,\left\langle\bar{e}_{k}+\mu \bar{e}_{l}\right\rangle\right\}$ for some $\lambda, \mu \in \mathbb{F}_{4}^{*}$ and some $i, j, k, l$ satisfying $\{i, j, k, l\}=\{1,2,3,4\}$.
(vii) If $F$ is a $\mathbb{G}_{3}$-hex, then $X_{F}=\left\langle\bar{e}_{i}, \bar{e}_{j}, \bar{e}_{k}\right\rangle \cap U$ for three mutually distinct $i, j, k \in\{1,2,3,4\}$.
(viii) If $F$ is an $\mathbb{H}_{3}$-hex, then $X_{F}=\alpha \cap U$ where $\alpha$ is a plane of $\operatorname{PG}(3,4)$ disjoint from $\left\{\left\langle\bar{e}_{1}\right\rangle,\left\langle\bar{e}_{2}\right\rangle,\left\langle\bar{e}_{3}\right\rangle,\left\langle\bar{e}_{4}\right\rangle\right\}$. So, $\left|X_{F}\right|=6$ and $X_{F}$ contains a unique point of each of the lines $\left\langle\bar{e}_{i}, \bar{e}_{j}\right\rangle, i, j \in\{1,2,3,4\}$ with $i \neq j$.
(ix) If $F \cong Q(5,2) \times \mathbb{L}_{3}$, then $X_{F}=\left\{\left\langle\bar{e}_{i}+\lambda \bar{e}_{j}\right\rangle\right\} \cup\left\{\left\langle\bar{e}_{l}\right\rangle,\left\langle\bar{e}_{k}+\mu \bar{e}_{l}\right\rangle \mid \mu \in \mathbb{F}_{4}\right\}$ for some $\lambda \in \mathbb{F}_{4}^{*}$ and some $i, j, k, l$ satisfying $\{i, j, k, l\}=\{1,2,3,4\}$.
(x) If $F \cong W(2) \times \mathbb{L}_{3}$, then $X_{F}=\left\{\left\langle\bar{e}_{i}+\lambda \bar{e}_{j}\right\rangle,\left\langle\bar{e}_{i}+\mu \bar{e}_{k}\right\rangle,\left\langle\lambda \bar{e}_{j}+\mu \bar{e}_{k}\right\rangle,\left\langle\bar{e}_{l}\right\rangle\right\}$ for some $\lambda, \mu \in \mathbb{F}_{4}^{*}$ and some $i, j, k, l$ satisfying $\{i, j, k, l\}=\{1,2,3,4\}$.

## 4 Structure of $\mathbb{G}_{4}$ with respect to an $\mathbb{H}_{3}$-hex

In this section, $H$ denotes a given $\mathbb{H}_{3}$-hex of $\mathbb{G}_{4}$.
Lemma 4.1 Let $x \in \Gamma_{2}(H)$ and $Q$ a quad of $H$ such that $\Gamma_{2}(x) \cap Q$ is an ovoid of $Q$. Then:
(1) $\langle x, Q\rangle$ is a hex of $\mathbb{G}_{4}$;
(2) if $Q$ is a $W(2)$-quad, then $\langle x, Q\rangle \cong \mathbb{G}_{3}$;
(3) if $Q$ is a grid-quad, then $Q$ is a grid-quad of type II and $\langle x, Q\rangle \cong \mathbb{H}_{3}$.

Proof. (1) Let $x_{1}$ and $x_{2}$ be two distinct points of $\Gamma_{2}(x) \cap Q$ and let $x_{3}$ be a common neighbour of $x_{1}$ and $x_{2}$. Then $x_{3} \in Q \backslash \Gamma_{2}(x)$ has distance 3 from $x$ and $\left\langle x, x_{3}\right\rangle$ is a hex. Now, $x_{1}$ and $x_{2}$ are two points on a geodesic path from $x_{3}$ to $x$. Hence, $\left\langle x, x_{1}, x_{2}\right\rangle \subseteq\left\langle x, x_{3}\right\rangle$. On the other hand, since $x_{3}$ is a common neighbour of $x_{1}$ and $x_{2}$, we also have $\left\langle x, x_{3}\right\rangle \subseteq\left\langle x, x_{1}, x_{2}\right\rangle$. Hence, $\left\langle x, x_{1}, x_{2}\right\rangle=\left\langle x, x_{3}\right\rangle$. Since $x_{1}$ and $x_{2}$ are two points of $Q$ at distance 2 from each other, $Q=\left\langle x_{1}, x_{2}\right\rangle$. It follows that $\langle x, Q\rangle=\left\langle x, x_{1}, x_{2}\right\rangle=\left\langle x, x_{3}\right\rangle$ is a hex.
(2) Since $x \in \Gamma_{2}(Q)$, the $W(2)$-quad $Q$ is not big in the hex $\langle x, Q\rangle$. Among the near hexagons which can occur as hex in $\mathbb{G}_{4}$, only $\mathbb{G}_{3}$ has nonbig $W(2)$-quads (recall Lemma 3.1(3)). It follows that $\langle x, Q\rangle \cong \mathbb{G}_{3}$.
(3) The grid-quad $Q$ is contained in the $\mathbb{H}_{3}$-hex $H$. Hence, by Lemma 3.1(11), $Q$ is a grid-quad of type II. Since $x \in \Gamma_{2}(Q)$, the grid-quad $Q$ of type II is not big in the hex $\langle x, Q\rangle$. Among the near hexagons which can occur as hex in $\mathbb{G}_{4}$, only $\mathbb{G}_{3}$ and $\mathbb{H}_{3}$ have non-big grid-quads. By Lemma 3.1(12), a $\mathbb{G}_{3}$-hex cannot contain grid-quads of type II. Hence, $\langle x, Q\rangle \cong \mathbb{H}_{3}$.

Remark. If $(x, Q)$ is a point-quad pair of a dense near hexagon such that $\mathrm{d}(x, Q)=2$, then $\Gamma_{2}(x) \cap Q$ is an ovoid of $Q$ since every line of $Q$ contains a unique point nearest to (and hence at distance 2 from) $x$.

Proposition 4.2 It holds that $|H|=105,\left|\Gamma_{1}(H)\right|=3360,\left|\Gamma_{2}(H)\right|=5040$ and $\left|\Gamma_{i}(H)\right|=0$ for every $i \geq 3$. If $x \in \Gamma_{2}(H)$, then there are two possibilities:
(a) $\Gamma_{2}(x) \cap H$ is an ovoid of $a W(2)$-quad $Q$ of $H$ and $\langle x, Q\rangle \cong \mathbb{G}_{3}$;
(b) $\Gamma_{2}(x) \cap H$ is an ovoid of a grid-quad of type II of $H$ and $\langle x, Q\rangle \cong \mathbb{H}_{3}$.

Proof. Suppose $y \in \Gamma_{i}(H)$ with $i \geq 3$. For every line $L$ of $H$, we have $\mathrm{d}(y, L) \leq 3$ since $L$ contains a unique point nearest to $y$. Hence $i=3$ and $\left|\Gamma_{3}(y) \cap L\right|=1$ for every line $L$ of $H$. It follows that $\Gamma_{3}(y) \cap H$ is an ovoid of $H$. But this is impossible since $H$ has no ovoids by [13, Lemma 5.5]. Hence, $\left|\Gamma_{i}(H)\right|=0$ for every $i \geq 3$. Clearly, $|H|=105,\left|\Gamma_{1}(H)\right|=|H| \cdot(22-6) \cdot 2=$ 3360 and $\left|\Gamma_{2}(H)\right|=8505-|H|-\left|\Gamma_{1}(H)\right|=5040$.

Suppose $x \in \Gamma_{2}(H)$. Applying Proposition 1.1 to the classical valuation $f$ of $\mathbb{G}_{4}$ with $O_{f}=\{x\}$, we find that the map $g: H \rightarrow \mathbb{N} ; y \mapsto \mathrm{~d}(x, y)-2$ is a valuation of $H$. The valuation $g$ is not classical since each of its values is at most 2. (A classical valuation of a dense near hexagon has maximal value equal to 3.) By Section 2, there are three possibilities:
(a) $O_{g}=\Gamma_{2}(x) \cap H$ is an ovoid in a $W(2)$-quad $Q$ of $H$;
(b) $O_{g}=\Gamma_{2}(x) \cap H$ is an ovoid in a grid-quad $Q$ of $H$;
(c) $O_{g}=\Gamma_{2}(x) \cap H$ is a set of 7 points and $G_{g}$ is a Fano-plane.

If case (a) occurs, then $\langle x, Q\rangle \cong \mathbb{G}_{3}$ by Lemma $4.1(2)$. If case (b) occurs, then $Q$ is a grid-quad of type II and $\langle x, Q\rangle \cong \mathbb{H}_{3}$ by Lemma 4.1(3).

We will now prove that case (c) cannot occur. Suppose the contrary. Let $u$ denote an arbitrary point of $O_{g}$ and let $Q_{1}, Q_{2}$ and $Q_{3}$ denote the
three grid-quads of $H$ through $u$. These grid-quads are special with respect to $g$ by Lemma 2.3(iii). Hence, $\Gamma_{2}(x) \cap Q_{i}$ is an ovoid of $Q_{i}$ for every $i \in$ $\{1,2,3\}$. By Lemma 4.1(3), the grid-quads $Q_{1}, Q_{2}$ and $Q_{3}$ have type II and $\left\langle x, Q_{1}\right\rangle \cong\left\langle x, Q_{2}\right\rangle \cong\left\langle x, Q_{3}\right\rangle \cong \mathbb{H}_{3}$. In the near hexagon $\left\langle x, Q_{1}\right\rangle \cong \mathbb{H}_{3}$, the quad $\langle x, u\rangle$ is one of the three quads through $x$ which meet $Q_{1}$. It follows that $\langle x, Q\rangle$ is a grid-quad. By Lemma 3.1(11), $\langle x, u\rangle$ is a grid-quad of type II. By Lemma 3.1(7), every line of $\langle x, u\rangle$ is an ordinary line. Let $L$ be one of the two (ordinary) lines of $\langle x, u\rangle$ through $u$. By Lemma 3.1(8), $L$ is contained in a unique $Q(5,2)$-quad $Q$. By Lemma 3.1(13), $Q \cap H$ is a line $L^{\prime}$. Since $Q_{1}, Q_{2}$ and $Q_{3}$ determine a partition of the lines of $H$ through $u$, we have $L^{\prime} \subseteq Q_{i}$ for precisely one $i \in\{1,2,3\}$. Now, the $\mathbb{H}_{3}$-hex $\left\langle x, Q_{i}\right\rangle$ contains $L^{\prime}$ and $L \subseteq\langle x, u\rangle$. So, the $Q(5,2)$-quad $Q=\left\langle L, L^{\prime}\right\rangle$ would be contained in the $\mathbb{H}_{3}$-hex $\left\langle x, Q_{i}\right\rangle$, clearly a contradiction, since $\mathbb{H}_{3}$ has only grid-quads and $W(2)$-quads.

Definition. A point $x$ of $\Gamma_{2}(H)$ is said to be of type (a), respectively (b), if case (a), respectively case (b), of Proposition 4.2 occurs.

Lemma 4.3 Let $H^{\prime}$ be a hex meeting $H$ in a quad $Q$. Then $\Gamma_{2}(H) \cap H^{\prime}=$ $\Gamma_{2}(Q) \cap H^{\prime}$.

Proof. Suppose $x \in \Gamma_{2}(H) \cap H^{\prime}$. Then $x$ has distance at least 2 from $Q$. Since $x$ and $Q$ are contained in $H^{\prime}$, every point of $Q$ has distance at most 3 from $x$. Hence, for every line $L$ of $Q, \mathrm{~d}(x, L) \leq 2$ since $L$ contains a unique point nearest to $x$. It follows that $x \in \Gamma_{2}(Q) \cap H^{\prime}$.

Conversely, suppose that $x \in \Gamma_{2}(Q) \cap H^{\prime}$. Then $x \notin H$ since $H \cap H^{\prime}=Q$. Suppose $x \in \Gamma_{1}(H)$. Then $x$ is classical with respect to $H$ and $\mathrm{d}(x, y)=$ $1+\mathrm{d}\left(\pi_{H}(x), y\right)$ for every point $y \in H$. It follows that the point $\pi_{H}(x)$ is collinear with every point of the ovoid $\Gamma_{2}(x) \cap Q$ of $Q$. This implies that $\pi_{H}(x) \in Q$. But this is in contradiction with $\pi_{H}(x) \sim x \in \Gamma_{2}(Q)$. It follows that $x \in \Gamma_{2}(H) \cap H^{\prime}$.

Lemma 4.4 In $\Gamma_{2}(H)$, there are 3360 points of type (a) and 1680 points of type (b).

Proof. In a given $\mathbb{G}_{3}$-hex, there are 120 points at distance 2 from any of its $W(2)$-quads. There are $28 W(2)$-quads in $H$ and each such quad is contained in a unique $\mathbb{G}_{3}$-hex by Lemma 3.1(9). Lemma 4.3 now implies that the total number of points of type (a) in $\Gamma_{2}(H)$ is equal to $28 \cdot 1 \cdot 120=3360$.

In a given $\mathbb{H}_{3}$-hex, there are 24 points at distance 2 from any of its gridquads. Now, there are 35 grid-quads (of type II) in $H$ and each of these grid-quads is contained in precisely $2 \mathbb{H}_{3}$-hexes distinct from $H$ (see Lemma
$3.1(11)+(12))$. Lemma 4.3 now implies that the number of points of type (b) in $\Gamma_{2}(H)$ is equal to $35 \cdot 2 \cdot 24=1680$.
(CHECK: The total number of points of $\Gamma_{2}(H)$ is indeed equal to $3360+$ $1680=5040$ as shown in Proposition 4.2).

Lemma 4.5 (Chapter 7 of [5]) Suppose one of the following cases occurs: (i) $Q$ is a grid-quad of $\mathcal{S} \cong \mathbb{H}_{3}$; (ii) $Q$ is a $W(2)$-quad of $\mathcal{S} \cong \mathbb{G}_{3}$. Let $x$ be a point of $\mathcal{S}$ at distance 2 from $Q$. Then every line of $\mathcal{S}$ through $x$ has a unique point in common with $\Gamma_{1}(Q)$.

Let $S$ denote the set of lines of $\mathbb{G}_{4}$ contained in $\Gamma_{2}(H)$.
Lemma 4.6 Let $x$ be a point of $\Gamma_{2}(H)$ and let $Q$ be the quad $\left\langle\Gamma_{2}(x) \cap H\right\rangle$. Then the lines through $x$ contained in $S$ are precisely the lines through $x$ not contained in the hex $\langle x, Q\rangle$. If $x$ has type (a), then precisely 10 lines through $x$ are contained in $S$. If $x$ has type (b), then precisely 16 lines through $x$ are contained in $S$.

Proof. If $x$ is a point of type (a), then $Q \cong W(2)$ and $\langle x, Q\rangle \cong \mathbb{G}_{3}$. If $x$ is a point of type (b), then $Q$ is a grid-quad and $\langle x, Q\rangle \cong \mathbb{H}_{3}$. By Lemmas 4.3 and 4.5, every line through $x$ contained in $\langle x, Q\rangle$ contains a point of $\Gamma_{1}(H)$. Conversely, suppose that $L$ is a line through $x$ containing a point $y \in \Gamma_{1}(H)$. Then $y$ is classical with respect to $H$ and the point $\pi_{H}(y)$ lies at distance 2 from $x$. Hence, $\pi_{H}(y) \in Q$ and $L \subseteq\left\langle x, \pi_{H}(y)\right\rangle \subseteq\langle x, Q\rangle$.

So, the number of lines through $x$ contained in $S$ is equal to the number of lines through $x$ not contained in the hex $\langle x, Q\rangle$. If $x$ is a point of type (a), then $x$ is contained in $22-12=10$ lines of $S$. If $x$ is a point of type (b), then $x$ is contained in $22-6=16$ lines of $S$.

From Lemmas 4.4 and 4.6, we readily obtain:
Corollary $4.7|S|=\frac{1}{3}[3360 \cdot 10+1680 \cdot 16]=20160$.
Lemma 4.8 Let $L=\left\{x_{1}, x_{2}, x_{3}\right\}$ be a line of $S$. For every $i \in\{1,2,3\}$, put $Q_{i}:=\left\langle\Gamma_{2}\left(x_{i}\right) \cap H\right\rangle$ and $H_{i}:=\left\langle x_{i}, Q_{i}\right\rangle$. Then $H_{1}, H_{2}$ and $H_{3}$ are mutually disjoint hexes.

Proof. By symmetry, it suffices to show that $H_{1} \cap H_{2}=\emptyset$. Suppose to the contrary that $u$ is a point of $H_{1} \cap H_{2}$. Every point on a shortest path between $u \in H_{1} \cap H_{2}$ and $x_{1} \in H_{1}$ belongs to $H_{1}$. If $x_{1} \notin H_{2}$, then since $x_{1}$ is classical with respect to $H_{2}$, the point $x_{2}=\pi_{H_{2}}\left(x_{1}\right)$ lies on such a shortest path. Hence, $x_{1} \in H_{2}$ or $x_{2} \in H_{1}$. So, the line $x_{1} x_{2}$ is contained in $H_{1}$ or $H_{2}$.

Lemma 4.5 then implies that $L$ contains a point of $\Gamma_{1}(H)$. This contradicts the fact that $L \in S$.

Lemma 4.9 Let $L=\left\{x_{1}, x_{2}, x_{3}\right\}$ be a line of $S$, put $Q_{i}=\left\langle\Gamma_{2}\left(x_{i}\right) \cap H\right\rangle$ and $H_{i}=\left\langle x_{i}, Q_{i}\right\rangle$. If $x_{1}$ is of type (a), then $x_{2}$ and $x_{3}$ have the same type and $\mathcal{R}_{H_{1}}\left(H_{2}\right)=H_{3}$.
Proof. By Proposition $4.2, Q_{1} \cong W(2)$ and $H_{1} \cong \mathbb{G}_{3}$. So, $H_{1}$ is big in $\mathbb{G}_{4}$. By Lemma 4.8, $H_{1}$ and $H_{2}$ are mutually disjoint. Let $H_{3}^{\prime}$ be the reflection of $H_{2}$ about $H_{1}$ (in the near octagon $\mathbb{G}_{4}$ ) and let $Q_{3}^{\prime}$ denote the reflection of $Q_{2}$ about $Q_{1}$ (in the near hexagon $H$ ). Then $Q_{3}^{\prime} \cong Q_{2}, H_{3}^{\prime} \cong H_{2}$ and $Q_{3}^{\prime} \subset H_{3}$. Since $x_{2}$ is a point of $H_{2}$ at distance 2 from the quad $Q_{2}$ of $H_{2}$, $x_{3}=\mathcal{R}_{H_{1}}\left(x_{2}\right)$ is a point of $H_{3}^{\prime}=\mathcal{R}_{H_{1}}\left(H_{2}\right)$ at distance 2 from $Q_{3}^{\prime}=\mathcal{R}_{H_{1}}\left(Q_{2}\right)$. So, $\Gamma_{2}\left(x_{3}\right) \cap Q_{3}^{\prime}$ is an ovoid of $Q_{3}^{\prime}$. This implies that $Q_{3}=Q_{3}^{\prime}$ and $H_{3}=H_{3}^{\prime}$. Since $H_{3}^{\prime} \cong H_{2}, x_{3}$ is of the same type as $x_{2}$.

Lemma 4.10 Every point $x$ of type $(a)$ of $\Gamma_{2}(H)$ is contained in precisely 6 lines of $S$ which only contain points of type (a).

Proof. Put $Q:=\left\langle\Gamma_{2}(x) \cap H\right\rangle$.
Let $\left\{x, x_{1}, x_{2}\right\}$ be a line of $S$ through $x$ which only contains points of type (a) and let $Q_{i}=\left\langle\Gamma_{2}\left(x_{i}\right) \cap H\right\rangle, i \in\{1,2\}$. Then by Lemmas 4.8 and 4.9, the $W(2)$-quads $Q, Q_{1}$ and $Q_{2}$ are mutually disjoint and $Q_{2}$ is the reflection of $Q_{1}$ about $Q$ (in the near hexagon $H$ ).

Let $Q^{\prime}$ be a $W(2)$-quad of $H$ disjoint from $Q$ and let $H^{\prime}$ denote the unique $\mathbb{G}_{3}$-hex through $Q^{\prime}$ (recall Lemma 3.1(9)). We prove that $\langle x, Q\rangle \cap H^{\prime}=\emptyset$. Suppose to the contrary that $\langle x, Q\rangle \cap H^{\prime}$ contains a point $u$. If $u \in H$, then $u \in Q=\langle x, Q\rangle \cap H$ and $u \in Q^{\prime}=H^{\prime} \cap H$, a contradiction. If $u \in \Gamma_{1}(H)$, then $u \notin \Gamma_{2}(Q) \cup \Gamma_{2}\left(Q^{\prime}\right)$ by Lemma 4.3 and hence $\pi_{H}(u) \in Q \cap Q^{\prime}$, a contradiction. If $u \in \Gamma_{2}(H)$, then $u \in \Gamma_{2}(Q) \cap \Gamma_{2}\left(Q^{\prime}\right)$ and hence $\Gamma_{2}(u) \cap H \subseteq Q \cap Q^{\prime}$, again a contradiction. So, the big $\mathbb{G}_{3}$-hexes $\langle x, Q\rangle$ and $H^{\prime}$ are disjoint. Hence, the line $x \pi_{H^{\prime}}(x)$ belongs to $S$ by Lemma 4.6. Since $x$ and $\pi_{H^{\prime}}(x)$ are points of type (a), also the third point of $x \pi_{H^{\prime}}(x)$ has type (a) by Lemma 4.9. So, the $W(2)$-quad $Q^{\prime}$ determines a line of $S$ through $x$ which only consists of points of type (a). If we denote by $Q^{\prime \prime} \cong W(2)$ the reflection of $Q^{\prime}$ about $Q$ (in $H)$ and by $H^{\prime \prime}$ the unique $\mathbb{G}_{3}$-hex through $Q^{\prime \prime}$, then $H^{\prime \prime}=\mathcal{R}_{H^{\prime}}(\langle x, Q\rangle)$ and $x \pi_{H^{\prime}}(x)=x \pi_{H^{\prime \prime}}(x)$. So, the $W(2)$-quads $Q^{\prime}$ and $Q^{\prime \prime}$ determine the same line of $S$ through $x$.

Since there are $12 W(2)$-quads in $H$ disjoint with $Q$, it follows from the above discussion that there are $\frac{12}{2}=6$ lines of $S$ through $x$ containing only points of type (a).

From Lemmas 4.4 and 4.10, we readily obtain:

Corollary 4.11 There are $\frac{3360 \cdot 6}{3}=6720$ lines of $S$ containing precisely three points of type (a).

Lemma 4.12 There are 13440 lines of $S$ containing one point of type (a) and two points of type (b).

Proof. Let $x$ be one of the 3360 points of type (a). By Lemmas 4.6, 4.9 and $4.10, x$ is contained in $10-6=4$ lines of $S$ which contain a unique point of type (a). Hence, the required number is equal to $3360 \cdot 4=13440$.

By Corollary 4.7, Corollary 4.11 and Lemma 4.12, we obtain:
Corollary 4.13 There are two types of lines in $S$ :
(1) Lines of $S$ only containing points of type (a).
(2) Lines of $S$ containing a unique point of type (a) and two points of type (b).

Recall that a hyperplane of a point-line geometry is a proper subspace meeting each line.

Proposition 4.14 Let $X$ denote the set of points of $\mathbb{G}_{4}$ consisting of the points of $H$, the points of $\Gamma_{1}(H)$ and the points of type $(a)$ of $\Gamma_{2}(H)$. Then $X$ is a hyperplane of $\mathbb{G}_{4}$.

Proof. We need to prove that every line $L$ containing a point $x$ of type (b) of $\Gamma_{2}(H)$ intersects $X$ in a unique point. Put $Q:=\left\langle\Gamma_{2}(x) \cap H\right\rangle$. Then $Q$ is a grid-quad of type II and $\langle x, Q\rangle \cong \mathbb{H}_{3}$.

If $L$ is not contained in $\langle x, Q\rangle$, then $L \in S$ by Lemma 4.6. Corollary 4.13 then implies that $|L \cap X|=1$.

If $L$ is contained in $\langle x, Q\rangle$, then $L$ contains a unique point $y$ of $\Gamma_{1}(Q)$ by Lemma 4.5. Let $z \in \Gamma_{2}(Q)$ denote the third point on the line $L$. By Lemma 4.3 applied to the hex $H^{\prime}=\langle x, Q\rangle, z \in \Gamma_{2}(H)$. Since $z \in \Gamma_{2}(Q)$ and $Q$ are contained in the hex $\langle x, Q\rangle, \Gamma_{2}(z) \cap Q$ is an ovoid of $Q$. It follows that $\Gamma_{2}(z) \cap H=\Gamma_{2}(z) \cap Q$. Since $Q$ is a grid, $z$ is of point of type (b) and $y$ is the unique point of $X$ contained in $L$.

## 5 A new class of valuations of $\mathbb{G}_{4}$

Let $H$ denote a hex of $\mathbb{G}_{4}$ isomorphic to $\mathbb{H}_{3}$ and let $f$ denote a valuation of Fano-type of $H$. Recall that every point of $\Gamma_{1}(H)$ is classical with respect to $H$. Lemma 2.3(i)+(ii) allows us to define the following function $\bar{f}$ from the point-set of $\mathbb{G}_{4}$ to $\mathbb{N}$ :
(i) If $x \in H$, then we define $\bar{f}(x):=f(x)$.
(ii) If $x \in \Gamma_{1}(H)$, then we define $\bar{f}(x):=1+f\left(\pi_{H}(x)\right)$.
(iii) If $x$ is a point of type (a) of $\Gamma_{2}(H)$, then $\bar{f}(x):=\mathrm{d}\left(x, x^{*}\right)$, where $x^{*}$ is the unique point of $O_{f}$ contained in the $W(2)$-quad $\left\langle\Gamma_{2}(x) \cap H\right\rangle$.
(iv) Let $x$ be a point of type (b) of $\Gamma_{2}(H)$ such that $\left|O_{f} \cap Q\right|=3$, where $Q$ is the unique grid-quad of $H$ containing $\Gamma_{2}(x) \cap H$. Then $\bar{f}(x):=2$ if $\Gamma_{2}(x) \cap\left(O_{f} \cap Q\right) \neq \emptyset$ and $\bar{f}(x):=1$ otherwise.
(v) Let $x$ be a point of type (b) of $\Gamma_{2}(H)$ such that $\left|O_{f} \cap Q\right|=0$ where $Q$ is the unique grid-quad of $H$ containing $\Gamma_{2}(x) \cap H$. Let $X$ denote the ovoid of $Q$ consisting of all points with $f$-value 1 . We define $\bar{f}(x):=3$ if $\Gamma_{2}(x) \cap X \neq \emptyset$ and $\bar{f}(x):=2$ otherwise.

Proposition 5.1 The map $\bar{f}$ is a valuation of $\mathbb{G}_{4}$.
Proof. Recall that a function from the point-set of $\mathbb{G}_{4}$ to $\mathbb{N}$ is a valuation of $\mathbb{G}_{4}$ if and only if it satisfies properties (V1) and (V2). Clearly, $\bar{f}$ satisfies property (V1). It remains to show that $\bar{f}$ also satisfies property (V2). Let $L$ be an arbitrary line of $\mathbb{G}_{4}$. We can distinguish 6 possibilities by corollary 4.13:
(1) $L$ is contained in $H$. Then $L$ satisfies property (V2) with respect to $\bar{f}$ since $L$ satisfies property (V2) with respect to $f$.
(2) $L$ intersects $H$ in a unique point $x_{L}$. Then $\bar{f}(x)=f\left(x_{L}\right)+1=\bar{f}\left(x_{L}\right)+1$ for every point $x$ of $L \backslash\left\{x_{L}\right\}$. So, $L$ satisfies property (V2).
(3) $L \subseteq \Gamma_{1}(H)$. Then $\pi_{H}(L):=\left\{\pi_{H}(x) \mid x \in L\right\}$ is a line of $H$ parallel with $L$. For every point $x$ of $L, \bar{f}(x)=f\left(\pi_{H}(x)\right)+1$. Since $\pi_{H}(L)$ satisfies property (V2) with respect to $f, L$ satisfies property (V2) with respect to $\bar{f}$.
(4) $L \cap \Gamma_{1}(H) \neq \emptyset$ and $L \cap \Gamma_{2}(H) \neq \emptyset$. Let $x$ denote an arbitrary point of $L \cap \Gamma_{2}(H)$ and let $Q$ denote the unique quad of $H$ containing $\Gamma_{2}(x) \cap H$. Then $\langle x, Q\rangle$ is a hex containing $L$.

From the definition of $\bar{f}$, we see that there exists a constant $\epsilon \in\{-1,0\}$ such that the map $u \mapsto \bar{f}(u)+\epsilon$ defines a valuation $f^{\prime}$ of $\langle x, Q\rangle$. If $x$ is a point of type (a), then $\epsilon=0$ and $f^{\prime}$ is a classical valuation of $\langle x, Q\rangle \cong \mathbb{G}_{3}$ by Lemma 2.3(i). If $x$ is a point of type (b), then $\epsilon=0$ if $\left|O_{f} \cap Q\right|=3$ and $\epsilon=-1$ if $\left|O_{f} \cap Q\right|=0$. Moreover, by Lemma 2.2, $f^{\prime}$ is a valuation of grid-type of $\langle x, Q\rangle \cong \mathbb{H}_{3}$.

By the previous paragraph, the line $L \subseteq\langle x, Q\rangle$ satisfies property (V2) with respect to $\bar{f}$.
(5) $L \subseteq \Gamma_{2}(H)$ and every point of $L$ is of type (a). Put $L=\left\{x_{1}, x_{2}, x_{3}\right\}$ and let $Q_{i}, i \in\{1,2,3\}$, denote the unique $W(2)$-quad of $H$ containing
$O_{i}=\Gamma_{2}\left(x_{i}\right) \cap H$. The set $O_{i}$ is an ovoid of $Q_{i}$. Put $H_{i}:=\left\langle x_{i}, Q_{i}\right\rangle, i \in\{1,2,3\}$. By Lemmas 4.8 and 4.9, $H_{1}, H_{2}$ and $H_{3}$ are three mutually disjoint $\mathbb{G}_{3}$-hexes, $\mathcal{R}_{H_{1}}\left(H_{2}\right)=H_{3}$ and $\mathcal{R}_{Q_{1}}\left(Q_{2}\right)=Q_{3}$ (reflection about the big $W(2)$-quad $Q_{1}$ in the $\mathbb{H}_{3}$-hex $H$ ). So, every line meeting $Q_{1}$ and $Q_{2}$ also meets $Q_{3}$. We have $\mathcal{R}_{H_{1}}\left(O_{2}\right)=\mathcal{R}_{H_{1}}\left(\Gamma_{2}\left(x_{2}\right) \cap Q_{2}\right)=\Gamma_{2}\left(x_{3}\right) \cap Q_{3}=O_{3}$. In a similar way, one can prove that $O_{3}=\mathcal{R}_{H_{2}}\left(O_{1}\right)$. It follows that $O_{1} \cup O_{2} \cup O_{3}$ is the union of 5 lines which meet $Q_{1}, Q_{2}$ and $Q_{3}$. Let $u_{i}^{*}, i \in\{1,2,3\}$, denote the unique point of $Q_{i}$ with $f$-value 0 (recall Lemma 2.3(i)). Since every two points of $O_{f}$ lie at distance 2 from each other, $\mathrm{d}\left(u_{1}^{*}, u_{2}^{*}\right)=2$. Since $u_{1}^{*}$ is classical with respect to $Q_{2}$, the unique point $v$ of $Q_{2}$ collinear with $u_{1}^{*}$ is collinear with $u_{2}^{*}$. Let $w$ denote the point of the line $u_{1}^{*} v$ distinct from $u_{1}^{*}$ and $v$. The quad $\left\langle u_{1}^{*}, u_{2}^{*}\right\rangle$ contains the line $u_{1}^{*} v$ and hence contains the point $w \in Q_{3}$. Since the local space of $H$ at the point $w$ is a Fano plane minus a point, the quads $\left\langle u_{1}^{*}, u_{2}^{*}\right\rangle$ and $Q_{3}$ meet in a line. Since $u_{1}^{*}, u_{2}^{*} \in O_{f}$, the quad $\left\langle u_{1}^{*}, u_{2}^{*}\right\rangle$ of $H$ is special with respect to $f$. So, $\left\langle u_{1}^{*}, u_{2}^{*}\right\rangle$ is a grid and the line $\left\langle u_{1}^{*}, u_{2}^{*}\right\rangle \cap Q_{3}$ contains a unique point of $O_{f}$ which necessarily coincides with $u_{3}^{*}$. The points $u_{1}^{*}$, $\pi_{Q_{1}}\left(u_{2}^{*}\right)$ and $\pi_{Q_{1}}\left(u_{3}^{*}\right)$ of $Q_{1}$ form a line of $Q_{1}$ which intersects $O_{1}$ in a unique point. It follows that $O_{1} \cup O_{2} \cup O_{3}$ has a unique point $u^{*}$ in common with $\left\{u_{1}^{*}, u_{2}^{*}, u_{3}^{*}\right\}$. If $i \in\{1,2,3\}$ such that $u^{*}=u_{i}^{*}$, then $\bar{f}\left(x_{i}\right)=2$ and $\bar{f}\left(x_{j}\right)=3$ for all $j \in\{1,2,3\} \backslash\{i\}$. This proves that $L$ satisfies property (V2).
(6) $L \subseteq \Gamma_{2}(H), L$ contains a unique point $x_{1}$ of type (a) and two points $x_{2}$ and $x_{3}$ of type (b). Let $Q_{1}$ denote the unique $W(2)$-quad of $H$ containing all points of $\Gamma_{2}\left(x_{1}\right) \cap H$ and put $H_{1}:=\left\langle x_{1}, Q_{1}\right\rangle$. Let $G_{i}, i \in\{2,3\}$, denote the grid-quad of $H$ containing all points of $\Gamma_{2}\left(x_{i}\right) \cap H$ and put $H_{i}:=\left\langle x_{i}, G_{i}\right\rangle$. Then $H_{1} \cong \mathbb{G}_{3}$ and $H_{2} \cong H_{3} \cong \mathbb{H}_{3}$. Moreover, by Lemmas 4.8 and 4.9, $H_{1}, H_{2}$ and $H_{3}$ are mutually disjoint, $\mathcal{R}_{H_{1}}\left(H_{2}\right)=H_{3}$ and $\mathcal{R}_{Q_{1}}\left(G_{2}\right)=G_{3}$. Put $G_{1}:=\pi_{Q_{1}}\left(G_{2}\right)=\pi_{Q_{1}}\left(G_{3}\right)$. We have $\mathcal{R}_{H_{1}}\left(\Gamma_{2}\left(x_{2}\right) \cap G_{2}\right)=\Gamma_{2}\left(x_{3}\right) \cap G_{3}$. Moreover, $\pi_{Q_{1}}\left(\Gamma_{2}\left(x_{2}\right) \cap G_{2}\right)=\pi_{Q_{1}}\left(\Gamma_{2}\left(x_{3}\right) \cap G_{3}\right)=\Gamma_{2}\left(x_{1}\right) \cap G_{1}$ since every line connecting a point of $\Gamma_{2}\left(x_{2}\right) \cap G_{2} \subseteq \Gamma_{3}\left(x_{1}\right)$ and $\Gamma_{2}\left(x_{3}\right) \cap G_{3} \subseteq \Gamma_{3}\left(x_{1}\right)$ contains a unique point nearest to $x_{1}$. We distinguish four possibilities (cf. Lemma 2.4):
(i) $\left|G_{2} \cap O_{f}\right|=\left|G_{3} \cap O_{f}\right|=0$, the unique point $x^{*}$ in $O_{f} \cap Q_{1}$ is contained in $G_{1}$ and $\mathrm{d}\left(x^{*}, x_{1}\right)=2$. Then the unique line through $x^{*}$ meeting $G_{2}$ and $G_{3}$ intersects $G_{2}$ and $G_{3}$ in points with $f$-value 1 belonging respectively to $\Gamma_{2}\left(x_{2}\right)$ and $\Gamma_{2}\left(x_{3}\right)$. It follows that $\bar{f}\left(x_{1}\right)=2$ and $\bar{f}\left(x_{2}\right)=\bar{f}\left(x_{3}\right)=3$. So, $L$ satisfies property (V2).
(ii) $\left|G_{2} \cap O_{f}\right|=\left|G_{3} \cap O_{f}\right|=0$, the unique point $x^{*}$ in $O_{f} \cap Q_{1}$ is contained in $G_{1}$ and $\mathrm{d}\left(x^{*}, x_{1}\right)=3$. Hence, the ovoid $\Gamma_{2}\left(x_{1}\right) \cap G_{1}$ of $G_{1}$ contains two points with $f$-value 1 and one point with $f$-value 2 (recall Lemma 2.3(i)). Since each of the three lines meeting $\Gamma_{2}\left(x_{1}\right) \cap G_{1}, \Gamma_{2}\left(x_{2}\right) \cap G_{2}$ and $\Gamma_{2}\left(x_{3}\right) \cap G_{3}$
contains a unique point with smallest $f$-value, there exists an $i \in\{2,3\}$ such that (a) the ovoid $\Gamma_{2}\left(x_{i}\right) \cap G_{i}$ contains two points with $f$-value 2 and 1 point with $f$-value 1, and (b) the ovoid $\Gamma_{2}\left(x_{5-i}\right) \cap G_{5-i}$ contains three points with $f$-value 2. It follows that $\bar{f}\left(x_{1}\right)=3, \bar{f}\left(x_{i}\right)=3$ and $\bar{f}\left(x_{5-i}\right)=2$. So, $L$ satisfies property (V2).
(iii) There exists an $i \in\{2,3\}$ such that $\left|G_{i} \cap O_{f}\right|=3$ and $\left|G_{5-i} \cap O_{f}\right|=0$. Moreover, we assume that $\mathrm{d}\left(x_{1}, x^{*}\right)=2$, where $x^{*}$ is the unique point in $O_{f} \cap Q_{1}$. (Recall $x^{*} \notin G_{1}$.) Since $\left\{x^{*}\right\} \cup\left(\Gamma_{2}\left(x_{1}\right) \cap G_{1}\right)$ is contained in the ovoid $\Gamma_{2}\left(x_{1}\right) \cap Q_{1}$ of $Q_{1}$, no point of $\Gamma_{2}\left(x_{1}\right) \cap G_{1}$ is collinear with $x^{*}$. So, $\Gamma_{2}\left(x_{1}\right) \cap G_{1}$ only contains points with $f$-value 2 (recall Lemma 2.3(i)). Since every line meeting $\Gamma_{2}\left(x_{1}\right) \cap G_{1}, \Gamma_{2}\left(x_{2}\right) \cap G_{2}$ and $\Gamma_{2}\left(x_{3}\right) \cap G_{3}$ has a unique point with smallest $f$-value and $G_{i}$ contains only points with $f$-value 0 or $1, \Gamma_{2}\left(x_{i}\right) \cap G_{i}$ only contains points with $f$-value 1 and $\Gamma_{2}\left(x_{5-i}\right) \cap G_{5-i}$ only contains points with $f$-value 2. It follows that $\bar{f}\left(x_{1}\right)=2, \bar{f}\left(x_{i}\right)=1$ and $\bar{f}\left(x_{5-i}\right)=2$. This proves that $L$ satisfies property (V2) with respect to $\bar{f}$.
(iv) There exists an $i \in\{2,3\}$ such that $\left|G_{i} \cap O_{f}\right|=3$ and $\left|G_{5-i} \cap O_{f}\right|=0$. Moreover, we assume that $\mathrm{d}\left(x_{1}, x^{*}\right)=3$ where $x^{*}$ is the unique point in $O_{f} \cap Q_{1}$. (Recall $x^{*} \notin G_{1}$.) Then $\Gamma_{2}\left(x_{1}\right) \cap G_{1} \subseteq \Gamma_{2}\left(x_{1}\right) \cap Q_{1}$ contains at least one point with $f$-value 1 (collinear with $x^{*}$ ). The unique line through each such point meeting $G_{2}$ and $G_{3}$ contains a unique point with smallest $f$-value. Hence, $\Gamma_{2}\left(x_{i}\right) \cap G_{i}$ contains at least one point with $f$-value 0 and $\Gamma_{2}\left(x_{5-i}\right) \cap G_{5-i}$ contains at least one point with $f$-value 1 (recall that every point of $G_{i}$ has $f$-value 0 or 1). It follows that $\bar{f}\left(x_{1}\right)=3, \bar{f}\left(x_{i}\right)=2$ and $\bar{f}\left(x_{5-i}\right)=3$. This proves that $L$ satisfies property (V2).

The valuation $\bar{f}$ of $\mathbb{G}_{4}$ defined above is called a valuation of Fano-type of $\mathbb{G}_{4}$.

## 6 The classification of the valuations of $\mathbb{G}_{4}$

### 6.1 Some lemmas

During the classification of the valuations of $\mathbb{G}_{4}$, we will need the following three properties which hold for valuations of general near polygons:

Lemma 6.1 ([11]) Let $f$ be a valuation of a dense near $2 n$-gon $\mathcal{S}$.
(i) $f$ is a classical valuation if and only if there exists a point with value $n$.
(ii) If $d\left(x, O_{f}\right) \leq 2$, then $f(x)=d\left(x, O_{f}\right)$.
(iii) No two distinct special quads intersect in a line.

Now, suppose that $f$ is a valuation of $\mathbb{G}_{4}$.

Lemma 6.2 If $x, y \in O_{f}$, then $d(x, y)$ is even.
Proof. By Property $(\mathrm{V} 2), \mathrm{d}(x, y) \neq 1$. Suppose $\mathrm{d}(x, y)=3$. Let $H$ denote the unique hex through $x$ and $y$. If $f^{\prime}$ denotes the valuation of $H$ induced by $f$ (recall Proposition 1.1), then $O_{f^{\prime}}$ contains two points at distance 3 from each other. This is impossible since none of the near hexagons $\mathbb{G}_{3}, W(2) \times \mathbb{L}_{3}$, $Q(5,2) \times \mathbb{L}_{3}, \mathbb{H}_{3}$ has such valuations.

Lemma 6.3 If there exists a $\mathbb{G}_{3}$-hex $H$ such that $\left|H \cap O_{f}\right|=15$, then $O_{f}=$ $H \cap O_{f}$.
Proof. Since $\left|H \cap O_{f}\right|=15$, the valuation $f^{\prime}$ of $H \cong \mathbb{G}_{3}$ induced by $f$ is non-classical. Suppose $x \in O_{f} \backslash H$. Since $H$ is big in $\mathbb{G}_{4}, x$ is classical with respect to $H$. The point $\pi_{H}(x)$ has value 1 and hence is contained in a unique quad $Q$ of $H$ which is special with respect to $f^{\prime}$ (recall Lemma 2.1(iii)). If $y$ is a point of $Q \cap O_{f}$ at distance 2 from $\pi_{H}(x)$, then $\mathrm{d}(x, y)=3$, contradicting Lemma 6.2.

Lemma 6.4 If $x$ and $y$ are two different points of $O_{f}$, then $d(x, y)=2$.
Proof. Suppose the contrary. Then $\mathrm{d}(x, y)=4$ by Lemma 6.2. Let $H$ denote an arbitrary $\mathbb{G}_{3}$-hex through $x$. Since $y \in O_{f} \backslash H$, the valuation induced in $H$ is classical by Lemma 6.3 (recall that $\left|O_{g}\right|=15$ for every nonclassical valuation $g$ of $\left.\mathbb{G}_{3}\right)$. Hence, $f\left(\pi_{H}(y)\right)=\mathrm{d}\left(x, \pi_{H}(y)\right)=3$. On the other hand, since $\mathrm{d}\left(\pi_{H}(y), y\right)=1$ and $f(y)=0$, it holds that $f\left(\pi_{H}(y)\right)=1$, a contradiction.

Lemma 6.5 One of the following cases occurs:
(A) $\left|O_{f}\right|=1$;
(B) There exists a unique $\mathbb{G}_{3}$-hex $H$ such that $O_{f} \subseteq H$ and $\left|H \cap O_{f}\right|=15$;
(C) $\left|O_{f}\right| \geq 2$ and every special quad is a grid-quad of type II.

Proof. Suppose $\left|O_{f}\right| \geq 2$ and let $x_{1}$ and $x_{2}$ denote two distinct points of $O_{f}$. Then $\mathrm{d}\left(x_{1}, x_{2}\right)=2$ by Lemma 6.4. Let $Q$ denote the unique special quad through $x_{1}$ and $x_{2}$. Then $Q$ is not isomorphic to $Q(5,2)$ since this generalized quadrangle has no ovoids (Payne and Thas [17]). If $Q$ is a $W(2)$-quad or a grid-quad of type I , then $Q$ is contained in a unique $\mathbb{G}_{3}$-hex $H$, see Lemma $3.1(9)+(11)$. Since $Q \cap O_{f} \subseteq H \cap O_{f}$, the valuation of $H$ induced by $f$ is non-classical and hence $\left|H \cap O_{f}\right|=15$ by Lemma 2.1. By Lemma 6.3, it then follows that $O_{f}=H \cap O_{f}$. The lemma is now clear.

### 6.2 Treatment of case (A) of Lemma 6.5

Proposition 6.6 If $f$ is a valuation of $\mathbb{G}_{4}$ such that $\left|O_{f}\right|=1$, then $f$ is a classical valuation.

Proof. Put $O_{f}=\{x\}$ and let $H$ denote an arbitrary $\mathbb{G}_{3}$-hex through $x$. By Lemma 3.1(5)+(6), there exists a unique special line $L$ through $x$ not contained in $H$. Let $x^{\prime}$ denote an arbitrary point of $L \backslash\{x\}$. By Lemmas $3.1(5)+(6)$, there exists a unique $\mathbb{G}_{3}$-hex $H^{\prime}$ through $x^{\prime}$ not containing the special line $L$. We will show that the valuation $f^{\prime}$ of $H^{\prime}$ induced by $f$ is classical. Suppose the contrary and let $Q$ denote a grid-quad of $H^{\prime}$ through $x^{\prime} \in O_{f^{\prime}}$ which is special with respect to $f^{\prime}$. (Such a grid-quad exists by Lemma 2.1(v).) By Lemma 3.1(12), $Q$ is a grid-quad of type I. By Lemma 3.1(11), $Q$ is contained in a unique $Q(5,2) \times \mathbb{L}_{3}$-hex $H^{\prime \prime}$. By Lemma $3.1(4)+(5), H^{\prime \prime}$ has two special lines though $x^{\prime}$. One of these lines is contained in the grid-quad $Q$ of type I. The other special line $L^{\prime}$ cannot be contained in $H^{\prime}$ since otherwise $H^{\prime \prime}=\langle Q, L\rangle=H^{\prime}$, which is clearly absurd. Since there is only 1 special line through $x^{\prime}$ not contained in the $\mathbb{G}_{3}$-hex $H^{\prime}$ (Lemmas $3.1(5)+(6)$ ), we must have $L^{\prime}=L$. Now, the valuation of $H^{\prime \prime}=\langle L, Q\rangle$ induced by $f$ contains a unique point with value 0 (namely $x$ ) and a point with value 1 at distance 3 from it (which is contained in $\Gamma_{2}\left(x^{\prime}\right) \cap Q$ ). But $Q(5,2) \times \mathbb{L}_{3}$ does not have valuations of this type. So, we have a contradiction. It follows that the valuation induced in $H^{\prime}$ is classical. This implies that every point of $H^{\prime}$ at distance 3 from $x^{\prime}$ has value 4. By Lemma 6.1(i), it then follows that $f$ is classical.

### 6.3 Treatment of case (B) of Lemma 6.5

Proposition 6.7 If $f$ is a valuation of $\mathbb{G}_{4}$ such that $O_{f}$ is a set of 15 points in $a \mathbb{G}_{3}$-hex $H$ of $\mathbb{G}_{4}$, then $f$ is the extension of a non-classical valuation of $\mathbb{G}_{3}$.

Proof. Let $f^{\prime}$ denote the valuation of $H$ induced by $f$. Then $f^{\prime}$ is a nonclassical valuation of $H$ with $O_{f^{\prime}}=O_{f}$. Hence, $f(x)=f^{\prime}(x)$ for every point $x \in H$. Now, let $x$ be an arbitrary point of $\mathbb{G}_{4}$ not contained in $H$. Recall that the $\mathbb{G}_{3}$-hex $H$ is big in $\mathbb{H}_{4}$. So, $x$ is collinear with the point $\pi_{H}(x)$ of $H$. Let $Q$ denote an arbitrary $Q(5,2)$-quad of $H$ through $\pi_{H}(x)$. Among the near hexagons which can occur as hex in $\mathbb{G}_{4}$, only $\mathbb{G}_{3}$ and $Q(5,2) \times \mathbb{L}_{3}$ have $Q(5,2)$-quads. It follows that the hex $\langle x, Q\rangle=\left\langle x \pi_{H}(x), Q\right\rangle$ is isomorphic to $\mathbb{G}_{3}$ or $Q(5,2) \times \mathbb{L}_{3}$. The hex $\langle x, Q\rangle$ contains a unique point of $O_{f}$, namely the unique point of $O_{f}$ in $Q$ (recall Lemma 2.1(iv)). Now, all valuations of the near hexagons $\mathbb{G}_{3}$ and $Q(5,2) \times \mathbb{L}_{3}$ which contain a unique point with value

0 are classical. In particular, the valuation induced in $\langle x, Q\rangle$ by $f$ is classical. Hence, $f(x)=\mathrm{d}\left(x, O_{f} \cap Q\right)=1+\mathrm{d}\left(\pi_{H}(x), O_{f} \cap Q\right)=1+f^{\prime}\left(\pi_{H}(x)\right)$, where the latter equality follows from Lemma 2.1(iv). This proves that $f$ is the extension of $f^{\prime}$.

### 6.4 Treatment of case (C) of Lemma 6.5

In this subsection, we suppose that $f$ is a valuation of $\mathbb{G}_{4}$ such that $\left|O_{f}\right| \geq 2$ and such that every special quad is a grid-quad of type II. By Lemma 6.4, every two distinct points of $O_{f}$ are contained in a unique special quad. Since a special grid-quad contains three points of $O_{f}$, we have $\left|O_{f}\right| \geq 3$.

Lemma 6.8 It holds that $\left|O_{f}\right|>3$.
Proof. Suppose to the contrary that $\left|O_{f}\right|=3$. Let $Q$ denote the unique special grid-quad of type II and put $\left\{x_{1}, x_{2}, x_{3}\right\}=Q \cap O_{f}$. By Lemma 3.1(12), there exists a $Q(5,2) \times \mathbb{L}_{3}$-hex $F$ through $Q$. This hex contains precisely 2 special lines through $x_{1}$ by Lemmas $3.1(4)+(5)$. So, $F$ has an ordinary line $L$ through $x_{1}$ not contained in $Q$. Let $y \in L \backslash\left\{x_{1}\right\}$. By Lemmas 3.1(6)+(8), there exists a $\mathbb{G}_{3}$-hex $H^{\prime}$ through $y$ not containing the line $L$. Let $f^{\prime}$ denote the valuation of $H^{\prime}$ induced by $f$. Since $\pi_{H^{\prime}}\left(\left\{x_{1}, x_{2}, x_{3}\right\}\right) \subseteq O_{f^{\prime}}, f^{\prime}$ is nonclassical. By Lemma 2.1(v), there exists a $W(2)$-quad $Q^{\prime}$ of $H^{\prime}$ through $y$ which is special with respect to $f^{\prime}$. Now, by Lemma 3.1(9), $Q^{\prime}$ is contained in $1 \mathbb{G}_{3}$-hex (namely $H^{\prime}$ ), $1 W(2) \times \mathbb{L}_{3}$-hex (namely the hex $\left\langle Q^{\prime}, M\right\rangle$ where $M$ is the unique special line through $y$ not contained in $H^{\prime}$ ) and three $\mathbb{H}_{3}$-hexes. Hence, $\left\langle L, Q^{\prime}\right\rangle$ is isomorphic to $\mathbb{H}_{3}$. This implies that $\left\langle L, Q^{\prime}\right\rangle$ does not contain $Q$ since $\langle L, Q\rangle \cong Q(5,2) \times \mathbb{L}_{3}$. It follows that the valuation of $\left\langle L, Q^{\prime}\right\rangle \cong \mathbb{H}_{3}$ induced by $f$ contains a unique point with value 0 (namely $x_{1}$ ) and a point with value 1 at distance 3 from it (which is contained in $\Gamma_{2}(y) \cap Q^{\prime}$ ). This is impossible, since $\mathbb{H}_{3}$ does not have such valuations.

Lemma 6.9 $O_{f}$ is a set of 7 points in an $\mathbb{H}_{3}$-hex of $\mathbb{G}_{4}$.
Proof. Let $x$ denote an arbitrary point of $O_{f}$. By Lemmas 6.4 and 6.8, there are two distinct special grid-quads $G_{1}$ and $G_{2}$ (of type II) through $x$. By Lemma 6.1(iii), $G_{1} \cap G_{2}=\{x\}$. Let $u_{1}$ be an arbitrary point of $\left(O_{f} \cap G_{1}\right) \backslash\{x\}$. By Lemma 6.4, $u_{1}$ has distance 2 from every point of $O_{f} \cap G_{2}$. If $\mathrm{d}\left(u_{1}, G_{2}\right)=1$, then $u_{1}$ is classical with respect to $G_{2}$ and all points of $O_{f} \cap G_{2}$ would be collinear with $\pi_{G_{2}}\left(u_{1}\right)$, clearly a contradiction. Hence, $\mathrm{d}\left(u_{1}, G_{2}\right)=2$. Since every line of $G_{2}$ contains a unique point nearest to $u_{1}$, we have $G_{2} \backslash O_{f} \subseteq \Gamma_{3}\left(u_{1}\right)$. Now, let $u_{2}$ be an arbitrary point of $G_{2} \backslash O_{f}$. Then $\left\langle u_{1}, u_{2}\right\rangle$ is a hex. Since $O_{f} \cap G_{2} \subseteq \Gamma_{2}\left(u_{1}\right)$, there are two distinct points
$v_{1}$ and $v_{2}$ of $O_{f} \cap G_{2}$ collinear with $u_{2}$ which are on a geodesic path from $u_{2}$ to $u_{1}$. Hence, $G_{2}=\left\langle v_{1}, v_{2}\right\rangle \subseteq\left\langle u_{1}, u_{2}\right\rangle$. In particular, $x \in\left\langle u_{1}, u_{2}\right\rangle$. Since $x, u_{1} \in\left\langle u_{1}, u_{2}\right\rangle$, we have $G_{1}=\left\langle x, u_{1}\right\rangle \subseteq\left\langle u_{1}, u_{2}\right\rangle$. So, $H:=\left\langle G_{1}, G_{2}\right\rangle$ is a hex. By Lemma $3.1(12), H$ is isomorphic to either $\mathbb{H}_{3}$ or $Q(5,2) \times \mathbb{L}_{3}$. (Recall that $G_{1}$ and $G_{2}$ are grids of type II). Now, in the near hexagon $Q(5,2) \times \mathbb{L}_{3}$ any two distinct grid-quads through the same point meet each other in a line. Since $G_{1} \cap G_{2}=\{x\}$, we necessarily have $H \cong \mathbb{H}_{3}$. Since $\left|G_{1} \cap O_{f}\right|=\left|G_{2} \cap O_{f}\right|=3$, the valuation $f_{H}$ of $H$ induced by $f$ must be of Fano-type. Hence, $\left|O_{f} \cap H\right|=7$.

We show that $\Gamma_{1}(H) \cap O_{f}=\emptyset$. Suppose to the contrary that $y$ is a point of $\Gamma_{1}(H) \cap O_{f}$. Then $y$ is classical with respect to $H$. Since $f(y)=0$, $f\left(\pi_{H}(y)\right)=1$ and hence by Lemma 2.3(iv) $\pi_{H}(y)$ is contained in a unique quad $Q$ of $H$ which is special with respect to $f_{H}$. Any point of $Q \cap O_{f_{H}}=$ $Q \cap O_{f}$ at distance 2 from $\pi_{H}(y)$ lies at distance 3 from $y$, contradicting Lemma 6.4. Hence, $\Gamma_{1}(H) \cap O_{f}=\emptyset$.

We show that $f(y) \geq 2$ for every point $y$ of type (a) of $\Gamma_{2}(H)$. Let $Q$ denote the $W(2)$-quad of $H$ containing all points of $\Gamma_{2}(y) \cap H$ and let $H^{\prime}$ be the $\mathbb{G}_{3}$-hex $\langle y, Q\rangle$. Let $u$ denote the unique point of $O_{f} \cap Q$ (recall Lemma $2.3(\mathrm{i})$ ) and let $L$ be a line of $Q$ through $u$. If the valuation $f_{H^{\prime}}$ of $H^{\prime}$ induced by $f$ is not classical, then by Lemma 2.1(v) there exists a quad of $H^{\prime}$ through $L$ which is special with respect to $f_{H^{\prime}}$. This implies that there is a point of $O_{f_{H^{\prime}}} \subseteq O_{f}$ contained in $\Gamma_{1}(H)$, a contradiction. Hence, $f_{H^{\prime}}$ is a classical valuation of $H^{\prime}$. It follows that $f(y)=f_{H^{\prime}}(y)=\mathrm{d}(y, u) \geq 2$.

We show that $f(y) \geq 1$ for every point $y$ of type (b) of $\Gamma_{2}(H)$. By Lemma 4.6 there exists a line $L \in S$ through $y$ and this line contains a unique point $u$ of type (a) by Corollary 4.13. Since $f(u) \geq 2$, we have $f(y) \geq 1$.

Let $H$ denote the unique $\mathbb{H}_{3}$-hex of $\mathbb{G}_{4}$ containing all points of $O_{f}$ and let $f^{\prime}$ be the valuation of $H$ induced by $f$. By Lemma 6.9, $f^{\prime}$ is a valuation of Fano-type of $H$.

Proposition 6.10 The valuation $f$ is obtained from $f^{\prime}$ in the way as described in Section 5.

Proof. Let $x$ denote an arbitrary point of $\mathbb{G}_{4}$.
If $x \in H$, then $\mathrm{d}\left(x, O_{f}\right) \leq 2$ and hence $f(x)=\mathrm{d}\left(x, O_{f}\right)=\mathrm{d}\left(x, O_{f^{\prime}}\right)=f^{\prime}(x)$ by Lemma 6.1(ii).
If $x \in \Gamma_{1}(H)$ such that $\mathrm{d}\left(\pi_{H}(x), O_{f}\right) \leq 1$, then $\mathrm{d}\left(x, O_{f}\right) \leq 2$ and hence $f(x)=\mathrm{d}\left(x, O_{f}\right)=1+\mathrm{d}\left(\pi_{H}(x), O_{f}\right)=1+f^{\prime}\left(\pi_{H}(x)\right)$ by Lemma 6.1(ii).
Let $x \in \Gamma_{1}(H)$ such that $\mathrm{d}\left(\pi_{H}(x), O_{f}\right)=2$, or equivalently, such that $f^{\prime}\left(\pi_{H}(x)\right)=2$. Let $H^{\prime}$ denote an arbitrary $\mathbb{G}_{3}$-hex through the line $x \pi_{H}(x)$.

Then $H^{\prime} \cap H$ is a $W(2)$-quad $Q$ by Lemma 3.1(14). The hex $H^{\prime}$ contains a unique point $y$ with $f$-value 0 , namely the unique point of $O_{f}$ in $Q$ (recall Lemma 2.3(i)). Hence, the valuation induced in $H^{\prime}$ is classical. Since $\mathrm{d}\left(\pi_{H}(x), O_{f}\right)=2$, we have $\mathrm{d}\left(\pi_{H}(x), y\right)=2$. It follows that $f(x)=\mathrm{d}(x, y)=1+\mathrm{d}\left(\pi_{H}(x), y\right)=3=1+f^{\prime}\left(\pi_{H}(x)\right)$.
Let $x$ denote a point of type (a) of $\Gamma_{2}(H)$. Let $Q$ denote the $W(2)$-quad of $H$ containing all points of $\Gamma_{2}(x) \cap H$ and let $x^{*}$ denote the unique point of $O_{f}$ in $Q$. The hex $\langle x, Q\rangle$ is isomorphic to $\mathbb{G}_{3}$ and contains a unique point of $O_{f}$, namely $x^{*}$. Hence, the valuation induced in $\langle x, Q\rangle$ is classical. It follows that $f(x)=\mathrm{d}\left(x, x^{*}\right)$.
Let $x$ denote a point of type (b) of $\Gamma_{2}(H)$ such that $\left|O_{f} \cap Q\right|=3$, where $Q$ is the unique grid-quad of $H$ containing $\Gamma_{2}(x) \cap H$. The hex $\langle x, Q\rangle$ is isomorphic to $\mathbb{H}_{3}$ and the valuation of $\langle x, Q\rangle$ induced by $f$ is of grid-type. It follows from Lemma 2.2 that $f(x)=2$ if $\Gamma_{2}(x) \cap O_{f} \cap Q \neq \emptyset$ and $f(x)=1$ otherwise.

Let $x$ denote a point of type (b) of $\Gamma_{2}(H)$ such that $\left|O_{f} \cap Q\right|=0$, where $Q$ is the unique grid-quad of $H$ containing $\Gamma_{2}(x) \cap H$. By Lemma 2.3(ii), the points with $f$-value 1 determine an ovoid of $Q$. So, the grid-quad $Q$ is special with respect to the valuation $f^{\prime}$ of $\langle x, Q\rangle \cong \mathbb{H}_{3}$ induced by $f$. This implies that the valuation $f^{\prime}$ is either of grid-type or of Fano-type. We will show that the latter possibility cannot occur.

Suppose that $f^{\prime}$ is a valuation of Fano-type. Let $u$ denote one of the three points of $Q$ with $f$-value 1. By Lemma 2.3(iv), there exists a point $v \notin Q$ of $O_{f}$ collinear with $u$. Let $G \neq Q$ denote a grid-quad of $\langle x, Q\rangle$ through $u$ (which is special with respect to $f^{\prime}$ ). Then $G \cap Q=\{u\}$. Let $w$ be a point of $G \cap \Gamma_{2}(u)$. If $w \in \Gamma_{1}(Q)$, then $w$ is classical with respect to $Q, \pi_{Q}(w)$ would be a common neighbour of $u$ and $w$, and the quad $G=\langle u, w\rangle$ would contain the line $u \pi_{Q}(w)$ of $Q$, a contradiction. So, $w \in \Gamma_{2}(Q)$. By Lemma 4.3 applied to the hexes $\langle x, Q\rangle$ and $H$, we see that $w \in \Gamma_{2}(H)$. So, there exists a unique hex through $w$ meeting $H$ in a quad and this hex coincides with $\langle x, Q\rangle$. This implies that the hex $\langle v u, G\rangle \neq\langle x, Q\rangle$ intersects $H$ in the line $u v$. It follows that the valuation induced in $\langle v u, G\rangle$ contains a unique point with value 0 (namely $v$ ) and a point with value 1 at distance 3 from it (which is contained in $\left.\Gamma_{2}(u) \cap G\right)$. Among the near hexagons which can occur as hex in $\mathbb{G}_{4}$, only $W(2) \times \mathbb{L}_{3}$ has such valuations. So, $\langle v u, G\rangle \cong W(2) \times \mathbb{L}_{3}$ and the valuation induced in $\langle v u, G\rangle$ is semi-classical. But in a $W(2) \times \mathbb{L}_{3}$-hex, every grid-quad is of type I, while the grid-quad $G$ has type II since it is contained in the $\mathbb{H}_{3}$-hex $\langle x, Q\rangle$ (recall Lemma 3.1(11)+(12)). So, we have a contradiction and the valuation $f^{\prime}$ must be of grid-type.

By Lemma 2.2 it now follows $f(x)=3$ if $\Gamma_{2}(x) \cap Q$ has a point with $f^{\prime}$-value 1 and $f(x)=2$ otherwise.

This proves the proposition.

### 6.5 A lemma

Recall that by Section 1.1, the near polygon $\mathbb{G}_{n}$ can be isometrically embedded into the dual polar space $D H(2 n-1,4)$.

Lemma 6.11 Let $F$ be a hex of $\mathbb{G}_{4}$ and let $f$ be a valuation of $F$. Suppose that one of the following cases occurs: $(i) F \cong \mathbb{H}_{3}$ and $f$ is a valuation of Fano-type of $F$; (ii) $F \cong \mathbb{G}_{3}$ and $f$ is a non-classical valuation of $F$. Suppose also that $\mathbb{G}_{4}$ is isometrically embedded into the dual polar space $\operatorname{DH}(7,4)$. Then there exists a unique point $x \in D H(7,4) \backslash \mathbb{G}_{4}$ such that $O_{f} \subseteq \Gamma_{1}(x)$.

Proof. For every convex subspace $A$ of $\mathbb{G}_{4}$, there exists a unique convex subspace $\bar{A}$ of $D H(7,4)$ containing $A$ and having the same diameter as $A$. If $A$ has diameter $\delta$ and if $x_{1}$ and $x_{2}$ are two points of $A$ at distance $\delta$ from each other, then $\bar{A}$ is the unique convex subspace of $D H(7,4)$ containing $x_{1}$ and $x_{2}$.

Let $Q$ be a quad of $F$ which is special with respect to $f$. We moreover assume that $Q$ is a $W(2)$-quad if we are in case (ii) of the lemma. Put $Q \cap O_{f}=\left\{x_{1}, x_{2}, \ldots, x_{k}\right\}$, where $k=3$ (case (i)) or $k=5$ (case (ii)). Let $y$ be an arbitrary point of $O_{f} \backslash Q$. Then $\mathrm{d}\left(y, x_{i}\right)=2$ for every $i \in\{1, \ldots, k\}$. If $\mathrm{d}(y, Q)=1$, then $y$ is classical with respect to $Q$ and all points of the ovoid $Q \cap O_{f}=\left\{x_{1}, \ldots, x_{k}\right\}$ of $Q$ would be collinear with $\pi_{Q}(y)$, clearly a contradiction. Hence, $\mathrm{d}(y, Q)=2$. Since every point of $\bar{Q} \backslash Q$ is collinear with some point of $Q$, we have $y \notin \bar{Q}$. Since the quad $\bar{Q}$ of $\operatorname{DH}(7,4)$ is big in the hex $\bar{F}$ of $D H(7,4)$, this implies that $\mathrm{d}(y, \bar{Q})=1$. Since $\mathrm{d}\left(y, x_{i}\right)=2$ and $y$ is classical with respect to $\bar{Q}$, we have $\mathrm{d}\left(\pi_{\bar{Q}}(y), x_{i}\right)=1$ for every $i \in\{1, \ldots, k\}$. For every $i \in\{1, \ldots, k\}$, let $Q_{i}$ denote the unique quad of $\mathbb{G}_{4}$ through $y$ and $x_{i}$. Since $y, x_{i} \in Q_{i} \cap O_{f}, Q_{i}$ is special with respect to $f$. So, $Q_{i}$ is either a $(3 \times 3)$-grid or a $W(2)$-quad and there exists a unique ovoid $O_{i}$ of $Q_{i}$ containing $y$ and $x_{i}$. Now, the $k$ quads $Q_{1}, \ldots, Q_{k}$ are all the quads through $y$ which are special with respect to $f$. Since any two distinct points of $O_{f}$ lie at distance 2 from each other, we necessarily have $O_{f}=O_{1} \cup O_{2} \cup \cdots \cup O_{k}$.

We prove that $\pi_{\bar{Q}}(y) \notin \mathbb{G}_{4}$. Suppose to the contrary that $\pi_{\bar{Q}}(y) \in \mathbb{G}_{4}$. Since $\pi_{\bar{Q}}(y)$ is collinear with the points $x_{1}, \ldots, x_{k}$, we would then have that $\pi_{\bar{Q}}(y) \in Q$. This is impossible since $\mathrm{d}(y, Q)=2$. Hence, $\pi_{\bar{Q}}(y) \notin \mathbb{G}_{4}$.

Since $\pi_{\bar{Q}}(y)$ is collinear with the points $y$ and $x_{i}, i \in\{1, \ldots, k\}, \pi_{\bar{Q}}(y)$ is contained in $\overline{Q_{i}}$. So, $\Gamma_{1}\left(\pi_{\bar{Q}}(y)\right) \cap Q_{i}$ is an ovoid of $Q_{i}$ containing $y$ and $x_{i}$. It follows that $\Gamma_{1}\left(\pi_{\bar{Q}}(y)\right) \cap Q_{i}=O_{i}$. Hence, $O_{f}=O_{1} \cup O_{2} \cup \cdots \cup O_{k} \subseteq \Gamma_{1}\left(\pi_{\bar{Q}}(y)\right)$.

Conversely, suppose $z$ is a point of $D H(7,4) \backslash \mathbb{G}_{4}$ such that $O_{f} \subseteq \Gamma_{1}(z)$. Since $z$ is collinear with the points $x_{1}, \ldots, x_{k}$, we have $z \in \bar{Q}$. Since $z$ is collinear with $y$. We necessarily have $z=\pi_{\bar{Q}}(y)$.

### 6.6 The valuations of $\mathbb{G}_{4}$ are induced by valuations of DH(7, 4)

Let the near octagon $\mathbb{G}_{4}$ be isometrically embedded in $D H(7,4)$. For every point $x$ of $D H(7,4)$, the classical valuation $g_{x}$ of $D H(7,4)$ with $O_{g_{x}}=\{x\}$ induces a valuation $f_{x}$ of $\mathbb{G}_{4}$. It holds that $\max \left\{f_{x}(u) \mid u \in \mathbb{G}_{4}\right\}=4-\mathrm{d}\left(x, \mathbb{G}_{4}\right)$ in view of the following result which holds for general dense near polygons.

Lemma 6.12 (Proposition 2.2 of [14]) Let $\mathcal{S}$ be a dense near $2 n$-gon and let $F=\left(\mathcal{P}^{\prime}, \mathcal{L}^{\prime}, \mathrm{I}^{\prime}\right)$ be a dense near $2 n$-gon which is fully and isometrically embedded in $\mathcal{S}$. Let $x$ be a point of $\mathcal{S}$ and let $f_{x}$ denote the valuation of $F$ induced by the classical valuation $g_{x}$ of $\mathcal{S}$ with $O_{g_{x}}=\{x\}$, then $d(x, F)=$ $n-M$, where $M$ is the maximal value attained by $f_{x}$.

If $x \in \mathbb{G}_{4}$, then $f_{x}$ is a classical valuation of $\mathbb{G}_{4}$ and $O_{f_{x}}=\{x\}$. If $x \notin \mathbb{G}_{4}$, then $f_{x}$ is not classical and hence is either the extension of a non-classical valuation of a $\mathbb{G}_{3}$-hex or is a valuation of Fano-type.

Proposition 6.13 Let $f$ be a valuation of $\mathbb{G}_{4}$. Then there exists a unique point $x$ of $D H(7,4)$ such that $f=f_{x}$.

Proof. Obviously, the proposition holds if $f$ is classical. The required point $x$ is then the unique point contained in $O_{f}$.

Suppose now that $f$ is non-classical. By the classification of the valuations of $\mathbb{G}_{4}$, we then know that $F:=\left\langle O_{f}\right\rangle$ is either an $\mathbb{H}_{3}$-hex or a $\mathbb{G}_{3}$-hex of $\mathbb{G}_{4}$. Moreover, if $f^{\prime}$ denotes the valuation of $F$ induced by $f$, then $O_{f^{\prime}}=O_{f}, f^{\prime}$ is a valuation of Fano-type of $F$ if $F \cong \mathbb{H}_{3}$ and $f^{\prime}$ is a non-classical valuation of $F$ if $F \cong \mathbb{G}_{3}$. By Lemma 6.11, there exists a unique point $x^{*} \in D H(7,4) \backslash \mathbb{G}_{4}$ such that $O_{f^{\prime}} \subseteq \Gamma_{1}\left(x^{*}\right)$. Then $O_{f} \subseteq O_{f_{x^{*}}}$. Hence, $O_{f}=O_{f_{x^{*}}}$ and $f=f_{x^{*}}$ by the classification of the valuations of $\mathbb{G}_{4}$.

Conversely, suppose that $f=f_{x}$ for some point $x$ of $D H(7,4)$. Since $f$ is non-classical, its maximal value is equal to 3 . Lemma 6.12 then implies that $\mathrm{d}\left(x, \mathbb{G}_{4}\right)=1$. We have $O_{f}=\Gamma_{1}(x) \cap \mathbb{G}_{4}$. Since $O_{f} \subseteq \Gamma_{1}(x)$, Lemma 6.11 implies that $x=x^{*}$.

By Proposition 6.13, the number of valuations of $\mathbb{G}_{4}$ is equal to the number of points of $D H(7,4)$. The number of classical valuations of $\mathbb{G}_{4}$ is equal to the number of points of $\mathbb{G}_{4}$, i.e., equal to 8505. The number of valuations of $\mathbb{G}_{4}$ which are extensions of non-classical valuations in $\mathbb{G}_{3}$-hexes is equal to $\left(\# \mathbb{G}_{3}\right.$-hexes $) \times\left(\#\right.$ non-classical valuations in a $\mathbb{G}_{3}$-hex $)=84 \cdot 486=40824$. The number of valuations of Fano-type of $\mathbb{G}_{4}$ is equal to ( $\# \mathbb{H}_{3}$-hexes) $\times(\#$ valuations of Fano-type in an $\mathbb{H}_{3}$-hex $)=2178 \cdot 30=65610$. The number $8505+40824+65610=114939$ is indeed equal to the total number of points of DH(7,4).

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