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Determining the fill rate for a periodic review inventory policy with capacitated replenishments, lost sales and zero lead time

Thomas Dubois [a], Georges Allaert [b], Frank Witlox [a]

[a] Department of Geography, Ghent University, Ghent, Belgium
[b] Center for Mobility and Spatial Planning, Ghent University, Ghent, Belgium

Abstract

In this paper we consider a periodic review order-up-to inventory system with capacitated replenishments, lost sales and zero lead time. We consider discrete demand. It is shown that the initial stock levels of the different review periods form a Markov chain and we determine the transition matrix. Furthermore we study for what probability mass functions of the review period demand the Markov chain has a unique stationary distribution. Finally we present a method to determine the fill rate.

Keywords: Inventory; Periodic review; Fill rate; Capacitated replenishment; Lost sales

1. Introduction

In several publications the fill rate is discussed for periodic review inventory systems with uncapacitated replenishment. For example Johnson et al. [1] study different fill rate expressions for inventory systems with backorders and normally distributed demand and compare these expressions experimentally via simulation experiments. The fill rate of an uncapacitated periodic review inventory system with backorders and continuous period demand is also studied in [2], [3] and [4]. Sobel [2] discusses besides single-stage systems also multistage systems and similar as in [1] the lead time is assumed to be a multiple of the review period. This is not assumed in Zhang et al. [3] and Silver et al. [4]. In [2] and [3] general continuous demand and normal demand are considered, [1] and [4] focus on normal demand. Guijarro et al. [5] discuss fill rate definitions and expressions for uncapacitated periodic review inventory systems with lost sales and discrete demand. In this paper however, periodic review inventory systems with a limited replenishment capacity are studied. Unlike [1], [2], [3], [4] and [5], in this paper the lead time is assumed to be negligible. In a part of [2], capacity is also considered, but in the context of multistage systems with process limitations. In [6] and [7] finite horizon fill rates are considered and compared with the infinite horizon fill rate.

We consider a single-item inventory system that applies a periodic review order-up-to inventory policy with lost sales and zero lead time. Because of the lost sales assumption and the zero lead time assumption, the inventory position (number of products on hand minus number of products backlogged plus number of products on order) equals the stock level (number of products on hand). In such inventory policy the stock level is reviewed periodically and every review an order is placed to raise the stock level to a fixed level, the order-up-to level s (a positive integer). We assume the demand during one review period (period between two reviews) to be discrete with a given probability mass function. We consider a review period to begin when the order is placed and to end just before the next order is placed. Following characteristics are assumed for the inventory system under study: (i) the order is placed immediately after review; (ii) the lead time is zero, i.e. the order arrives immediately after the order is placed; (iii) the demands during different review periods are independent and identical distributed; (iv) the demand during a particular review period is independent of every stock level at the beginning of a review period that precedes that review period or coincides with that review period; (v) unsatisfied demands result in lost sales; and (vi) replenishment is capacitated with capacity c (a positive integer), i.e. if more than c products are ordered, only c are delivered.

In this paper we determine the fill rate of a periodic review inventory system with capacitated replenishments. A similar problem was already studied by Mapes [8], who determined the service level of a capacitated periodic review inventory system approximately by simulation. In this paper a new method to determine the fill rate is presented which is exact given the used fill rate definition and the above stated six assumptions. Similarly as in [9], we define the fill rate of a periodic review inventory system as the proportion of the expected satisfied demand to the expected demand (see (23) for the exact formula). Another definition used in literature for the fill rate (e.g. in [2] and [3]) is the expectation of the proportion of the satisfied demand to the demand. According to [6] and [3], both definitions agree if an infinite horizon is considered.

2. Determination of the fill rate

In this section we will determine the fill rate β of a periodic review order-up-to inventory system with order-up-to level s and replenishment capacity c. We assume $c<s$ because when c is greater than or equal to s replenishment is not capacitated. Let D_t be the random variable associated with the demand during the review period t, I_t the random variable associated with the stock level at the beginning of review period t, f_D the probability mass function of D_t (with the set of the integers as domain and value zero for negative integers) and f_{It} the probability mass function of I_t, for all $t\in\{1,2,...\}$. We assume the stock level at the beginning of the first review period to be c, $c+1$, ... or s. Because of the used inventory policy, the following holds:

$$I_t = \min\{s, \max\{I_{t-1} - D_{t-1}, 0\} + c\}, \text{ for all } t\in\{2,3,...\}. \quad (1)$$

We continue by first proving four theorems and then presenting a method to find the fill rate based on these theorems. For (finite state) Markov chain theory we refer to [10], chapter 4.

Theorem 1. $I_1, I_2, I_3, ...$ is a Markov chain.

Proof. For proving theorem 1, we need to proof the following:

$$P(I_t = i_t | I_{t-1} = i_{t-1} \cap I_{t-2} = i_{t-2} \cap ... \cap I_1 = i_1) = P(I_t = i_t | I_{t-1} = i_{t-1}), \text{ for all } t\in\{2,3,...\} \text{ and}$$

for all $i_1,...,i_t\in\{c,c+1,...,s\}$ for which $P(I_{t-1}=i_{t-1}\cap...\cap I_1=i_1)\neq 0$ and $P(I_{t-1}=i_{t-1})\neq 0$ (2)

We start with the definition of conditional probability and (1) and then use assumption (iv).

For all $t\in\{2,3,...\}$ and for all $i_1,...,i_t\in\{c,c+1,...,s\}$ for which $P(I_{t-1}=i_{t-1}\cap...\cap I_1=i_1)\neq 0$ and $P(I_{t-1}=i_{t-1})\neq 0$:

$$P(I_t = i_t | I_{t-1} = i_{t-1} \cap I_{t-2} = i_{t-2} \cap ... \cap I_1 = i_1)$$

$$= \frac{P(\min\{s, \max\{I_{t-1} - D_{t-1}, 0\} + c\} = i_t \cap I_{t-1} = i_{t-1} \cap ... \cap I_1 = i_1)}{P(I_{t-1} = i_{t-1} \cap ... \cap I_1 = i_1)} \quad (3)$$

$$= \frac{P(\min\{s, \max\{i_{t-1} - D_{t-1}, 0\} + c\} = i_t \cap I_{t-1} = i_{t-1} \cap ... \cap I_1 = i_1)}{P(I_{t-1} = i_{t-1} \cap ... \cap I_1 = i_1)} \quad (4)$$

$$= P(\min\{s, \max\{i_{t-1} - D_{t-1}, 0\} + c\} = i_t). \quad (5)$$

Similarly, for all $t\in\{2,3,...\}$ and for all $i_1,...,i_t\in\{c,c+1,...,s\}$ for which $P(I_{t-1}=i_{t-1}\cap...\cap I_1=i_1)\neq 0$ and $P(I_{t-1}=i_{t-1})\neq 0$:

$$P(I_t = i_t | I_{t-1} = i_{t-1}) = \frac{P(\min\{s, \max\{I_{t-1} - D_{t-1}, 0\} + c\} = i_t \cap I_{t-1} = i_{t-1})}{P(I_{t-1} = i_{t-1})} \quad (6)$$

$$= \frac{P(\min\{s, \max\{i_{t-1} - D_{t-1}, 0\} + c\} = i_t \cap I_{t-1} = i_{t-1})}{P(I_{t-1} = i_{t-1})} \quad (7)$$

$$= P(\min\{s, \max\{i_{t-1} - D_{t-1}, 0\} + c\} = i_t)\ . \quad (8)$$

Combination of (5) and (8) yields (2), which completes the proof. □

Theorem 2. The element at row i and column j of the transition matrix $\boldsymbol{P}$ of Markov chain $I_1, I_2, I_3, \ldots$ with states $c, c+1, \ldots, s$ is:

$$p_{ij} = \sum_{k=0}^{\infty} f_D(k)\delta\left(\min\{s, \max\{c-1+i-k, 0\}+c\} - c + 1 - j\right), \text{ for all } i,j \in \{1,2,\ldots,s-c+1\} \quad (9)$$

with $\delta(x)=1$ if $x=1$ and $\delta(x)=0$ if $x \neq 1$ for every integer x.

Proof. For all $i,j \in \{1,2,\ldots,s-c+1\}$ and for all $t \in \{2,3,\ldots\}$ for which $P(I_{t-1}=c-1+i) \neq 0$:

$$p_{ij} = P\left(I_t = c-1+j \,|\, I_{t-1} = c-1+i\right) \quad (10)$$

$$= P(\min\{s, \max\{c-1+i-D_{t-1}, 0\}+c\} = c-1+j) \quad (11)$$

$$= \sum_{k=0}^{\infty} f_D(k)\delta\left(\min\{s, \max\{c-1+i-k, 0\}+c\} - c + 1 - j\right) \quad (12)$$

For getting (10) we applied the definition of transition matrix and for getting (11) we used (8). □

Theorem 3.

- If $f_D(c) \neq 1$, then for the Markov chain $I_1, I_2, I_3, \ldots$ the following matrix equation in the variable $[f_I(c)\ f_I(c+1)\ \ldots\ f_I(s)]^T$, with $0 \leq f_I(c) \leq 1$, $0 \leq f_I(c+1) \leq 1$, ... and $0 \leq f_I(s) \leq 1$, has a unique solution

$$\begin{bmatrix} 1 & 1 & 1 & \ldots & 1 \\ p_{12} & p_{22}-1 & p_{32} & \ldots & p_{s-c+1\ 2} \\ p_{13} & p_{23} & p_{33}-1 & \ldots & p_{s-c+1\ 3} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ p_{1\ s-c+1} & p_{2\ s-c+1} & p_{3\ s-c+1} & \ldots & p_{s-c+1\ s-c+1}-1 \end{bmatrix} \begin{bmatrix} f_I(c) \\ f_I(c+1) \\ f_I(c+2) \\ \ldots \\ f_I(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \ldots \\ 0 \end{bmatrix} \quad (13)$$

and for these $f_I(c), f_I(c+1), \ldots$ and $f_I(s)$

$$\lim_{n \to \infty} f_{In}(i) = f_I(i), \text{ for all } i \in \{c, c+1, \ldots, s\}. \quad (14)$$

- If $f_D(c)=1$, then $f_{It}(i) = f_{I1}(i)$ for all $t \in \{1,2,\ldots\}$ and for all $i \in \{c, c+1, \ldots, s\}$.

Proof. The transition matrix of the Markov chain is

$$\boldsymbol{P} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1\ s-c+1} \\ p_{21} & p_{22} & \cdots & p_{2\ s-c+1} \\ \ldots & \ldots & \ldots & \ldots \\ p_{s-c-1\ 1} & p_{s-c-1\ 2} & \cdots & p_{s-c+1\ s-c+1} \end{bmatrix}. \quad (15)$$

Because of theorem 2, we get

$$
\boldsymbol{P}=\begin{bmatrix}
\sum_{k=c+m}^{\infty} f_D(k)+f_D(c+m-1)+\ldots+f_D(c) & f_D(c-1) & f_D(c-2) & \ldots & f_D(c-m+1) & \sum_{k=-\infty}^{c-m} f_D(k) \\
\sum_{k=c+m}^{\infty} f_D(k)+f_D(c+m-1)+\ldots+f_D(c+1) & f_D(c) & f_D(c-1) & \ldots & f_D(c-m+2) & f_D(c-m+1)+\sum_{k=-\infty}^{c-m} f_D(k) \\
\sum_{k=c+m}^{\infty} f_D(k)+f_D(c+m-1)+\ldots+f_D(c+2) & f_D(c+1) & f_D(c) & \ldots & f_D(c-m+3) & f_D(c-m+2)+f_D(c-m+1)+\sum_{k=-\infty}^{c-m} f_D(k) \\
\ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
\sum_{k=c+m}^{\infty} f_D(k)+f_D(c+m-1) & f_D(c+m-2) & f_D(c+m-3) & \ldots & f_D(c) & f_D(c-1)+\ldots+f_D(c-m+1)+\sum_{k=-\infty}^{c-m} f_D(k) \\
\sum_{k=c+m}^{\infty} f_D(k) & f_D(c+m-1) & f_D(c+m-2) & \ldots & f_D(c+1) & f_D(c)+\ldots+f_D(c-m+1)+\sum_{k=-\infty}^{c-m} f_D(k)
\end{bmatrix}, \text{ with } m=s-c. \quad (16)
$$

Case 1: $f_D(c)\neq 1$ and $f_D(x)=0$ for all $x\in\{0,1,...,c-1\}$.

By studying (16) we conclude that $\boldsymbol{P}$ is a lower triangular matrix and for every state the probability to go to state c in a number of steps is positive and the probability to go from state i to state j in a number steps is zero if $i<j$. Therefore state c is recurrent and the other states are transient.

Case 2: $f_D(c)\neq 1$ and $f_D(x)=0$ for all $x\in\{c+1,c+2,...\}$.

By studying (16) we conclude that $\boldsymbol{P}$ is a upper triangular matrix and for every state the probability to go to state s in a number of steps is positive and the probability to go from state i to state j in a number steps is zero if $j<i$. Therefore state s is recurrent and the other states are transient.

Case 3: if $f_D(c)\neq 1$ and $f_D(x)\neq 0$ for at least one integer x smaller than c and $f_D(x)\neq 0$ for at least one integer x larger than c.

Similarly with case 1 and 2, for every state the probability to go to state c in a number of steps is positive and for every state the probability to go to state s in a number of steps is positive. Therefore s and c are in the same communication class and every state communicates with c or is a transient state. We conclude that also in this case there is one recurrent communication class and all other states are transient.

In case 1, 2 and 3 the recurrent communication class is aperiodic because in case 1 and 3 the probability for going from state c to state c in one step is positive, and in case 2 and 3 the probability for going from state s to state s in one step is positive. Application of theorem 6A on page 118 of [10] yields the following two statements:

- there is a unique left probability eigenvector of $\boldsymbol{P}$ with eigenvalue 1 (17)

(this vector is called the stationary distribution of the Markov chain)

- let $[f_I(c) \quad f_I(c+1) \quad \ldots \quad f_I(s)]$ be this vector, then

$$\lim_{n\to\infty}(\boldsymbol{P})^n = \begin{bmatrix} f_I(c) & f_I(c+1) & \ldots & f_I(s) \\ f_I(c) & f_I(c+1) & \ldots & f_I(s) \\ \ldots & \ldots & \ldots & \ldots \\ f_I(c) & f_I(c+1) & \ldots & f_I(s) \end{bmatrix}. \tag{18}$$

(18) allows us to calculate the following limit, what results in the proof of (14).

$$\lim_{n\to\infty}\left[f_{In}(c) \quad f_{In}(c+1) \quad \ldots \quad f_{In}(s)\right] = \lim_{n\to\infty}\left(\left[f_{I1}(c) \quad f_{I1}(c+1) \quad \ldots \quad f_{I1}(s)\right](\boldsymbol{P})^n\right) \tag{19}$$

$$= \left[f_I(c) \quad f_I(c+1) \quad \ldots \quad f_I(s)\right] \tag{20}$$

It is equivalent with (17) that $\left[f_I(c) \quad f_I(c+1) \quad \ldots \quad f_I(s)\right]^T$ the unique right probability eigenvector is of $\boldsymbol{P}^T$ with eigenvalue 1. Therefore the following matrix equation has a unique probability vector solution.

$$\begin{bmatrix} p_{11}-1 & p_{21} & \ldots & p_{s-c+1\,1} \\ p_{12} & p_{22}-1 & \ldots & p_{s-c+1\,2} \\ \ldots & \ldots & \ldots & \ldots \\ p_{1\,s-c+1} & p_{2\,s-c+1} & \ldots & p_{s-c+1\,s-c+1}-1 \\ 1 & 1 & \ldots & 1 \end{bmatrix} \begin{bmatrix} f_I(c) \\ f_I(c+1) \\ \ldots \\ f_I(s) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \ldots \\ 0 \\ 1 \end{bmatrix} \tag{21}$$

Considered as a system of linear equations, (21) contains s-c+2 equations. The first equation in this system of linear equations equals the opposite of the sum of the other equations except the last one, because for all $i\in\{1,2,\ldots,s-c+1\}$

$$\sum_{k=1}^{s-c+1} p_{ik} = 1. \tag{22}$$

Omitting the first equation in (21) therefore yields an equivalent matrix equation. With the first equation omitted and the last equation first we obtain matrix equation (13) which is equivalent with (21).

Case 4: if $f_D(c)=1$.

Then $\boldsymbol{P}$ is the $(s\text{-}c+1)\times(s\text{-}c+1)$ identity matrix. Therefore $f_{It}(i)=f_{I1}(i)$, for all $t\in\{1,2,\ldots\}$ and for all $i\in\{c,c+1,\ldots,s\}$. □

Using the notation of this section the infinite horizon fill rate β of the periodic review inventory system under study is:

$$\beta = \lim_{n\to\infty}\frac{E\left(\min\{I_1,D_1\}+\min\{I_2,D_2\}+\ldots+\min\{I_n,D_n\}\right)}{E\left(D_1+D_2+\ldots+D_n\right)}. \tag{23}$$

Theorem 4.

$$\lim_{n\to\infty}\frac{E\left(\min\{I_1,D_1\}+\min\{I_2,D_2\}+\ldots+\min\{I_n,D_n\}\right)}{E\left(D_1+D_2+\ldots+D_n\right)} = 1-\frac{\sum_{i=c}^{s} f_I(i)\sum_{j=i+1}^{\infty}(j-i)f_D(j)}{\sum_{j=1}^{\infty} jf_D(j)} \tag{24}$$

with $f_I(i)=\lim_{n\to\infty} f_{In}(i)$, for all $i\in\{c,c+1,\ldots,s\}$.

Proof. $\lim_{n\to\infty}\frac{E\left(\min\{I_1,D_1\}+\min\{I_2,D_2\}+\ldots+\min\{I_n,D_n\}\right)}{E\left(D_1+D_2+\ldots+D_n\right)}$

$$= \lim_{n\to\infty}\frac{E(\min\{I_1,D_1\})+E(\min\{I_2,D_2\})+\ldots+E(\min\{I_n,D_n\})}{nE\left(D_1\right)} \tag{25}$$

$$= \frac{\lim_{n\to\infty} E(\min\{I_n,D_n\})}{E\left(D_1\right)} \tag{26}$$

$$= \frac{\lim_{n\to\infty} E(D_n - \max\{D_n - I_n, 0\})}{E(D_1)} \tag{27}$$

$$= 1 - \frac{\lim_{n\to\infty} E(\max\{D_n - I_n, 0\})}{E(D_1)} \tag{28}$$

$$= 1 - \frac{\lim_{n\to\infty} \sum_{i=c}^{s} \sum_{j=0}^{\infty} \max\{j-i, 0\} f_{In}(i) f_D(j)}{\sum_{j=0}^{\infty} j f_D(j)} \tag{29}$$

$$= 1 - \frac{\sum_{i=c}^{s} \sum_{j=i+1}^{\infty} (j-i) f_I(i) f_D(j)}{\sum_{j=1}^{\infty} j f_D(j)} \tag{30}$$

$$= 1 - \frac{\sum_{i=c}^{s} f_I(i) \sum_{j=i+1}^{\infty} (j-i) f_D(j)}{\sum_{j=1}^{\infty} j f_D(j)} \tag{31}$$

For getting (26) we used:

$$\lim_{n\to\infty} \frac{f(1) + f(2) + \ldots + f(n)}{n} = \lim_{n\to\infty} f(n) \tag{32}$$

with f a function from the natural numbers to the real numbers for which the sequence $f(1), f(2), \ldots$ converges. Furthermore, we used assumption (iii) for getting (25), (28) and (29) and assumption (iv) for getting (29). □

Theorems 1, 2, 3 and 4 put forward a method to determine the fill rate of the studied inventory system. First we construct the matrix $\boldsymbol{P}$ with the aid of theorem 2 or (16). Subsequently, if $f_D(c) \neq 1$, we solve matrix equation (13). According to theorem 3, this matrix equation has a unique probability vector solution and this solution gives us $\lim_{n\to\infty} f_{In}(c)$, $\lim_{n\to\infty} f_{In}(c+1)$, ... and $\lim_{n\to\infty} f_{In}(s)$. Notice that $\lim_{n\to\infty} f_{In}(c)$, $\lim_{n\to\infty} f_{In}(c+1)$, ... and $\lim_{n\to\infty} f_{In}(s)$ do not depend on the stock level at the beginning of the first review period, if $f_D(c) \neq 1$. If $f_D(c) = 1$ however, $\lim_{n\to\infty} f_{In}(i)$ equals $f_{I1}(i)$ for every state i. Finally we use these limits to calculate the fill rate of the inventory system with the aid of theorem 4.

3. Conclusion and further research

We proved four theorems that allow us to determine the fill rate of every periodic review order-up-to inventory system with capacitated replenishments, lost sales and zero lead time, for any demand probability mass function. The method is exact given the used definitions and assumptions. Extensions of this research are the study of capacitated periodic review inventory systems with positive lead times and the study of a series of periodic review inventory systems with joint capacitated replenishments. Allowing the lead time to be positive increases the complexity. For example, if the positive lead time is smaller than the review period, the stock level at the beginning of review period t will depend on the stock level at the beginning of review period t-1, the demand during review period t-1 and the demand during review period t-1 before replenishment, for the zero lead time case, see (1), the stock level at the beginning of review period t depends only on the stock level at the beginning of review period t-1 and the demand during review period t-1. If the lead time is greater than the review period but not greater than two review periods, the stock level at the beginning of review period t will depend also on the stock level at the beginning of review period t-2, therefore the Markov chain will have order 2, for the zero lead time case, the order is 1.

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