Pseudo-embeddings and pseudo-hyperplanes

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Abstract

We generalize some known results regarding hyperplanes and projective embeddings of point-line geometries with three points per line to geometries with an arbitrary but finite number of points on each line. In order to generalize these results, we need to introduce the new notions of pseudo-hyperplane, (universal) pseudo-embedding, pseudo-embedding rank and pseudo-generating rank. We prove several connections between these notions and address the problem of the existence of (certain) pseudo-embeddings. We apply this to several classes of point-line geometries. We also determine the pseudo-embedding rank and the pseudo-generating rank of the projective space $\mathrm{PG}(n, 4)$ and the affine space $\mathrm{AG}(n, 4)$.

1 Introduction

The aim of this section is to introduce some new notions and to state the main results which will be proved in Section 2. The results presented here are natural analogues of well-known results about hyperplanes and projective embeddings of point-line geometries. Several notions defined here, like pseudo-hyperplane, (universal) pseudo-embedding and pseudo-embedding rank, have their natural analogues in the known theory for hyperplanes and ordinary embeddings of point-line geometries. The terminology is chosen in such a way that the natural analogue corresponding to the notion "pseudo-x" is precisely the notion "x". In case all lines have three points, the notions "pseudo-x" and "x" coincide and several of the described results coincide with some known results about hyperplanes and ordinary embeddings of point-line geometries obtained by Ronan [21].

Throughout this section, $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$. For a line L of $\mathcal{S}$, we denote by $\mathcal{P}_L$ the set of points of $\mathcal{S}$ incident with L. We suppose that $3 \leq |\mathcal{P}_L| < \infty$ for every line $L \in \mathcal{L}$.

A *pseudo-hyperplane* of $\mathcal{S}$ is a proper subset H of $\mathcal{P}$ such that $|\mathcal{P}_L \cap (\mathcal{P} \setminus H)|$ is even for every line L of $\mathcal{S}$. In the case $\mathcal{S}$ is finite, the definition of pseudo-hyperplane can be

rephrased in coding theoretical terms: a proper subset H of $\mathcal{P}$ is a pseudo-hyperplane if and only if the characteristic vector of its complement $\mathcal{P} \setminus H$ belongs to the dual code of $\mathcal{S}$. Here, the code of $\mathcal{S}$ is defined as the subspace of $\mathbb{F}_2^{\mathcal{P}}$ generated by the characteristic vectors of the lines.

A pseudo-hyperplane of $\mathcal{S}$ has a nice intersection pattern with the lines of $\mathcal{S}$. There is a vast literature about sets of points of finite point-line geometries which have certain intersection patterns with respect to lines. In this context it is worth mentioning the pioneering work of Tallini-Scafati [24, 25, 26] on this topic. The pseudo-hyperplanes of certain point-line geometries have been studied before in the literature under a different name. This is the case for geometries with three points per line, where the pseudo-hyperplanes are precisely the hyperplanes. In this context it is also worth mentioning the work of Hirschfeld & Hubaut [14] and Sherman [22], who obtained a classification of all pseudo-hyperplanes (also known as sets of odd type) of $\mathrm{PG}(n,4)$.

If H_1 and H_2 are two distinct pseudo-hyperplanes of $\mathcal{S}$, then the complement $\overline{H_1 \Delta H_2} := \mathcal{P} \setminus (H_1 \Delta H_2)$ of the symmetric difference $H_1 \Delta H_2$ of H_1 and H_2 is again a pseudo-hyperplane of $\mathcal{S}$. The fact that $\overline{H_1 \Delta H_2}$ is again a pseudo-hyperplane is a well-known fact for hyperplanes of point-line geometries with three points per line and was an important tool in the papers of Hirschfeld & Hubaut [14] and Sherman [22] to obtain their desired classification results.

Suppose V is a vector space over the field $\mathbb{F}_2$ of order 2. A *pseudo-embedding* of $\mathcal{S}$ into the projective space $\Sigma = \mathrm{PG}(V)$ is a mapping e from $\mathcal{P}$ to the point set of Σ satisfying: (1) $< e(\mathcal{P}) >_\Sigma = \Sigma$; (2) if L is a line of $\mathcal{S}$ and $\mathcal{P}_L = \{x_1, x_2, \ldots, x_k\}$, then the points $e(x_1), e(x_2), \ldots, e(x_{k-1})$ of Σ are linearly independent and $e(x_k) = < \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_{k-1} >$ where $\bar{v}_i$, $i \in \{1, 2, \ldots, k-1\}$, is the unique vector of V for which $e(x_i) = < \bar{v}_i >_\Sigma$. If moreover e is an injective mapping, then the pseudo-embedding $e : \mathcal{S} \to \Sigma$ is called *faithful*. Observe that in the definition of the notion pseudo-embedding, the ordering $x_1, x_2, \ldots, x_k$ given to the points of the line L is not essential. Two pseudo-embeddings $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ of $\mathcal{S}$ are called *isomorphic* ($e_1 \cong e_2$) if there exists an isomorphism $\phi : \Sigma_1 \to \Sigma_2$ such that $e_2 = \phi \circ e_1$.

A connection between pseudo-hyperplanes and pseudo-embeddings is described in the following theorem.

Theorem 1.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry and suppose that $3 \leq |\mathcal{P}_L| < \infty$ for every line $L \in \mathcal{L}$. Suppose $e : \mathcal{S} \to \Sigma$ is a pseudo-embedding of $\mathcal{S}$ and Π is a hyperplane of Σ. Then $H_\Pi := e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a pseudo-hyperplane of $\mathcal{S}$.*

If a pseudo-hyperplane H of $\mathcal{S}$ is obtained from a pseudo-embedding e as described in Theorem 1.1, then H is said to *arise from* e. If e is a pseudo-embedding of $\mathcal{S}$, then $\mathcal{H}_e$ denotes the set of all pseudo-hyperplanes of $\mathcal{S}$ arising from e.

Suppose $e : \mathcal{S} \to \Sigma$ is a pseudo-embedding of $\mathcal{S}$ and α is a subspace of Σ satisfying the following two properties:

(Q1) if x is a point of $\mathcal{S}$, then $e(x) \notin \alpha$;

(Q2) if L is a line of $\mathcal{S}$, then $\alpha \cap < e(x_1), e(x_2), \ldots, e(x_k) >_\Sigma = \emptyset$ where $\{x_1, x_2, \ldots, x_k\} = \mathcal{P}_L$.

Then a new pseudo-embedding $e/\alpha : \mathcal{S} \to \Sigma/\alpha$ can be defined which maps each point x of $\mathcal{S}$ to the point $< \alpha, e(x) >$ of the quotient projective space Σ/α. This new pseudo-embedding e/α is called a *quotient* of e. If $\alpha \neq \emptyset$, then e/α is called a *proper quotient* of e. If $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ are two pseudo-embeddings of $\mathcal{S}$, then we say that $e_1 \geq e_2$ if e_2 is isomorphic to a quotient of e_1. A pseudo-embedding $\widetilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ is called *universal* if $\widetilde{e} \geq e$ for any pseudo-embedding e of $\mathcal{S}$. The following can be said about universal pseudo-embeddings.

Theorem 1.2 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry and suppose that $3 \leq |\mathcal{P}_L| < \infty$ for every line $L \in \mathcal{L}$.*

(1) *If $\mathcal{S}$ admits a pseudo-embedding, then $\mathcal{S}$ admits a universal pseudo-embedding. This universal pseudo-embedding is uniquely determined, up to isomorphism. If $\mathcal{S}$ admits a faithful pseudo-embedding, then the universal pseudo-embedding of $\mathcal{S}$ is also faithful.*

(2) *Let V be a vector space over the field $\mathbb{F}_2$ with a basis B whose vectors are indexed by the elements of $\mathcal{P}$, say $B = \{\bar{v}_x \,|\, x \in \mathcal{P}\}$. Let W denote the subspace of V generated by all vectors of the form $\bar{v}_{x_1} + \bar{v}_{x_2} + \cdots + \bar{v}_{x_k}$ where $\{x_1, x_2, \ldots, x_k\} = \mathcal{P}_L$ for some line L of $\mathcal{S}$. If $\mathcal{S}$ admits at least one pseudo-embedding, then the map $\tilde{e}$ which associates with each point $x \in \mathcal{P}$ the subspace $\{\bar{v}_x + W, W\}$ of V/W defines a pseudo-embedding of $\mathcal{S}$ into $\mathrm{PG}(V/W)$ which is isomorphic to the universal pseudo-embedding of $\mathcal{S}$.*

If $\tilde{e} : \mathcal{S} \to \mathrm{PG}(\widetilde{V})$ is the universal pseudo-embedding of $\mathcal{S}$, then the dimension of the vector space $\widetilde{V}$ is called the *pseudo-embedding rank* of $\mathcal{S}$ and denoted by $er^*(\mathcal{S})$.

The universal (pseudo-)embeddings of certain point-line geometries with three points on each line have been studied in the literature. In this context, it is worth mentioning the conjectures of Andries Brouwer regarding the dimensions of the universal embeddings of finite symplectic and Hermitian dual polar spaces over $\mathbb{F}_2$ and their final solutions by Blokhuis & Brouwer [2] and Li [17, 18].

The following theorem describes a fundamental connection between pseudo-hyperplanes and universal pseudo-embeddings.

Theorem 1.3 *If $\mathcal{S}$ admits at least one pseudo-embedding, then every pseudo-hyperplane of $\mathcal{S}$ arises from the universal pseudo-embedding $\tilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ of $\mathcal{S}$. Moreover, the formula $H = \tilde{e}^{-1}(\tilde{e}(\mathcal{P}) \cap \Pi)$ determines a one-to-one correspondence between the pseudo-hyperplanes H of $\mathcal{S}$ and the hyperplanes Π of $\widetilde{\Sigma}$.*

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Observe that the empty set is a pseudo-hyperplane of $\mathcal{S}$ if and only if every line of $\mathcal{S}$ has an even number of points. If this is the case and $\mathcal{S}$ admits a pseudo-embedding, then by Theorem 1.3, the universal pseudo-embedding $\tilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ embeds $\mathcal{S}$ in the complement of a uniquely determined hyperplane of $\widetilde{\Sigma}$.

The following theorem addresses the existence problem for (certain) pseudo-embeddings of $\mathcal{S}$. Before we can state this theorem, we need to introduce a number of properties for sets of pseudo-hyperplanes. More precisely, we consider the following properties for a set $\mathcal{H}$ of pseudo-hyperplanes of $\mathcal{S}$.

(A1) If L is a line of $\mathcal{S}$ for which $|\mathcal{P}_L|$ is odd, then for every point x of L, there exists a pseudo-hyperplane of $\mathcal{H}$ intersecting $\mathcal{P}_L$ in $\{x\}$.

(A2) If L is a line of $\mathcal{S}$ for which $|\mathcal{P}_L|$ is even, then for any two distinct points x_1 and x_2 of L, there exists a pseudo-hyperplane of $\mathcal{H}$ intersecting $\mathcal{P}_L$ in $\{x_1, x_2\}$.

(A3) For any two distinct points x_1 and x_2 of $\mathcal{S}$, there exists a pseudo-hyperplane of $\mathcal{H}$ containing x_1, but not x_2.

(A4) If H_1 and H_2 are two distinct elements of $\mathcal{H}$, then also $\overline{H_1 \Delta H_2}$ belongs to $\mathcal{H}$.

(A5) For every point x of $\mathcal{S}$, there exists a pseudo-hyperplane of $\mathcal{H}$ not containing x.

Observe that Property (A5) is a consequence of Properties (A1) and (A2) if there is at least one line incident with x. Clearly, a pseudo-embedding e is faithful if and only if $\mathcal{H}_e$ satisfies property (A3).

Theorem 1.4 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry and suppose that $3 \leq |\mathcal{P}_L| < \infty$ for every line $L \in \mathcal{L}$. Then:*

(1) *$\mathcal{S}$ admits a pseudo-embedding if and only if the set of all pseudo-hyperplanes of $\mathcal{S}$ satisfies Properties (A1) and (A2) above.*

(2) *$\mathcal{S}$ admits a faithful pseudo-embedding if and only if the set of all pseudo-hyperplanes of $\mathcal{S}$ satisfies Properties (A1), (A2) and (A3) above.*

(3) *If e is a pseudo-embedding of $\mathcal{S}$, then $\mathcal{H}_e$ satisfies the conditions (A1), (A2), (A4) and (A5) above. Conversely, if $\mathcal{H}$ is a finite set of pseudo-hyperplanes of $\mathcal{S}$ satisfying the conditions (A1), (A2), (A4) and (A5), then there exists up to isomorphism a unique pseudo-embedding e of $\mathcal{S}$ for which $\mathcal{H}_e = \mathcal{H}$.*

The claim in Theorem 1.4(3) might also be valid for certain infinite sets $\mathcal{H}$ of pseudo-hyperplanes of $\mathcal{S}$. For instance, this claim is certainly valid if $\mathcal{H}$ is the set of all pseudo-hyperplanes of $\mathcal{S}$, with the corresponding pseudo-embedding e being the universal pseudo-embedding of $\mathcal{S}$. However, as we shall see later, there are counterexamples to that claim if $\mathcal{H}$ is allowed to be infinite. The construction of counterexamples relies on the known

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fact that for an infinite-dimensional vector space V, the dimension of the dual space V^* is always bigger than the dimension of V (contrary to the finite-dimensional case, where we always have that $\dim(V) = \dim(V^*)$).

A *pseudo-subspace* of $\mathcal{S}$ is a set X of points of $\mathcal{S}$ such that $|\mathcal{P}_L \cap (\mathcal{P} \setminus X)| \neq 1$ for every line L of $\mathcal{S}$. If X_i, $i \in I$, is a family of pseudo-subspaces of $\mathcal{S}$ (for some index set I), then the fact that $|\mathcal{P}_L \cap (\mathcal{P} \setminus X_i)| \neq 1$ for every $i \in I$, implies that also $|\mathcal{P}_L \cap (\mathcal{P} \setminus \bigcap_{i \in I} X_i)| \neq 1$. Hence, the intersection of a number of pseudo-subspaces of $\mathcal{S}$ is again a pseudo-subspace of $\mathcal{S}$. If X is a set of points of $\mathcal{S}$, then $[X]^*$ denotes the intersection of all pseudo-subspaces containing X. Clearly, the set $[X]^*$ is well-defined since there always exists a pseudo-subspace containing X, namely the whole set of points. The set $[X]^*$ is the smallest pseudo-subspace of $\mathcal{S}$ which contains the set X and is called the *pseudo-subspace generated by X*. If $[X]^* = \mathcal{P}$, then we will also say that X is a *pseudo-generating set* of $\mathcal{S}$. The minimal size of a pseudo-generating set of $\mathcal{S}$ is called the *pseudo-generating rank* of $\mathcal{S}$ and is denoted by $gr^*(\mathcal{S})$.

If X is a set of points of $\mathcal{S}$, then the pseudo-subspace $[X]^*$ of $\mathcal{S}$ generated by X can also be obtained in the following recursive way. Put $X_0 := X$ and for every $i \in \mathbb{N}$, we define $X_{i+1} := X_i \cup \left(\bigcup_{L \in \mathcal{L}_i} \mathcal{P}_L \right)$ where $\mathcal{L}_i$ is the set of all lines $L \in \mathcal{L}$ for which $|\mathcal{P}_L \cap (\mathcal{P} \setminus X_i)| = 1$. Clearly, $\bigcup_{i \in \mathbb{N}} X_i$ is a pseudo-subspace of $\mathcal{S}$ and every pseudo-subspace of $\mathcal{S}$ containing X must also contain $\bigcup_{i \in \mathbb{N}} X_i$. Hence, $[X]^* = \bigcup_{i \in \mathbb{N}} X_i$.

In practice it can be hard to determine whether a given pseudo-embedding of $\mathcal{S}$ is universal or to determine whether a given pseudo-generating set has minimal size $gr^*(\mathcal{S})$. The following theorem can help in achieving these goals.

Theorem 1.5 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry and suppose that $3 \leq |\mathcal{P}_L| < \infty$ for every line $L \in \mathcal{L}$. Suppose $\mathcal{S}$ admits a pseudo-embedding. Then:*
  (1) We have $er^*(\mathcal{S}) \leq gr^*(\mathcal{S})$.
  (2) If there exists a pseudo-embedding $e : \mathcal{S} \to \mathrm{PG}(V)$ and a pseudo-generating set X of $\mathcal{S}$ such that $|X| = \dim(V) < \infty$, then $er^*(\mathcal{S}) = gr^*(\mathcal{S}) = \dim(V)$ and e is isomorphic to the universal pseudo-embedding of $\mathcal{S}$.

A result, similar to Theorem 1.5, is known for full projective embeddings and generating sets of point-line geometries having an arbitrary, not necessarily finite, number of points on each line. As told above, for point-line geometries with three points per line several of the above-stated results are known. The results stated in Theorems 1.1, 1.2, 1.3 and 1.4(1)+(2) have basically been proved by Ronan [21] for point-line geometries having three points on each line.

In Section 3, we consider several classes of point-line geometries and prove or disprove that they admit a pseudo-embedding. Our discussion includes the projective spaces, affine spaces, polar spaces, dual polar spaces, generalized polygons, dense near polygons, some geometries related to quadrics of finite projective spaces and the (restricted) ovoid geometries (of dense near polygons). At the end of section 3, we give some applications of the theory of pseudo-embeddings and pseudo-hyperplanes to the near hexagon $\mathbb{E}_2$.

We show there that the hyperplanes of $\mathbb{E}_2$ are precisely those sets of points of $\mathbb{E}_2$ which intersect each ovoid of a quad in either one, three or five points. We also prove there that the ovoid geometry of $\mathbb{E}_2$ is not fully embeddable in a projective space.

In Section 4, we determine the pseudo-generating ranks and the pseudo-embedding ranks of the projective space $\mathrm{PG}(n, 4)$ and the affine space $\mathrm{AG}(n, 4)$. We prove that $gr^*(\mathrm{AG}(n, 4)) = er^*(\mathrm{AG}(n, 4)) = n^2 + n + 1$ and $gr^*(\mathrm{PG}(n, 4)) = \frac{1}{3}(n + 1)(n^2 + 2n + 3)$. A classification of the pseudo-hyperplanes of $\mathrm{PG}(n, 4)$ was obtained by Sherman [22], and from this classification it follows that also $er^*(\mathrm{PG}(n, 4)) = \frac{1}{3}(n + 1)(n^2 + 2n + 3)$.

The existence of pseudo-embeddings and/or the knowledge of the precise values of the pseudo-embedding and pseudo-generating ranks remain open for several classes of point-line geometries. We hope to be able to address some of these problems in future work. Our main aim was to lay the basis of the theory and to illustrate it with a number of examples. We hope that the theory of pseudo-embeddings and pseudo-hyperplanes will find further applications in related areas. Some of these applications will occur in the forthcoming paper [11], where we will study the pseudo-hyperplanes and pseudo-embeddings of generalized quadrangles of order $(3, t)$, together with some of their applications to so-called m-ovoids and tight sets.

2 Proofs of the main theorems

In this section, we prove all the theorems which we stated in Section 1. Throughout, we suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a point-line geometry with the property that $3 \leq |\mathcal{P}_L| < \infty$ for every line $L \in \mathcal{L}$.

Lemma 2.1 *Let the vector spaces V, W and the vectors $\bar{v}_x$, $x \in \mathcal{P}$, be as defined in Theorem 1.2.*

(1) *If U is a hyperplane of V containing W, then the set of all points x of $\mathcal{P}$ for which $\bar{v}_x \in U$ is a pseudo-hyperplane of $\mathcal{S}$.*

(2) *If H is a pseudo-hyperplane of $\mathcal{S}$, then there exists a unique hyperplane U of V such that H consists of all points $x \in \mathcal{P}$ for which $\bar{v}_x \in U$. This hyperplane U contains W.*

Proof. (1) Observe that there certainly exists a point x of $\mathcal{S}$ for which $\bar{v}_x \notin U$. Notice that $\bar{v}_1 + \bar{v}_2 \in U$ for all $\bar{v}_1, \bar{v}_2 \in V \setminus U$. If L is a line of $\mathcal{S}$ with $\mathcal{P}_L = \{x_1, x_2, \ldots, x_k\}$, then the fact that $\bar{v}_{x_1} + \bar{v}_{x_2} + \cdots + \bar{v}_{x_k} \in W \subseteq U$ implies that there are an even number of $i \in \{1, 2, \ldots, k\}$ for which $\bar{v}_{x_i} \notin U$. So, the set of all points x of $\mathcal{P}$ for which $\bar{v}_x \in U$ is a pseudo-hyperplane of $\mathcal{S}$.

(2) Suppose H is a pseudo-hyperplane of $\mathcal{S}$. Let U denote the hyperplane of V consisting of all vectors $\sum_{x \in \mathcal{P}} Y_x \bar{v}_x$ for which $\sum_{x \in \mathcal{P} \setminus H} Y_x = 0$. (Notice that this is well-defined since only a finite number of the coordinates Y_x, $x \in \mathcal{P}$, are distinct from 0.) Clearly, $x \in H \Leftrightarrow \bar{v}_x \in U$. To prove that U contains W, we must show that $\bar{v}_{x_1} + \bar{v}_{x_2} +$

$\cdots + \bar{v}_{x_k} \in U$ if $\{x_1, x_2, \ldots, x_k\} = \mathcal{P}_L$ for some line L of $\mathcal{S}$. But clearly this holds since an even number of points of L do not belong to H.

Suppose $U' \neq U$ is another hyperplane of V such that H consists of all point $x \in \mathcal{P}$ for which $\bar{v}_x \in U'$. Let U'' denote the unique hyperplane of V through $U \cap U'$ distinct from U and U'. The fact that $\bar{v}_x \in U' \Leftrightarrow x \in H \Leftrightarrow \bar{v}_x \in U$ would imply that U'' contains all vectors $\bar{v}_x$, $x \in \mathcal{P}$, clearly a contradiction. ∎

(I) We prove of Theorem 1.1.

We have that $H_\Pi \neq \mathcal{P}$. For, if H_Π would be equal to $\mathcal{P}$, then we would have $\Sigma = <e(\mathcal{P})> = <e(H_\Pi)> = <e(\mathcal{P}) \cap \Pi> \subseteq \Pi$, which is clearly impossible.

We prove that $|\mathcal{P}_L \cap (\mathcal{P} \setminus H_\Pi)|$ is even for every line L of $\mathcal{S}$. Put $\mathcal{P}_L = \{x_1, x_2, \ldots, x_k\}$. Let V be a vector space over $\mathbb{F}_2$ such that $\Sigma = \mathrm{PG}(V)$ and let $\bar{v}_i$, $i \in \{1, 2, \ldots, k\}$, be the unique vector of V such that $e(x_i) = <\bar{v}_i>$. Since e is a pseudo-embedding, the vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{k-1}$ are linearly independent and $\bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_k = \bar{o}$. Put $A = \Pi \cap <e(x_1), e(x_2), \ldots, e(x_k)>$. If $A = <e(x_1), e(x_2), \ldots, e(x_k)>$, then $\{x_1, x_2, \ldots, x_k\} \subseteq H_\Pi$ and so $|\mathcal{P}_L \cap (\mathcal{P} \setminus H_\Pi)| = 0$ is even. Suppose now that A is a hyperplane of $<e(x_1), e(x_2), \ldots, e(x_k)>$ corresponding to some hyperplane U of $<\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{k-1}>$. Similarly, as in the proof of Lemma 2.1, the fact that $\bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_k = \bar{o} \in U$ implies that there are an even number of $i \in \{1, 2, \ldots, k\}$ for which $\bar{v}_i \notin U$. This implies that $|\mathcal{P}_L \cap (\mathcal{P} \setminus H_\Pi)|$ must be even.

(II) We prove Theorem 1.2. Let V, W, $\bar{v}_x$ $(x \in \mathcal{P})$ and $\tilde{e}$ be as defined in the statement of Theorem 1.2 and put $\widetilde{V} := V/W$.

Our first aim is to prove that if $\mathcal{S}$ has a pseudo-embedding $e : \mathcal{S} \to \mathrm{PG}(V')$, then $\tilde{e}$ is a pseudo-embedding of $\mathcal{S}$ and $\tilde{e} \geq e$. This will prove that if $\mathcal{S}$ admits pseudo-embeddings, then $\tilde{e}$ is a universal pseudo-embedding.

For every point x of $\mathcal{S}$, let $\bar{v}'_x$ denote the unique vector of V' such that $e(x) = <\bar{v}'_x>$. Since $B = \{\bar{v}_x \mid x \in \mathcal{P}\}$ is a basis of V, the map $\bar{v}_x \mapsto \bar{v}'_x$ extends in a unique way to a linear map θ_1 from V to V'.

Now, let L be an arbitrary line of $\mathcal{S}$ and put $\mathcal{P}_L = \{x_1, x_2, \ldots, x_k\}$. Since e is a pseudo-embedding, we have $\theta_1(\bar{v}_{x_1} + \bar{v}_{x_2} + \cdots + \bar{v}_{x_k}) = \bar{v}'_{x_1} + \bar{v}'_{x_2} + \cdots + \bar{v}'_{x_k} = \bar{o}$. Hence, we have $W \subseteq \ker(\theta_1)$. So, θ_1 induces a linear map θ_2 from $\widetilde{V}$ to V' if we define $\theta_2(\bar{v} + W) := \theta_1(\bar{v})$ for every $\bar{v} \in V$.

Now, put $U := \ker(\theta_2)$ and let α be the subspace of $\mathrm{PG}(\widetilde{V})$ corresponding to U. For every point x of $\mathcal{S}$, the fact that $\theta_2(\bar{v}_x + W) = \theta_1(\bar{v}_x) = \bar{v}'_x \neq \bar{o}$ implies that $\bar{v}_x + W \notin U$, in particular $\bar{v}_x \notin W$. We also note that if L is a line of $\mathcal{S}$ with $\mathcal{P}_L = \{x_1, x_2, \ldots, x_k\}$ and $i \in \{1, 2, \ldots, k-1\}$, then $(\bar{v}_{x_1} + W) + (\bar{v}_{x_2} + W) + \cdots + (\bar{v}_{x_k} + W) = W$ and $(\bar{v}_{x_1} + W) + (\bar{v}_{x_2} + W) + \cdots + (\bar{v}_{x_i} + W)$ does not belong to U (and hence is also distinct from W). Indeed, if this latter claim was not true, then the application of the map θ_2 would yield that $\bar{v}'_{x_1} + \bar{v}'_{x_2} + \cdots + \bar{v}'_{x_i} = \bar{o}$, which is in contradiction with the fact that e is a pseudo-embedding. We conclude that $\tilde{e}$ is a pseudo-embedding of $\mathcal{S}$ and that the subspace α of $\mathrm{PG}(\widetilde{V})$ satisfies the Properties (Q1) and (Q2) of Section 1 (with respect to $\tilde{e}$).

The map θ_2 induces a linear isomorphism θ_3 between the vector spaces $\widetilde{V}/U = \widetilde{V}/ker(\theta_2)$ and $Im(\theta_2) = V'$ by defining $\theta_3(\bar{v} + ker(\theta_2)) = \theta_2(\bar{v})$ for every $\bar{v} \in \widetilde{V}$. This linear isomorphism induces an automorphism ϕ between the projective spaces $\mathrm{PG}(\widetilde{V})/\alpha$ and $\mathrm{PG}(V')$ by defining $\phi(<\alpha, <\bar{v}>>) := <\theta_2(\bar{v})>$ for every $\bar{v} \in \widetilde{V} \setminus U$. Clearly, for every point x of $\mathcal{S}$, we have $\phi \circ \tilde{e}/\alpha(x) = \phi(<<\bar{v}_x + W>, \alpha>) = <\theta_2(\bar{v}_x + W)> = <\theta_1(\bar{v}_x)> = <\bar{v}'_x> = e(x)$. Hence, $\phi \circ \tilde{e}/\alpha = e$. It follows that e is isomorphic to a quotient of $\tilde{e}$.

Our next aim is to prove that if there exists a pseudo-embedding of $\mathcal{S}$ then there exists up to isomorphism a unique universal pseudo-embedding of $\mathcal{S}$ (which is then necessarily isomorphic to $\tilde{e}$). This is true if $\widetilde{V}$ is finite-dimensional by an obvious counting argument on the dimensions of the universal pseudo-embeddings spaces, but we wish to give an argument which holds in general. Suppose $e_1 : \mathcal{S} \to \Sigma_1$ and $e_2 : \mathcal{S} \to \Sigma_2$ are two universal pseudo-embeddings of $\mathcal{S}$ and suppose e_1 and e_2 are not isomorphic. Then e_2 is isomorphic to a proper quotient of e_1 and e_1 is isomorphic to a proper quotient of e_2. We conclude that e_1 is isomorphic to a proper quotient of itself, i.e. $e_1 \cong e_1/\alpha$ for some nonempty subspace α of Σ_1 satisfying properties (Q1) and (Q2). Now, let Π be a hyperplane of Σ_1 not containing α and let H_Π be the pseudo-hyperplane $e_1^{-1}(e_1(\mathcal{P}) \cap \Pi)$ of $\mathcal{S}$. If Π' is a hyperplane of Σ_1 distinct from Π, then the pseudo-hyperplane $H_{\Pi'} = e_1^{-1}(e_1(\mathcal{P}) \cap \Pi')$ is distinct from H_Π. Indeed, if $H_{\Pi'}$ were equal to H_Π and Π'' is the unique hyperplane of Σ_1 through $\Pi \cap \Pi'$ distinct from Π and Π', then the pseudo-hyperplane $H_{\Pi''}$ of $\mathcal{S}$ would coincide with the whole point set $\mathcal{P}$, clearly a contradiction. The fact that $\alpha \not\subseteq \Pi$ and $H_{\Pi'} \neq H_\Pi$ implies that the pseudo-hyperplane $H_\Pi \in \mathcal{H}_{e_1}$ cannot arise from the pseudo-embedding e_1/α. So, e_1 and e_1/α cannot be isomorphic leading to our desired contradiction.

All claims of Theorem 1.2 have now been verified, except for the claim regarding the faithfulness of the universal pseudo-embedding. But also this is obvious. If the universal pseudo-embedding $\tilde{e}$ of $\mathcal{S}$ is not faithful, then any pseudo-embedding of $\mathcal{S}$ (which necessarily arises as quotient of $\tilde{e}$) is also not faithful.

(III) Observe that Theorem 1.3 is an immediate consequence of Lemma 2.1 and Theorem 1.2(2).

(IV) We prove Theorem 1.4. Let the vector spaces V, W and the vectors $\bar{v}_x$, $x \in \mathcal{P}$, be as defined in Theorem 1.2.

Lemma 2.2 *If $e : \mathcal{S} \to \mathrm{PG}(V')$ is a pseudo-embedding of $\mathcal{S}$, then for every line L of $\mathcal{S}$ and every set X of points of L for which $|\mathcal{P}_L| - |X| \neq 0$ is even, there exists a pseudo-hyperplane of $\mathcal{H}_e$ which intersects $\mathcal{P}_L$ in X.*

Proof. Put $|X| = i$ and $\mathcal{P}_L = \{x_1, x_2, \ldots, x_k\}$ such that $X = \{x_1, x_2, \ldots, x_i\}$. Then $i \leq k - 2$. Let $\bar{v}_i$, $i \in \{1, 2, \ldots, k\}$, be the vector of V' such that $e(x_i) = <\bar{v}_i>$. Since e is a pseudo-embedding, the vectors $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_{k-1}$ are linearly independent and $\bar{v}_k = \bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_{k-1}$. Now, consider the hyperplane A of $<e(x_1), e(x_2), \ldots, e(x_k)>$

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with equation $Y_{i+1} + Y_{i+2} + \cdots + Y_{k-1} = 0$, where $< Y_1\bar{v}_1 + Y_2\bar{v}_2 + \cdots + Y_{k-1}\bar{v}_{k-1} >$ denotes a generic point of $< e(x_1), e(x_2), \ldots, e(x_k) >$. Since $k - i$ is even, the points $x_1, x_2, \ldots, x_i$ are the only points x of L for which $e(x) \in A$. Now, let Π be a hyperplane of $\mathrm{PG}(V')$ intersecting $< e(x_1), e(x_2), \ldots, e(x_k) >$ in A and put $H_\Pi := e^{-1}(e(\mathcal{P}) \cap \Pi)$. Then H_Π is a pseudo-hyperplane of $\mathcal{H}_e$ and $H_\Pi \cap \mathcal{P}_L = X$. ∎

Observe that Lemma 2.2 remains valid if $X = \mathcal{P}_L$ and $< e(\mathcal{P}_L) > \neq \mathrm{PG}(V')$.

(a) Suppose $e : \mathcal{S} \to \Sigma$ is a pseudo-embedding of $\mathcal{S}$. Then $\mathcal{H}_e$ satisfies Properties (A1) and (A2) by Lemma 2.2.

Suppose H_1 and H_2 are two distinct elements of $\mathcal{H}_e$. Let Π_i, $i \in \{1, 2\}$, be the hyperplane of Σ such that $H_i = e^{-1}(e(\mathcal{P}) \cap \Pi_i)$. Then $\Pi_1 \neq \Pi_2$. Let Π_3 denote the unique hyperplane of Σ through $\Pi_1 \cap \Pi_2$ distinct from Π_1 and Π_2. Then $\overline{H_1 \Delta H_2} = e^{-1}(e(\mathcal{P}) \cap \Pi_3)$. So, $\mathcal{H}_e$ satisfies Property (A4).

Let x be an arbitrary point of $\mathcal{S}$ and let Π be an arbitrary hyperplane of Σ not containing $e(x)$. Then the pseudo-hyperplane $e^{-1}(e(\mathcal{P}) \cap \Pi)$ of $\mathcal{H}_e$ does not contain x, proving that $\mathcal{H}_e$ also satisfies Property (A5).

(b) The set $\mathcal{H}^*$ of all pseudo-hyperplanes of $\mathcal{S}$ clearly satisfies Property (A4). If the set $\mathcal{H}^*$ satisfies Properties (A1) and (A2), then it also satisfies Property (A5). For, take a point x. If x is incident with some line L of $\mathcal{S}$, then there is some pseudo-hyperplane of $\mathcal{S}$ not containing x by Properties (A1) and (A2). If x is an isolated point of $\mathcal{S}$, then $H \setminus \{x\} \in \mathcal{H}^*$ for every $H \in \mathcal{H}^*$.

(c) Suppose $\mathcal{H}$ is one of the following sets of pseudo-hyperplanes: (i) a finite set of pseudo-hyperplanes of $\mathcal{S}$ satisfying Properties (A1), (A2), (A4) and (A5); (ii) the set of all pseudo-hyperplanes of $\mathcal{S}$. In case (ii), we will moreover assume that $\mathcal{H}$ satisfies Properties (A1), (A2) and hence also (A4) and (A5) by (b). We will prove that there exists a pseudo-embedding e of $\mathcal{S}$ for which $\mathcal{H}_e = \mathcal{H}$.

After applying Property (A4) a suitable number of times, we readily see that Properties (A1) and (A2) imply the following.

(∗) For every line L of $\mathcal{S}$ and every set X of points of L for which $|\mathcal{P}_L| - |X| \neq 0$ is even, there exists a pseudo-hyperplane $H \in \mathcal{H}$ which intersects $\mathcal{P}_L$ in X.

By Lemma 2.1, there exists for every pseudo-hyperplane $H \in \mathcal{H}$ a unique hyperplane U_H of V such that H consists of all points x of $\mathcal{S}$ for which $\bar{v}_x \in U_H$. This hyperplane U_H is moreover unique. Also, $W \subseteq U_H$. Observe that if H_1 and H_2 are two distinct pseudo-hyperplanes of $\mathcal{H}$, then $U_{\overline{H_1 \Delta H_2}}$ is the unique hyperplane of V through $U_{H_1} \cap U_{H_2}$ distinct from U_{H_1} and U_{H_2}. Let W' denote the intersection of all hyperplanes U_H, $H \in \mathcal{H}$. Then $W \subseteq W'$. If $\mathcal{H}$ consists of all pseudo-hyperplanes of $\mathcal{S}$, then $W' = W$ and $\{U_H \mid H \in \mathcal{H}\}$ is the set of all hyperplanes of V through W' by Lemma 2.1. Also in the case $\mathcal{H}$ is finite, $\{U_H \mid H \in \mathcal{H}\}$ is the set of all pseudo-hyperplanes of V through W'.

Property (A5) implies that $\bar{v}_x \notin W'$ for every point x of $\mathcal{S}$. Now, let L be an arbitrary line of $\mathcal{S}$ and put $\mathcal{P}_L = \{x_1, x_2, \ldots, x_k\}$. We prove that the vectors $\bar{v}_{x_1} + W', \bar{v}_{x_2} + W', \ldots, \bar{v}_{x_{k-1}} + W'$ of V/W' are linearly independent and that $(\bar{v}_{x_1} + W') + (\bar{v}_{x_2} + W') +$

$\cdots + (\bar{v}_k + W') = W'$. The latter property certainly holds since $\bar{v}_{x_1} + \bar{v}_{x_2} + \cdots + \bar{v}_{x_k} \in W \subseteq W'$. Suppose that the former property does not hold and consider a linearly dependent collection of $\bar{v}_{x_1} + W', \bar{v}_{x_2} + W', \ldots, \bar{v}_{k-1} + W'$ of minimal size. Without loss of generality, we may suppose that $\bar{v}_{x_1} + W', \bar{v}_{x_2} + W', \ldots, \bar{v}_{x_i} + W'$ is such a collection of minimal size. Then $(\bar{v}_{x_1} + W') + (\bar{v}_{x_2} + W') + \cdots + (\bar{v}_{x_i} + W')$ necessarily is equal to W', i.e. $\bar{v}_{x_1} + \bar{v}_{x_2} + \cdots + \bar{v}_{x_i} \in W'$. Clearly, $2 \le i \le k - 1$. The fact that $\bar{v}_{x_1} + \bar{v}_{x_2} + \cdots + \bar{v}_{x_i} \in W'$ implies that every pseudo-hyperplane of $\mathcal{H}$ containing $x_1, x_2, \ldots, x_{i-1}$ also contains x_i. By Property $(*)$, there exists a pseudo-hyperplane $H \in \mathcal{H}$ which intersects $\mathcal{P}_L$ in either $\{x_1, x_2, \ldots, x_{i-1}\}$ or $\{x_1, x_2, \ldots, x_{i-1}, x_k\}$. So, H contains $x_1, x_2, \ldots, x_{i-1}$ but not x_i. As told before, this is impossible.

From the above discussion, we now know that the map e which associates with each point x of $\mathcal{S}$, the point $\{\bar{v}_x + W', W'\}$ of the projective space $\mathrm{PG}(V/W')$ is a pseudo-embedding of $\mathcal{S}$ for which $\mathcal{H}_e = \mathcal{H}$.

(c) We prove that if $\mathcal{H}$ is a set of pseudo-hyperplanes of $\mathcal{S}$, then there exists, up to isomorphism, at most one pseudo-embedding e of $\mathcal{S}$ such that $\mathcal{H} = \mathcal{H}_e$. Suppose e_1 and e_2 are two pseudo-embeddings of $\mathcal{S}$ such that $\mathcal{H} = \mathcal{H}_{e_1} = \mathcal{H}_{e_2}$. Let $\tilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ denote the universal pseudo-embedding of $\mathcal{S}$ and let α_i, $i \in \{1, 2\}$, be a subspace of $\widetilde{\Sigma}$ satisfying (Q1) and (Q2) such that $e_i \cong \tilde{e}/\alpha_i$. We prove that $\alpha_1 = \alpha_2$. If $\alpha_1 \ne \alpha_2$, then there exists a hyperplane Π of $\widetilde{\Sigma}$ which contains precisely one of α_1, α_2. This implies that the pseudo-hyperplane $\tilde{e}^{-1}(\Pi \cap \tilde{e}(\mathcal{P}))$ of $\mathcal{S}$ arises from precisely one of $\tilde{e}/\alpha_1, \tilde{e}/\alpha_2$. But this is impossible since $\mathcal{H}_{\tilde{e}/\alpha_1} = \mathcal{H} = \mathcal{H}_{\tilde{e}/\alpha_2}$. Hence, $\alpha_1 = \alpha_2$ and $e_1 \cong \tilde{e}/\alpha_1 = \tilde{e}/\alpha_2 \cong e_2$.

(d) Claims (1) and (3) of Theorem 1.4 have now been proved. Claim (2) is also obvious. A pseudo-embedding e of $\mathcal{S}$ is faithful if and only if it satisfies Property (A3). So, if the set of all pseudo-hyperplanes satisfies Properties (A1), (A2) and (A3), then $\mathcal{S}$ admits a pseudo-embedding (by Claim (1)) and the universal pseudo-embedding of $\mathcal{S}$ must then be faithful.

Remark. If we take a closer look to the above proof, then we readily see why we had to impose that $\mathcal{H}$ is finite. If the hyperplanes U_H, $H \in \mathcal{H}$, of V do not constitute all the hyperplanes of V through W', then there exist hyperplanes arising from the constructed pseudo-embedding which do not belong to $\mathcal{H}$. If this is the case, then the claim of the theorem must be false. The situation where the hyperplanes U_H, $H \in \mathcal{H}$, do not constitute all the hyperplanes of V through W' cannot occur if W' has finite co-dimension in V, but it can occur if W' has infinite co-dimension in V, as we are now going to show.

Let $\mathcal{S}$ be a point-line geometry with the property that the number of points on each line is finite and at least three, and suppose $\mathcal{S}$ admits a pseudo-embedding in an infinite-dimensional projective space. There are plenty of point-line geometries which satisfy this condition (e.g., every point-line geometry with three points on each line which admits a full projective embedding in an infinite-dimensional projective space). Let $\tilde{e} : \mathcal{S} \to \mathrm{PG}(\widetilde{V})$ denote the universal pseudo-embedding of $\mathcal{S}$, and let $\mathcal{B} = \{\bar{e}_i \mid i \in I\}$ be a basis of $\widetilde{V}$ where I is some infinite index set. Let $\mathcal{V}$ denote the set of all hyperplanes of $PG(\widetilde{V})$ whose equation is of the form $\sum_{i \in I} a_i X_i = 0$, where only a finite number of the constants

a_i, $i \in I$, are distinct from 0. (Here, X_i, $i \in I$, are the coordinates of a generic point of $PG(\widetilde{V})$ with respect to the basis $\mathcal{B}$). Put $H_i := \tilde{e}^{-1}(\tilde{e}(\mathcal{P}) \cap \Pi_i)$ for every $i \in I$. Then the set $\mathcal{H} = \{ H_i \mid i \in I \}$ satisfies the properties (A1), (A2), (A4) and (A5). (The fact that $\mathcal{H}$ satisfies Properties (A1) and (A2) follows from a refinement of the arguments in the proof of Lemma 2.2.) Now, the intersection of the hyperplanes Π_i, $i \in I$, is empty and $PG(\widetilde{V})$ has hyperplanes not belonging to $\mathcal{V}$, namely those whose equation is of the form $\sum_{i \in I} a_i X_i = 0$, where an infinite number of the constants a_i, $i \in I$, are distinct from 0. So, we have constructed our desired counterexamples.

Observe that the above construction is heavily based on the well-known fact that the dimension $\dim(Z^*)$ of the dual space Z^* of an infinite-dimensional vector space Z is always bigger than $\dim(Z)$.

(V) We prove Theorem 1.5.

(1) Let $\tilde{e} : \mathcal{S} \to \widetilde{\Sigma}$ denote the universal pseudo-embedding of $\mathcal{S}$ and let X be a pseudo-generating set of size $gr^*(\mathcal{S})$ of $\mathcal{S}$. Put $X_0 := X$ and for every $i \in \mathbb{N}$, we define $X_{i+1} := X_i \cup \left(\bigcup_{L \in \mathcal{L}_i} \mathcal{P}_L \right)$ where $\mathcal{L}_i$ is the set of all lines $L \in \mathcal{L}$ for which $|\mathcal{P}_L \cap (\mathcal{P} \setminus X_i)| = 1$. Then $[X]^* = \bigcup_{i \in \mathbb{N}} X_i = \mathcal{P}$. In order to prove that $er^*(\mathcal{S}) \leq gr^*(\mathcal{S})$, it suffices to prove that $< \tilde{e}(X) >= \widetilde{\Sigma} =< \tilde{e}([X]^*) >$ or that $< \tilde{e}(X_i) >=< \tilde{e}(X) >$ for every $i \in \mathbb{N}$. We will prove this by induction on i. Obviously, the claim holds if $i = 0$. Next, suppose that $< \tilde{e}(X_i) >=< \tilde{e}(X) >$ for a certain $i \in \mathbb{N}$. If $L \in \mathcal{L}_i$, then $\tilde{e}(x) \notin< \tilde{e}(X_i) >$ for at most one point x of L. Since e is a pseudo-embedding, $\tilde{e}(x) \in< \tilde{e}(X_i) >$ for every $x \in \mathcal{P}_L$. It follows that $< \tilde{e}(X_{i+1}) >=< \tilde{e}(X_i) >=< \tilde{e}(X) >$. This is what we needed to prove.

(2) By (1), we have $\dim(V) \leq er^*(\mathcal{S}) \leq gr^*(\mathcal{S}) \leq |X|$. Since $\dim(V) = |X| < \infty$, we have $er^*(\mathcal{S}) = gr^*(\mathcal{S}) = \dim(V)$. If e were not isomorphic to the universal pseudo-embedding of $\mathcal{S}$, then $e : \mathcal{S} \to PG(V)$ would be a proper quotient of $\tilde{e} : \mathcal{S} \to \widetilde{\Sigma} = PG(\widetilde{V})$ and hence $\dim(V) < \dim(\widetilde{V}) = er^*(\mathcal{S}) = \dim(V)$, a contradiction.

3 Examples of point-line geometries (not) admitting a pseudo-embedding

In this section, we consider several classes of point-line geometries and prove or disprove that they admit a pseudo-embedding. We start by discussing some easy examples of point-line geometries which admit a pseudo-embedding.

1. Let $\mathcal{S}$ be the partial linear space without lines having a unique point. Then $\mathcal{S}$ admits a pseudo-embedding in the projective space $PG(0, 2)$.

2. Suppose $\mathcal{S}$ is the finite partial linear space consisting of one line L and $s + 1 \geq 3$ points which are all incident with L. Then clearly $\mathcal{S}$ has a pseudo-embedding in the projective space $PG(s - 1, 2)$.

3. Suppose $\mathcal{S}$ is the finite partial linear space consisting of $k \geq 2$ lines $L_1, L_2, \ldots, L_k$ which are incident with the same point x^* and that every point of $\mathcal{S}$ distinct from x^* is incident with precisely one of the lines $L_1, L_2, \ldots, L_k$. Assuming that the line L_i, $i \in \{1, 2, \ldots, k\}$, is incident with precisely $s_i + 1 \geq 3$ points, it is readily seen that $\mathcal{S}$ has a pseudo-embedding in $\mathrm{PG}(s_1 + s_2 + \cdots + s_k - k, 2)$.

All these pseudo-embeddings are universal. This is easily derived from Theorem 1.5(2): in the first example, $\mathcal{S}$ has a pseudo-generating set of size 1; in the second example, $\mathcal{S}$ has a pseudo-generating set of size s; in the last example, $\mathcal{S}$ has a pseudo-generating set of size $s_1 + s_2 + \cdots + s_k - k + 1$. The following two constructions for pseudo-embeddings are also obvious.

4. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry and suppose that $3 \leq |\mathcal{P}_L| < \infty$ for every line $L \in \mathcal{L}$. Let $\mathcal{P}' \subseteq \mathcal{P}$, $\mathcal{L}' \subseteq \mathcal{L}$ and $\mathrm{I}' = \mathrm{I} \cap (\mathcal{P}' \times \mathcal{L}')$ such that for every line L of $\mathcal{L}'$, we have $\mathcal{P}_L \subseteq \mathcal{P}'$. Then every (faithful) pseudo-embedding $e : \mathcal{S} \to \Sigma$ of $\mathcal{S}$ will induce a (faithful) pseudo-embedding of $\mathcal{S}' = (\mathcal{P}', \mathcal{L}', \mathrm{I}')$ into a subspace of Σ.

5. Let $\mathcal{S}_i = (\mathcal{P}_i, \mathcal{L}_i, \mathrm{I}_i)$, $i \in I$, be a collection of point-line geometries (for some index set I) and let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the disjoint union of the $\mathcal{S}_i$'s. Assuming that all sets $\mathcal{P}_i$, $i \in \mathrm{I}$, are mutually disjoint as well as all sets $\mathcal{L}_i$, $i \in \mathrm{I}$, we can define $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ by putting $\mathcal{P} := \bigcup_{i \in \mathrm{I}} \mathcal{P}_i$, $\mathcal{L} := \bigcup_{i \in \mathrm{I}} \mathcal{L}_i$ and $\mathrm{I} := \bigcup_{i \in \mathrm{I}} \mathrm{I}_i$. Clearly, $gr^*(\mathcal{S}) = \sum_{i \in I} gr^*(\mathcal{S}_i)$. Moreover, $\mathcal{S}$ admits a (faithful) pseudo-embedding if and only if each $\mathcal{S}_i$, $i \in I$, admits a (faithful) pseudo-embedding. If this is the case, then $er^*(\mathcal{S}) = \sum_{i \in I} er^*(\mathcal{S}_i)$.

Proposition 3.1 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a finite point-line geometry with the property that every line of $\mathcal{S}$ is incident with at least three points and that every point of $\mathcal{S}$ is incident with at least three lines. Suppose both $\mathcal{S}$ and the point-line dual $\mathcal{S}^D$ of $\mathcal{S}$ admit pseudo-embeddings. Then $er^*(\mathcal{S}^D) = er^*(\mathcal{S}) + |\mathcal{L}| - |\mathcal{P}|$.*

Proof. Let M be an incidence matrix of $\mathcal{S}$. So, the rows of M are indexed by the points of $\mathcal{S}$ and the columns of M are indexed by the lines of $\mathcal{S}$. If $p \in \mathcal{P}$ and $L \in \mathcal{L}$, then the (p, L)-entry of M is equal to 1 if $(p, L) \in \mathrm{I}$ and equal to 0 otherwise. Then M^T is an incidence matrix of $\mathcal{S}^D$. By Theorem 1.2(2), $er^*(\mathcal{S}) = |\mathcal{P}| - rank_{\mathbb{F}_2}(M)$ and $er^*(\mathcal{S}^D) = |\mathcal{L}| - rank_{\mathbb{F}_2}(M^T) = |\mathcal{L}| - rank_{\mathbb{F}_2}(M)$. Hence, $er^*(\mathcal{S}^D) = er^*(\mathcal{S}) + |\mathcal{L}| - |\mathcal{P}|$. ∎

Definition. A *hyperplane* of a point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a set H of points, distinct from $\mathcal{P}$, such that for every line L of $\mathcal{S}$, the intersection $H \cap \mathcal{P}_L$ is either a singleton or $\mathcal{P}_L$. A *full projective embedding* $e : \mathcal{S} \to \Sigma$ of $\mathcal{S}$ into a projective space Σ is a map e from $\mathcal{P}$ to the point set of a projective space Σ satisfying: (i) $< e(\mathcal{P}) >_\Sigma = \Sigma$; (ii) if L is a line of $\mathcal{S}$ and $\mathcal{P}_L = \{x_1, x_2, \ldots, x_k\}$, then $e(x_1), e(x_2), \ldots, e(x_k)$ are mutually distinct and $\{e(x_1), e(x_2), \ldots, e(x_k)\}$ is a line of Σ. If e is injective, then the full projective embedding e is called *faithful*. If $e : \mathcal{S} \to \Sigma$ is a full projective embedding, then for every hyperplane Π of Σ, the set $e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a hyperplane of $\mathcal{S}$. Any hyperplane of $\mathcal{S}$ which can be obtained in this way is said to *arise from e*.

Proposition 3.2 *Let $\mathcal{S}$ be a point-line geometry with the property that the number of points on each line is finite and odd. Then:*

(1) *If for every line L and every point x of L, there exists a hyperplane of $\mathcal{S}$ intersecting $\mathcal{P}_L$ in $\{x\}$, then $\mathcal{S}$ admits pseudo-embeddings. If $\mathcal{S}$ admits pseudo-embeddings and for every two distinct points x_1 and x_2 of $\mathcal{S}$, there exists a hyperplane of $\mathcal{S}$ containing x_1 but not x_2, then $\mathcal{S}$ also admits faithful pseudo-embeddings.*

(2) *If $\mathcal{S}$ admits a (faithful) full projective embedding e in a (possibly nondesarguesian) projective space, then $\mathcal{S}$ admits (faithful) pseudo-embeddings.*

Proof. Claim (1) is an immediate consequence of Theorem 1.4(1)+(2) taking into account that every hyperplane of $\mathcal{S}$ is also a pseudo-hyperplane. Claim (2) is a consequence of Claim (1) if we take those hyperplanes of $\mathcal{S}$ into account which arise from the projective embedding e. ∎

Observe that Proposition 3.2(2) can be used to construct many examples of point-line geometries which admit a faithful pseudo-embedding.

Proposition 3.3 (1) *Every (possibly nondesarguesian) projective space of dimension at least two whose constant line size is finite and odd admits faithful pseudo-embeddings.*

(2) *Every (possibly nondesarguesian) affine space $\mathcal{A}$ of dimension at least two whose constant line size $q \geq 4$ is finite and even admits faithful pseudo-embeddings.*

Proof. Claim (1) is an immediate consequence of Proposition 3.2. As for Claim (2), we observe that every proper set of points of $\mathcal{A}$ that is the union of an even number of parallel hyperplanes of $\mathcal{A}$ is a pseudo-hyperplane of $\mathcal{A}$. The set $\mathcal{H}$ of all pseudo-hyperplanes of $\mathcal{A}$ which arise in this way satisfies Properties (A1), (A2) and (A3). So, $\mathcal{A}$ admits faithful pseudo-embeddings by Theorem 1.4(2). ∎

Proposition 3.4 (1) *Let $\mathcal{S}$ be a projective space of dimension at least two whose constant line size $q + 1 \geq 4$ is finite and even. Then the empty set is the only pseudo-hyperplane of $\mathcal{S}$. As a consequence, $\mathcal{S}$ admits no pseudo-embeddings.*

(2) *Let $\mathcal{A}$ be an affine space of dimension at least two whose constant line size is finite and odd. Then $\mathcal{A}$ has no pseudo-hyperplanes. As a consequence, $\mathcal{A}$ does not admit pseudo-embeddings.*

Proof. (1) Suppose H is a pseudo-hyperplane of the projective space $\mathcal{S}$, distinct from the empty set. Then there exists a point $x \in H$ and a point $y \notin H$. Let π be an arbitrary plane through the line xy. In the plane π, there are $q + 1$ lines through x and each of these lines contains besides x an odd number of points of H. Since $q + 1$ is even, $|(H \cap \pi) \setminus \{x\}|$ must be even and hence $|H \cap \pi|$ is odd. On the other hand, in the plane π there are $q + 1$ lines through $y \notin H$ and each of these lines contains an even number of points of H. Hence, $|H \cap \pi|$ must be even, a contradiction. Hence, the empty set is the only pseudo-hyperplane of $\mathcal{S}$. Theorem 1.1 or Theorem 1.4(1) then implies that $\mathcal{S}$ does not admit pseudo-embeddings.

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(2) Suppose $\mathcal{A}$ is obtained from a projective space $\mathcal{S}$ by removing a hyperplane Π. If H were a pseudo-hyperplane of $\mathcal{A}$, then $H \cup \Pi$ would be a pseudo-hyperplane of $\mathcal{S}$ which is impossible by Claim (1). Again it follows that $\mathcal{S}$ cannot admit pseudo-embeddings. ∎

Proposition 3.5 *Let $\mathcal{S}$ be a connected point-line geometry, distinct from a point, having the property that the number of points on each line is finite and at least three. Suppose that every line of $\mathcal{S}$ is contained in a full subgeometry which is a finite projective plane or affine plane of odd order. Then $\mathcal{S}$ has no nonempty pseudo-hyperplanes and hence also no pseudo-embeddings.*

Proof. By connectedness of $\mathcal{S}$, it suffices to prove that if a pseudo-hyperplane H contains a point x, then it also contains any point $y \neq x$ collinear with x. This would imply that H is the whole point set which is clearly impossible.

Consider a full subgeometry π through the line xy which is either a finite projective plane or affine plane of odd order. Now, $H \cap \pi$ is either π or a pseudo-hyperplane of π. The latter possibility cannot occur by Proposition 3.4. So, $\pi \subseteq H$ and $y \in H$ as we needed to prove. ∎

Proposition 3.5 can be used to prove the nonexistence of pseudo-embeddings for many point-line geometries, like certain polar spaces of rank at least three, Grassmannians, half-spin geometries, exceptional geometries, etc. We only state the explicit result here for finite polar spaces of rank at least three. Later, we will also discuss the existence problem for pseudo-embeddings of polar spaces of rank two, i.e. for pseudo-embeddings of generalized quadrangles.

Corollary 3.6 *Let $\mathcal{S}$ be one of the polar spaces $W(2n-1,q)$, $Q(2n,q)$, $Q^+(2n-1,q)$, $Q^-(2n+1,q)$, $H(2n-1,q^2)$ or $H(2n,q^2)$, where $n \geq 3$. Then $\mathcal{S}$ admits pseudo-embeddings if and only if q is even.*

Proof. If q is even, then pseudo-embeddings exist by Proposition 3.2. If q is odd, then no pseudo-embeddings exist by Proposition 3.5. ∎

We now discuss the existence problem for pseudo-embeddings of certain classes of near polygons.

Definitions. A partial linear space $\mathcal{S}$ is called a *near polygon* if for every line L and every point x, there exists a unique point on L nearest to x. Here, distances $\mathrm{d}(\cdot, \cdot)$ are measured in the collinearity graph Γ of $\mathcal{S}$. If d is the diameter of Γ, then the near polygon is called a *near $2d$-gon*. A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbors. Generalized $2d$-gons (Van Maldeghem [27]) and dual polar spaces of rank $d \geq 2$ (Cameron [8]) are examples of near $2d$-gons. A dual polar space with at least three points on each line is a dense near polygon.

Let $k \in \mathbb{N} \setminus \{0, 1\}$, $s_1, s_2, \ldots, s_k \in \mathbb{N} \setminus \{0, 1\}$ and let X_i, $i \in \{1, 2, \ldots, k\}$, be a set of size $s_i + 1$. Then we can define a near polygon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ whose point set $\mathcal{P}$ is equal

to the cartesian product $X_1 \times X_2 \times \cdots \times X_k$, whose line set $\mathcal{L}$ consists of all sets of the form $\{x_1\} \times \{x_2\} \times \cdots \times \{x_{i-1}\} \times X_i \times \{x_{i+1}\} \times \cdots \times \{x_k\}$ where $i \in \{1, 2, \ldots, k\}$ and $x_j \in X_j$ for every $j \in \{1, 2, \ldots, k\} \setminus \{i\}$, with incidence being containment. The partial linear space $\mathcal{S}$ is also denoted by $Ham(s_1 + 1, s_2 + 1, \ldots, s_k + 1)$ and is called a *Hamming near polygon*.

Proposition 3.7 *Let $k \in \mathbb{N} \setminus \{0, 1\}$ and $s_1, s_2, \ldots, s_k \in \mathbb{N} \setminus \{0, 1\}$. Then the Hamming near polygon $\mathcal{S} = Ham(s_1 + 1, s_2 + 1, \ldots, s_k + 1)$ admits a pseudo-embedding. Moreover, we have $gr^*(\mathcal{S}) = er^*(\mathcal{S}) = \prod_{1 \leq i \leq k} s_i$.*

Proof. Let $\mathcal{S} = Ham(s_1 + 1, s_2 + 1, \ldots, s_k + 1)$ be constructed from sets X_i, $i \in \{1, 2, \ldots, k\}$, as described before this proposition. Recall that $s_i + 1 = |X_i|$ for every $i \in \{1, 2, \ldots, k\}$. Let x_i^* be a fixed element of X_i and put $Y_i := X_i \setminus \{x_i^*\}$ for every $i \in \{1, 2, \ldots, k\}$.

We prove that $Y_1 \times Y_2 \times \cdots \times Y_k$ is a pseudo-generating set of $\mathcal{S}$. For every point $p = (x_1, x_2, \ldots, x_k)$ of $\mathcal{S}$, let $N(p)$ denote the number of $i \in \{1, 2, \ldots, k\}$ for which $x_i = x_i^*$. We prove by induction on $N(p) \in \{0, 1, \ldots, k\}$ that $p \in [Y_1 \times Y_2 \times \cdots \times Y_k]^*$. If $N(p) = 0$, then $p \in Y_1 \times Y_2 \times \cdots \times Y_k \subseteq [Y_1 \times Y_2 \times \cdots \times Y_k]^*$. Suppose therefore that $i = N(p) > 0$ and that $p' \in [Y_1 \times Y_2 \times \cdots \times Y_k]^*$ for every point p' of $\mathcal{S}$ for which $N(p') < N(p)$. Now, there exists a line L through p such that $N(p') = N(p) - 1$ for every $p' \in L \setminus \{p\}$. By the induction hypothesis, $L \setminus \{p'\} \subseteq [Y_1 \times Y_2 \times \cdots \times Y_k]^*$ and hence also $p \in [Y_1 \times Y_2 \times \cdots \times Y_k]^*$.

Now, let V be a vector space of dimension $\prod_{1 \leq i \leq k} s_i$ over $\mathbb{F}_2$ with a basis B which is indexed by the elements of $Y_1 \times Y_2 \times \cdots \times Y_k$, say $B = \{\bar{v}_p \mid p \in Y_1 \times Y_2 \times \cdots \times Y_k\}$. For every point $p = (x_1, x_2, \ldots, x_k)$ of $\mathcal{S}$, let $A(p)$ denote the set of all $(y_1, y_2, \ldots, y_k) \in Y_1 \times Y_2 \times \cdots \times Y_k$ such that $y_i = x_i$ for every $i \in \{1, 2, \ldots, k\}$ for which $x_i \in Y_i$. Observe that if $p \in Y_1 \times Y_2 \times \cdots \times Y_k$, then $A(p) = \{p\}$ and $\bar{v}_p = \sum_{p' \in A(p)} \bar{v}_{p'}$. Now, for every $p \in X_1 \times X_2 \times \cdots \times X_k \setminus Y_1 \times Y_2 \times \cdots \times Y_k$, define $\bar{v}_p := \sum_{p' \in A(p)} \bar{v}_{p'}$. One readily sees that if $L = \{x_1, x_2, \ldots, x_{s+1}\}$ is a line of $\mathcal{S}$, then $\bar{v}_{x_1}, \bar{v}_{x_2}, \ldots, \bar{v}_{x_s}$ are linearly independent and $\bar{v}_{x_1} + \bar{v}_{x_2} + \cdots + \bar{v}_{x_{s+1}} = \bar{o}$. Hence, the map $p \mapsto <\bar{v}_p>$ defines a pseudo-embedding $\tilde{e}$ of $\mathcal{S}$ into $PG(V)$.

Now, since $|Y_1 \times Y_2 \times \cdots \times Y_k| = \dim(V) = \prod_{1 \leq i \leq k} s_i < \infty$, Theorem 1.5(2) implies that $gr^*(\mathcal{S}) = er^*(\mathcal{S}) = \prod_{1 \leq i \leq k} s_i$ and that $\tilde{e}$ is universal. $\blacksquare$

Proposition 3.8 *Let $\mathcal{S}$ be a near $2d$-gon with the property that the number of points on each line is finite and odd. Suppose also that every geodesic path in $\mathcal{S}$ can be extended to a geodesic path of length d. Then $\mathcal{S}$ admits faithful pseudo-embeddings.*

Proof. Let L be an arbitrary line of $\mathcal{S}$ and x an arbitrary point of L. Let y be an arbitrary point of $\mathcal{P}_L \setminus \{x\}$. Then the path y, x of length 1 can be extended to a geodesic path connecting y with a point z at distance d from y. Let H_z denote the set of points of $\mathcal{S}$ at distance at most $d - 1$ from z. Then H_z is a hyperplane of $\mathcal{S}$ which is called the *singular hyperplane of $\mathcal{S}$ with deepest point z*. The point x is the unique point of L contained in H_z.

Let x_1 and x_2 be two arbitrary distinct points of $\mathcal{S}$. Then any geodesic path between x_2 and x_1 can be extended to a geodesic path connecting x_2 with a point x_3 at distance d from x_2. The singular hyperplane H_{x_3} contains the point x_1 but not the point x_2.

Proposition 3.2(1) now implies that $\mathcal{S}$ admits faithful pseudo-embeddings. ∎

Remarks. (1) If $\mathcal{S}$ is a near $2d$-gon with three points on each line having the property that every geodesic path in $\mathcal{S}$ can be extended to a geodesic path of length d, then Proposition 3.8 implies that $\mathcal{S}$ has faithful full projective embeddings. This fact was already known, see Brouwer & Shpectorov [6] and De Bruyn [10, Theorem 3.11].

(2) Suppose $\mathcal{S}$ is a finite near $2d$-gon having precisely three points on each line. Suppose that every geodesic path in $\mathcal{S}$ can be extended to a geodesic path of length d. Let $\mathcal{H}'$ denote the set of all singular hyperplanes of $\mathcal{S}$. Then $\mathcal{H}'$ satisfies Properties (A1), (A2), (A3) and (A5). Let $\mathcal{H}$ denote the smallest set of hyperplanes of $\mathcal{S}$ which contains $\mathcal{H}'$ and satisfies Property (A4). By Theorem 1.4(3), there exists up to isomorphism a unique (faithful) full projective embedding e for which $\mathcal{H} = \mathcal{H}_e$. This full embedding is precisely the *near polygon embedding* of $\mathcal{S}$ as described in Brouwer & Shpectorov [6], see also Brouwer, Cohen, Hall & Wilbrink [3, p. 350].

Proposition 3.9 *Let $\mathcal{S}$ be a generalized $2d$-gon or a dense near $2d$-gon where $d \in \mathbb{N} \setminus \{0, 1\}$. Suppose the number of points on each line of $\mathcal{S}$ is finite and odd. Then $\mathcal{S}$ admits faithful pseudo-embeddings.*

Proof. This follows from Proposition 3.8 and the fact that every geodesic path can be extended to a geodesic path of maximal length d. For dense near polygons, the existence of such a geodesic path of maximal length is a consequence of the theory which has been developed for these geometries, see Brouwer and Wilbrink [7] or Chapter 2 of De Bruyn [9]. ∎

Proposition 3.10 *Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a near $2d$-gon with the property that the number of points on each line is finite, even and at least four. Then $\mathcal{S}$ admits faithful pseudo-embeddings.*

Proof. Let L be an arbitrary line of $\mathcal{S}$ and let x_1, x_2 be two arbitrary distinct points of L. For every point x of $\mathcal{S}$, let $\pi_L(x)$ denote the unique point of L nearest to x. Let H denote the set of all points x of $\mathcal{S}$ for which $\pi_L(x) \in \{x_1, x_2\}$. Then $H \neq \mathcal{P}$ and $H \cap \mathcal{P}_L = \{x_1, x_2\}$. We prove that H is a pseudo-hyperplane of $\mathcal{S}$. Let M be an arbitrary line of $\mathcal{S}$ and put $\delta = \mathrm{d}(L, M)$. There are two possibilities for the mutual position of the lines L and M, see Brouwer and Wilbrink [7, Lemma 1] or De Bruyn [9, Theorem 1.3].

(a) Suppose there exist unique points $l^* \in \mathcal{P}_L$ and $m^* \in \mathcal{P}_M$ such that $\mathrm{d}(l, m) = \mathrm{d}(l, l^*) + \mathrm{d}(l^*, m^*) + \mathrm{d}(m^*, m)$ for all $l \in \mathcal{P}_L$ and $m \in \mathcal{P}_M$. Then $\mathrm{d}(l^*, m^*) = \delta$ and $\pi_L(m) = l^*$ for every $m \in \mathcal{P}_M$. If $l^* \in \{x_1, x_2\}$, then M is completely contained in H. If $l^* \notin \{x_1, x_2\}$, then M has no points in common with H.

(b) Suppose every point of L lies at distance δ from M and every point of M lies at distance δ from L. Then the map $\mathcal{P}_M \to \mathcal{P}_L; x \mapsto \pi_L(x)$ is a bijection. Let y_i, $i \in \{1, 2\}$,

be the unique point of M at distance δ from x_i. Then $\pi_L(y_i) = x_i$. The points y_1 and y_2 are the only points of M which are contained in H.

By the above, M contains an even number of points of $\mathcal{P} \setminus H$. So, H is a pseudo-hyperplane of $\mathcal{S}$ which has only the points x_1 and x_2 in common with L.

Now, let x_1 and x_2 be two arbitrary distinct points of $\mathcal{S}$, let L be an arbitrary line through x_1 containing a point x_2' at distance $\mathrm{d}(x_1, x_2) - 1$ from x_2, let x_3 be a point of L distinct from x_1 and x_2', and let H denote the set of all points x of $\mathcal{S}$ for which $\pi_L(x) \in \{x_1, x_3\}$. Then H is a pseudo-hyperplane of $\mathcal{S}$ which contains x_1, but not x_2.

By Theorem 1.4(2) we can now conclude that $\mathcal{S}$ admits faithful pseudo-embeddings. ∎

The following is a consequence of Propositions 3.9 and 3.10.

Corollary 3.11 (1) *Let $\mathcal{S}$ be a generalized $2d$-gon, $d \geq 2$, with the property that the number of points on each line is finite and at least three. Then $\mathcal{S}$ admits faithful pseudo-embeddings.*

(2) Every dense near polygon with a finite number of points on each line admits faithful pseudo-embeddings.

(3) Every finite dual polar space with at least three points on each line admits faithful pseudo-embeddings.

Other examples of point-line geometries which admit a pseudo-embedding are related to quadrics of finite projective spaces. We refer to Hirschfeld and Thas [15, Chapter 22] for a discussion of the basic properties of such quadrics. The following lemma will be useful in our discussion.

Lemma 3.12 *Let V be a 4-dimensional vector space over $\mathbb{F}_2$ and let $Q = \{< \bar{v}_1 >, < \bar{v}_2 >, \ldots, < \bar{v}_5 >\}$ be a nonsingular elliptic quadric of $\mathrm{PG}(V) \cong \mathrm{PG}(3, 2)$. Then the vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3, \bar{v}_4$ are linearly independent and $\bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_5 = \bar{o}$.*

Proof. With respect to a suitable basis of V, the elliptic quadric Q has equation $X_0^2 + X_0 X_1 + X_1^2 + X_2 X_3 = 0$. The five points of Q are $(1, 0, 1, 1)$, $(0, 1, 1, 1)$, $(1, 1, 1, 1)$, $(0, 0, 0, 1)$ and $(0, 0, 1, 0)$. Any four of these points are linearly independent and $(1, 0, 1, 1) + (0, 1, 1, 1) + (1, 1, 1, 1) + (0, 0, 0, 1) + (0, 0, 1, 0) = (0, 0, 0, 0)$. ∎

Suppose now that Q is a quadric of the projective space $\mathrm{PG}(n, 2)$, $n \geq 3$, and let $Sing(Q)$ denote the set of all singular points of Q. These are points $x \in Q$ which the property that every line L of $\mathrm{PG}(n, 2)$ through x intersects Q in either $\{x\}$ or L. The set $Sing(Q)$ is a subspace of $\mathrm{PG}(n, 2)$. Let A denote the set of all 3-dimensional subspaces α of $\mathrm{PG}(n, 2)$ which intersect Q in a non-singular elliptic quadric of α. We suppose that $A \neq \emptyset$. Then we can define the point-line geometry $\mathcal{S}_Q$ whose points are the elements of $Q \setminus Sing(Q)$ and whose lines are all the elements of A, with incidence derived from $\mathrm{PG}(n, 2)$. One can readily verify that the set $Q \setminus Sing(Q)$ generates the whole projective space $\mathrm{PG}(n, 2)$. By Lemma 3.12, the inclusion $Q \subset \mathrm{PG}(n, 2)$ defines a pseudo-embedding of $\mathcal{S}_Q$ into $\mathrm{PG}(n, 2)$.

In the following table, we list the pseudo-embedding rank of $\mathcal{S}_Q$ in case Q is a quadric of $PG(n, 2)$ with $n \leq 7$ and $\dim(Sing(Q)) \leq 0$. We calculated these dimensions with the aid of GAP [12].

Quadric Q	$er^*(\mathcal{S}_Q)$	Quadric Q	$er^*(\mathcal{S}_Q)$	Quadric Q	$er^*(\mathcal{S}_Q)$
$Q^-(3, 2)$	4	$Q(6, 2)$	7	$pQ(4, 2)$	12
$Q(4, 2)$	10	$Q^+(7, 2)$	8	$pQ^+(5, 2)$	16
$Q^+(5, 2)$	14	$Q^-(7, 2)$	8	$pQ^-(5, 2)$	8
$Q^-(5, 2)$	6	$pQ^-(3, 2)$	6	$pQ(6, 2)$	9

Other examples of point-line geometries which admit pseudo-embeddings are the (restricted) ovoid-geometries of certain classes of point-line geometries.

Definitions. An *ovoid* of a partial linear space $\mathcal{S}$ is a set of points containing a unique point of every line of $\mathcal{S}$. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a partial linear space and let $\mathcal{Q}$ be a set of full subgeometries of $\mathcal{S}$ isomorphic to $W(2)$. Let $\mathcal{O}$ be the set of all ovoids of the elements of $\mathcal{Q}$. Then $\mathcal{S}' = (\mathcal{P}, \mathcal{O}, \mathrm{I}')$, where I' is the incidence relation on $\mathcal{P} \times \mathcal{O}$ defined by inclusion, is called the *ovoid-geometry* of $(\mathcal{S}, \mathcal{Q})$. In the special case that $\mathcal{Q}$ is the set of all full subgeometries of $\mathcal{S}$ isomorphic to $W(2)$, then $\mathcal{S}'$ is called the *ovoid-geometry* of $\mathcal{S}$.

A class of point-line geometries which admit a natural family of full subgeometries isomorphic to $W(2)$ are the dense near polygons with three points per line. Suppose $\mathcal{S}$ is a dense near polygon with three points per line. If x and y are two points of $\mathcal{S}$ at distance 2 from each other, then x and y are contained in a unique quad $Q(x, y)$ by Shult and Yanushka [23, Proposition 2.5] (see also De Bruyn [9, Theorem 1.20]). The full subgeometry of $\mathcal{S}$ induced on the set $Q(x, y)$ is isomorphic to either the (3×3)-grid $Ham(3, 3)$, the generalized quadrangle $W(2)$ or the generalized quadrangle $Q(5, 2)$. If we take for $\mathcal{Q}$ the set of all $W(2)$-quads together with all $W(2)$-subquadrangles of the $Q(5, 2)$-quads, then we obtain the ovoid-geometry of $\mathcal{S}$. If we take for $\mathcal{Q}$ only the $W(2)$-quads, then we call the ovoid-geometry of $(\mathcal{S}, \mathcal{Q})$ also the *restricted ovoid-geometry* of $\mathcal{S}$.

In Chapter 6 of De Bruyn [9], several classes of dense near polygons with three points per line were described. Among the examples discussed there, the near polygons $DW(2n - 1, 2)$, $DH(2n - 1, 2)$, $\mathbb{G}_n$, $\mathbb{H}_n$, $\mathbb{I}_n$, $\mathbb{E}_2$ and $\mathbb{E}_3$ ($n \geq 2$) are suitable candidates for considering the (restricted) ovoid-geometries. The ovoid-geometries and restricted ovoid geometries are identical for the dense near polygons $DW(2n - 1, 2)$, $\mathbb{H}_n$, $\mathbb{I}_n$ and $\mathbb{E}_2$.

Lemma 3.13 *Let $e : W(2) \to PG(V)$ be a faithful full projective embedding of the generalized quadrangle $W(2)$, let $O = \{x_1, x_2, x_3, x_4, x_5\}$ be an ovoid of $W(2)$ and let $\bar{v}_i$, $i \in \{1, 2, 3, 4, 5\}$, be the unique nonzero vector of V for which $e(x_i) = <\bar{v}_i>$. Then the vectors $\bar{v}_1, \bar{v}_2, \bar{v}_3$ and $\bar{v}_4$ are linearly independent and $\bar{v}_1 + \bar{v}_2 + \cdots + \bar{v}_5 = \bar{o}$.*

Proof. There are two possibilities for the faithful full projective embedding e of $W(2)$.

(1) Suppose $e = \tilde{e}$ where $\tilde{e} : W(2) \to \widetilde{\Sigma}$ is the universal embedding of $W(2)$. Then the image of e is a nonsingular parabolic quadric $Q(4, 2)$ of $\widetilde{\Sigma} \cong \mathrm{PG}(4, 2)$. The ovoid O is a hyperplane of $W(2)$ and arises from $\tilde{e}$. Hence, there exists a hyperplane Π of $\mathrm{PG}(4, 2)$ such that $e(O) = \Pi \cap Q(4, 2)$. Now, $\Pi \cap Q(4, 2)$ is a nonsingular elliptic quadric of Π. Lemma 3.12 yields the desired result.

(2) Suppose $e = \tilde{e}/k$, where k is the kernel of the parabolic quadric $Q(4, 2)$ of $\widetilde{\Sigma} \cong \mathrm{PG}(4, 2)$. Every hyperplane of $\mathrm{PG}(4, 2)$ through k is tangent to $Q(4, 2)$. Hence, the hyperplane Π as defined in part (1) does not contain k. This implies that $e(O) = \tilde{e}/k(O)$ remains a nonsingular elliptic quadric in a 3-dimensional space. Again, Lemma 3.12 yields the desired result. ∎

The following is an immediate corollary of Lemma 3.13.

Corollary 3.14 *Let $\mathcal{S}$ be a partial linear space which admits a faithful full projective embedding e and let $\mathcal{Q}$ be any set of full subgeometries of $\mathcal{S}$ isomorphic to $W(2)$. Then e induces a faithful pseudo-embedding of the ovoid-geometry of $(\mathcal{S}, \mathcal{Q})$.*

The following is an immediate consequence of Proposition 3.9 and Corollary 3.14.

Corollary 3.15 *Let $\mathcal{S}$ be a dense near polygon with three points per line. Then the ovoid-geometry of $\mathcal{S}$ and the restricted ovoid-geometry of $\mathcal{S}$ admit faithful pseudo-embeddings.*

The fact that the (restricted) ovoid-geometry of a given dense near polygon with three points per line admits a pseudo-embedding could also follow from Proposition 3.2 if we knew in advance that this (restricted) ovoid-geometry admitted a full projective embedding. Although the following lemma shows that this is the case for the ovoid-geometry of the generalized quadrangle $W(2)$, we show with a counter example that this claim is false in general.

Lemma 3.16 *The ovoid-geometry of $W(2)$ admits a faithful full projective embedding in $\mathrm{PG}(2, 4)$.*

Proof. Let H be a hyperoval of the projective plane $\mathrm{PG}(2, 4)$. Let $\mathcal{Q}$ be the point-line geometry whose points are the points of $\mathrm{PG}(2, 4)$ not contained in H and whose lines are those lines of $\mathrm{PG}(2, 4)$ which contain two points of H, with incidence being derived from $\mathrm{PG}(2, 4)$. Then $\mathcal{Q}$ is a generalized quadrangle of order 2 and hence is isomorphic to $W(2)$. The 6 lines of $\mathrm{PG}(2, 4)$ disjoint from H correspond to the 6 ovoids of $\mathcal{Q} \cong W(2)$. So, we have realized a faithful full projective embedding of the ovoid-geometry of $W(2)$ in $\mathrm{PG}(2, 4)$. ∎

Consider the dense near polygon $\mathbb{E}_2$ whose points are the blocks of the unique Steiner system $S(5, 8, 24)$ and whose lines are the triples of mutually disjoint blocks of $S(5, 8, 24)$, with incidence being containment. This near hexagon was first constructed by Shult and Yanushka [23, p.40]. The uniqueness of the Steiner system $S(5, 8, 24)$ is due to Witt [28].

We refer to Beth, Jungnickel and Lenz [1] for the elementary properties of this Witt design which we will use later.

All quads of $\mathbb{E}_2$ define $W(2)$-subquadrangles. So, the ovoid-geometry $\mathcal{S}$ of $\mathbb{E}_2$ is identical to the restricted ovoid-geometry of $\mathbb{E}_2$. By Corollary 3.15, $\mathcal{S}$ admits a pseudo-embedding. With the aid of GAP [12] we calculated that $er^*(\mathcal{S}) = 23$.

The dense near polygon $\mathbb{E}_2$ also admits faithful full projective embeddings. The vector dimension $er(\mathbb{E}_2)$ of the universal embedding of $\mathbb{E}_2$, the so-called *embedding rank of* $\mathbb{E}_2$, was determined by Brouwer, Cohen, Hall and Wilbrink [3, p. 350] with the aid of GAP. They found that $er(\mathbb{E}_2) = 23$. Because $er^*(\mathcal{S}) = er(\mathbb{E}_2) = 23$, we have the following.

Proposition 3.17 *The universal embedding of the dense near polygon $\mathbb{E}_2$ induces a pseudo-embedding of the ovoid-geometry of $\mathbb{E}_2$ which is universal.*

By Theorem 1.3 and Proposition 3.17, we obtain

Corollary 3.18 *A set X of points of $\mathbb{E}_2$, distinct from the whole point set, is a hyperplane of $\mathbb{E}_2$ if and only if it intersects each ovoid of a quad of $\mathbb{E}_2$ in either 1, 3 or 5 points.*

A classification of the hyperplanes of $\mathbb{E}_2$ can be found in Brouwer, Cuypers and Lambeck [4]. The ovoid-geometry $\mathcal{S}$ of $\mathbb{E}_2$ has many subgeometries admitting a full projective embedding.

- Since $\mathbb{E}_2$ has many $W(2)$-subquadrangles, $\mathcal{S}$ has many full subgeometries isomorphic to the ovoid-geometry of $W(2)$. All these subgeometries admit a full projective embedding by Lemma 3.16.
- If x_1, x_2 and x_3 are three distinct points of the Steiner system $S(5, 8, 24)$, then the set of 21 blocks of $S(5, 8, 24)$ through $\{x_1, x_2, x_3\}$ is a subspace of $\mathcal{S}$ and the full subgeometry induced on that subspace is isomorphic to $PG(2, 4)$. Obviously, this subgeometry has a full projective embedding in $PG(2, 4)$.

Despite the above observations, $\mathcal{S}$ itself does not admit full projective embeddings as we are going to prove now.

Proposition 3.19 *The ovoid-geometry of $\mathbb{E}_2$ does not admit full projective embeddings.*

Proof. Suppose $e : \mathcal{S} \to \Sigma$ is a full projective embedding of the ovoid-geometry $\mathcal{S}$ of $\mathbb{E}_2$. Let H be a hyperplane of $\mathcal{S}$ arising from e. We prove that H is an ovoid of $\mathbb{E}_2$. Since H is a hyperplane of $\mathcal{S}$, it is also a pseudo-hyperplane of $\mathcal{S}$ and hence a hyperplane of $\mathbb{E}_2$ by Corollary 3.18. If Q is a quad of $\mathbb{E}_2$, then $Q \cap H$ is either Q or a hyperplane of the subgeometry $\mathcal{Q} \cong W(2)$ of $\mathbb{E}_2$ induced on Q. By Payne and Thas [20, Theorem 2.3.1], one of the following possibilities then occurs.

(1) $Q \subseteq H$.

(2) $Q \cap H$ consists of the 7 points of Q which are collinear with or equal to a given point x^* of Q. But then H intersects 4 ovoids of $\mathcal{Q}$ in precisely 3 points, in contradiction with the fact that H is a hyperplane of $\mathcal{S}$.

(3) $Q \cap H$ is a (3×3)-subgrid of $\mathcal{Q}$. Then H intersects each of the 6 ovoids of $\mathcal{Q}$ in precisely three points, in contradiction with the fact that H is a hyperplane of $\mathcal{S}$.

(4) $Q \cap H$ is an ovoid of $\mathcal{Q}$.

So, every quad Q of $\mathbb{E}_2$ intersects H in either Q or an ovoid of $\mathcal{Q}$. If every quad Q intersects H in an ovoid of $\mathcal{Q}$, then H itself must be an ovoid of $\mathbb{E}_2$. Suppose therefore that there exists some quad Q^* which is contained in H. Then every quad must be contained in H because of the following two facts: (1) if Q_1 and Q_2 are two quads which intersect in a line, then $Q_1 \subseteq H$ implies that $Q_2 \subseteq H$; (2) for every quad Q of H, there exist quads $Q_0, Q_1, \ldots, Q_k$ for some $k \geq 0$ such that $Q_0 = Q^*$, $Q_k = Q$ and $Q_i \cap Q_{i-1}$ is a line for every $i \in \{1, 2, \ldots, k\}$. But since H is not the whole point-set, not every quad can be contained in H.

So, every hyperplane H arising from e must be an ovoid. Now, let H_1 and H_2 be two distinct hyperplanes of $\mathcal{S}$ arising from e and let H_3 denote the complement of the symmetric difference of H_1 and H_2. Then H_1, H_2 and H_3 are mutually distinct. Since H_i, $i \in \{1, 2, 3\}$, is an ovoid of $\mathbb{E}_2$, there exists by Brouwer and Lambeck [5, p. 105] (see also De Bruyn [9, Section 6.6.2]) a unique point x_i of $S(5, 8, 24)$ such that H_i consists of all 253 blocks through x_i. Now, we have $H_1 \cap H_2 = H_1 \cap H_2 \cap H_3$. But this is impossible since $H_1 \cap H_2$ consists of all 77 blocks of $S(5, 8, 24)$ containing $\{x_1, x_2\}$ and $H_1 \cap H_2 \cap H_3$ consists of all 21 blocks of $S(5, 8, 24)$ containing $\{x_1, x_2, x_3\}$. ∎

4 The pseudo-embedding and pseudo-generating ranks of $\mathrm{PG}(n, 4)$ and $\mathrm{AG}(n, 4)$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry and suppose that $3 \leq |\mathcal{P}_L| < \infty$ for every line L of $\mathcal{S}$. A set of points of $\mathcal{S}$ is said to be of *even type* [resp. *odd type*] if it has an even [resp. odd] number of points in common with each line of $\mathcal{S}$. If $|\mathcal{P}_L|$ is odd for every $L \in \mathcal{L}$, then the pseudo-hyperplanes of $\mathcal{S}$ are precisely the sets of odd type of $\mathcal{S}$ distinct from $\mathcal{P}$. If $|\mathcal{P}_L|$ is even for every $L \in \mathcal{L}$, then the pseudo-hyperplanes of $\mathcal{S}$ are precisely the sets of even type of $\mathcal{S}$ distinct from $\mathcal{P}$.

We have seen in Proposition 3.3(1) that the projective space $\mathrm{PG}(n, 4)$, $n \geq 0$, admits a pseudo-embedding. If $d = er^*(\mathrm{PG}(n, 4))$, then we know by Theorem 1.3 that the number of sets of odd type of $\mathrm{PG}(n, 4)$ is equal to 2^d. Sherman [22] obtained a classification of the sets of odd type of $\mathrm{PG}(n, 4)$. By Sherman [22, Corollary 1, p.550], we know that $d = \frac{1}{3}(n + 1)(n^2 + 2n + 3)$. So, we have

Proposition 4.1 *For every $n \geq 0$, we have $er^*(\mathrm{PG}(n, 4)) = \frac{1}{3}(n + 1)(n^2 + 2n + 3)$.*

Remark. Another approach could be the following. By Theorem 1.2(2) we know that if $n \geq 1$ then $er^*(\mathrm{PG}(n, 4)) = \frac{4^{n+1} - 1}{3} - \mathrm{rank}_{\mathbb{F}_2}(M)$ where M is an incidence matrix of $\mathrm{PG}(n, 4)$. The ranks of incidence matrices involving subspaces of finite projective spaces have been studied by many people. Complicated formulas which enable to compute

rank$_{\mathbb{F}_2}(M)$ for any $n \geq 1$ can be found in Hamada [13, Theorem 1] or Inamdar and Sastry [16, Theorem 2.13]. These formulas are not so easy to work with. In fact, it seems even hard to derive a closed expression for rank$_{\mathbb{F}_2}(M)$ from these formulas.

Since $er^*(\mathrm{PG}(n,4)) = \frac{1}{3}(n+1)(n^2+2n+3)$, the projective space $\mathrm{PG}(n,4)$ admits a pseudo-embedding (the universal one) in an $\frac{n(n^2+3n+5)}{3}$-dimensional projective space over $\mathbb{F}_2$. Pseudo-embeddings do however exist in projective spaces of smaller dimension. Indeed, in Proposition 4.2 below, we prove that $\mathrm{PG}(n,4)$ admits a pseudo-embedding in a projective space of dimension $n^2 + 2n$.

Every possibly degenerate Hermitian variety of $\mathrm{PG}(n,4)$ is a set of odd type of $\mathrm{PG}(n,4)$. It is straightforward to verify that the set $\mathcal{H}$ of all possibly degenerate Hermitian varieties of $\mathrm{PG}(n,4)$, distinct from the whole point set, satisfies Properties (A1), (A2), (A3), (A4) and (A5). So, by Theorem 1.4(3), there exists, up to isomorphism, a unique (faithful) pseudo-embedding e of $\mathrm{PG}(n,4)$ for which $\mathcal{H}_e = \mathcal{H}$. This pseudo-embedding is precisely the Hermitian Veronese map ν first described by Lunardon [19]. With respect to certain reference systems in $\mathrm{PG}(n,4)$ and $\mathrm{PG}(n^2+2n,2)$, ν maps every point $(X_0, X_1, \ldots, X_n)$ of $\mathrm{PG}(n,4)$ to the point $(X_0^3, X_1^3, \ldots, X_n^3, X_iX_j^2 + X_jX_i^2, \delta X_iX_j^2 + \delta^2 X_jX_i^2 \,|\, 0 \leq i < j \leq n)$ of $\mathrm{PG}(n^2+2n,2)$.

Proposition 4.2 *The Hermitian Veronese map ν of $\mathrm{PG}(n,4)$ is a faithful pseudo-embedding of $\mathrm{PG}(n,4)$. The pseudo-hyperplanes arising from ν are precisely the possibly degenerate Hermitian varieties of $\mathrm{PG}(n,4)$, distinct from the whole point set.*

Proof. The verification of this proposition is straightforward. In order to prove that ν maps the points of a line L of $\mathrm{PG}(n,4)$ to a set of points of $\mathrm{PG}(n^2+2n,2)$ having the right configuration, one could choose a reference system in $\mathrm{PG}(n,4)$ with respect to which the line is given by the equation $X_2 = X_3 = \cdots = X_n = 0$. ∎

We already know $er^*(\mathrm{PG}(n,4))$. We now calculate $er^*(\mathrm{AG}(n,4))$.

Proposition 4.3 *Let $q \geq 4$ be even and $n \geq 1$. Then $er^*(\mathrm{AG}(n,q)) = er^*(\mathrm{PG}(n,q)) - er^*(\mathrm{PG}(n-1,q))$.*

Proof. Notice first that $er^*(\mathrm{AG}(n,q))$ and $er^*(\mathrm{PG}(n,q))$ are defined by Proposition 3.3. Suppose $\mathrm{AG}(n,q)$ is the affine space obtained from $\mathrm{PG}(n,q)$ by removing a hyperplane Π from $\mathrm{PG}(n,q)$. The set V of all sets of odd type of $\mathrm{PG}(n,q)$ can be given the structure of an $\mathbb{F}_2$-vector space by defining $0 \cdot H = \mathcal{P}$, $1 \cdot H = H$ and $H_1 + H_2 = \overline{H_1 \Delta H_2}$ for all $H, H_1, H_2 \in V$. By Theorem 1.3, we have $\dim(V) = er^*(\mathrm{PG}(n,q))$.

Let p be a point of $\mathrm{PG}(n,q)$ not contained in Π. If G is a set of odd type of $\Pi \cong \mathrm{PG}(n-1,q)$, then the cone pG with top p and basis G is easily seen to be a set of odd type of $\mathrm{PG}(n,q)$. Notice that if G_1 and G_2 are two distinct sets of odd type of Π, then $pG_1 + pG_2 = p(G_1 + G_2)$. Now, consider the following subspaces of V:

- the subspace V_1 of V consisting of all sets of odd type of $\mathrm{PG}(n,4)$ containing Π;

- the subspace V_2 of V consisting of all sets of odd type of $\mathrm{PG}(n,4)$ of the form pG where G is some set of odd type of Π.

Notice that $\dim(V_2) = er^*(\mathrm{PG}(n-1,q))$. We prove that $V = V_1 \oplus V_2$. If $H \in V$, then $H = [H + p(H \cap \Pi)] + p(H \cap \Pi)$, where $H + p(H \cap \Pi) \in V_1$ and $p(H \cap \Pi) \in V_2$. If $pG \in V_1$ where G is some set of odd type of Π, then $G = \Pi$ and hence $pG = \mathcal{P}$, proving that $V_1 \cap V_2 = \{\mathcal{P}\}$. Hence, $V = V_1 \oplus V_2$.

So, we have $\dim(V_1) = er^*(\mathrm{PG}(n,q)) - er^*(\mathrm{PG}(n-1,q))$. Now, there are $2^{\dim(V_1)}$ sets of odd type of $\mathrm{PG}(n,q)$ which contain Π. A set X of points of $\mathrm{AG}(n,q)$ is a set of even type of $\mathrm{AG}(n,q)$ if and only if $X \cup \Pi$ is a set of odd type of $\mathrm{PG}(n,q)$. Hence, there are $2^{\dim(V_1)}$ sets of even type of $\mathrm{AG}(n,q)$. By Theorem 1.3, this implies that $er^*(\mathrm{AG}(n,q)) = \dim(V_1) = er^*(\mathrm{PG}(n,q)) - er^*(\mathrm{PG}(n-1,q))$. ∎

The following is an immediate corollary of Propositions 4.1 and 4.3.

Corollary 4.4 *For every $n \geq 0$, we have $er^*(\mathrm{AG}(n,4)) = n^2 + n + 1$.*

Our next aim is to determine the pseudo-generating ranks of $\mathrm{PG}(n,4)$ and $\mathrm{AG}(n,4)$ for every $n \geq 0$.

Lemma 4.5 (1) *The geometry $\mathrm{PG}(n,4)$, $n \geq 0$, has a pseudo-generating set of size $\frac{1}{3}(n+1)(n^2 + 2n + 3)$.*
  (2) *If Π is a hyperplane of $\mathrm{PG}(n,4)$, $n \geq 0$, then there exists a set X of points of $\mathrm{PG}(n,4)$ such that $X \cap \Pi = \emptyset$, $|X| = n^2 + n + 1$ and $\Pi \cup X$ is a pseudo-generating set of $\mathrm{PG}(n,4)$.*

Proof. We will prove the lemma by induction on $n \geq 0$. If $n = 0$, then putting $X = \{x\}$, where x is the unique point of $\mathrm{PG}(0,4)$, we see that Claims (1) and (2) of the lemma are valid. If $n = 1$, then any set of 4 points of $\mathrm{PG}(1,4)$ is a pseudo-generating set of $\mathrm{PG}(1,4)$. If $n = 1$ and Π is a hyperplane of $\mathrm{PG}(1,4)$, then for any set X of three points of $\mathrm{PG}(1,4)$ disjoint from Π, the set $\Pi \cup X$ is a pseudo-generating set of $\mathrm{PG}(1,4)$. So, Claims (1) and (2) of the lemma are also valid if $n = 1$. In the sequel, we suppose that $n \geq 2$ and that the lemma holds for smaller values of n.

Let Π_1 be an arbitrary hyperplane of $\mathrm{PG}(n,4)$. Then by the induction hypothesis, there exists a pseudo-generating set X_1 of size $\frac{1}{3}n(n^2 + 2)$ of $\Pi_1 \cong \mathrm{PG}(n-1,4)$. If we can prove that there exists a set X_2 of points of $\mathrm{PG}(n,4)$ such that $X_2 \cap \Pi_1 = \emptyset$, $|X_2| = n^2 + n + 1$ and $X_1 \cup X_2$ is a pseudo-generating set of $\mathrm{PG}(n,4)$, then since $|X_1 \cup X_2| = \frac{1}{3}n(n^2 + 2) + n^2 + n + 1 = \frac{1}{3}(n+1)(n^2 + 2n + 3)$, we see that Claims (1) and (2) of the lemma are valid.

Let Π_2 be a hyperplane of $\mathrm{PG}(n,4)$ distinct from Π_1. By the induction hypothesis, there exists a subset X_2' of size $n^2 - n + 1$ of $\Pi_2 \setminus \Pi_1$ such that $(\Pi_1 \cap \Pi_2) \cup X_2'$ is a pseudo-generating set of Π_2. Clearly, $[X_1 \cup X_2']^* = \Pi_1 \cup \Pi_2$.

Let V be an $(n+1)$-dimensional vector space over $\mathbb{F}_4$ such that $\mathrm{PG}(n,4) = \mathrm{PG}(V)$. Let Π_3, Π_4 and Π_5 denote the three hyperplanes of $\mathrm{PG}(n,4)$ through $\Pi_1 \cap \Pi_2$ distinct

from Π_1 and Π_2. We can choose an ordered basis $B = (\bar{e}_0, \bar{e}_1, \ldots, \bar{e}_n)$ of V such that Π_3 has equation $x_0 = 0$, Π_4 has equation $x_1 = 0$ and Π_5 has equation $x_0 = x_1$. Here, $(x_0, x_1, \ldots, x_n)$ denote the coordinates of a point of $\mathrm{PG}(n, 4)$. Let $\delta \in \mathbb{F}_4 \setminus \mathbb{F}_2$. We define $r_1 :=< \bar{e}_0 >$ and $r_2 :=< \bar{e}_1 >$. For every $i \in \{1, 2, \ldots, n-1\}$, we also define $p_i :=< \bar{e}_0 + \bar{e}_{i+1} >$ and $q_i :=< \bar{e}_0 + \delta \bar{e}_{i+1} >$. Put $X_2'' := \{r_1, r_2, p_1, q_1, p_2, q_2, \ldots, p_{n-1}, q_{n-1}\}$ and $X_2 := X_2' \cup X_2''$. Then $|X_2''| = 2n$, $|X_2| = n^2 + n + 1$ and no point of X_2 is contained in Π_1. We will now prove that $[X_1 \cup X_2]^* = \mathrm{PG}(n, 4)$. Since $[X_1 \cup X_2']^* = \Pi_1 \cup \Pi_2$, it suffices to prove that each point of the set $(\Pi_3 \cup \Pi_4 \cup \Pi_5) \setminus (\Pi_1 \cap \Pi_2)$ belongs to $[X_1 \cup X_2]^*$, or equivalently, that for every $\bar{w} \in W :=< \bar{e}_2, \bar{e}_3, \ldots, \bar{e}_{n+1} >$, the points $< \bar{e}_0 + \bar{w} >$, $< \bar{e}_1 + \bar{w} >$ and $< \bar{e}_0 + \bar{e}_1 + \bar{w} >$ belong to $[X_1 \cup X_2]^*$. We will prove this by induction on the *weight* of $\bar{w}$ which is defined as the total number of nonzero coordinates of $\bar{w}$ with respect to the ordered basis B. During the proof, we will also make use of the following observation:

($*$) If L is a line of $\mathrm{PG}(n, 4)$ disjoint from $\Pi_1 \cap \Pi_2$ such that at least two of the points $L \cap \Pi_3$, $L \cap \Pi_4$, $L \cap \Pi_5$ belong to $[X_1 \cup X_2]^*$, then since $L \cap \Pi_1 \subseteq [X_1 \cup X_2]^*$ and $L \cap \Pi_2 \subseteq [X_1 \cup X_2]^*$, the line L must be completely contained in $[X_1 \cup X_2]^*$.

Suppose first that the weight of $\bar{w}$ is equal to 0, i.e. $\bar{w} = \bar{o}$. We need to prove that the points $r_1 =< \bar{e}_0 >$, $r_2 =< \bar{e}_1 >$ and $r_3 :=< \bar{e}_0 + \bar{e}_1 >$ belong to $[X_1 \cup X_2]^*$. Clearly, $r_1, r_2 \in [X_1 \cup X_2]^*$ since $r_1, r_2 \in X_2$. The fact that r_3 belongs to $[X_1 \cup X_2]^*$ follows by applying observation ($*$) to the unique line through the points r_1 and r_2.

We now prove the claim in the case the weight of $\bar{w}$ is equal to 1. Let $i \in \{1, 2, \ldots, n-1\}$. Since $< \bar{e}_0 + \bar{e}_{i+1} >\in [X_0 \cup X_2]^*$, we have $< \bar{e}_0 + \bar{e}_1 + \bar{e}_{i+1} >\in [X_0 \cup X_2]^*$ by applying observation ($*$) to the unique line through the points $< \bar{e}_0 + \bar{e}_{i+1} >$ and r_2. If we apply observation ($*$) to the unique line through the points $< \bar{e}_0 + \bar{e}_1 + \bar{e}_{i+1} >$ and $< \bar{e}_0 >$, then we find that $< \bar{e}_1 + \bar{e}_{i+1} >\in [X_0 \cup X_2]^*$. In a similar way, by starting from the point $q_i =< \bar{e}_0 + \delta \bar{e}_{i+1} >\in [X_0 \cup X_2]^*$ instead of the point $p_i =< \bar{e}_0 + \bar{e}_{i+1} >$, one can also prove that $< \bar{e}_0 + \bar{e}_1 + \delta \bar{e}_{i+1} >$ and $< \bar{e}_1 + \delta \bar{e}_{i+1} >$ belong to $[X_0 \cup X_2]^*$. Now, let $\bar{u}, \bar{u}', \bar{u}''$ be vectors such that $\{\bar{u}, \bar{u}', \bar{u}''\} = \{\bar{e}_0, \bar{e}_1, \bar{e}_0 + \bar{e}_1\}$. If we apply observation ($*$) to the unique line through the points $< \bar{u}' + \bar{e}_{i+1} >$ and $< \bar{u}'' + \delta \bar{e}_{i+1} >$, we find that $< \bar{u} + (1+\delta)\bar{e}_{i+1} >\in [X_0 \cup X_2]^*$. Summarizing, we can conclude that the claim is valid for vectors $\bar{w}$ of weight 1.

Suppose finally that the weight k of $\bar{w}$ is at least 2. We need to prove that for every $\bar{u} \in \{\bar{e}_0, \bar{e}_1, \bar{e}_0 + \bar{e}_1\}$, the point $< \bar{u} + \bar{w} >$ belongs to $[X_1 \cup X_2]^*$. Put $\bar{w} := \bar{w}_1 + \bar{w}_2$ where $\bar{w}_1$ is a vector of weight $k-1$ of W and $\bar{w}_2$ is a vector of weight 1 of W. Let $\bar{u}'$ and $\bar{u}''$ be vectors of V such that $\{\bar{u}, \bar{u}', \bar{u}''\} = \{\bar{e}_0, \bar{e}_1, \bar{e}_0 + \bar{e}_1\}$. By the induction hypothesis, the points $< \bar{u}' + \bar{w}_1 >$ and $< \bar{u}'' + \bar{w}_2 >$ belong to $[X_1 \cup X_2]^*$. Hence, by applying observation ($*$) to the unique line through the points $< \bar{u}' + \bar{w}_1 >$ and $< \bar{u}'' + \bar{w}_2 >$, we find that $< \bar{u} + \bar{w} >\in [X_1 \cup X_2]^*$, as we needed to prove. ∎

Proposition 4.6 *For every $n \geq 0$, we have $gr^*(\mathrm{PG}(n, 4)) = \frac{1}{3}(n+1)(n^2 + 2n + 3)$ and $gr^*(\mathrm{AG}(n, 4)) = n^2 + n + 1$.*

Proof. By Theorem 1.5(2), Proposition 4.1 and Lemma 4.5(1), we have $gr^*(\mathrm{PG}(n,4)) = \frac{1}{3}(n+1)(n^2+2n+3)$. Let $\mathrm{AG}(n,4)$ denote the affine space obtained by removing a hyperplane Π from $\mathrm{PG}(n,4)$. By Lemma 4.5(2), there exists a set X of n^2+n+1 points of $\mathrm{AG}(n,4)$ such that $X \cup \Pi$ is a pseudo-generating set of $\mathrm{PG}(n,4)$. A set Y of points of $\mathrm{AG}(n,4)$ is a pseudo-subspace of $\mathrm{AG}(n,4)$ if and only if $Y \cup \Pi$ is a pseudo-subspace of $\mathrm{PG}(n,4)$. Since every pseudo-subspace of $\mathrm{PG}(n,4)$ containing $X \cup \Pi$ coincides with the whole point set of $\mathrm{PG}(n,4)$, every pseudo-subspace of $\mathrm{AG}(n,4)$ containing X coincides with the whole point set of $\mathrm{AG}(n,4)$. So, X is a pseudo-generating set of $\mathrm{AG}(n,4)$. By Theorem 1.5(2) and Corollary 4.4, we then know that $gr^*(\mathrm{AG}(n,4)) = n^2+n+1$. ∎

Acknowledgment

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