The hyperplanes of the glued near hexagon $Q(5,2) \otimes Q(5,2)$

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Abstract

With the aid of the computer algebra system GAP, we show that the glued near hexagon $Q(5,2) \otimes Q(5,2)$ has 16 isomorphism classes of hyperplanes. We give at least one explicit construction for a representative of each isomorphism class and we list several properties of such a representative.

Keywords: (glued) near hexagon, hyperplane, universal embedding
MSC2010: 05E20, 51A45, 51E12

1 Introduction

A partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with nonempty point set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$ is called a *near polygon* if for every point p and every line L, there exists a unique point on L nearest to p. Here, distances $\mathrm{d}(\cdot,\cdot)$ are measured in the collinearity graph Γ of $\mathcal{S}$. If $d \in \mathbb{N}$ is the diameter of Γ, then the near polygon is also called a *near* $2d$*-gon*. A near 0-gon is a point and a near 2-gon is a line. Near quadrangles are usually called *generalized quadrangles* (GQ's). A near polygon is called *dense* if every line is incident with at least three points and if every two points at distance 2 have at least two common neighbors. Near polygons were introduced in Shult & Yanushka [18].

A *hyperplane* of a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a set of points, distinct from $\mathcal{P}$, intersecting each line in either a singleton or the whole line. If $\mathcal{S}$ admits a full projective embedding and $\mathcal{S}$ is slim (i.e. every line of $\mathcal{S}$ is incident with precisely three points), then by Ronan [17, Corollary 2, p.180], there exists a natural bijective correspondence between the hyperplanes of $\mathcal{S}$ and the hyperplanes of the so-called universal embedding space of $\mathcal{S}$. This fact makes the study of the hyperplanes of fully embeddable slim partial linear spaces somewhat special and interesting.

This paper is part of an ongoing project to classify all hyperplanes of all slim dense near hexagons. By Brouwer et al. [1], there are up to isomorphism eleven such near hexagons. For three of these near hexagons (namely those which are a direct product of a line of size three with one of the three GQ's of order $(2, t)$), the classification of the hyperplanes is a trivial problem. For five other near hexagons, a complete classification of the hyperplanes is available in the literature, see Brouwer, Cuypers & Lambeck [2] for the M_{24} near hexagon $\mathbb{E}_2$, Pralle [16] (see also De Bruyn [5]) for the dual polar space $DW(5,2)$, De Bruyn & Pralle [7, 8] for the dual polar space $DH(5,4)$, De Bruyn & Pralle [9] for the near hexagon $\mathbb{H}_3$ on the 2-factors of the complete graph K_8 and De Bruyn & Shpectorov [10] for the $U_4(3)$ near hexagon $\mathbb{E}_3$. The present paper deals with the hyperplanes of the glued near hexagon $Q(5,2) \otimes Q(5,2)$.

In future work [6, 11], we will also deal with the two remaining slim dense near hexagons, namely the near hexagon $\mathbb{E}_1$ related to the extended ternary Golay code and the near hexagon $\mathbb{G}_3$. It should be noted that the slim dense near hexagons are not the only slim near polygons whose hyperplanes might be worthwhile to classify. In the paper [14], Frohardt & Johnson classified all hyperplanes of the two generalized hexagons of order $(2,2)$.

As told before, the aim of the present paper is to obtain a complete classification of the hyperplanes of the glued near hexagon $Q(5,2) \otimes Q(5,2)$. The following is our main result.

Main Theorem. *The glued near hexagon $Q(5,2) \otimes Q(5,2)$ has up to isomorphism* 16 *hyperplanes.*

The hyperplanes of the glued near hexagon $Q(5,2) \otimes Q(5,2)$ will be classified in Section 3 with the aid of the computer algebra system GAP [15]. Some of the basic properties of the 16 nonisomorphic hyperplanes of $Q(5,2) \otimes Q(5,2)$ can be found in Table 1 of Section 3.

Another goal that we have is to give at least one explicit representative for each of the 16 isomorphism classes of hyperplanes. We achieve this goal in the following way.

In Section 4 we describe four classes of hyperplanes which we call the *basic hyperplanes* of $Q(5,2) \otimes Q(5,2)$. These hyperplanes include the singular hyperplanes, the extensions of the $W(2)$-subquadrangles of the $Q(5,2)$-quads, the so-called hyperplanes of valuation type and certain hyperplanes with 171 points.

It is well-known that if H_1 and H_2 are two distinct hyperplanes of a slim partial linear space, then the complement $H_1 * H_2$ of the symmetric difference $H_1 \Delta H_2$ of H_1 and H_2 is again a hyperplane. Using this "$*$-operator" (which is commutative and associative), we will be able in Section 5 to describe the twelve remaining hyperplanes of $Q(5,2) \otimes Q(5,2)$ in terms of the basic hyperplanes. For some of the hyperplanes of $Q(5,2) \otimes Q(5,2)$, we will be able to give more than one construction.

Before we can start the actual classification of the hyperplanes, we need to discuss some of the basic structural properties of the glued near hexagon $Q(5,2) \otimes Q(5,2)$. This will be done in Section 2. An understanding of the structure of the near hexagon $Q(5,2) \otimes Q(5,2)$ is indispensable to understand some of the constructions for the hyperplanes.

2 The glued near hexagon $Q(5,2)\otimes Q(5,2)$ and its properties

Let $H(5,4)$ be a nonsingular Hermitian variety of $\mathrm{PG}(5,4)$. Associated with $H(5,4)$ there is a dual polar space $DH(5,4)$. This is the point-line geometry whose points are the planes of $\mathrm{PG}(5,4)$ contained in $H(5,4)$ and whose lines are the lines of $\mathrm{PG}(5,4)$ contained in $H(5,4)$, with incidence being reverse containment.

Suppose α is a plane of $\mathrm{PG}(5,4)$ intersecting $H(5,4)$ in a unital of α. Then the planes of $H(5,4)$ meeting α form a subspace $\mathcal{P}_\alpha$ of $DH(5,4)$ and the point-line geometry $\mathcal{S}_\alpha$ induced on $\mathcal{P}_\alpha$ is a slim dense near hexagon by Brouwer et al. [1]. This slim dense near hexagon, which we denote by $Q(5,2)\otimes Q(5,2)$, belongs to the family of the glued near hexagons introduced in De Bruyn [3]. If ζ is the Hermitian polarity of $\mathrm{PG}(5,4)$ associated with $H(5,4)$, then also the plane α^ζ intersects $H(5,4)$ in a unital of α^ζ and we have that $\mathcal{P}_\alpha = \mathcal{P}_{\alpha^\zeta}$. Indeed, every plane of $H(5,4)$ that meets α must also meet α^ζ.

Let $G \cong P\Gamma U(6,2)$ denote the automorphism group of $DH(5,4)$. The setwise stabilizer of $\mathcal{P}_\alpha$ in G induces a group of automorphisms of $\mathcal{S}_\alpha \cong Q(5,2)\otimes Q(5,2)$ which is in fact the full group of automorphisms of $\mathcal{S}_\alpha$. The setwise stabilizer of α in G also induces a group of automorphisms of $\mathcal{S}_\alpha$, but this group has index 2 in the full group of automorphisms of $\mathcal{S}_\alpha$. Indeed, there are elements of G that interchange the planes α and α^ζ. The automorphism groups of general glued near hexagons were studied in De Bruyn [4]. An explicit description of the automorphisms of $Q(5,2)\otimes Q(5,2)$ can easily be extracted from the discussion in [4].

In the rest of this section, we describe some basic properties of the glued near hexagon $Q(5,2)\otimes Q(5,2) = (\mathcal{P},\mathcal{L},\mathrm{I})$ that will be useful later.

The glued near hexagon $Q(5,2)\otimes Q(5,2)$ has 243 points and every point is incident with precisely nine lines. If $i \in \mathbb{N}$ and $x \in \mathcal{P}$, then $\Gamma_i(x)$ denotes the set of points at distance i from x. We denote $\{x\}\cup\Gamma_1(x)$ also by $x^\perp$. If $i\in\mathbb{N}$ and $\emptyset \neq X \subseteq \mathcal{P}$, then $\Gamma_i(X)$ denotes the set of points at distance i from X, i.e. the set of all points y for which $\min\{\mathrm{d}(y,x)\,|\,x\in X\} = i$.

As it is the case for every dense near polygon (see Shult and Yanushka [18, Proposition 2.5]) every two points x and y at distance 2 are contained in a unique convex subspace $\langle x,y\rangle$ of diameter 2, called a *quad*. This quad is isomorphic to either the (3×3)-grid or the generalized quadrangle $Q(5,2)$.

If Q is a $Q(5,2)$-quad, then every point of $Q(5,2)\otimes Q(5,2)$ has distance at most 1 from Q. Moreover, for every point x, there exists a (necessarily unique) point $\pi_Q(x)\in Q$ such that $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_Q(x)) + \mathrm{d}(\pi_Q(x),y)$ for every $y\in Q$. The point $\pi_Q(x)$ is called the *projection of x onto Q*. If $x\in Q$, then we define $\mathcal{R}_Q(x) := x$. If $x\notin Q$, then $\mathrm{d}(x,\pi_Q(x)) = 1$ and we denote by $\mathcal{R}_Q(x)$ the point on the line $x\pi_Q(x)$ distinct from x and $\pi_Q(x)$. The map $\mathcal{R}_Q : \mathcal{P}\to\mathcal{P}$ defines an automorphism of $Q(5,2)\otimes Q(5,2)$.

Suppose G is a grid-quad. Then every point $x\in\Gamma_1(G)$ is collinear with a unique point of G which we denote by $\pi_G(x)$. If $x\in\Gamma_2(G)$, then $\Gamma_2(x)\cap G$ is an *ovoid* of G, i.e. a set of three points meeting each line of G in a unique point.

Every point x of $Q(5,2)\otimes Q(5,2)$ is contained in precisely two $Q(5,2)$-quads. These two

$Q(5,2)$-quads meet in a line L_x. The lines L_x, $x \in \mathcal{P}$, form a spread S^* of $Q(5,2) \otimes Q(5,2)$. The group of automorphisms of $Q(5,2) \otimes Q(5,2)$ stabilizing each line of S^* is a cyclic group of order 3 and acts regularly on each line of S^*. The glued near hexagon $Q(5,2) \otimes Q(5,2)$ has two partitions T_1 and T_2 in $Q(5,2)$-quads. If $Q_1 \in T_1$ and $Q_2 \in T_2$, then $Q_1 \cap Q_2$ is a line belonging to S^*. Every line of S^* is contained in precisely two $Q(5,2)$-quads and no grid-quads. Every line of $\mathcal{S}$ not belonging to S^* is contained in precisely four grid-quads and a unique $Q(5,2)$-quad. For every $Q(5,2)$-quad Q, the set S_Q of lines of S^* that are contained in Q is a *regular spread* of Q. This means that if L_1 and L_2 are two distinct lines of S_Q, then the unique line L_3 for which $L_1 \cup L_2 \cup L_3$ defines a (3×3)-subgrid of Q also belongs to S_Q.

3 The hyperplanes of $Q(5,2) \otimes Q(5,2)$

In the computer algebra system GAP [15], there are build many models of permutation groups, including a model of the permutation representation of $P\Gamma U(6,2)$ on the set $\{1,2,\ldots,891\}$ that is equivalent with the permutation representation of $Aut(DH(5,4))$ on the point set of $DH(5,4)$. One can easily identify those subsets of size 3 of $\{1,2,\ldots,891\}$ that correspond to the lines of $DH(5,4)$. In this way, we find a computer model of the dual polar space $DH(5,4)$.

Subsequently, we determined a subset X of $\{1,2,\ldots,891\}$ that corresponds with a subspace of $DH(5,4)$ on which the induced subgeometry is isomorphic to $Q(5,2) \otimes Q(5,2)$. There are a number of ways in which this goal can be achieved. One way goes as follows. Take three quads Q_1, Q_2 and Q_3 of $DH(5,4)$ for which the corresponding points x_1, x_2 and x_3 of $H(5,4)$ generate a plane α intersecting $H(5,4)$ in a unital of α. Then the smallest subspace of $DH(5,4)$ containing $Q_1 \cup Q_2 \cup Q_3$ consists of all planes of $H(5,4)$ meeting α and hence the geometry induced on that subspace is isomorphic to $Q(5,2) \otimes Q(5,2)$.

Once we found the set X, we have in fact also a computer model for the glued near hexagon $Q(5,2) \otimes Q(5,2)$. We can also easily implement a permutation representation of the automorphism group of $Q(5,2) \otimes Q(5,2)$ on the point set $\mathcal{P}$ of $Q(5,2) \otimes Q(5,2)$. Indeed, we just have to ask GAP to calculate the setwise stabilizer of X in $P\Gamma U(6,2)$.

Now, let V be a vector space over $\mathbb{F}_2$ having a basis B indexed by the points of $Q(5,2) \otimes Q(5,2)$, say $B = \{\bar{v}_x \,|\, x \in \mathcal{P}\}$. Let W denote the subspace of V generated by all vectors of the form $\bar{v}_{x_1} + \bar{v}_{x_2} + \bar{v}_{x_3}$ where $\{x_1, x_2, x_3\}$ is some line of $Q(5,2) \otimes Q(5,2)$ and consider the quotient vector space V/W. Then the map $x \in \mathcal{P} \mapsto \langle \bar{v}_x + W \rangle$ defines a full projective embedding $\widetilde{e}$ of $Q(5,2) \otimes Q(5,2)$ into $\mathrm{PG}(V/W)$ which is isomorphic to the universal embedding of $Q(5,2) \otimes Q(5,2)$. If Π is a hyperplane of $\mathrm{PG}(V/W)$, then the set $H_\Pi := \widetilde{e}^{-1}(\widetilde{e}(\mathcal{P}) \cap \Pi)$ is a hyperplane of $Q(5,2) \otimes Q(5,2)$. By Ronan [17], we know that the correspondence $\Pi \leftrightarrow H_\Pi$ defines a bijection between the set of hyperplanes of $\mathrm{PG}(V/W)$ and the set of hyperplanes of $Q(5,2) \otimes Q(5,2)$.

We have implemented the universal embedding in GAP and found that $\dim(V/W) = |\mathcal{P}| - \dim(W) = 18$. (There are also computer free methods for calculating this dimension.) So, $Q(5,2) \otimes Q(5,2)$ must have $2^{18} - 1$ hyperplanes. Subsequently, we have used the following method for enumerating all hyperplanes of $Q(5,2) \otimes Q(5,2)$.

Type	N	v	$[De,Si,Su]$	$[de,si,ov]$	Line distribution	Stabilizer	O_1	O_2
1	243	115	[2,16,0]	[16,224,192]	$2^{64}5^{32}9^{19}$	$((SL(2,3) \times SL(2,3)) : C_2) : C_2$	5	1
2	432	171	[6,0,12]	[150,270,12]	$6^{36}7^{108}9^{27}$	$(((C_3 \times C_3) : C_2) \times S_3 \times S_3) : C_2$	3	1
3	648	99	[0,18,0]	[12,108,312]	$1^{18}2^{72}9^{9}$	$C_2 \times ((((C_3 \times C_3) : Q_8) : C_3) : C_2)$	3	1
4	648	147	[4,6,8]	[72,312,48]	$4^{48}5^{12}7^{72}9^{15}$	$((C_3 \times SL(2,3)) : C_2) \times S_3$	5	1
5	2592	123	[0,6,12]	[34,234,164]	$1^{6}2^{6}4^{36}5^{54}6^{18}9^{3}$	$S_3 \times S_3 \times S_3$	6	3
6	2916	147	[2,0,16]	[88,264,80]	$5^{48}6^{48}7^{48}9^{3}$	$(C_2 \times GL(2,3)) : C_2$	4	1
7	5832	131	[0,2,16]	[52,244,136]	$1^{2}4^{16}5^{72}6^{40}9^{1}$	$GL(2,3) : C_2$	7	4
8	7776	107	[0,14,4]	[6,190,236]	$1^{2}2^{48}3^{24}4^{16}6^{4}7^{12}9^{1}$	$(C_6 \times S_3) : C_2$	9	7
9	11664	139	[2,4,12]	[62,278,92]	$2^{4}4^{24}5^{44}6^{24}7^{36}9^{7}$	$D_8 \times S_3$	11	4
10	15552	115	[0,10,8]	[12,236,184]	$1^{2}2^{23}3^{12}4^{34}5^{31}6^{7}7^{6}$	$S_3 \times S_3$	14	12
11	15552	123	[1,9,8]	[26,258,148]	$2^{18}3^{12}4^{36}5^{24}6^{6}7^{24}9^{3}$	$C_2 \times ((C_3 \times C_3) : C_2)$	13	6
12	23328	131	[1,5,12]	[44,268,120]	$2^{4}3^{8}4^{32}5^{38}6^{20}7^{28}9^{1}$	D_{24}	16	8
13	34992	115	[0,10,8]	[16,224,192]	$1^{2}2^{24}3^{16}4^{32}5^{24}6^{8}7^{8}9^{1}$	D_{16}	14	11
14	46656	115	[0,10,8]	[12,236,184]	$1^{2}2^{23}3^{12}4^{34}5^{31}6^{7}7^{6}$	D_{12}	18	17
15	46656	123	[0,6,12]	[30,246,156]	$2^{9}3^{12}4^{30}5^{45}6^{21}7^{6}$	D_{12}	16	14
16	46656	123	[0,6,12]	[30,246,156]	$2^{9}3^{12}4^{30}5^{45}6^{21}7^{6}$	D_{12}	17	13

Table 1: The hyperplanes of $Q(5,2) \otimes Q(5,2)$

T = type of hyperplane H; N = Number of hyperplanes of Type T; $v := |H|$; De [Si, resp. Su] denotes the total number of $Q(5,2)$-quads that are deep [singular, resp. subquadrangular] with respect to H; de [si, resp. ov] denotes the total number of grid-quads that are deep [singular, resp. ovoidal] with respect to H ; if $a_1^{e_1} a_2^{e_2} \cdots a_k^{e_k}$ is the line distribution, then there are exactly e_i points in H that are incident with precisely a_i lines contained in H ($i \in \{1, 2, \ldots, k\}$); O_1 [resp. O_2] denotes the total number of orbits of the stabilizer of H on H [resp. the complement of H].

Step 1. Put $N := 0$.

Step 2. Let GAP choose a random hyperplane Π of V through W and let GAP calculate the corresponding hyperplane H of $Q(5,2) \otimes Q(5,2)$. Calculate the stabilizer S_H of H (in the full automorphism group G of $Q(5,2) \otimes Q(5,2)$). Then the total number of hyperplanes isomorphic to H is equal to $N_H := \frac{|G|}{|S_H|}$. Verify whether H is isomorphic to one of the previous obtained hyperplanes. If this is not the case, then put $N := N + N_H$.

Step 3. If $N < 2^{18} - 1$, then go again to Step 2. If $N = 2^{18} - 1$, then we have found all hyperplanes and we are done.

In this way, we found that $Q(5,2) \otimes Q(5,2)$ has up to isomorphism 16 hyperplanes. We have also written various procedures in GAP to find various properties of these hyperplanes. These properties can be found in Table 1 where we have ordered the hyperplanes according to the sizes of their orbits and the number of points they contain. Perhaps the contents of columns [De,Si,Su] and [de,si,ov] still need more explanation.

Suppose Q is a quad and H is a hyperplane of $Q(5,2) \otimes Q(5,2)$. The quad Q is called *deep* with respect to H if $Q \subseteq H$. It is called *singular* with respect to H if $Q \cap H = x^\perp \cap Q$ for some point $x \in Q$. A $Q(5,2)$-quad Q is *subquadrangular* with respect to H is $Q \cap H$ is a $W(2)$-subquadrangle of Q. A grid-quad Q is called *ovoidal* with respect to H if $Q \cap H$ an ovoid of Q. Every $Q(5,2)$-quad is deep, singular or subquadrangular with respect to H. Every grid-quad is deep, singular or ovoidal with respect to H.

4 The basic hyperplanes

4.1 The singular hyperplanes

Since $Q(5,2) \otimes Q(5,2)$ is a dense near hexagon, the set H_x of points of $Q(5,2) \otimes Q(5,2)$ at distance at most 2 from a given point x is a hyperplane of $Q(5,2) \otimes Q(5,2)$. This hyperplane is called the *singular hyperplane of* $Q(5,2) \otimes Q(5,2)$ *with deepest point* x. The singular hyperplanes can easily be found in Table 1. (We must have $v = 115$ and $De = 2$.)

Description 1a. The hyperplanes of Type 1 occurring in Table 1 are precisely the singular hyperplanes.

4.2 The extensions of the $W(2)$-subquadrangles of the $Q(5,2)$-quads

Let Q be a $Q(5,2)$-quad and σ a $W(2)$-subquadrangle of Q. Then the set of points of $Q(5,2) \otimes Q(5,2)$ at distance at most 1 from σ is a hyperplane H_σ of $Q(5,2) \otimes Q(5,2)$, the so-called *extension of* σ. There are precisely three lines in S^* that are contained in σ. These three lines form a (3×3)-grid G_σ. The extensions of the $W(2)$-subquadrangles can easily be found in Table 1. (We must have $v = 147$ and $De = 4$.)

Description 4a. The hyperplanes of Type 4 occurring in Table 1 are precisely the extensions of the $W(2)$-subquadrangles of the $Q(5,2)$-quads.

4.3 The hyperplanes of valuation type

A *valuation* of $Q(5,2) \otimes Q(5,2)$ is a map f from the point set $\mathcal{P}$ of $Q(5,2) \otimes Q(5,2)$ to the set of nonnegative integers satisfying the following properties.

(V1) There exists a point x with value $f(x) = 0$.

(V2) Every line L of $Q(5,2) \otimes Q(5,2)$ is incident with a unique point x_L such that $f(x) = f(x_L) + 1$ for every point x of L distinct from x_L.

(V3) Every point x of $\mathcal{S}$ is contained in a (necessarily unique) convex subspace F_x for which the following holds:

- $f(y) \leq f(x)$ for every point $y \in F_x$;
- if $y \in F_x$ and $z \sim y$ such that $f(z) = f(y) - 1$, then $z \in F_x$.

Valuations of general dense near polygons was introduced by De Bruyn & Vandecasteele [12].

If f is a valuation of $Q(5,2) \otimes Q(5,2)$, then O_f denotes the set of points with value 0. For every point x of $Q(5,2) \otimes Q(5,2)$, the map $f : \mathcal{P} \to \mathbb{N}; y \mapsto \mathrm{d}(x,y)$ is a valuation of $Q(5,2) \otimes Q(5,2)$. Any valuation of $Q(5,2) \otimes Q(5,2)$ that can be obtained in this way is called *classical*.

By De Bruyn & Vandecasteele [13, Section 7.2], the near hexagon $Q(5,2) \otimes Q(5,2)$ has two types of valuations. There are 243 classical valuations and 648 non-classical valuations. Supposing that $Q(5,2) \otimes Q(5,2)$ is embedded into the dual polar space $DH(5,4)$ as described in Section 2, every non-classical valuation is obtained as follows. If x is one of the 648 points of $DH(5,4)$ not belonging to $\mathcal{P}$, then the map $\mathcal{P} \to \mathbb{N}; y \mapsto \mathrm{d}(x,y) - 1$ is a non-classical valuation of $Q(5,2) \otimes Q(5,2)$.

If f is a valuation of $Q(5,2) \otimes Q(5,2)$, then by Property (V2), the set of points of $Q(5,2) \otimes Q(5,2)$ with non-maximal value is a hyperplane H_f of $Q(5,2) \otimes Q(5,2)$. A hyperplane of $Q(5,2) \otimes Q(5,2)$ is said to be of *valuation type* if it is associated with a non-classical valuation of $Q(5,2) \otimes Q(5,2)$.

Suppose f is a non-classical valuation of $Q(5,2) \otimes Q(5,2)$. Then $|O_f| = 9$ and the maximal value of f is 2. Every $Q(5,2)$-quad contains a unique point of O_f and is singular with respect to H_f. If $f(x) = 1$, then x is collinear with either one or two points of O_f. If x is collinear with a unique point $y \in O_f$, then $F_x = xy \in S^*$. If x is collinear with two distinct points $y_1, y_2 \in O_f$, then none of the lines xy_1, xy_2 belongs to S^* and $F_x = \langle xy_1, xy_2 \rangle$ is a grid-quad. A standard counting now yields that $|H_f| = |O_f| + |O_f| \cdot 1 \cdot 2 + \frac{|O_f| \cdot 8 \cdot 2}{2} = 99$. Consulting Table 1, we then have:

Description 3a. The hyperplanes of Type 3 occurring in Table 1 are precisely the hyperplanes of valuation type.

4.4 A certain hyperplane with 171 points

Let $\widetilde{e}$ denote the universal embedding of $Q(5,2) \otimes Q(5,2)$ into $\widetilde{\Sigma} = \mathrm{PG}(17,2)$. Suppose $\{Q_1, Q_2, Q_3, R_1, R_2, R_3\}$ is a set of quads of $Q(5,2) \otimes Q(5,2)$ such that $Q_1, Q_2, Q_3 \in T_1$, $R_1, R_2, R_3 \in T_2$, $Q_3 = \mathcal{R}_{Q_2}(Q_1)$ and $R_3 = \mathcal{R}_{R_2}(R_1)$. Now, $A := Q_1 \cup Q_2 \cup Q_3 \cup R_1 \cup R_2 \cup R_3$ is a subspace of $Q(5,2) \otimes Q(5,2)$.

We prove that $\langle \widetilde{e}(A) \rangle$ is 15-dimensional. Let Q_4 be a quad of T_1 distinct from Q_1, Q_2 and Q_3. The generating rank of $Q(5,2)$ is equal to 6 and we can take a generating set X_i, $i \in \{1,2,4\}$, of Q_i containing two points of $Q_i \cap R_1$, two points of $Q_i \cap R_2$ and two points outside $R_1 \cup R_2 \cup R_3$. Then $X_1 \cup X_2 \cup X_4$ is a generating set of size 18 of $Q(5,2) \otimes Q(5,2)$ implying that $\widetilde{e}(X_1 \cup X_2 \cup X_4)$ is a set of 18 linearly independent points of $\widetilde{\Sigma}$. If u and v are the two points of X_4 not contained in $R_1 \cup R_2 \cup R_3$, then the subspace of $Q(5,2) \otimes Q(5,2)$ generated by the 16 points of $(X_1 \cup X_2 \cup X_4) \setminus \{u, v\}$ is precisely A. So, we must indeed have that $\langle \widetilde{e}(A) \rangle$ is 15-dimensional.

Since $\langle \widetilde{e}(A) \rangle$ is 15-dimensional, there are precisely three hyperplanes H_1, H_2 and H_3 containing A. Now, let $i \in \{1,2,3\}$. Obviously, the only $Q(5,2)$-quads that can be contained in H_i are the quads Q_1, Q_2, Q_3, R_1, R_2 and R_3 themselves. (Four distinct quads of the same T_j, $j \in \{1,2\}$, generate the whole of $Q(5,2) \otimes Q(5,2)$.) So, every $Q(5,2)$-quad distinct from Q_1, Q_2, Q_3, R_1, R_2 and R_3 intersects H_i in a $W(2)$-subquadrangle. This implies that H_i contains precisely $3 \cdot 27 + 6 \cdot 15 = 171$ points. If L is a line of S^* not contained in any of the quads Q_1, Q_2, Q_3, R_1, R_2 and R_3, then $H_i \cap L$ is a singleton. So, the group of automorphisms of $Q(5,2) \otimes Q(5,2)$ stabilizing each line of S^* (which is isomorphic to C_3) acts regularly on the set $\{H_1, H_2, H_3\}$, implying that the hyperplanes H_1, H_2 and H_3 are mutually isomorphic. The set $(Q_1 \cup Q_2 \cup Q_3) \cap (R_1 \cup R_2 \cup R_3)$ determines a $(3 \times 3 \times 3)$-subcube of $Q(5,2) \otimes Q(5,2)$. This $(3 \times 3 \times 3)$-cube is called the *base cube* of the hyperplane H_i. Since $|H_i| = 171$, Table 1 immediately yields:

Description 2a. The hyperplanes of Type 2 occurring in Table 1 are precisely the hyperplanes with 171 points defined above.

5 Constructions of other hyperplanes

5.1 Introduction

In Section 4, we were already able to give an explicit description for a representative for four of the sixteen isomorphism classes of hyperplanes of $Q(5,2) \otimes Q(5,2)$. The aim of the present section is to achieve the same goal for the remaining twelve isomorphism classes of hyperplanes.

The procedure we have chosen to achieve this goal is as follows. We consider an arbitrary hyperplane H whose stabilizer S_H has at most ten orbits on the set $\mathcal{P}$ of points of $Q(5,2) \otimes Q(5,2)$. By Table 1, we then know that H has Type i, where $i \leq 6$. We already know what H is if $i \in \{1,2,3,4\}$. Later we also give explicit constructions for the hyperplanes of Type 5 and 6. For each of the six possibilities for H, we explicitly

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	18	–	–	32	64	–	–	–	–	128	–	–	–	–	–	–
2	–	–	–	–	–	–	–	72	27	36	108	–	–	–	–	–
3	–	–	9	–	–	–	18	–	72	–	–	–	–	144	–	–
4	12	–	–	15	–	–	–	48	–	96	–	72	–	–	–	–
5	6	–	–	–	3	–	36	6	–	84	–	36	–	18	54	–
6	–	–	–	–	–	3	–	–	–	–	–	48	96	48	48	–
7	–	–	2	–	16	–	17	–	–	24	16	–	–	48	48	72
8	–	4	–	4	2	–	–	17	–	36	24	48	36	–	36	36
9	–	1	4	–	–	–	–	–	30	–	20	–	48	80	24	36
10	2	1	–	4	14	–	9	18	–	42	18	36	–	18	36	45
11	–	3	–	–	–	–	6	12	15	18	27	24	36	48	36	18
12	–	–	–	2	4	6	–	16	–	24	16	31	24	36	36	48
13	–	–	–	–	–	8	–	8	16	–	16	16	35	56	40	48
14	–	–	2	–	1	3	6	–	20	6	16	18	42	54	39	36
15	–	–	–	–	3	3	6	6	6	12	12	18	30	39	54	54
16	–	–	–	–	–	–	9	6	9	15	6	24	36	36	54	48

Table 2: The "action" of the singular hyperplanes on the hyperplanes of a given type

describe the orbits for the action of S_H on $\mathcal{P}$, and for each such orbit O, we determine which kind of hyperplane $H * H_x$ is where x is an arbitrary point of O. This goal will be achieved with the aid of GAP, although sometimes purely theoretical arguments can also yield the desired conclusions. In this way, we are able to give explicit descriptions for representatives of eleven of the twelve remaining isomorphism classes. Only a description for the hyperplanes of Type 16 is still missing. Such a description will arise from the study of those hyperplanes that have the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1, x_2 and x_3 are three mutually opposite points of $Q(5,2) \otimes Q(5,2)$.

The procedure sketched in the previous paragraph will often lead to several equivalent descriptions of the same hyperplane. It should be mentioned that the above procedure could also be applied to hyperplanes H whose stabilizer S_H has more than ten orbits on $\mathcal{P}$. Our general feeling is however that the description of the orbits will then be more complicated as well as the descriptions for the corresponding hyperplanes. The procedure sketched in the previous paragraph is already sufficient to achieve all our goals.

One extremely helpful tool in our investigations is the information provided by Table 2. With the aid of GAP, we determined for each hyperplane H of Type i the number of points x for which $H * H_x$ is a hyperplane of Type j. This number can be found as entry (i,j) in Table 2. In almost all cases, the information given by Table 2 provides a clue on how to construct a particular hyperplane.

5.2 Hyperplanes of the form $H_{x_1} * H_{x_2}$ where x_1 and x_2 are two distinct points

Let x_1 and x_2 be two distinct points of $Q(5,2) \otimes Q(5,2)$. By Table 1, the stabilizer of H_{x_1} has six orbits on the points of $Q(5,2) \otimes Q(5,2)$. This leads to the following five possibilities for x_2:

- $\mathrm{d}(x_1, x_2) = 1$ and $x_1 x_2 \in S^*$;
- $\mathrm{d}(x_1, x_2) = 1$ and $x_1 x_2 \notin S^*$;
- $\mathrm{d}(x_1, x_2) = 2$ and $\langle x_1, x_2 \rangle$ is a grid-quad;
- $\mathrm{d}(x_1, x_2) = 2$ and $\langle x_1, x_2 \rangle$ is a $Q(5,2)$-quad;
- $\mathrm{d}(x_1, x_2) = 3$.

If $\mathrm{d}(x_1, x_2) = 1$ and x_3 is the third point on the line $x_1 x_2$, then $H_{x_1} * H_{x_2} = H_{x_3}$. If $\mathrm{d}(x_1, x_2) = 2$ and $Q := \langle x_1, x_2 \rangle$ is a $Q(5,2)$-quad, then the complement (in Q) of the symmetric difference of $x_1^{\perp} \cap Q$ and $x_2^{\perp} \cap Q$ is a $W(2)$-subquadrangle σ and we have that $H_{x_1} * H_{x_2} = H_{\sigma}$. So, we have the following alternative construction for one of the basic hyperplanes.

Description 4b. The hyperplanes of Type 4 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2}$ where x_1 and x_2 are two points at distance 2 from each other such that $\langle x_1, x_2 \rangle$ is a $Q(5,2)$-quad.

The remaining two possibilities for the point x_2 give rise to two new hyperplanes.

Description 5a. The hyperplanes of Type 5 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2}$ where x_1 and x_2 are two points of $Q(5,2) \otimes Q(5,2)$ at distance 2 from each other such that $\langle x_1, x_2 \rangle$ is a grid-quad.

Description 10a. The hyperplanes of Type 10 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2}$ where x_1 and x_2 are two opposite points of $Q(5,2) \otimes Q(5,2)$.

5.3 Hyperplanes of the form $H_f * H_x$ where f is a non-classical valuation and x is a point

Let f be a non-classical valuation and x be a point of $Q(5,2) \otimes Q(5,2)$. By Table 1, the stabilizer of H_f has four orbits on the points of $Q(5,2) \otimes Q(5,2)$. This leads to the following four possibilities for x:

- $f(x) = 0$;
- $f(x) = 1$ and $F_x \in S^*$;
- $f(x) = 1$ and F_x is a grid-quad;
- $x \notin H_f$.

Suppose f is induced by a point y of $DH(5,4)$ into which $Q(5,2) \otimes Q(5,2)$ is isometrically embedded. If $f(x) = 0$ then $y \sim x$ and $H_f * H_x$ is the hyperplane of valuation type induced by the point z of the line yx distinct from x and y. The three other possibilities for x give rise to new hyperplanes.

Description 7a. The hyperplanes of Type 7 occurring in Table 1 are precisely the hyperplanes of the form $H_f * H_x$ where f is a non-classical valuation of $Q(5,2) \otimes Q(5,2)$ and x is a point of $Q(5,2) \otimes Q(5,2)$ for which $f(x) = 1$ and $F_x \in S^*$.

Description 9a. The hyperplanes of Type 9 occurring in Table 1 are precisely the hyperplanes of the form $H_f * H_x$ where f is a non-classical valuation of $Q(5,2) \otimes Q(5,2)$ and x is a point of $Q(5,2) \otimes Q(5,2)$ for which $f(x) = 1$ and F_x is a grid-quad.

Description 14a. The hyperplanes of Type 14 occurring in Table 1 are precisely the hyperplanes of the form $H_f * H_x$ where f is a non-classical valuation of $Q(5,2) \otimes Q(5,2)$ and x is a point of $Q(5,2) \otimes Q(5,2)$ not belonging to H_f.

5.4 Hyperplanes of the form $H_\sigma * H_x$ where x is a point and σ a $W(2)$-subquadrangle

Let σ be a $W(2)$-subquadrangle of a $Q(5,2)$-quad Q of $Q(5,2) \otimes Q(5,2)$ and let x be a point of $Q(5,2) \otimes Q(5,2)$. Recall that the three lines of S^* contained in σ define a (3×3)-grid G_σ. By Table 1, the stabilizer of H_σ has six orbits on the set of points of $Q(5,2) \otimes Q(5,2)$. This leads to the following six possibilities for x:

- $x \in G_\sigma$;
- $x \in \sigma \setminus G_\sigma$;
- $x \in Q \setminus \sigma$;
- $x \notin Q$ and $\pi_Q(x) \in G_\sigma$;
- $x \notin Q$ and $\pi_Q(x) \in \sigma \setminus G_\sigma$;
- $x \notin Q$ and $\pi_Q(x) \in Q \setminus \sigma$.

If $x \in \sigma$, then the complement (in Q) of the symmetric difference of $x^\perp \cap Q$ and σ is a $W(2)$-subquadrangle σ' and we have that $H_x * H_\sigma = H_{\sigma'}$. If $x \in Q \setminus \sigma$, then the complement (in Q) of the symmetric difference of $x^\perp \cap Q$ and σ is equal to $y^\perp \cap Q$ for some point $y \in (\Gamma_2(x) \cap Q) \setminus \sigma$, and we have that $H_x * H_\sigma = H_y$. For one other possibility, we already encountered the corresponding hyperplane.

Description 10b. The hyperplanes of Type 10 occurring in Table 1 are precisely the hyperplanes of the form $H_\sigma * H_x$ where σ is a $W(2)$-subquadrangle of a $Q(5,2)$-quad Q and x is a point not belonging to Q for which $\pi_Q(x) \notin \sigma$.

The two other possibilities give rise to new hyperplanes.

Description 8a. The hyperplanes of Type 8 occurring in Table 1 are precisely the hyperplanes of the form $H_\sigma * H_x$ where σ is a $W(2)$-subquadrangle of a $Q(5,2)$-quad Q and x is a point not belonging to Q for which $\pi_Q(x) \in \sigma \setminus G_\sigma$.

Description 12a. The hyperplanes of Type 12 occurring in Table 1 are precisely the hyperplanes of the form $H_\sigma * H_x$ where σ is a $W(2)$-subquadrangle of a $Q(5,2)$-quad Q and x is a point not belonging to Q for which $\pi_Q(x) \in G_\sigma$.

5.5 Hyperplanes of the form $H * H_x$ where x is a point and H a basic hyperplane with 171 points

Let H be a basic hyperplane with 171 points and x a point of $Q(5,2) \otimes Q(5,2)$. By Table 1, the stabilizer of H has four orbits on the set of points of $Q(5,2) \otimes Q(5,2)$. This leads to the following four possibilities for the point x:

- x belongs to the base cube of H;
- x belongs to precisely one deep quad of H (and hence not to the base cube of H);
- $x \in H$ and x does not belong to any of the six deep quads of H;
- $x \notin H$.

Three of the four possibilities for x give rise to hyperplanes we already considered. The other possibility gives rise to a new hyperplane.

Description 8b. The hyperplanes of Type 8 occurring in Table 1 are precisely the hyperplanes of the form $H * H_x$ where H is a basic hyperplane with 171 points and x is a point not belonging to H.

Description 9b. The hyperplanes of Type 9 occurring in Table 1 are precisely the hyperplanes of the form $H * H_x$ where H is a basic hyperplane with 171 points and x is a point of H belonging to its base cube.

Description 10c. The hyperplanes of Type 10 occurring in Table 1 are precisely the hyperplanes of the form $H * H_x$ where H is a basic hyperplane with 171 points and x is a point of H not belonging to any of the 6 deep quads.

Description 11a. The hyperplanes of Type 11 occurring in Table 1 are precisely the hyperplanes of the form $H * H_x$ where H is a basic hyperplane with 171 points and x is a point of H belonging to precisely one deep quad of H.

5.6 Hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1, x_2 and x_3 are three points for which $\langle x_1, x_2 \rangle$ is a grid-quad

Suppose x_1, x_2 and x_3 are three points of $Q(5,2) \otimes Q(5,2)$ such that $\mathrm{d}(x_1, x_2) = 2$ and $G := \langle x_1, x_2 \rangle$ is a grid-quad. Let a_1 and a_2 denote the two common neighbors of x_1 and x_2, let b_1, b_2, b_3 and b_4 denote the four points of $\langle x_1, x_2 \rangle$ at distance 1 from one of x_1, x_2 and distance 2 from the other, and let c denote the unique point of $\langle x_1, x_2 \rangle$ at distance 2 from x_1 and x_2. Without loss of generality, we may suppose that $\{a_1, b_1, x_2\}$, $\{b_2, c, b_4\}$, $\{x_1, b_3, a_2\}$, $\{a_1, b_2, x_1\}$, $\{b_1, c, b_3\}$ and $\{x_2, b_4, a_2\}$ are the six lines that are contained in G.

We know that $H_{x_1} * H_{x_2}$ is a hyperplane of Type 5. The points x_1 and x_2 are not uniquely determined by the hyperplane $H_{x_1} * H_{x_2}$. Indeed, we have $H_{x_1} * H_{x_2} = (H_{x_1} * H_{a_2}) * (H_{a_2} * H_{x_2}) = H_{b_3} * H_{b_4}$. Similarly, $H_{x_1} * H_{x_2} = (H_{x_1} * H_{a_1}) * (H_{a_1} * H_{x_2}) = H_{b_1} * H_{b_2}$. There are precisely 2592 hyperplanes of Type 5 while there are 7776 unordered pairs $\{y_1, y_2\}$ of points for which $\langle y_1, y_2 \rangle$ is a grid-quad. So, if $H_{y_1} * H_{y_2} = H_{x_1} * H_{x_2}$, then $\{y_1, y_2\}$ is equal to either $\{x_1, x_2\}$, $\{b_1, b_2\}$ and $\{b_3, b_4\}$.

By Table 1, the stabilizer of $H_{x_1} * H_{x_2}$ has nine orbits on the points of $Q(5,2) \otimes Q(5,2)$. This leads to the following nine possibilities for x_3:

- $x_3 \in \{x_1, x_2, b_1, b_2, b_3, b_4\}$;
- $x_3 \in \{a_1, a_2, c\}$;
- $x_3 \in \Gamma_1(G)$, $\pi_G(x_3) \in \{x_1, x_2, b_1, b_2, b_3, b_4\}$ and $x_3\pi_G(x_3) \in S^*$;
- $x_3 \in \Gamma_1(G)$, $\pi_G(x_3) \in \{x_1, x_2, b_1, b_2, b_3, b_4\}$ and $x_3\pi_G(x_3) \notin S^*$;
- $x_3 \in \Gamma_1(G)$, $\pi_G(x_3) \in \{a_1, a_2, c\}$ and $x_3\pi_G(x_3) \in S^*$;
- $x_3 \in \Gamma_1(G)$, $\pi_G(x_3) \in \{a_1, a_2, c\}$ and $x_3\pi_G(x_3) \notin S^*$;
- $x_3 \in \Gamma_2(G)$ and $\Gamma_2(x_3) \cap G = \{a_1, a_2, c\}$;
- $x_3 \in \Gamma_2(G)$ and $\Gamma_2(x_3) \cap G$ is equal to $\{x_1, b_1, b_4\}$ or $\{x_2, b_2, b_3\}$;
- $x_3 \in \Gamma_2(G)$ and $\Gamma_2(x_3) \cap G$ is equal to either $\{x_1, x_2, c\}$, $\{a_2, b_1, b_2\}$ or $\{a_1, b_3, b_4\}$.

If $x_3 \in \{x_1, x_2, b_1, b_2, b_3, b_4\}$, then $H_{x_1} * H_{x_2} * H_{x_3}$ must be a singular hyperplane. If $x_3 \in \{a_1, a_2, c\}$, then $H_{x_1} * H_{x_2} * H_{x_3}$ must be a hyperplane of Type 5, and if $x_3 \in \Gamma_1(G)$ and $\pi_G(x_3) \in \{x_1, x_2, b_1, b_2, b_3, b_4\}$, then $H_{x_1} * H_{x_2} * H_{x_3}$ must be a hyperplane of Type 10. The other five possibilities give rise to four alternative descriptions and one new description for certain hyperplanes of $Q(5,2) \otimes Q(5,2)$.

Description 8c. The hyperplanes of Type 8 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1, x_2 and x_3 are points such that $G := \langle x_1, x_2 \rangle$ is a grid-quad, $x_3 \in \Gamma_1(G)$, $\pi_G(x_3) \in (\Gamma_1(x_1) \cap \Gamma_1(x_2)) \cup (\Gamma_2(x_1) \cap \Gamma_2(x_2))$ and $x_3\pi_G(x_3) \in S^*$.

Description 12b. The hyperplanes of Type 12 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1, x_2 and x_3 are points such that $G := \langle x_1, x_2 \rangle$ is a grid-quad, $x_3 \in \Gamma_1(G)$, $\pi_G(x_3) \in (\Gamma_1(x_1) \cap \Gamma_1(x_2)) \cup (\Gamma_2(x_1) \cap \Gamma_2(x_2))$ and $x_3\pi_G(x_3) \notin S^*$.

Description 14b. The hyperplanes of Type 14 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1, x_2 and x_3 are points such that $G := \langle x_1, x_2 \rangle$ is a grid-quad, $x_3 \in \Gamma_2(G)$ and $\Gamma_2(x_3) \cap G = (\Gamma_1(x_1) \cap \Gamma_1(x_2)) \cup (\Gamma_2(x_1) \cap \Gamma_2(x_2) \cap G)$.

Description 7b. The hyperplanes of Type 7 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1, x_2 and x_3 are points such that $G := \langle x_1, x_2 \rangle$ is a grid-quad, $x_3 \in \Gamma_2(G)$ and $|\Gamma_2(x_3) \cap G \cap \{x_1, x_2\}| = 1$.

Description 15a. The hyperplanes of Type 15 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1, x_2 and x_3 are points such that $G := \langle x_1, x_2 \rangle$ is a grid-quad, $x_3 \in \Gamma_2(G)$ and $\Gamma_2(x_3) \cap G$ is the unique ovoid of G through $\{x_1, x_2\}$.

5.7 Hyperplanes of the form $H_{\sigma_1} * H_{\sigma_2}$ and $H_{\sigma_1} * H_{\sigma_2} * H_x$ where x is a point and σ_1, σ_2 are two distinct $W(2)$-subquadrangles intersecting in a line

Suppose σ_1 and σ_2 are two distinct $W(2)$-subquadrangles of $Q(5,2) \otimes Q(5,2)$ intersecting in a line L, and let x be an arbitrary point of $Q(5,2) \otimes Q(5,2)$. Let Q_i, $i \in \{1,2\}$, be the unique $Q(5,2)$-quad of $Q(5,2) \otimes Q(5,2)$ containing σ_i. Then $Q_1 \cap Q_2 = L \in S^*$. One can easily prove that the hyperplane $H_{\sigma_1} * H_{\sigma_2}$ has precisely two deep $Q(5,2)$-quads and $|\mathcal{P}| - |H_{\sigma_1}| - |H_{\sigma_2}| + 2 \cdot |H_{\sigma_1} \cap H_{\sigma_2}| = 243 - 147 - 147 + 2 \cdot 99 = 147$ points. Consulting Table 1, we then know the following.

Description 6a. The hyperplanes of Type 6 occurring in Table 1 are precisely the hyperplanes of the form $H_{\sigma_1} * H_{\sigma_2}$ where σ_1 and σ_2 are two distinct $W(2)$-subquadrangles intersecting in a line.

Suppose $L = \{x_1, x_2, x_3\}$. For every $i \in \{1,2,3\}$, we have $H_{x_i} * H_{\sigma_1} = H_{\sigma_1^{(i)}}$ and $H_{x_i} * H_{\sigma_2} = H_{\sigma_2^{(i)}}$ for certain $W(2)$-subquadrangles $\sigma_1^{(i)} \subseteq Q_1$ and $\sigma_2^{(i)} \subseteq Q_2$ through L. We have $H_{\sigma_1} * H_{\sigma_2} = (H_{\sigma_1} * H_{x_i}) * (H_{\sigma_2} * H_{x_i}) = H_{\sigma_1^{(i)}} * H_{\sigma_2^{(i)}}$. Put $\sigma_1^{(0)} := \sigma_1$ and $\sigma_2^{(0)} := \sigma_2$. The number of unordered pairs of $W(2)$-subquadrangles intersecting in a line is precisely 11664 while there are only 2916 hyperplanes of Type 6. So, if σ and σ' are $W(2)$-subquadrangles intersecting in a line such that $H_\sigma * H_{\sigma'} = H_{\sigma_1} * H_{\sigma_2}$, then $\{\sigma, \sigma'\}$ is equal to $\{\sigma_1^{(i)}, \sigma_2^{(i)}\}$ for precisely one $i \in \{0,1,2,3\}$. For every point $y \in Q_1 \setminus L$, there exists a unique $i \in \{0,1,2,3\}$ such that $y \in G_{\sigma_1^{(i)}}$. So, without loss of generality, we may suppose that the following holds:

(∗) if $x \notin Q_1 \cup Q_2$, then $\pi_{Q_1}(x) \in G_{\sigma_1}$.

By Table 1, the stabilizer of the hyperplane $H_{\sigma_1} * H_{\sigma_2}$ has five orbits on the points of $Q(5,2) \otimes Q(5,2)$. These orbits correspond to the following choices for x (taking into account that assumption (∗) holds):

- $x \in L$;
- $x \in (Q_1 \cup Q_2) \setminus L$;
- $x \notin Q_1 \cup Q_2$, $x \in H_{\sigma_1} * H_{\sigma_2}$ and $\pi_{Q_2}(x) \in G_{\sigma_2}$;
- $x \notin Q_1 \cup Q_2$, $x \in H_{\sigma_1} * H_{\sigma_2}$ and $\pi_{Q_2}(x) \in \sigma_2 \setminus G_{\sigma_2}$;
- $x \notin H_{\sigma_1} * H_{\sigma_2}$.

If $x \in L$, then $H_{\sigma_1} * H_{\sigma_2} * H_x$ is a hyperplane of Type 6 since it is equal to the hyperplane $H_{\sigma_1^{(i)}} * H_{\sigma_2}$ where $i \in \{1,2,3\}$ such that $x = x_i$. One of remaining four possibilities leads to a new type of hyperplane.

Description 13a. The hyperplanes of Type 13 occurring in Table 1 are precisely the hyperplanes of the form $H_{\sigma_1} * H_{\sigma_2} * H_x$ where σ_1 and σ_2 are two $W(2)$-subquadrangles intersecting in a line and x is a point not contained in $H_{\sigma_1} * H_{\sigma_2}$.

For the other possibilities, we can say the following (verified with GAP).

Description 12c. The hyperplanes of Type 12 occurring in Table 1 are precisely the hyperplanes of the form $H_{\sigma_1} * H_{\sigma_2} * H_x$ where σ_1 and σ_2 are two distinct $W(2)$-subquadrangles intersecting in a line L and $x \notin L$ is a point belonging to the $Q(5,2)$-quad that contains σ_1.

Description 14c. The hyperplanes of Type 14 occurring in Table 1 are precisely the hyperplanes of the form $H_{\sigma_1} * H_{\sigma_2} * H_x$ where σ_i, $i \in \{1,2\}$, is a $W(2)$-subquadrangle of some $Q(5,2)$-quad Q_i and x is a point such that $\sigma_1 \cap \sigma_2 = Q_1 \cap Q_2$ is a line, $x \notin Q_1 \cup Q_2$, $\pi_{Q_1}(x) \in G_{\sigma_1}$ and $\pi_{Q_2}(x) \in G_{\sigma_2}$.

Description 14d. The hyperplanes of Type 14 occurring in Table 1 are precisely the hyperplanes of the form $H_{\sigma_1} * H_{\sigma_2} * H_x$ where σ_i, $i \in \{1,2\}$, is a $W(2)$-subquadrangle of some $Q(5,2)$-quad Q_i and x is a point such that $\sigma_1 \cap \sigma_2 = Q_1 \cap Q_2$ is a line, $x \notin Q_1 \cup Q_2$, $\pi_{Q_1}(x) \in \sigma_1 \setminus G_{\sigma_1}$ and $\pi_{Q_2}(x) \in \sigma_2 \setminus G_{\sigma_2}$.

Description 15b. The hyperplanes of Type 15 occurring in Table 1 are precisely the hyperplanes of the form $H_{\sigma_1} * H_{\sigma_2} * H_x$ where σ_i, $i \in \{1,2\}$, is a $W(2)$-subquadrangle of some $Q(5,2)$-quad Q_i and x is a point such that $\sigma_1 \cap \sigma_2 = Q_1 \cap Q_2$ is a line, $x \notin Q_1 \cup Q_2$, $\pi_{Q_1}(x) \in G_{\sigma_1}$ and $\pi_{Q_2}(x) \in \sigma_2 \setminus G_{\sigma_2}$.

The following description of the Type 12 hyperplanes can be deduced from Description 12c given above.

Description 12d. The hyperplanes of Type 12 occurring in Table 1 are precisely the hyperplanes of the form $H_{\sigma_1} * H_{\sigma_2}$ where σ_i, $i \in \{1,2\}$, is a $W(2)$-subquadrangle of a $Q(5,2)$-quad Q_i such that $Q_1 \cap Q_2$ is a line contained in σ_2 but not in σ_1.

5.8 Hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1, x_2 and x_3 are three mutually distinct points

Suppose x_1, x_2 and x_3 are three mutually distinct points of $Q(5,2) \otimes Q(5,2)$. We wish to determine which kind of hyperplane $H_{x_1} * H_{x_2} * H_{x_3}$ is. If two of these points lie at distance 1 from each other, then $H_{x_1} * H_{x_2} * H_{x_3}$ is of the form $H_{y_1} * H_{y_2}$, a type of hyperplane we already examined. If two of these points lie at distance 2 and are contained in a $Q(5,2)$-quad, then $H_{x_1} * H_{x_2} * H_{x_3}$ is of the form $H_\sigma * H_y$, a type of hyperplane we have also already examined. Also the case where there are two points at distance 2 contained in a grid-quad has already been examined. It remains to consider the case where x_1, x_2 and x_3 are mutually opposite points.

For every $i \in \{1,2\}$, let Q_i and R_i be the unique $Q(5,2)$-quads through x_i. Without loss of generality, we may suppose that $Q_1, Q_2 \in T_1$ and $R_1, R_2 \in T_2$. Then $Q_1 \cap Q_2 = R_1 \cap R_2 = \emptyset$. Put $Q_3 := \mathcal{R}_{Q_2}(Q_1) = \mathcal{R}_{Q_1}(Q_2) \in T_1$ and $R_3 := \mathcal{R}_{R_2}(R_1) = \mathcal{R}_{R_1}(R_2) \in T_2$. The line $Q_3 \cap R_3$ contains a unique point opposite to x_1 and x_2. We denote this point by $\mu(x_1, x_2)$. We also define $\Omega_1(x_1, x_2) = \{Q_1, Q_2, Q_3, R_1, R_2, R_3\}$ and $\Omega_2(x_1, x_2) = \{Q_3, R_3\}$.

There are three $Q(5,2)$-quads $Q \in T_1$ for which $\{\pi_Q(x_1), \pi_Q(x_2), \pi_Q(\mu(x_1,x_2))\}$ is a line. Indeed, the points x_1, $\pi_{R_1}(x_2)$ and $\pi_{R_1}(\mu(x_1,x_2))$ form an ovoid of the (3×3)-subgrid $R_1 \cap (Q_1 \cup Q_2 \cup Q_3)$ and if u is one of the three common neighbors of these three points, then the unique $Q(5,2)$-quad Q_u of T_1 through u satisfies this condition since $\pi_{Q_u}(x_1) = u$, $\pi_{Q_u}(x_2) = \pi_{R_2}(u)$ and $\pi_{Q_u}(\mu(x_1,x_2)) = \pi_{R_3}(u)$. Each of the three $Q(5,2)$-quads of T_1 that arise in this way must be deep with respect to the hyperplane $H_{x_1} * H_{x_2} * H_{\mu(x_1,x_2)}$. In a similar fashion, one can show that there are three $Q(5,2)$-quads $R \in T_2$ that are deep with respect to $H_{x_1} * H_{x_2} * H_{\mu(x_1,x_2)}$. This implies that $H_{x_1} * H_{x_2} * H_{\mu(x_1,x_2)}$ is a basic hyperplane with 171 points. In this way, we obtain a more direct construction for these basic hyperplanes, i.e. a construction that does not rely on the universal embedding $\widetilde{e}$ of $Q(5,2) \otimes Q(5,2)$.

Description 2b. The hyperplanes of Type 2 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1 and x_2 are two opposite points and $x_3 = \mu(x_1,x_2)$.

The following more direct construction for the Type 11 hyperplanes can be given.

Description 11b. The hyperplanes of Type 11 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1 and x_2 are two opposite points and x_3 is a point of $\Gamma_3(x_1) \cap \Gamma_3(x_2) \setminus \{\mu(x_1,x_2)\}$ contained in one of the two quads of $\Omega_2(x_1,x_2)$.

We have not yet considered the case where the point x_3 is not contained in any quad of $\Omega_1(x_1,x_2)$. This possibility leads to our final type of hyperplane.

Description 16a. The hyperplanes of Type 16 occurring in Table 1 are precisely the hyperplanes of the form $H_{x_1} * H_{x_2} * H_{x_3}$ where x_1, x_2 and x_3 are mutually opposite points of $Q(5,2) \otimes Q(5,2)$ such that x_3 is not contained in any of the six quads of $\Omega_1(x_1,x_2)$.

References

[1] A. E. Brouwer, A. M. Cohen, J. I. Hall and H. A. Wilbrink. Near polygons and Fischer spaces. *Geom. Dedicata* 49 (1994), 349–368.

[2] A. E. Brouwer, H. Cuypers and E. W. Lambeck. The hyperplanes of the M_{24} near polygon. *Graphs Combin.* 18 (2002), 415–420.

[3] B. De Bruyn. On near hexagons and spreads of generalized quadrangles. *J. Algebraic Combin.* 11 (2000), 211–226.

[4] B. De Bruyn. On the number of nonisomorphic glued near hexagons. *Bull. Belg. Math. Soc. Simon Stevin* 7 (2000), 493–510.

[5] B. De Bruyn. The hyperplanes of $DW(5,2^h)$ which arise from embedding. *Discrete Math.* 309 (2009), 304–321.

[6] B. De Bruyn. The hyperplanes of the near hexagon $\mathbb{G}_3$. In preparation.

[7] B. De Bruyn and H. Pralle. The exceptional hyperplanes of $DH(5,4)$. *European J. Combin.* 28 (2007), 1412–1418.

[8] B. De Bruyn and H. Pralle. The hyperplanes of $DH(5,q^2)$. *Forum Math.* 20 (2008), 239–264.

[9] B. De Bruyn and H. Pralle. The hyperplanes of the near hexagon on the 2-factors of the complete graph K_8. *Discrete Math.* 308 (2008), 5656–5671.

[10] B. De Bruyn and S. Shpectorov. The hyperplanes of the $U_4(3)$ near hexagon. *Graphs Combin.* 26 (2010), 647–671.

[11] B. De Bruyn and S. Shpectorov. The hyperplanes of the near hexagon related to the extended ternary Golay code. In preparation.

[12] B. De Bruyn and P. Vandecasteele. Valuations of near polygons. *Glasg. Math. J.* 47 (2005), 347–361.

[13] B. De Bruyn and P. Vandecasteele. The distance-2-sets of the slim dense near hexagons. *Ann. Comb.* 10 (2006), 193–210.

[14] D. Frohardt and P. Johnson. Geometric hyperplanes in generalized hexagons of order $(2,2)$. *Comm. Algebra* 22 (1994), 773–797.

[15] The GAP Group, GAP – Groups, Algorithms, and Programming, Version 4.4.12; 2008. (http://www.gap-system.org)

[16] H. Pralle. The hyperplanes of $DW(5,2)$. *Experiment. Math.* 14 (2005), 373–384.

[17] M. A. Ronan. Embeddings and hyperplanes of discrete geometries. *European J. Combin.* 8 (1987), 179–185.

[18] E. Shult and A. Yanushka. Near n-gons and line systems. *Geom. Dedicata* 9 (1980), 1–72.