Non-classical hyperplanes of $DW(5, q)$

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Abstract

The hyperplanes of the symplectic dual polar space $DW(5, q)$ arising from embedding, the so-called classical hyperplanes of $DW(5, q)$, have been determined earlier in the literature. In the present paper, we classify non-classical hyperplanes of $DW(5, q)$. If q is even, then we prove that every such hyperplane is the extension of a non-classical ovoid of a quad of $DW(5, q)$. If q is odd, then we prove that every non-classical ovoid of $DW(5, q)$ is either a semi-singular hyperplane or the extension of a non-classical ovoid of a quad of $DW(5, q)$. If $DW(5, q)$, q odd, has a semi-singular hyperplane, then q is not a prime number.

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1 Introduction

The hyperplanes of the finite symplectic dual polar space $DW(5, q)$ that arise from some projective embedding, the so-called classical hyperplanes of $DW(5, q)$, have explicitly been determined earlier in the literature, see Cooperstein & De Bruyn [5], De Bruyn [7] and Pralle [21]. In the present paper, we give a rather complete classification for the non-classical hyperplanes of $DW(5, q)$. There are two standard constructions for such hyperplanes.

(1) Suppose x is a point of $DW(5, q)$ and O is a set of points of $DW(5, q)$ at distance 3 from x such that every line at distance 2 from x has a unique point in common with O. Then $x^\perp \cup O$ is a non-classical hyperplane of $DW(5, q)$, the so-called semi-singular hyperplane with deepest point x.

(2) Suppose Q is a quad of $DW(5, q)$. Then the points and lines contained in Q define a generalized quadrangle $\widetilde{Q}$ isomorphic to $Q(4, q)$. If O is a non-classical ovoid of $\widetilde{Q}$, then the set of points of $DW(5, q)$ at distance at most 1 from O is a non-classical hyperplane of $DW(5, q)$, the so-called *extension of* O. Several classes of non-classical ovoids of $Q(4, q)$ are known, see Section 2.2 for a discussion.

The following is our main result.

Theorem 1.1 (1) *If q is even, then every non-classical hyperplane of $DW(5,q)$ is the extension of a non-classical ovoid of a quad of $DW(5,q)$.*

(2) *If q is odd, then every non-classical hyperplane of $DW(5,q)$ is either a semi-singular hyperplane or the extension of a non-classical ovoid of a quad of $DW(5,q)$.*

Up to present, no semi-singular hyperplane of $DW(5,q)$ is known to exist. If a semi-singular hyperplane of $DW(5,q)$ exists, then q must be odd (Theorem 3.11) and not a prime (Corollary 3.10).

The lines and quads through a given point x of $DW(5,q)$ define a projective plane isomorphic to $\mathrm{PG}(2,q)$ which we denote by $Res(x)$. If H is a hyperplane of $DW(5,q)$ and x is a point of H, then $\Lambda_H(x)$ denotes the set of lines through x contained in H. We regard $\Lambda_H(x)$ as a set of points of $Res(x)$. If $\Lambda_H(x)$ is the whole set of points of $Res(x)$, then x is called *deep with respect to H*.

The dual polar space $DW(5,q)$ has a nice full projective embedding e in the projective space $\mathrm{PG}(13,q)$, which is called the *Grassmann embedding* of $DW(5,q)$, see e.g. Cooperstein [4, Proposition 5.1]. A hyperplane of $DW(5,q)$ whose image under e is contained in a hyperplane of of $\mathrm{PG}(13,q)$ is said to arise from e. For a proof of the following proposition, we refer to Pasini [16, Theorem 9.3] or Cardinali & De Bruyn [3, Corollary 1.5].

Proposition 1.2 *If H is a hyperplane of $DW(5,q)$ arising from the Grassmann embedding of $DW(5,q)$, then for every point x of H, $\Lambda_H(x)$ is one of the following sets of points of $Res(x)$:* (1) *a point;* (2) *a line;* (3) *the union of two distinct lines;* (4) *a nonsingular conic;* (5) *the whole set of points of $Res(x)$.*

If $q \neq 2$, then the Grassmann embedding of $DW(5,q)$ is the so-called absolutely universal embedding of $DW(5,q)$ (Cooperstein [4, Theorem B], Kasikova & Shult [12, Section 4.6], Ronan [22]), implying that the classical hyperplanes of $DW(5,q)$ are precisely those hyperplanes arising from the Grassmann embedding. Combining Theorem 1.1 with Proposition 1.2, we easily find:

Corollary 1.3 *If H is a hyperplane of $DW(5,q)$, $q \neq 2$, then for every point x of H, $\Lambda_H(x)$ is one of the following sets of points of $Res(x)$:* (1) *the empty set;* (2) *a point;* (3) *a line;* (4) *the union of two distinct lines;* (5) *a nonsingular conic;* (6) *the whole set of points of $Res(x)$. If $\Lambda_H(x)$ is the empty set, then H is a semi-singular hyperplane whose deepest point lies at distance* 3 *from x. If H is not a semi-singular hyperplane, then case* (1) *cannot occur.*

The conclusion of Corollary 1.3 is false for the dual polar space $DW(5,2)$. If x is a point of $DW(5,2)$, then for every set Y of points of $Res(x) \cong \mathrm{PG}(2,2)$, there exists a hyperplane H through x such that $\Lambda_H(x) = Y$, see Pralle [21, Table 1].

If $n \geq 4$, then the symplectic dual polar space $DW(2n-1,q)$ has many full subgeometries isomorphic to $DW(5,q)$. So, Corollary 1.3 reveals information on the local structure of any hyperplane of any symplectic dual polar space $DW(2n-1,q)$, where $q \neq 2$ and $n \geq 4$.

Theorem 1.1 will be proved in Section 3. In Section 2, we give the basic definitions (including some of the notions already mentioned above) and basic properties which will play a role in the proof of Theorem 1.1.

2 Preliminaries

2.1 The dual polar space $DW(5,q)$

Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a point-line geometry with nonempty point-set $\mathcal{P}$, line set $\mathcal{L}$ and incidence relation $\mathrm{I} \subseteq \mathcal{P} \times \mathcal{L}$. A set $H \subsetneq \mathcal{P}$ is called a *hyperplane* of $\mathcal{S}$ if every line of $\mathcal{S}$ has either one or all of its points in H. A *full projective embedding* of $\mathcal{S}$ is an injective mapping e from $\mathcal{P}$ to the point-set of a projective space Σ satisfying (i) $\langle e(\mathcal{P}) \rangle_\Sigma = \Sigma$; (ii) $\{e(x) \,|\, (x, L) \in \mathrm{I}\}$ is a line of Σ for every line L of $\mathcal{S}$. If $e : \mathcal{S} \to \Sigma$ is a projective embedding of $\mathcal{S}$ and Π is a hyperplane of Σ, then $e^{-1}(e(\mathcal{P}) \cap \Pi)$ is a hyperplane of $\mathcal{S}$. A hyperplane of $\mathcal{S}$ is said to be *classical* if it is of the form $e^{-1}(e(\mathcal{P}) \cap \Pi)$, where e is some full projective embedding of $\mathcal{S}$ into a projective space Σ and Π is some hyperplane of Σ.

Distances $\mathrm{d}(\cdot, \cdot)$ in $\mathcal{S}$ will be measured in its collinearity graph. If x is a point of $\mathcal{S}$ and $i \in \mathbb{N}$, then $\Gamma_i(x)$ denotes the set of points of $\mathcal{S}$ at distance i from x. Similarly, if X is a nonempty set of points and $i \in \mathbb{N}$, then $\Gamma_i(X)$ denotes the set of all points at distance i from X, i.e. the set of all points y for which $\min\{\mathrm{d}(y, x) \,|\, x \in X\} = i$.

Let $W(5,q)$ be the polar space whose subspaces are the subspaces of $\mathrm{PG}(5,q)$ that are totally isotropic with respect to a given symplectic polarity of $\mathrm{PG}(5,q)$, and let $DW(5,q)$ denote the associated dual polar space. The points and lines of $DW(5,q)$ are the totally isotropic planes and lines of $\mathrm{PG}(5,q)$, with incidence being reverse containment. The dual polar space $DW(5,q)$ belongs to the class of *near polygons* introduced by Shult and Yanushka in [23]. This means that for every point x and every line L, there exists a unique point on L nearest to x. The maximal distance between two points of $DW(5,q)$ is equal to 3. The dual polar space $DW(5,q)$ has $(q+1)(q^2+1)(q^3+1)$ points, $q+1$ points on each line and q^2+q+1 lines through each point.

If x and y are two points of $DW(5,q)$ at distance 2 from each other, then the smallest convex subspace $\langle x, y \rangle$ of $DW(5,q)$ containing x and y is called a *quad*. A quad Q of $DW(5,q)$ consists of all totally isotropic planes of $W(5,q)$ that contain a given point x_Q of $W(5,q)$. Any two lines L and M of $DW(5,q)$ that meet in a unique point are contained in a unique quad. We denote this quad by $\langle L, M \rangle$. Obviously, we have $\langle L, M \rangle = \langle x, y \rangle$ where x and y are arbitrary points of $L \setminus M$ and $M \setminus L$, respectively. The points and lines of $DW(5,q)$ that are contained in a given quad Q define a point-line geometry $\widetilde{Q}$ isomorphic to the generalized quadrangle $Q(4,q)$ of the points and lines of a nonsingular parabolic quadric of $\mathrm{PG}(4,q)$. If Q is a quad of $DW(5,q)$ and x is a point not contained in Q, then Q contains a unique point $\pi_Q(x)$ collinear with x and $\mathrm{d}(x,y) = 1 + \mathrm{d}(\pi_Q(x), y)$ for every point y of Q. If Q_1 and Q_2 are two distinct quads of $DW(5,q)$, then $Q_1 \cap Q_2$ is either empty or a line of $DW(5,q)$. If $Q_1 \cap Q_2 = \emptyset$, then the map $Q_1 \to Q_2; x \mapsto \pi_{Q_2}(x)$ is an isomorphism between $\widetilde{Q_1}$ and $\widetilde{Q_2}$.

2.2 Hyperplanes of $Q(4,q)$

By Payne and Thas [18, 2.3.1], every hyperplane of the generalized quadrangle $Q(4,q)$ is either the perp $x^\perp$ of a point x, a $(q+1)\times(q+1)$-subgrid or an ovoid. An ovoid of $Q(4,q)$ is *classical* if it is an elliptic quadric $Q^-(3,q)\subseteq Q(4,q)$. For many values of q, non-classical ovoids of $Q(4,q)$ do exist: (i) $q=p^h$ with p an odd prime and $h\geq 2$ [11]; (ii) $q=2^{2h+1}$ with $h\geq 1$ [26]; (iii) $q=3^{2h+1}$ with $h\geq 1$ [11]; (iv) $q=3^h$ with $h\geq 3$ [24]; (v) $q=3^5$ [19]. For several prime powers q, it is known that all ovoids of $Q(4,q)$ are classical:

Proposition 2.1 • **([2, 15])** *Every ovoid of $Q(4,4)$ is classical.*
- **([13, 14])** *Every ovoid of $Q(4,16)$ is classical.*
- **([1])** *Every ovoid of $Q(4,q)$, q prime, is classical.*

A set $\mathcal{G}$ of hyperplanes of $Q(4,q)$ is called a *pencil of hyperplanes* if every point of $Q(4,q)$ is contained in either 1 or all elements of $\mathcal{G}$. The following lemma is precisely Lemma 3.2 and Corollary 3.3 of De Bruyn [8].

Lemma 2.2 *If G_1 and G_2 are two distinct classical hyperplanes of $Q(4,q)$, then through every point x of $Q(4,q)$ not contained in $G_1\cup G_2$, there exists a unique classical hyperplane G_x satisfying $G_x\cap G_1=G_1\cap G_2=G_2\cap G_x$. As a consequence, any two distinct classical hyperplanes of $Q(4,q)$ are contained in a unique pencil of classical hyperplanes of $Q(4,q)$.*

2.3 Hyperplanes of $DW(5,q)$

Since $DW(5,q)$ is a near polygon, the set of points of $DW(5,q)$ at distance at most 2 from a given point x is a hyperplane of $DW(5,q)$, the so-called *singular hyperplane with deepest point x*. If O is a set of points of $DW(5,q)$ at distance 3 from a given point x such that every line at distance 2 from x has a unique point in common with O, then $x^\perp\cup O$ is a hyperplane of $DW(5,q)$, a so-called *semi-singular hyperplane of $DW(5,q)$ with deepest point x*. If Q is a quad of $DW(5,q)$ and G is a hyperplane of $\widetilde{Q}\cong Q(4,q)$, then $Q\cup\{x\in\Gamma_1(Q)\,|\,\pi_Q(x)\in G\}$ is a hyperplane of $DW(5,q)$, the so-called *extension* of G.

If H is a hyperplane of $DW(5,q)$ and Q is a quad, then either $Q\subseteq H$ or $Q\cap H$ is a hyperplane of $Q\cong Q(4,q)$. If $Q\subseteq H$, then Q is called a *deep quad*. If $Q\cap H=x^\perp\cap Q$ for some point $x\in Q$, then Q is called *singular* with respect to H and x is called the *deep point* of Q. The quad Q is called *ovoidal* (respectively, *subquadrangular*) with respect to H if and only if $Q\cap H$ is an ovoid (respectively, a $(q+1)\times(q+1)$-subgrid) of Q. A hyperplane H of $DW(5,q)$ is called *locally singular* (*locally subquadrangular*, respectively *locally ovoidal*) if every non-deep quad of $DW(5,q)$ is singular (subquadrangular, respectively ovoidal) with respect to H. A hyperplane that is locally singular, locally ovoidal or locally subquadrangular is also called a *uniform hyperplane*. In the following proposition, we collect a number of known results which we will need to invoke later in the proof of the Main Theorem.

Proposition 2.3 (1) *The dual polar space $DW(5,q)$, $q \neq 2$, has no locally subquadrangular hyperplanes.*

(2) *The dual polar space $DW(5,q)$ has no locally ovoidal hyperplanes.*

(3) *Every nonuniform hyperplane of $DW(5,q)$ admits a singular quad.*

Proposition 2.3(1) is due to Pasini & Shpectorov [17]. Locally ovoidal hyperplanes of $DW(5,q)$ are just ovoids and cannot exist by Thomas [25, Theorem 3.2], see also Cooperstein and Pasini [6]. Proposition 2.3(3) is due to Pralle [20].

The classical hyperplanes of the dual polar space $DW(5,q)$ have already been classified in the literature. The dual polar space $DW(5,q)$, $q \neq 2$, has six isomorphism classes of classical hyperplanes by Cooperstein & De Bruyn [5] and De Bruyn [7]. This fact is not true if $q = 2$. The dual polar space $DW(5,2)$ has twelve isomorphism classes of hyperplanes by Pralle [21], see also De Bruyn [7, Section 9]. Observe that all these hyperplanes are classical by Ronan [22, Corollary 2]. By De Bruyn [8], the classical hyperplanes of $DW(5,q)$ can be characterized as follows.

Proposition 2.4 *The classical hyperplanes of $DW(5,q)$ are precisely those hyperplanes H of $DW(5,q)$ that satisfy the following property: if Q is an ovoidal quad, then $Q \cap H$ is a classical ovoid of Q.*

2.4 Hyperbolic sets of quads of $DW(5,q)$

As in Section 2.1, let $W(5,q)$ be the polar space associated with a symplectic polarity of $\mathrm{PG}(5,q)$. If L is a hyperbolic line of $\mathrm{PG}(5,q)$ (i.e. a line of $\mathrm{PG}(5,q)$ that is not a line of $W(5,q)$), then the set of the $q+1$ (mutually disjoint) quads of $DW(5,q)$ corresponding to the points of L satisfy the property that every line that meets at least two of its members meets each of its members in a unique point. Any set of $q+1$ quads that is obtained in this way will be called a *hyperbolic set of quads* of $DW(5,q)$. Every two disjoint quads Q_1 and Q_2 of $DW(5,q)$ are contained in a unique hyperbolic set of quads of $DW(5,q)$. We will denote this hyperbolic set of quads by $\mathcal{H}(Q_1,Q_2)$. Considering all the lines meeting Q_1 and Q_2, we easily see that the following holds.

Lemma 2.5 *Let $\{Q_1,Q_2,\ldots,Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$ and let H be a hyperplane of $DW(5,q)$ such that $H \cap Q_1$ and $\pi_{Q_1}(H \cap Q_2)$ are distinct hyperplanes of $\widetilde{Q_1}$. Then $\{\pi_{Q_1}(H \cap Q_i) \,|\, 1 \leq i \leq q+1\}$ is a pencil of hyperplanes of $\widetilde{Q_1}$.*

3 Proof of Theorem 1.1

Throughout this section, we suppose that H is an arbitrary hyperplane of $DW(5,q)$. In De Bruyn [9], we classified for every field $\mathbb{K}$ of size at least three the hyperplanes of $DW(5,\mathbb{K})$ containing a quad. The main theorem of [9] implies the following:

Proposition 3.1 *Every non-classical hyperplane of $DW(5,q)$, $q \neq 2$, containing a quad is the extension of a non-classical ovoid of a quad.*

We have already mentioned above that every hyperplane of $DW(5,2)$ is classical by Ronan [22, Corollary 2]. Since we are interested in the classification of all non-classical hyperplanes of $DW(5,q)$, we may by the above assume that the following holds:

Assumption: We have $q \geq 3$ and the hyperplane H does not contain quads.

We denote by v the total number of points of H and by l the total number of lines of $DW(5,q)$ contained in H. In Section 3.1, we prove that there are only three possible values for v, namely $q^5+q^3+q^2+q+1$, $q^5+q^4+q^3+q^2+2q+1$ or $q^5+q^4+q^3+q^2+q+1$. In Section 3.2, we prove that if $v = q^5+q^3+q^2+q+1$, then H is a semi-singular hyperplane. We also prove there that semi-singular hyperplanes cannot exist if q is even. In [10] (see also Corollary 3.10), the nonexistence of semi-singular hyperplanes was already shown for prime values of q. In Section 3.3, we prove that the case $v = q^5+q^4+q^3+q^2+2q+1$ cannot occur and in Section 3.4, we prove that H must be classical if $v = q^5+q^4+q^3+q^2+q+1$. All these results together imply that Theorem 1.1 must hold.

3.1 The possible values of v

The following lemma is an immediate consequence of Proposition 2.3.

Lemma 3.2 *The hyperplane admits singular quads.*

Lemma 3.3 *We have* $l = \frac{v\cdot(q^2+q+1)-(q^2+1)(q^3+1)(q^2+q+1)}{q}$.

Proof. We count the number of lines not contained in H. There are $(q+1)(q^2+1)(q^3+1)-v$ points outside H and each of these points is contained in q^2+q+1 lines which contain a unique point of H. Hence, the total number of lines not contained in H is equal to $\frac{((q+1)(q^2+1)(q^3+1)-v)(q^2+q+1)}{q}$. Since the total number of lines of $DW(5,q)$ equals $(q^2+1)(q^3+1)(q^2+q+1)$, we have $l = (q^2+1)(q^3+1)(q^2+q+1) - \frac{((q+1)(q^2+1)(q^3+1)-v)(q^2+q+1)}{q} = \frac{v\cdot(q^2+q+1)-(q^2+1)(q^3+1)(q^2+q+1)}{q}$. □

Lemma 3.4 *If Q is a singular quad with deep point x, then one of the following cases occurs:*

(1) $x^\perp \cap H = x^\perp \cap Q$;

(2) *there exists a line L through x not contained in Q such that* $x^\perp \cap H = (x^\perp \cap Q) \cup L$;

(3) *there exists a quad R through x distinct from Q such that* $x^\perp \cap H = (x^\perp \cap Q) \cup (x^\perp \cap R)$;

(4) $x^\perp \subseteq H$.

Proof. Since $x^\perp \cap Q \subseteq x^\perp \cap H$, $|\Lambda_H(x)| \geq q+1$. If $|\Lambda_H(x)| \in \{q+1, q+2\}$, then either case (1) or (2) of the lemma occurs. Suppose therefore that $|\Lambda_H(x)| \geq q+3$ and let L_1 and L_2 be two distinct lines through x that are contained in H, but not in Q. Put $R := \langle L_1, L_2 \rangle$. Since $L_1 \subseteq R \cap H$, $L_2 \subseteq R \cap H$ and $R \cap Q \subseteq R \cap H$, R is singular with deep point x and hence every line of R through x is contained in H. So, $|\Lambda_H(x)| \geq 2q+1$.

If $|\Lambda_H(x)| = 2q + 1$, then obviously case (3) of the lemma occurs. Suppose therefore that $|\Lambda_H(x)| \geq 2q+2$. Then there exists a line $L_3 \subseteq H$ through x not contained in $Q \cup R$. If Q' is a quad through L_3 distinct from $\langle L_3, Q \cap R \rangle$, then since $Q' \cap Q \subseteq H$, $Q' \cap R \subseteq H$ and $L_3 \subseteq H$, Q' is singular with deep point x and hence every line of Q' through x is contained in H. It follows that all lines of $DW(5, q)$ through x are contained in H, except maybe for the $q - 1$ lines through x contained in $\langle L_3, Q \cap R \rangle$ and distinct from L_3 and $Q \cap R$. Let L' be one of these $q - 1$ lines and let Q'' be a quad through L' distinct from $\langle L_3, Q \cap R \rangle$. Since $q \geq 3$ lines of Q'' through x are contained in H, Q'' is singular with deep point x and hence also L' is contained in H. So, $x^{\perp} \subseteq H$ and case (4) of the lemma occurs. □

Lemma 3.5 *If Q is a singular quad with deep point x, then $|\Gamma_3(x) \cap H| = q^5$.*

Proof. Every point of $\Gamma_3(x) \cap H$ is collinear with a unique point of $\Gamma_2(x) \cap Q$. Conversely, every point u of $\Gamma_2(x) \cap Q$ is collinear with precisely q^2 points of $\Gamma_3(x) \cap H$. (One on each line through u not contained in Q.) Hence, $|\Gamma_3(x) \cap H| = |\Gamma_2(x) \cap Q| \cdot q^2 = q^5$. □

Lemma 3.6 *Suppose Q is a singular quad with deep point x.*

- *If case (1) of Lemma 3.4 occurs, then $v = q^5 + q^4 + q^3 + q^2 + q + 1$ and $l = q^5 + q^4 + q^3 + q^2 + q + 1$.*
- *If case (2) of Lemma 3.4 occurs, then $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$ and $l = (q^2 + q + 1)(q^3 + 2)$.*
- *If case (3) of Lemma 3.4 occurs, then $v = q^5 + q^4 + q^3 + q^2 + q + 1$ and $l = q^5 + q^4 + q^3 + q^2 + q + 1$.*
- *If case (4) of Lemma 3.4 occurs, then $v = q^5 + q^3 + q^2 + q + 1$ and $l = q^2 + q + 1$.*

Proof. Suppose case (1) of Lemma 3.4 occurs. Then x is contained in 1 singular quad that has x as deep point (namely Q) and $q^2 + q$ singular quads that do not have x as deep point. In this case, $|\Gamma_0(x) \cap H| = 1$, $|\Gamma_1(x) \cap H| = q^2 + q$, $|\Gamma_2(x) \cap H| = 1 \cdot 0 + (q^2 + q) \cdot q^2$ and $|\Gamma_3(x) \cap H| = q^5$. Hence, $v = 1 + (q^2 + q) + (q^2 + q) \cdot q^2 + q^5 = q^5 + q^4 + q^3 + q^2 + q + 1$.

Suppose case (2) of Lemma 3.4 occurs. Then x is contained in 1 singular quad with deep point equal to x, $q + 1$ subquadrangular quads and $q^2 - 1$ singular quads with deep point different from x. In this case, $|\Gamma_0(x) \cap H| = 1$, $|\Gamma_1(x) \cap H| = (q + 2)q = q^2 + 2q$, $|\Gamma_2(x) \cap H| = 1 \cdot 0 + (q + 1) \cdot q^2 + (q^2 - 1) \cdot q^2 = q^4 + q^3$ and $|\Gamma_3(x) \cap H| = q^5$. Hence, $v = 1 + (q^2 + 2q) + (q^4 + q^3) + q^5 = q^5 + q^4 + q^3 + q^2 + 2q + 1$.

Suppose case (3) of Lemma 3.4 occurs. Then x is contained in 2 singular quads with deep point x, $q-1$ singular quads with deep point different from x and q^2 subquadrangular quads. In this case, $|\Gamma_0(x) \cap H| = 1$, $|\Gamma_1(x) \cap H| = (2q + 1)q = 2q^2 + q$, $|\Gamma_2(x) \cap H| =$

$2 \cdot 0 + (q-1) \cdot q^2 + q^2 \cdot q^2 = q^4 + q^3 - q^2$ and $|\Gamma_3(x) \cap H| = q^5$. Hence, $v = 1 + (2q^2 + q) + (q^4 + q^3 - q^2) + q^5 = q^5 + q^4 + q^3 + q^2 + q + 1$.

Suppose case (4) of Lemma 3.4 occurs. Then x is contained in q^2+q+1 singular quads that have x as deep point. Hence, $v = |\Gamma_0(x) \cap H| + |\Gamma_1(x) \cap H| + |\Gamma_2(x) \cap H| + |\Gamma_3(x) \cap H| = 1 + q(q^2+q+1) + 0 + q^5 = q^5 + q^3 + q^2 + q + 1$.

In each of the four cases, the value of l can be derived from Lemma 3.3. □

By Lemmas 3.2, 3.4 and 3.6, we have:

Corollary 3.7 $v \in \{q^5+q^3+q^2+q+1, q^5+q^4+q^3+q^2+q+1, q^5+q^4+q^3+q^2+2q+1\}$.

We see that if case (2) of Lemma 3.4 occurs for one singular quad Q, then case (2) occurs for all singular quads Q. A similar remark holds applies to case (4) of Lemma 3.4.

3.2 The case $v = q^5 + q^3 + q^2 + q + 1$

Let Q^* denote a singular quad and x^* its deep point.

Lemma 3.8 *If $v = q^5+q^3+q^2+q+1$, then H is a semi-singular hyperplane of $DW(5,q)$ with deepest point x^*.*

Proof. If $v = q^5 + q^3 + q^2 + q + 1$, then case (4) of Lemma 3.4 occurs for the pair (Q^*, x^*). So, we have that $x^{*\perp} \subseteq H$ and $\Gamma_2(x^*) \cap H = \emptyset$ (no deep quad through x^*). Since $\Gamma_2(x^*) \cap H = \emptyset$, every line at distance 2 from x^* contains a unique point of $\Gamma_3(x^*) \cap H$. It follows that H is a semi-singular hyperplane of $DW(5,q)$ with deepest point x^*. □

The following proposition was proved in De Bruyn and Vandecasteele [10, Corollary 6.3].

Proposition 3.9 *If q is a prime power such that every ovoid of $Q(4,q)$ is classical, then $DW(5,q)$ does not have semi-singular hyperplanes.*

By Propositions 2.1 and 3.9, we have

Corollary 3.10 *If q is prime, then $DW(5,q)$ has no semi-singular hyperplanes.*

We will now use hyperbolic sets of quads of $DW(5,q)$ to prove the nonexistence of semi-singular hyperplanes of $DW(5,q)$, q even.

Theorem 3.11 *The dual polar space $DW(5,q)$, q even, has no semi-singular hyperplanes.*

Proof. Suppose H is a semi-singular hyperplane of $DW(5,q)$, q even, and as before let x^* denote the deepest point of H. Let Q be a quad through x^*, let G be a $(q+1) \times (q+1)$-subgrid of $\widetilde{Q}$ not containing x^*, let L_1 and L_2 be two disjoint lines of G and let Q_i, $i \in \{1,2\}$, be a quad through L_i distinct from Q. Then Q_1 and Q_2 are disjoint. Put $\mathcal{H} = \mathcal{H}(Q_1, Q_2)$. Every $Q_3 \in \mathcal{H}$ intersects Q in a line of G and hence $x^* \notin Q_3$. It follows that every $Q_3 \in \mathcal{H}$ is ovoidal with respect to H. Suppose $Q_3 \in \mathcal{H} \setminus \{Q_1\}$ and $x_3 \in Q_3 \cap H$ such that $x_1 = \pi_{Q_1}(x_3) \in Q_1 \cap H$. Then the line x_1x_3 is contained in H and hence $x^* \in x_1x_3$. But this is impossible, since no quad of $\mathcal{H}$ contains x^*. Hence, $\pi_{Q_1}(Q_3 \cap H)$ is disjoint from $Q_1 \cap H$. By Lemma 2.5, the set $\{\pi_{Q_1}(Q_3 \cap H) \,|\, Q_3 \in \mathcal{H}\}$ is a partition of Q_1 into ovoids. This is however impossible since the generalized quadrangle $Q(4, q)$, q even, has no partition in ovoids by Payne and Thas [18, Theorem 1.8.5]. □

3.3 The case $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$

We suppose that $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$ and $l = (q^2 + q + 1)(q^3 + 2)$. Recall that if Q is a singular quad and x is the deep point of Q, then case (2) of Lemma 3.4 occurs for the pair (Q, x).

Lemma 3.12 *Let Q be a singular quad, let x be the deep point of Q, let L be the line through x not contained in Q such that $x^\perp \cap H = (x^\perp \cap Q) \cup L$ and let y be a point of $L \setminus \{x\}$. Then there are $q + 1$ lines $L_1, L_2, \dots, L_{q+1}$ through y different from L that are contained in H. The $q + 2$ lines $L, L_1, L_2, \dots, L_{q+1}$ form a hyperoval of the projective plane $Res(y) \cong \mathrm{PG}(2, q)$. (Hence, q must be even.)*

Proof. The $q + 1$ quads $R_1, \dots, R_{q+1}$ through L determine a partition of the set of lines through y different from L. Each of these quads is subquadrangular. Hence, R_i, $i \in \{1, 2, \dots, q + 1\}$, contains a unique line $L_i \neq L$ through y that is contained in H.

For all $i, j \in \{1, 2, \dots, q + 1\}$ with $i \neq j$, the lines L, L_i and L_j are not contained in a quad since the quad $\langle L, L_i\rangle$ is subquadrangular. Suppose there exist mutually distinct $i, j, k \in \{1, 2, \dots, q + 1\}$ such that L_i, L_j and L_k are contained in a quad Q'. Then L is not contained in Q' and hence $Q \cap Q' = \emptyset$. Since L_i, L_j and L_k are contained in H, Q' is singular with deep point y. Let $z' \in Q' \setminus y^\perp$ and $z := \pi_Q(z')$. Since z and z' are not contained in H, the line zz' contains a unique point $z'' \in H$. Let Q'' denote the unique quad through z'' intersecting L in a point u. Then $Q'' \in \mathcal{H}(Q, Q')$. So, every point of $u^\perp \cap Q''$ is contained in a line joining a point of $y^\perp \cap Q'$ with a point of $x^\perp \cap Q$ and hence is contained in H. Since also $z'' \in H$, $Q'' \subseteq H$, contradicting the fact that there are no deep quads. □

Lemma 3.13 *There are four possible types of points in H:*

(A) points x for which $\Lambda_H(x)$ is the union of a line of $Res(x)$ and a point of $Res(x)$ not belonging to that line;

(B) points x for which $\Lambda_H(x)$ is a hyperoval of $Res(x)$;

(C) points x for which $|\Lambda_H(x)| = 2$;

(D) points x for which $\Lambda_H(x)$ is empty.

Moreover, we have:

(i) Every point of Type (A) has distance 1 from precisely $q^2 - 1$ points of Type (A), q points of Type (B) and $q + 1$ points of Type (C).

(ii) Every point of Type (B) has distance 1 from precisely $q + 2$ points of Type (A), $(q + 2)(q - 1)$ points of Type (B) and 0 points of Type (C).

(*iii*) *Every point of Type* (*C*) *has distance* 1 *from precisely* $2q$ *points of Type* (*A*), 0 *points of Type* (*B*) *and* 0 *points of Type* (*C*).

Proof. Suppose Q^* is a singular quad and x^* is its deep point. Consider the collinearity graph Γ of $DW(5,q)$ and let Γ_H denote the subgraph of Γ induced on the vertex set H. Suppose x is a point of H such that x and x^* belong to different connected components of Γ_H. We prove that $\Lambda_H(x)$ is empty. Suppose to the contrary that there exists a line L through x contained in H. If L meets Q^*, then $L \cap Q^*$ must be contained in $x^{*\perp}$, contradicting the fact that x^* and x belong to different connected components of Γ_H. So, L is disjoint from Q^*. Then $\pi_{Q^*}(L)$ meets $x^{*\perp}$ and hence x^* and x are connected by a path of Γ_H, again a contradiction.

Notice that by Lemma 3.6 and the fact that $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$, x^* is a point of Type (A). So, in order to prove the first part of the lemma, it suffices to verify that every vertex x of Type (X), $X \in \{A, B, C\}$, of Γ_H is adjacent with only vertices of Type (A), (B) or (C). As a by-product of our verification, also the conclusions of the second part of the lemma will be obtained.

First, suppose that x is a point of Type (A). Without loss of generality, we may suppose that $x = x^*$. Let L^* denote the unique line through x^* such that $x^{*\perp} \cap H = (x^{*\perp} \cap Q^*) \cup L^*$. By Lemma 3.12, every point of $L^* \setminus \{x^*\}$ has Type (B). Now, let L be a line through x^* contained in Q^*. Then $\langle L, L^* \rangle$ is a subquadrangular quad. Any quad through L different from $\langle L, L^* \rangle$ and Q^* is singular with deep point contained in $L \setminus \{x^*\}$. By Lemmas 3.4 and 3.6 and the fact that $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$, every point of $L \setminus \{x^*\}$ is the deep point of at most 1 such singular quad. Hence, $q - 1$ points of $L \setminus \{x^*\}$ have Type (A) and the remaining point of $L \setminus \{x^*\}$ has type (C).

Suppose x is a point of Type (C). Let L_1 and L_2 denote the two lines through x that are contained in H. Then $\langle L_1, L_2 \rangle$ is a subquadrangular quad. If Q is a quad through L_1 distinct from $\langle L_1, L_2 \rangle$, then Q is singular with deep point on $L_1 \setminus \{x\}$. By Lemmas 3.4 and 3.6 and the fact that $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$, every point of $L_1 \setminus \{x\}$ is the deep point of at most 1 such singular quad. It follows that every point of $L_1 \setminus \{x\}$ has Type (A). In a similar way, one shows that every point of $L_2 \setminus \{x\}$ has Type (A).

Suppose x is a point of Type (B). Let L be an arbitrary line through x contained in H. Every quad through L is subquadrangular. It follows that through every point $u \in L$ there are precisely $q + 2$ lines that are contained in H. If at least three of these lines are contained in a certain quad R, then R is singular with deep point u and hence u is of type (A). Otherwise, u is of type (B). By Lemma 3.12, there are two possibilities.

(1) L contains a unique point of Type (A) and q points of Type (B).

(2) L contains $q + 1$ points of Type (B).

We show that case (2) cannot occur. Suppose it does occur. Then $|\Gamma_0(L) \cap H| = q + 1$ and $|\Gamma_1(L) \cap H| = (q + 1)^2 q$. Each quad intersecting L in a unique point is either ovoidal or subquadrangular and contributes q^2 to the value of $|\Gamma_2(L) \cap H|$. Since every point of $\Gamma_2(L)$ is contained in a unique quad that intersects L in a unique point, $|\Gamma_2(L) \cap H| = (q+1)q^2 \cdot q^2$.

It follows that $|H| = |\Gamma_0(L) \cap H| + |\Gamma_1(L) \cap H| + |\Gamma_2(L) \cap H| = (q+1) + (q+1)^2 q + (q+1)q^4 = q^5 + q^4 + q^3 + 2q^2 + 2q + 1$, contradicting the fact that $|H| = q^5 + q^4 + q^3 + q^2 + 2q + 1$. $\square$

Now, let n_A, n_B, n_C respectively n_D, denote the total number of points of H of Type (A), (B), (C), respectively (D). Then by Lemma 3.13, we have $n_A \cdot q = n_B \cdot (q+2)$ and $n_A \cdot (q+1) = n_C \cdot 2q$. Hence,

$$n_B = \frac{n_A \cdot q}{q+2}, \quad (1)$$

$$n_C = \frac{n_A \cdot (q+1)}{2q}. \quad (2)$$

Now, counting in two different ways the number of pairs (x, L), with $x \in H$ and L a line through x contained in H, we obtain

$$n_A \cdot (q+2) + n_B \cdot (q+2) + n_C \cdot 2 = l \cdot (q+1) = (q^2+q+1)(q+1)(q^3+2). \quad (3)$$

From equations (1), (2) and (3), we find $n_A = \frac{(q^2+q+1)(q^3+2)q}{2q+1}$, $n_B = \frac{(q^2+q+1)(q^3+2)q^2}{(q+2)(2q+1)}$ and $n_C = \frac{(q^2+q+1)(q^3+2)(q+1)}{2(2q+1)}$. If $q = 3$, then $n_A \notin \mathbb{N}$. If $q \geq 4$, then

$$\begin{aligned} n_A + n_B + n_C &= (q^2+q+1)(q^3+2) \cdot \frac{5q^2+7q+2}{2(q+2)(2q+1)} \\ &> (q^5+q^4+q^3+q^2+2q+1) \cdot 1 \\ &= v, \end{aligned}$$

a contradiction. Hence, the case $v = q^5 + q^4 + q^3 + q^2 + 2q + 1$ cannot occur.

3.4 The case $v = q^5 + q^4 + q^3 + q^2 + q + 1$

Suppose $v = q^5 + q^4 + q^3 + q^2 + q + 1$.

Lemma 3.14 *There are five possible types of points in H:*
(*A*) *points x for which $|\Lambda_H(x)| = 1$;*
(*B*) *points x for which $\Lambda_H(x)$ is a line of Res(x);*
(*C*) *points x for which $\Lambda_H(x)$ is the union of two distinct lines of Res(x);*
(*D*) *points x for which $\Lambda_H(x)$ is an oval of Res(x);*
(*E*) *points x for which $\Lambda_H(x)$ is empty.*

Proof. Suppose Q^* is a singular quad and x^* is its deep point. Consider the collinearity graph Γ of $DW(5,q)$ and let Γ_H denote the subgraph of Γ induced on the vertex set H. Suppose x is a point of H such that x and x^* belong to different connected components of Γ_H. Then we prove that $\Lambda_H(x)$ is empty. Suppose to the contrary that there exists a line L through x contained in H. If L meets Q^*, then $L \cap Q^*$ must be contained in $x^{*\perp}$, contradicting the fact that x^* and x belong to different connected components of Γ_H. So,

L is disjoint from Q^*. Then $\pi_{Q^*}(L)$ meets $x^{*\perp}$ and hence x^* and x are connected by a path of Γ_H, again a contradiction.

By Lemmas 3.4 and 3.6 applied to the pair (Q^*, x^*), x^* is a point of Type (B) or (C). So, in order to prove the lemma, it suffices to prove that if x is a point of Type $(X) \in \{(A), (B), (C), (D)\}$ and y is a point of $H \setminus \{x\}$ collinear with x, then y is of Type (A), (B), (C) or (D). Put $L := xy$. Since x is of Type (A), (B), (C) or (D), one of the following two possibilities occurs:

(1) L is contained in $q+1$ singular quads with deep point on L.

(2) L is contained in a unique singular quad with deep point on L and q subquadrangular quads.

Observe that case (1) can only occur if x has Type (A), (B) or (C), while case (2) can only occur if x has Type (C) or (D).

Suppose case (1) occurs. Then $\Lambda_H(y)$ is the union of a number of lines of $Res(y)$ through a given point of $Res(y)$, union this point. Since every quad through y is singular, subquadrangular or ovoidal, every line of $Res(y)$ intersects $\Lambda_H(y)$ in either 0, 1, 2 or $q+1$ points. Notice also that the point y cannot be deep with respect to H, since otherwise Lemmas 3.4 and 3.6 applied to any singular quad through y would yield that $v = q^5 + q^3 + q^2 + q + 1$, which is impossible. It follows that y is of Type (A), (B) or (C).

If case (2) occurs, then there are two possibilities:

(2a) $\Lambda_H(y)$ is a line of $Res(y)$ + q extra points. By Lemma 3.4, y necessarily is a point of Type (C).

(2b) $|\Lambda_H(y)| = q+1$. If at least three of the points of $\Lambda_H(y)$ are collinear, then $\Lambda_H(y)$ is necessarily a line of $Res(y)$. But this is impossible since y is not the deep point of a singular quad through L. So, no three points of $\Lambda_H(y)$ are collinear. This implies that $\Lambda_H(y)$ is an oval of $Res(y)$, i.e. y is a point of Type (D). □

Definition. As we have already noticed in the proof of Lemma 3.14, every line $L \subseteq H$ must be contained in either $q+1$ singular quads or one singular quad and q subquadrangular quads. If all quads on L are singular, then L is said to be *special*.

Lemma 3.15 *If L is a special line, then L contains only points of Type (A), (B) and (C). Moreover, the number of points of Type (A) on L equals the number of points of Type (C) on L.*

Proof. Since every quad through L is singular, there are $(q+1)q$ lines contained in H that meet L in a unique point. Moreover, for every $y \in L$, $\Lambda_H(y)$ is the union of a number of lines of $Res(y)$, union the point of $Res(y)$ corresponding to L. It follows that every point of L is of Type (A), (B) or (C). Let n_1, n_2, respectively n_3, denote the number of points of Type (A), (B), respectively (C), contained in L. Then $n_1 + n_2 + n_3 = q+1$ and $n_1 \cdot 0 + n_2 \cdot q + n_3 \cdot 2q = q(q+1)$. It follows that $n_1 = n_3$. □

The proof of the following lemma is straightforward.

Lemma 3.16 *Every point of Type (A) is contained in a unique special line. Every point of Type (C) is contained in a unique special line.*

Let n_A, n_B, n_C, n_D, respectively n_E, denote the total number of points of H of Type (A), (B), (C), (D), respectively (E). The following is an immediate corollary of Lemmas 3.15 and 3.16.

Corollary 3.17 *We have $n_C = n_A$.*

Lemma 3.18 *We have $n_E = 0$.*

Proof. We count in two different ways the number of pairs (x, L) with $x \in H$ and L a line of H through x. We find

$$n_A \cdot 1 + n_B \cdot (q+1) + n_C \cdot (2q+1) + n_D \cdot (q+1) + n_E \cdot 0 = l(q+1).$$

Using the facts that $n_A = n_C$ and $l = (q^2+q+1)(q^3+1) = v$, we find $n_A+n_B+n_C+n_D = v$. Hence, $n_E = 0$. □

Lemma 3.19 *We have $n_D = \frac{2q^2}{q+1} n_A$.*

Proof. We count in two different ways the number of pairs (x, Q) where Q is a singular quad and x is its deep point. We find

$$Si = n_B + 2 \cdot n_C, \tag{4}$$

where Si denotes the total number of singular quads. We count in two different ways the number of pairs (x, Q) where Q is a singular quad and x is a point of $Q \cap H$ distinct from the deep point of Q. We find

$$(q+1)q \cdot Si = (q+1)n_A + q(q+1)n_B + (q-1)n_C + (q+1)n_D. \tag{5}$$

From (4) and (5) and the fact that $n_A = n_C$, it readily follows that $n_D = \frac{2q^2}{q+1} n_A$. □

Now, put $\delta := n_A$. Then we have $n_A = n_C = \delta$, $n_D = \frac{2q^2}{q+1} \cdot \delta$ and $n_B = (q^2+q+1)(q^3+1) - \frac{2(q^2+q+1)}{q+1} \cdot \delta$.

Lemma 3.20 *We have $0 \leq \delta \leq \lfloor \frac{1}{2}(q+1)(q^3+1) \rfloor$.*

Proof. This follows from the fact that $n_B \geq 0$. □

Remark. If $q \geq 4$ is even, then by De Bruyn [7], the dual polar space $DW(5, q)$ has up to isomorphism two hyperplanes not containing quads. The values of δ corresponding to these two hyperplanes are respectively equal to 0 and $\frac{q^3(q+1)}{2}$. If q is odd, then by Cooperstein and De Bruyn [5], the dual polar space $DW(5, q)$ has up to isomorphism two hyperplanes not containing quads. The values of δ corresponding to these two hyperplanes are respectively equal to $\frac{1}{2}(q+1)(q^3-1)$ and $\frac{1}{2}(q+1)(q^3+1)$. So, the lower and upper bounds in Lemma 3.20 can be tight.

Definition. Recall that if Q is a quad of $DW(5,q)$ then the points and lines of $DW(5,q)$ contained in Q bijectively correspond to the points and lines of $\mathrm{PG}(4,q)$ that are contained in a given nonsingular parabolic quadric $Q(4,q)$ of $\mathrm{PG}(4,q)$. A *conic* of Q is a set of $q+1$ points of Q that corresponds to a nonsingular conic of $Q(4,q)$, i.e. with a set of $q+1$ points of $Q(4,q)$ contained in a plane π of $\mathrm{PG}(4,q)$ intersecting $Q(4,q)$ in a nonsingular conic of π.

Lemma 3.21 *Let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ be a hyperbolic set of quads of $DW(5,q)$ such that Q_1 is ovoidal with respect to H and $|\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| \geq 2$. Then:*

(1) *$\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$ is a conic of Q_1.*

(2) *The number of ovoidal quads of $\{Q_1, \ldots, Q_{q+1}\}$ is bounded above by $\frac{q+1}{2}$. If the number of these ovoidal quads is precisely $\frac{q+1}{2}$, then the remaining $\frac{q+1}{2}$ quads of $\{Q_1, \ldots, Q_{q+1}\}$ are subquadrangular with respect to H.*

Proof. We first prove that $\pi_{Q_1}(Q_2 \cap H) \neq Q_1 \cap H$. Suppose to the contrary that $\pi_{Q_1}(Q_2 \cap H) = Q_1 \cap H$. Let u be a point of $Q_1 \setminus H$, let L be the unique line through u meeting each quad of $\{Q_1, Q_2, \ldots, Q_{q+1}\}$, let v denote the unique point of L contained in H, and let i be the unique element of $\{3, \ldots, q+1\}$ such that $v \in Q_i$. Now, since $Q_i \cap H$ contains $\pi_{Q_i}(Q_2 \cap H)$ and the point $v \in Q_i \setminus \pi_{Q_i}(Q_2 \cap H)$, we must have $Q_i \subseteq H$. This is however impossible since no quad is contained in H.

So, $\pi_{Q_1}(Q_2 \cap H) \neq Q_1 \cap H$. By Lemma 2.5, $\{\pi_{Q_1}(Q_i \cap H) \,|\, 1 \leq i \leq q+1\}$ is a pencil of hyperplanes of $\widetilde{Q_1}$. Let α_1, α_2, respectively α_3, denote the number of quads of $\{Q_1, \ldots, Q_{q+1}\}$ that are ovoidal, singular, respectively subquadrangular, with respect to H. Put $\beta := |\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| \geq 2$. We prove that $\beta = q+1$.

If $\alpha_1 = q+1$ and $\alpha_2 = \alpha_3 = 0$, then $(q+1)(q^2+1) = |Q_1| = \beta + (q+1)(q^2+1-\beta) = (q+1)(q^2+1) - q\beta < (q+1)(q^2+1)$, a contradiction. So, without loss of generality, we may suppose that Q_2 is not ovoidal with respect to H. If Q_2 is subquadrangular with respect to H, then $\beta = |\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| = q+1$. If Q_2 is singular with respect to H with deep point u such that $\pi_{Q_1}(u) \notin Q_1 \cap H$, then also $\beta = |\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| = q+1$. If Q_1 were singular with respect to H with deep point u such that $\pi_{Q_1}(u) \in Q_1 \cap H$, then $\beta = |\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)| = 1$, a contradiction. Hence, $\beta = q+1$ as claimed.

Now, we have $\alpha_1 + \alpha_2 + \alpha_3 = q+1$ and $(q+1)(q^2+1) = |Q_1| = (q+1) + \alpha_1(q^2-q) + \alpha_2 q^2 + \alpha_3(q^2+q) = (q+1) + (q+1)q^2 + q(\alpha_3 - \alpha_1)$, i.e. $\alpha_1 + \alpha_2 + \alpha_3 = q+1$ and $\alpha_1 = \alpha_3$. Hence, $\alpha_1 = \alpha_3 \leq \frac{q+1}{2}$. Moreover, if $\alpha_1 = \alpha_3 = \frac{q+1}{2}$, then $\alpha_2 = 0$. This proves claim (2).

Now, $\alpha_2 + \alpha_3 \geq \frac{q+1}{2}$. So, $\alpha_2 + \alpha_3 \geq 2$. Without loss of generality, we may suppose that the quads Q_2 and Q_3 are singular or subquadrangular with respect to H.

The points and lines contained in Q_1 can be identified (in a natural way) with the points and lines lying on a given nonsingular parabolic quadric $Q(4,q)$ of $\mathrm{PG}(4,q)$. Now, each of $\pi_{Q_1}(Q_2 \cap H)$ and $\pi_{Q_1}(Q_3 \cap H)$ is either a singular hyperplane or a subgrid of $\widetilde{Q_1}$

and hence arises by intersecting $Q(4,q)$ with a hyperplane of $\mathrm{PG}(4,q)$. Since $\pi_{Q_1}(Q_2 \cap H) \cap \pi_{Q_1}(Q_3 \cap H) = \pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$ is a set of $q+1$ mutually noncollinear points, $\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$ must be a conic of Q_1. □

Lemma 3.22 *If Q_1 is an ovoidal quad, then through every two points of $Q_1 \cap H$, there is a conic of Q_1 that is completely contained in $Q_1 \cap H$.*

Proof. Let x_1 and x_2 be two distinct points of $Q_1 \cap H$. By Lemmas 3.14 and 3.18, there exists a line L_i, $i \in \{1,2\}$ through x_i that is contained in H. Let Q_2 be a quad distinct from Q_1 that meets L_1 and L_2, and let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ be the unique hyperbolic set of quads of $DW(5,q)$ containing Q_1 and Q_2. Since $\{x_1, x_2\} \subseteq \pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$, Lemma 3.21 applies. We conclude that $\pi_{Q_1}(Q_2 \cap H) \cap (Q_1 \cap H)$ is a conic containing x_1 and x_2. □

Lemma 3.23 *For every quad Q_1 that is ovoidal with respect to H, there is a quad Q_2 disjoint from Q_1 that is singular with respect to H such that $\pi_{Q_1}(u) \notin Q_1 \cap H$ where u is the deepest point of the singular hyperplane $Q_2 \cap H$ of $\widetilde{Q_2}$.*

Proof. The number of points $x \in \Gamma_1(Q_1) \cap H$ for which $\pi_{Q_1}(x) \notin Q_1 \cap H$ is equal to $(|Q_1| - |Q_1 \cap H|) \cdot q^2 = q^3(q^2+1)$. Now, since $n_D = \frac{2q^2}{q+1}\delta \leq \frac{2q^2}{q+1} \cdot \frac{1}{2}(q+1)(q^3+1) = q^2(q^3+1) < q^3(q^2+1)$, there exists a point $y \in \Gamma_1(Q_1) \cap H$ not of type (D) for which $\pi_{Q_1}(y) \notin Q_1 \cap H$. Let $L \subseteq H$ be a special line through y and let z denote the unique point of L for which $\pi_{Q_1}(z) \in Q_1 \cap H$. By Lemma 3.14, there are at most two quads R through L for which z is the deep point of the singular hyperplane $R \cap H$ of $\widetilde{R}$. Hence, there exists a quad Q_2 through L for which the deep point u of the singular hyperplane $Q_2 \cap H$ of $\widetilde{Q_2}$ is distinct from z. Since u is not collinear with a point of $Q_1 \cap H$, Q_1 and Q_2 are disjoint. □

Lemma 3.24 *If Q_1 is ovoidal with respect to H, then $Q_1 \cap H$ is a classical ovoid of $\widetilde{Q_1}$.*

Proof. By Lemma 3.23, there exists a quad Q_{q+1} disjoint from Q_1 that is singular with respect to H such that $\pi_{Q_1}(u) \notin Q_1 \cap H$ where u is the deep point of the singular hyperplane $Q_{q+1} \cap H$ of $\widetilde{Q_{q+1}}$. Let $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ denote the unique hyperbolic set of quads of $DW(5,q)$ containing Q_1 and Q_{q+1}. By Lemma 3.21, we then have:

(1) $X := \pi_{Q_1}(Q_{q+1} \cap H) \cap (Q_1 \cap H)$ is a conic of Q_1;

(2) the number k of ovoidal quads of the set $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ is at most $\frac{q}{2}$.

Without loss of generality, we may suppose that $Q_1, \ldots, Q_k$ are the quads of $\{Q_1, Q_2, \ldots, Q_{q+1}\}$ that are ovoidal with respect to H. Since $(q+1) - \frac{q}{2} \geq 2$, Q_q and Q_{q+1} are not ovoidal with respect to H. By Lemmas 2.2 and 2.5, $\pi_{Q_1}(Q_q \cap H)$ and $\pi_{Q_1}(Q_{q+1} \cap H)$ are contained in a unique pencil of classical hyperplanes of $\widetilde{Q_1}$. Moreover, this pencil contains the hyperplanes $\pi_{Q_1}(Q_i \cap H)$, $i \in \{k+1, \ldots, q+1\}$. Let $A_1, \ldots, A_k$ denote the remaining elements of this pencil. Then $X \subseteq A_1 \cap \cdots \cap A_k$ and $A_1 \cup \cdots \cup A_k = \pi_{Q_1}(Q_1 \cap H) \cup \cdots \cup \pi_{Q_1}(Q_k \cap H)$. Now, $|A_1 \cup \cdots \cup A_k| \geq |X| + k(q^2+1-|X|) = (q+1) + k(q^2-q)$ and equality holds if and only if every A_j, $j \in \{1, \ldots, k\}$, is a classical ovoid of $\widetilde{Q_1}$. Now, since $|\pi_{Q_1}(Q_1 \cap H) \cup \cdots \cup \pi_{Q_1}(Q_k \cap H)| = |X| + k(q^2+1-|X|) = (q+1) + k(q^2-q)$, we can conclude that every A_j, $j \in \{1,\ldots,k\}$, is a classical ovoid of $\widetilde{Q_1}$.

Now, let $i \in \{1,\ldots,k\}$ and suppose there exists no $j \in \{1,\ldots,k\}$ such that $\pi_{Q_1}(Q_i \cap H) = A_j$. Then $X \subseteq \pi_{Q_1}(Q_i \cap H) \subseteq A_1 \cup \cdots \cup A_k$ and there exist two distinct $j_1, j_2 \in \{1,\ldots,k\}$ such that $\pi_{Q_1}(Q_i \cap H) \cap (A_{j_1} \setminus X) \neq \emptyset$ and $\pi_{Q_1}(Q_i \cap H) \cap (A_{j_2} \setminus X) \neq \emptyset$. Let y_1 be an arbitrary point of $\pi_{Q_1}(Q_i \cap H) \cap (A_{j_1} \setminus X)$ and let y_2 be an arbitrary point of $\pi_{Q_1}(Q_i \cap H) \cap (A_{j_2} \setminus X)$. By Lemma 3.22, there exists a conic C through y_1 and y_2 that is completely contained in $\pi_{Q_1}(Q_i \cap H)$ and hence also in $A_1 \cup \cdots \cup A_k$. Since $|C| = q+1$ and $k \leq \frac{q}{2}$, there exists a $j_3 \in \{1,\ldots,k\}$ such that $|C \cap A_{j_3}| \geq 3$. Since A_{j_3} is a classical ovoid of $\widetilde{Q_1}$, this necessarily implies that $C \subseteq A_{j_3}$, contradicting the fact that $y_1 \in A_{j_1} \setminus X$, $y_2 \in A_{j_2} \setminus X$ and $j_1 \neq j_2$. Hence, there exists a $j \in \{1,\ldots,k\}$ such that $\pi_{Q_1}(Q_i \cap H) = A_j$. This implies that the ovoid $Q_i \cap H$ of $\widetilde{Q_i}$ is classical. □

Corollary 3.25 *The hyperplane H is classical.*

Proof. This is an immediate corollary of Proposition 2.4 and Lemma 3.24. □

Remark. With the terminology of Cooperstein & De Bruyn [5] and De Bruyn [7], the hyperplane H is either a hyperplane of Type V or a hyperplane of Type VI.

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