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Functional properties of rearrangement invariant spaces defined in terms of oscillations

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Abstract

Function spaces whose definition involves the quantity $f^{**} - f^*$, which measures the oscillation of f^* , have recently attracted plenty of interest and proved to have many applications in various, quite diverse fields. Primary role is played by the spaces $S_p(w)$, with $0 < p < \infty$ and w a weight function on $(0, \infty)$, defined as the set of Lebesgue-measurable functions on \mathbb{R} such that $f^*(\infty) = 0$ and

$$\|f\|_{S_p(w)} := \left(\int_0^\infty (f^{**}(s) - f^*(s))^p w(s) ds \right)^{1/p} < \infty.$$

Some of the main open questions concerning these spaces relate to their functional properties, such as their lattice property, normability and linearity. We study these properties in this paper.

Keywords and phrases: Lattice property, normability, distribution function, decreasing rearrangement.

1 Introduction

In 1981, in order to obtain a Marcinkiewicz–type interpolation theorem for operators that are unbounded on L^∞ , C. Bennett, R. DeVore, and R. Sharpley in [4], introduced a new rearrangement-invariant space consisting of those measurable functions for which $f^{**} - f^*$ is bounded (where f^* is the decreasing rearrangement of f and $f^{**}(t) = t^{-1} \int_0^t f^*(s) ds$) which plays the role of “weak- L^∞ ” in the sense that it contains L^∞ and possesses appropriate interpolation properties. Moreover, since $f^{**} - f^*$ can be interpreted as some kind of measure of the oscillation of f^* , they proved that weak- $L^\infty(Q)$, (Q is a cube in \mathbb{R}^n) is, in fact, the rearrangement-invariant hull of $BMO(Q)$. Since then, the main ideas that $f^{**} - f^*$ might be useful as a replacement for f^* in certain contexts, and that a natural way to measure the oscillation of a decreasing function is provided by the quantity $f^{**} - f^*$, have been particularly fruitful, and have been applied in various problems. The principal difficulty which one meets when dealing with the functional $f^{**} - f^*$ is that this expression is not linear, this problem has been generally solved in two ways, obtaining equivalent expressions in terms of $f^{**} - f^*$ for the norms of classical spaces (usually generalizations of the classical Lorentz spaces $L^{p,q}$, see for example [5], [14], [6], and the references quoted therein) or obtaining some estimates for the difference $f^{**} - f^*$ without any connection with spaces (see [3] and [13]).

In 2003, J. Bastero, M. Milman and F. Ruiz (see [2]) combined both ideas in order to prove a sharp version of the Sobolev embedding theorem. First using a natural extension of the classical Lorentz spaces $L^{(p,q)}(\Omega)$ (Ω is an open subset of \mathbb{R}^n) introduced a new scale of spaces (conditions) that interpolate between L^∞ and the space weak- L^∞ of Bennett-DeVore-Sharpley, defined by¹

$$L(\infty, q)(\Omega) = \{f : t^{-1/q}(f^{**}(t) - f^*(t)) \in L_q(\Omega)\},$$

and second, proving the following new form of the Pólya-Szegő symmetrization principle

$$f^{**}(t) - f^*(t) \leq c_n(\nabla f)^{**}(t)t^{1/n}, \quad f \in C_0^\infty(\Omega) \quad (1)$$

they stated that the $L(\infty; q)$ spaces are natural target spaces for sharp endpoint Sobolev embedding theorems. In particular, when $|\Omega| < \infty$, they proved that (see also [17])

$$W_0^{1,n}(\Omega) \subset L(\infty, n)(\Omega) \subset BW_n(\Omega) \quad (2)$$

where $BW_n(\Omega)$ is the Maz’ya–Hansson–Brézis–Wainger space defined by the condition

$$\int_0^1 \left(\frac{f^*(t)}{\log(e/t)} \right)^n \frac{dt}{t} < \infty.$$

Notice that the first embedding of (2) follows readily from (1), and thus the proof of (2) is reduced to an embedding result for rearrangement-invariant spaces.

M. Milman and E. Pustylnik in the recent paper [18], (see also [19]), extending the methods developed in [2], to the case $k > 1$; obtained a unified method to prove the Sobolev embedding theorem and the corresponding sharp borderline cases. They started by showing that²

$$W_0^{k, n/k}(\Omega) \subset L(\infty, \frac{n}{k})(\Omega) \subsetneq BW_{\frac{n}{k}}(\Omega). \quad (3)$$

¹See [20], [21] and [22] for the role of $L(\infty, q)(\Omega)$ spaces in very actual problems of interpolation theory.

²The consideration of these spaces allows the authors treat the case $p = \frac{n}{k}$, $q = \infty$ in a unified form with the other cases that appear in the literature.

Moreover, sets of functions with finite quantities $\|f^{**} - f^*\|_E$ appear in this work for a large spectrum of spaces E in a very natural context of optimal Sobolev embeddings. In particular in the setting of rearrangement invariant spaces they extend results in [12] in the sense that if Y is a rearrangement invariant space which satisfy some mild conditions, then $W_0^{k;Y}(\Omega) \subset Y_n(\infty; k) = \{f : t^{-k/n}(f^{**}(t) - f^*(t)) \in Y\}$ and, in fact, $Y_n(\infty; k)$ is not larger (and in many cases essentially smaller) than any rearrangement invariant space $X(\Omega)$ such that $W_0^{k;Y}(\Omega) \subset X(\Omega)$.

The fractional case of (3) have been considered in [16] by proving a rearrangement inequalities that give a relation between the oscillation $f^{**} - f^*$ and the modulus of continuity of the function f , which is the suitable replacement for (1), in the fractional case, and allow the authors to derive applications to embedding of Besov spaces, using again a suitable function space E such that $\|f^{**} - f^*\|_E < \infty$ and following the method developed in [18].

In this paper, given w , a non-negative Lebesgue measurable function on \mathbb{R}^+ (briefly a weight), and given $0 < p < \infty$, we consider two types of weighted function spaces whose definition involves the quantity $f^{**} - f^*$:

- 1) The space $S_p(w)$, defined by those measurable functions f such that $f^*(\infty) = 0$ and

$$\|f\|_{S_p(w)} := \left(\int_0^\infty (f^{**}(s) - f^*(s))^p w(s) ds \right)^{1/p} < \infty,$$

and its weak version,

- 2) the space $S_{p,\infty}(w)$, defined by those measurable functions such that $f^*(\infty) = 0$ and

$$\|f\|_{S_{p,\infty}(w)} := \sup_{t>0} (f^{**}(t) - f^*(t)) t \left(\int_t^\infty \frac{w(s)}{s^p} ds \right)^{1/p} < \infty.$$

Obviously, $S_p(w)$ and $S_{p,\infty}(w)$ are invariant under rearrangement and $S_p(w) \subset S_{p,\infty}(w)$ (see Corollary 2.1).

Obviously $S_p(w) = \{f : f^{**}(t) - f^*(t) \in L^p(w)\}$, so our spaces are a particular case of the Milman-Pustylnik spaces by taking $E = L^p(w)$ (see [18]).

Example 1.1 *If $w(t) = 1/t\chi_{[0,1]}$, then*

$$\|f\|_{S_p(w)} = \left(\int_0^1 (f^{**}(t) - f^*(t))^p \frac{dt}{t} \right)^{1/p}$$

and we obtain the function spaces considered in [17], [13] and [2]. Notice that (see Subsection 3.1.3 below)

$$S_1(w) = L^\infty \quad \text{with} \quad \|f\|_{S_1(w)} = \|f\|_{L^\infty} - \|f^* \chi_{[0,1]}\|_{L^1}.$$

Similarly,

$$\|f\|_{S_{1,\infty}(1/t)} = \sup_{t>0} (f^{**}(t) - f^*(t))$$

is the Bennett-DeVore-Sharpely space Weak- L^∞ .

One of the principal difficulties which one meets when dealing with spaces defined in terms of $f^{**} - f^*$ is that this expression is not linear, thus functional properties like normability, lattice property or linearity are difficult to prove.

The purpose of this paper is to study functional properties of the spaces $S_p(w)$ and $S_{p,\infty}(w)$. In particular, we investigate whether or not these spaces have the lattice property, whether they are normable and whether they form linear sets (these problems were posed explicitly in the last section of [18]). Embedding properties between the function spaces $S_p(w)$ and the classical Lorentz spaces $\Lambda^p(v)$ and $\Gamma^p(v)$ will be considered in the forthcoming paper [8].

The paper is organized as follows: In Section 2, we provide some technical results involving the functional $f^{**} - f^*$. In Section 3, we characterize the weights w for which $S_p(w)$ (resp. $S_{p,\infty}(w)$) is a lattice, is a normed space and obtain necessary conditions to be a linear space. We actually prove that, in order to have the lattice property (and, likewise, in order to be normable), it is necessary for each of the spaces $S_p(w)$ and $S_{p,\infty}(w)$ to coincide with an appropriate classical Lorentz space of type ‘Gamma’ (see the definitions below). Finally, in Section 4, we describe the associate space of $S_p(w)$ and $S_{p,\infty}(w)$.

As usual, the symbol $f \simeq g$ will indicate the existence of a universal constant $c > 0$ (independent of all parameters involved) so that $(1/c)f \leq g \leq cf$, while the symbol $f \preceq g$ means that $f \leq cg$ (resp. $f \succeq g$, means that $f \geq cg$). We write $\|g\|_p$ to denote $\|g\|_{L^p}$ and given a weight w , we denote $W(t) := \int_0^t w(s) ds$.

In what follows in order to avoid some complication with the notation and since our results can be easily extended to \mathbb{R}^n we assume that the underlying measure space is \mathbb{R} .

2 Preliminaries and technical lemmas

Let $L^0 = L^0(\mathbb{R})$ be the space of all (equivalence classes of) Lebesgue measurable functions on \mathbb{R} . Given $f \in L^0$, its *distribution function* is defined, for $t > 0$, by

$$\lambda_f(t) = |\{x \in \mathbb{R} : |f(x)| > t\}|$$

(where $|\cdot|$ denotes the Lebesgue measure) and the *decreasing rearrangement* of f is defined by

$$f^*(t) = \inf\{s > 0 : \lambda_f(s) \leq t\}, \quad t > 0.$$

Also, f^{**} is the maximal function of f^* , i.e.

$$f^{**}(t) := Pf^*(t),$$

where P is the Hardy operator on $\mathbb{R}^+ := [0, \infty)$ defined by $Ph(t) := \frac{1}{t} \int_0^t h(s) ds$.

In the next lemmas, we collect some basic properties of the functional $f^{**} - f^*$ that will be useful in what follows.

Let us consider the cone

$$A = \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^+) \text{ positive, decreasing and such that } f(\infty) = 0 \right\}.$$

Given $f \in A$, let us define

$$Tf(t) := \frac{1}{t} \left(Pf\left(\frac{1}{t}\right) - f\left(\frac{1}{t}\right) \right), \quad (4)$$

For our later purpose, it is important to observe that, for every $f \in L^1_{loc}(\mathbb{R})$,

$$\frac{1}{t} \left(f^{**} \left(\frac{1}{t} \right) - f^* \left(\frac{1}{t} \right) \right) = Tf^*(t). \quad (5)$$

Lemma 2.1 *The operator T , defined by (4), is linear, satisfying*

$$T : A \rightarrow A,$$

and such that

$$1) \ T \circ T = Id.$$

$$2) \ T \text{ is self-adjoint.}$$

In particular, every $h \in A$ can be represented as Tf for some $f \in A$.

Proof: That $Tf \in A$ if $f \in A$, is an easy consequence of the following formula relating f^{**} , f^* and λ_f .

$$f^{**}(t) - f^*(t) = \frac{1}{t} \int_{f^*(t)}^{\|f\|_\infty} \lambda_f(s) ds. \quad (6)$$

To see (6), let $[x]^+ = \max(x, 0)$ then for all $y > 0$ we have that

$$\int_0^\infty [f^*(x) - y]^+ dx = \int_0^\infty \lambda_{[f^*-y]^+}(s) ds = \int_y^\infty \lambda_{f^*}(s) ds = \int_y^{\|f\|_\infty} \lambda_f(s) ds. \quad (7)$$

Inserting $y = f^*(t)$ in (7) and taking in account that f^* is decreasing we get

$$t(f^{**}(t) - f^*(t)) = \int_0^t (f^*(x) - f^*(t)) dx = \int_0^\infty [f^*(x) - f^*(t)]^+ dx = \int_{f^*(t)}^{\|f\|_\infty} \lambda_f(s) ds.$$

To prove 1), we observe that a simple computation shows that $Tf(t)$ is the derivative of the function $t \rightarrow t \int_0^{t^{-1}} f(s) ds$. Hence,

$$\int_0^{r^{-1}} Tf(s) ds = \int_0^{r^{-1}} \frac{\partial}{\partial s} \left(s \int_0^{s^{-1}} f(u) du \right) ds = \frac{1}{r} \int_0^r f(s) ds = Pf(r) \quad (8)$$

and then,

$$T(Tf)(t) = \frac{1}{t} \left(P(Tf) \left(\frac{1}{t} \right) - Tf \left(\frac{1}{t} \right) \right) = Pf(t) - Pf(t) + f(t) = f(t).$$

Finally, to see 2), if $f, g \in A$, using (6), Fubini's theorem and (8), we obtain

$$\begin{aligned} \int_0^\infty Tf(t)g(t) dt &= \int_0^\infty (Pf(t) - f(t))g \left(\frac{1}{t} \right) \frac{1}{t} dt = \int_0^\infty \left(\int_{f(t)}^{\|f\|_\infty} \lambda_f(s) ds \right) g \left(\frac{1}{t} \right) \frac{1}{t^2} dt \\ &= \int_0^{\|f\|_\infty} \lambda_f(u) \left(\int_{\lambda_f(u)}^\infty g \left(\frac{1}{t} \right) \frac{1}{t^2} dt \right) du \\ &= \int_0^{\|f\|_\infty} \lambda_f(u) \int_0^{1/\lambda_f(u)} g(t) dt du \\ &= \int_0^{\|f\|_\infty} \lambda_f(u) \int_0^{1/\lambda_f(u)} T(Tg)(t) dt du = \int_0^{\|f\|_\infty} \int_0^{\lambda_f(u)} (Tg)(t) dt du \\ &= \int_0^\infty f(t)Tg(t) dt. \quad \square \end{aligned}$$

Corollary 2.1 $S_p(w) \subset S_{p,\infty}(w)$.

Proof: Since $t(f^{**}(t) - f^*(t))$ is an increasing function of t , we have

$$t(f^{**}(t) - f^*(t)) \left(\int_t^\infty \frac{w(s)}{s^p} ds \right)^{1/p} \leq \left(\int_0^\infty (f^{**}(s) - f^*(s))^p w(s) ds \right)^{1/p},$$

and the result follows immediately. \square

Since, for any $D \subset \mathbb{R}$ with $|D| = t$,

$$\|\chi_D\|_{S_p(w)} = t \left(\int_t^\infty \frac{w(s)}{s^p} ds \right)^{1/p} = \|\chi_D\|_{S_{p,\infty}(w)},$$

$S_p(w) \neq \{0\}$ (resp. $S_{p,\infty}(w) \neq \{0\}$) if and only if $\int_t^\infty \frac{w(s)}{s^p} ds < \infty$. In particular, if $S_p(w)$ is non-trivial, it contains all characteristic functions of sets of finite measure. This suggests the following definition:

Definition 2.1 The fundamental function $\varphi_{p,w}$ of $S_p(w)$ (resp. $S_{p,\infty}(w)$) is given by

$$\varphi_{p,w}(t) := t \left(\int_t^\infty \frac{w(s)}{s^p} ds \right)^{1/p} \quad (t > 0).$$

Sometimes, it will be useful to get a description of $\|f\|_{S_p(w)}$ in terms of the distribution function of f , and, to this end, we need to work with the following operator, defined on decreasing functions,

$$A_p(g)(t) := \left(\frac{\int_t^\infty g(s) ds}{g(t)} \right)^{p-1} \varphi_{p,w}(g(t))^p. \quad (9)$$

Lemma 2.2 If $0 < p < \infty$,

$$\|f\|_{S_p(w)} = \left(p \int_0^{\|f\|_\infty} A_p(\lambda_f)(t) dt \right)^{1/p}.$$

Proof: Using (6) and Fubini's theorem, we obtain

$$\begin{aligned} \|f\|_{S_p(w)}^p &= \int_0^\infty (f^{**}(t) - f^*(t))^p w(t) dt = \int_0^\infty \left(\int_{f^*(t)}^{\|f\|_\infty} \lambda_f(s) ds \right)^p \frac{w(t)}{t^p} dt \\ &= p \int_0^\infty \left(\int_{f^*(t)}^\infty \left(\int_u^{\|f\|_\infty} \lambda_f(s) ds \right)^{p-1} \lambda_f(u) du \right) \frac{w(t)}{t^p} dt \\ &= p \int_0^{\|f\|_\infty} \left(\int_u^\infty \lambda_f(s) ds \right)^{p-1} \lambda_f(u) \left(\int_{\lambda_f(u)}^\infty \frac{w(s)}{s^p} ds \right) du \\ &= p \int_0^{\|f\|_\infty} \left(\frac{\int_u^\infty \lambda_f(s) ds}{\lambda_f(u)} \right)^{p-1} \varphi_{p,w}(\lambda_f(u))^p du. \quad \square \end{aligned}$$

The following class of weights will play an important role throughout the paper (cf. [1]).

Definition 2.2 Given $0 < p < \infty$, we shall say that w satisfies the reverse B_p -condition, $w \in RB_p$, if there is a constant $c > 0$ such that

$$\int_0^r w(s) ds \leq cr^p \int_r^\infty \frac{w(s)}{s^p} ds \quad (r > 0).$$

Via the change of variables $t \rightarrow 1/t$, one gets

$$w \in RB_p \iff t^{p-2}w\left(\frac{1}{t}\right) \in B_p;$$

we recall that $v \in B_p$ if

$$r^p \int_r^\infty \frac{v(s)}{s^p} ds \leq c \int_0^r v(s) ds.$$

Moreover $\|Pf\|_{\Lambda^p(v)} \simeq \|f\|_{\Lambda^p(v)}$ for all $f \in \Lambda^p(v)$ if and only if $v \in B_p$ (see [1]).

We end this section by recalling the definition of the classical Lorentz spaces:

$$\Lambda^p(w) = \left\{ f \in L^0 : \|f\|_{\Lambda^p(w)} = \left(\int_0^\infty f^*(s)^p w(s) ds \right)^{1/p} < \infty \right\}.$$

and

$$\Gamma^p(w) = \left\{ f \in L^0 : \|f\|_{\Gamma^p(w)} = \left(\int_0^\infty f^{**}(s)^p w(s) ds \right)^{1/p} < \infty \right\}.$$

and the weak type space

$$\Gamma^\infty(w) = \left\{ f \in L^0 : \|f\|_{\Gamma^\infty(w)} = \sup_{t>0} f^{**}(t)w(t) < \infty \right\}.$$

3 Functional properties.

As said in the introduction, we study, in this section, several functional properties of the spaces $S_p(w)$ and $S_{p,\infty}(w)$. In particular, we consider the problem of characterizing when $S_p(w)$ (resp. $S_{p,\infty}(w)$) satisfies one of the following properties:

- 1) to be a lattice,
- 2) to be a normed space.

Furthermore, we obtain a necessary condition for $S_p(w)$ and $S_{p,\infty}(w)$ to be a linear space.

We shall use the symbol $h \searrow$ to denote that h is decreasing on \mathbb{R}^+ , and if a Lebesgue-measurable function h is only defined on \mathbb{R}^+ , we shall denote by the same letter h , the extended function defined on \mathbb{R} by $h(x) = 0$ if $x < 0$.

3.1 The space $S_p(w)$.

3.1.1 The Lattice property.

It is our aim in this section to investigate when $S_p(w)$ has the lattice property, i.e. when there is a constant $c > 0$, such that

$$g \in S_p(w) \text{ and } \|g\|_{S_p(w)} \leq c\|f\|_{S_p(w)}, \text{ whenever } g \in L^0, f \in S_p(w) \text{ and } |g| \leq |f| \text{ a.e.}$$

Before formulating our main results, it will be convenient to present two preliminary ones:

Proposition 3.1 *Let $0 < p < \infty$. If $S_p(w)$ has the lattice property, then*

1) $\varphi_{p,w}$ is **quasi-increasing**, i.e. there is $c > 0$, such that

$$\varphi_{p,w}(x) \leq c\varphi_{p,w}(y), \text{ whenever } x \leq y.$$

2) $S_p(w) \subset \Gamma^\infty(\varphi_{p,w})$.

Proof: 1) Observe that if $x \leq y$, then $\chi_{[0,x]} \leq \chi_{[0,y]}$, and therefore

$$\varphi_{p,w}(x) = \|\chi_{[0,x]}\|_{S_p(w)} \leq c\|\chi_{[0,y]}\|_{S_p(w)} = c\varphi_{p,w}(y).$$

2) Given $f \in S_p(w)$, and $r > 0$, let us consider f^* and $f^*\chi_{[0,r]}$. Since $(f^*)^* = f^*$, $f^* \in S_p(w)$ and, obviously,

$$\|f^*\|_{S_p(w)} = \|f\|_{S_p(w)}.$$

On the other hand, since $f^*\chi_{[0,r]} \leq f^*$, and

$$\begin{aligned} \|f^*\chi_{[0,r]}\|_{S_p(w)}^p &= \int_0^r (f^*(s) - f^*(s))^p w(s) ds + \left(\int_0^r f^*(s) ds \right)^p \int_r^\infty \frac{w(s)}{s^p} ds \\ &\geq \left(\frac{1}{r} \int_0^r f^*(s) ds \right)^p \varphi_{p,w}(r)^p, \end{aligned}$$

we have, by the lattice property, that

$$\|f\|_{\Gamma^\infty(\varphi_{p,w})} = \sup_{r>0} \left(\frac{1}{r} \int_0^r f^*(s) ds \right) \varphi_{p,w}(r) \leq \sup_{r>0} \|f^*\chi_{[0,r]}\|_{S_p(w)} \leq c\|f\|_{S_p(w)}. \quad \square$$

Remark 3.1 *A function ψ is called pseudoconcave if it is equivalent on \mathbb{R}^+ to a concave strictly positive function. It is a well-known result (see, e.g. [15], Theorem 1.1 of Chapter 2) that ψ is pseudoconcave if and only if there is a constant c such that*

$$\psi(s) \leq c\psi(t) \quad \text{and} \quad \psi(t)/t \leq c\psi(s)/s$$

whenever $0 < s < t < \infty$.

Thus, if $\varphi_{p,w}$ is quasi-increasing, it follows readily that it is pseudoconcave and hence, there is a decreasing function v such that

$$\varphi_{p,w}(t) \simeq \varphi_{p,w}(0^+) + \int_0^t v(s) ds.$$

Theorem 3.1 *Let $0 < p < \infty$. Then, $S_p(w)$ has the lattice property if and only if there is $c > 0$, such that, for every decreasing functions $g_1 \leq g_2$,*

$$\int_0^\infty A_p(g_1)(t) dt \leq c \int_0^\infty A_p(g_2)(t) dt, \quad (10)$$

where the operator A_p is defined as in (9).

Proof: Assume first that $S_p(w)$ is a lattice. Let g_1, g_2 be decreasing such that $g_1 \leq g_2$ and let f_1 and f_2 satisfy $\lambda_{f_1} = g_1$ and $\lambda_{f_2} = g_2$. Then $\lambda_{f_1} \leq \lambda_{f_2}$, and hence $f_1^* \leq f_2^*$. Now, since $S_p(w)$ is a lattice,

$$\|f_1\|_{S_p(w)} \leq c \|f_2\|_{S_p(w)},$$

which, by Lemma 2.2, is equivalent to (10).

Conversely, if $f, g \in L^0$ with $|f| \leq |g|$ and $g \in S_p(w)$, we have, since $\lambda_f \leq \lambda_g$,

$$\int_0^\infty A_p(\lambda_f)(t) dt \leq c \int_0^\infty A_p(\lambda_g)(t) dt,$$

and Lemma 2.2 applies. □

The case $1 \leq p < \infty$.

Theorem 3.2 *$S_1(w)$ has the lattice property if and only if $\varphi_{1,w}$ is quasi-increasing. Moreover, there is a decreasing weight v such that, if $\varphi_{1,w}(0^+) = 0$, $S_1(w) = \Lambda^1(v)$, and if $\varphi_{1,w}(0^+) \neq 0$, then $S_1(w) = \Lambda^1(v) \cap L^\infty$.*

Proof: By Proposition 3.1, if $S_1(w)$ has the lattice property, $\varphi_{1,w}$ is quasi-increasing. Conversely, since

$$A_1(g)(t) = \varphi_{1,w}(g(t))$$

and $\varphi_{1,w}$ is quasi-increasing, Theorem 3.1 applies.

On the other hand, by Lemma 2.2,

$$\|f\|_{S_1(w)} = \int_0^{\|f\|_\infty} \varphi_{1,w}(\lambda_f(u)) du \simeq \int_0^\infty f^*(t)v(t) dt + \varphi_{1,w}(0^+)\|f\|_\infty,$$

where, by Remark 3.1,

$$\varphi_{1,w}(t) \simeq \varphi_{1,w}(0^+) + \int_0^t v(s) ds. \quad \square$$

Let us see now that, for $p > 1$, the relevant information of Proposition 3.1 is contained in the embedding $S_p(w) \subset \Gamma^\infty(\varphi_{p,w})$.

Theorem 3.3 *Let $1 < p < \infty$. The following statements are equivalent:*

- 1) $S_p(w)$ has the lattice property.
- 2) $S_p(w) \subset \Gamma^\infty(\varphi_{p,w})$.
- 3) $w \in RB_p$.

$$4) S_p(w) = \Gamma^p(w).$$

Proof: 1) \Rightarrow 2) is Proposition 3.1-2).

2) \Rightarrow 3) We first claim that

$$\sup_{r>0} \varphi_{p,w}(r) \sup_{g \searrow} \frac{\int_0^\infty g(s) \chi_{[0,1/r]}(s) ds}{\left(\int_0^\infty g(s)^p s^{p-2} w\left(\frac{1}{s}\right) ds\right)^{1/p}} < \infty. \quad (11)$$

Then, using (8), (5), and the change of variables $t \rightarrow \frac{1}{s}$, we have, for every $f^* \in A$ (therefore for every $f \in S_p(w)$) and every $r > 0$,

$$\frac{\int_0^r f^*(s) ds}{\|f\|_{S_p(w)}} = \frac{r \int_0^{r^{-1}} T f^*(s) ds}{\left(\int_0^\infty (T f^*(s))^p s^{p-2} w\left(\frac{1}{s}\right) ds\right)^{1/p}}.$$

Let c be the constant of the embedding in 2), that is,

$$c = \sup_{f \neq 0} \sup_{r>0} \frac{\varphi_{p,w}(r) f^{**}(r)}{\|f\|_{S_p(w)}}.$$

Fix $r > 0$. Then, by our hypothesis and Lemma 2.1,

$$c \geq \varphi_{p,w}(r) \sup_{f \in S_p(w)} \frac{\frac{1}{r} \int_0^r f^*(s) ds}{\|f\|_{S_p(w)}} = \varphi_{p,w}(r) \sup_{h \in A} \frac{\int_0^\infty h(s) \chi_{[0,1/r]}(s) ds}{\left(\int_0^\infty h(s)^p s^{p-2} w\left(\frac{1}{s}\right) ds\right)^{1/p}}.$$

Now, in order to complete the proof of (11), we only have to consider the situation when g is a constant function (since every $g \searrow$ can be represented as a sum of $h \in A$ and a constant). But then, we obviously have

$$\varphi_{p,w}(r) \sup_{g \searrow} \frac{\int_0^\infty g(s) \chi_{[0,1/r]}(s) ds}{\left(\int_0^\infty g(s)^p s^{p-2} w\left(\frac{1}{s}\right) ds\right)^{1/p}} = \left(\frac{\int_0^\infty \frac{w(s)}{s^p} ds}{\int_0^\infty \frac{w(s)}{s^p} ds}\right)^{1/p} \leq 1,$$

and (11) follows.

On the other hand, a result of Sawyer (see [23] Theorem 1) ensures that

$$\sup_{g \searrow} \frac{\int_0^\infty g(s) \chi_{[0,1/r]}(s) ds}{\left(\int_0^\infty g(s)^p s^{p-2} w\left(\frac{1}{s}\right) ds\right)^{1/p}} \simeq \left(\int_0^{1/r} \left(\frac{t}{\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds}\right)^{p'-1} dt\right)^{1/p'},$$

where as usual $1/p + 1/p' = 1$.

Summarizing, we have proved that

$$\varphi_{p,w}(r) \left(\int_0^{1/r} \left(\frac{t}{\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds}\right)^{p'-1} dt\right)^{1/p'} \leq c,$$

i.e.

$$\frac{1}{r} \left(\int_{1/r}^\infty \frac{w(s)}{s^p} ds\right)^{1/p} \left(\int_0^r \left(\frac{t}{\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds}\right)^{p'-1} dt\right)^{1/p'} \leq c, \quad r \in (0, \infty).$$

Finally, since

$$\int_{1/r}^\infty \frac{w(s)}{s^p} ds = \int_0^r s^{p-2} w\left(\frac{1}{s}\right) ds,$$

we get that

$$\frac{1}{r} \left(\int_0^r s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p} \left(\int_0^r \left(\frac{t}{\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds} \right)^{p'-1} dt \right)^{1/p'} \leq c,$$

which, by [23], Theorem 4, is equivalent to $s^{p-2}w(1/s) \in B_p$; that means $w \in RB_p$.

3) \Rightarrow 4) Since $w \in RB_p \Leftrightarrow s^{p-2}w(1/s) \in B_p$ and Tf^* is decreasing, we get

$$\begin{aligned} \left(\int_0^\infty (f^{**}(s) - f^*(s))^p w(s) ds \right)^{1/p} &= \left(\int_0^\infty (Tf^*(s))^p s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p} \\ &\simeq \left(\int_0^\infty \left(\frac{1}{s} \int_0^s Tf^*(u) du \right)^p s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p}, \end{aligned}$$

and, by (8),

$$\begin{aligned} \left(\int_0^\infty \left(\frac{1}{s} \int_0^s Tf^*(u) du \right)^p s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p} &= \left(\int_0^\infty \left(\int_0^{1/s} f^*(u) du \right)^p s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p} \\ &= \left(\int_0^\infty f^{**}(s)^p w(s) ds \right)^{1/p} = \|f\|_{\Gamma^p(w)}. \end{aligned}$$

4) \Rightarrow 1) is obvious. □

The case $0 < p < 1$.

In this case, the embedding $S_p(w) \subset \Gamma^\infty(\varphi_{p,w})$ does not give us any relevant information about the lattice property of $S_p(w)$, since, actually, the above embedding follows from the fact that $\varphi_{p,w}$ is quasi-increasing. To see this, let $\widehat{w}(s) = s^{p-2}w(1/s)$, and let us write $V(r) = \int_0^r \widehat{w}(s) ds$. Then, it is known (see [9] Theorem 4.2) that,

$$\Lambda^p(\widehat{w}) \subset \Gamma^\infty(V^{1/p}).$$

Using now the monotonicity of Tf^* and (8), we have

$$\begin{aligned} \|f\|_{S_p(w)} &\geq \sup_r \left(\frac{1}{r} \int_0^r Tf^*(t) dt \right) \left(\int_0^r \widehat{w}(t) dt \right)^{1/p} \\ &= \sup_r \left(\int_0^{1/r} f^*(t) dt \right) \left(\int_0^r \widehat{w}(t) dt \right)^{1/p} \\ &= \sup_r \left(\frac{1}{r} \int_0^r f^*(t) dt \right) r \left(\int_r^\infty \frac{w(t)}{t^p} dt \right)^{1/p} = \|f\|_{\Gamma^\infty(\varphi_{p,w})}. \end{aligned}$$

Lemma 3.1 *Let us assume that $S_p(w)$ has the lattice property. Then,*

$$\|f\|_{S_p(w)} \preceq \left(\int_0^{\|f\|_\infty} u^{p-1} \varphi_{p,w}(\lambda_f(u))^p du \right)^{1/p}.$$

Proof: Let $f \in S_p(w)$. Let $I_j = (2^j, 2^{j+1}]$ and let us define

$$h_1(s) = \lambda_f(2^{j+1}), \quad s \in I_j \quad \text{and} \quad h_2(s) = \lambda_f(2^j), \quad s \in I_j.$$

Then, if f_1 is such that $\lambda_{f_1}(s) = h_1(s)$, since $\lambda_{f_1}(s) \leq \lambda_f(s)$ implies $f_1^* \leq f^*$, we have

$$f_1 \in S_p(w).$$

Now $h_2(s) = h_1(s/2)$ and hence if

$$f_2(t) = \lambda_{h_2}(t) = |\{s : h_2(s) > t\}| = |\{s : h_1(s/2) > t\}| = 2|\{s : h_1(s) > t\}| = 2f_1(t),$$

we get

$$f_2 \in S_p(w).$$

Moreover, since $S_p(w)$ is a lattice,

$$\|f\|_{S_p(w)} \simeq \|f_1\|_{S_p(w)} \simeq \|f_2\|_{S_p(w)}.$$

Now, by Lemma 2.2, and since $0 < p < 1$,

$$\begin{aligned} \|f_2\|_{S_p(w)}^p &\simeq \int_0^{\|f_2\|_\infty} \left(\frac{\lambda_{f_2}(u)}{\int_u^\infty \lambda_{f_2}(s) ds} \right)^{1-p} \varphi_{p,w}(\lambda_{f_2}(u))^p du \\ &\leq \sum_{j \in \mathbb{Z}} \int_{2^j}^{2^{j+1}} \left(\frac{\lambda_f(2^j)}{\lambda_f(2^j) \int_u^{2^{j+1}} ds} \right)^{1-p} \varphi_{p,w}(\lambda_f(2^j))^p \chi_{[0, \|f_2\|_\infty]}(u) du \\ &\simeq \sum_{j \in \mathbb{Z}} 2^{jp} \varphi_{p,w}(\lambda_f(2^j))^p \chi_{[0, \|f_2\|_\infty]}(2^j) \preceq \int_0^{\|f\|_\infty} u^{p-1} \varphi_{p,w}(\lambda_f(u))^p du. \quad \square \end{aligned}$$

Theorem 3.4 *Let w be a positive and locally integrable weight. Then, $S_p(w)$ has the lattice property if and only if $w \in RB_p$.*

Proof: Using the same argument that in the proof of part 3) \Rightarrow 4) of Theorem 3.3, we get that if $w \in RB_p$ then

$$S_p(w) = \Gamma^p(w),$$

and $S_p(w)$ has the lattice property.

Conversely, if $S_p(w)$ has the lattice property, we have, using that w is locally integrable, that $\varphi_{p,w}(0^+) = 0$. To see this, observe that if $\varphi_{p,w}(0^+) = \lambda > 0$, then

$$\frac{\lambda^p}{t} \preceq \frac{\varphi_{p,w}(t)^p}{t} = t^{p-1} \int_t^\infty \frac{w(s)}{s^p} ds,$$

where the first inequality follows from the fact that $\varphi_{p,w}$ is quasi-increasing. Therefore,

$$\int_0^r w(s) ds + r^p \int_r^\infty \frac{w(s)}{s^p} ds \simeq \int_0^r t^{p-1} \int_t^\infty \frac{w(s)}{s^p} ds dt \succeq \int_0^r \frac{\lambda^p}{t} dt = \infty.$$

By Remark 3.1, we know that there is a decreasing weight v such that

$$\varphi_{p,w}(t) \simeq \int_0^t v(s) ds.$$

thus, using Lemma 3.1,

$$\|f\|_{S_p(w)} \preceq \left(\int_0^\infty u^{p-1} \varphi_{p,w}(\lambda_f(u))^p du \right)^{1/p} \simeq \left(\int_0^\infty u^{p-1} \left(\int_0^{\lambda_f(u)} v(s) ds \right)^p du \right)^{1/p},$$

or, equivalently,

$$\Lambda^p(V^{p-1}v) \subset S_p(w),$$

where $V^{p-1}(r) = \left(\int_0^r v(s) ds \right)^{p-1}$.

Let us now consider the function

$$F(x) = \left(Q\chi_{[0,r]}(x) \right)^{1/p-1} = \left(\log \frac{r}{x} \right)^{1/p-1} \chi_{[0,r]}(x),$$

where $Qf(x) := \int_x^\infty f(s)/s ds$. Since

$$(QF(x))^p \simeq Q\chi_{[0,r]}(x),$$

we get

$$\begin{aligned} \|QF\|_{\Lambda^p(V^{p-1}v)}^p &= \int_0^\infty (QF(x))^p V^{p-1}(x) v(x) dx \simeq \int_0^\infty Q\chi_{[0,r]}(x) V^{p-1}(x) v(x) dx \\ &= \int_0^\infty \chi_{[0,r]}(x) P(V^{p-1}v)(x) dx \simeq \int_0^r \frac{V(x)^p}{x} dx \\ &\simeq \int_0^r x^{p-1} \int_x^\infty \frac{w(s)}{s^p} ds dx \simeq \int_0^r w + r^p \int_r^\infty \frac{w(s)}{s^p} ds < \infty. \end{aligned}$$

On the other hand, since $F = F^*$, we have

$$(QF)^{**} - QF = P(QF) - QF = (P + Q)F - QF = PF,$$

which implies that

$$\|F\|_{\Lambda^p(w)} \leq \|PF\|_{\Lambda^p(w)} = \|QF\|_{S_p(w)} \preceq \|QF\|_{\Lambda^p(V^{p-1}v)},$$

i.e., there is a constant $A \geq 1$ such that

$$\int_0^r \left(\log \frac{r}{x} \right)^{1-p} w(x) dx \leq A \left(\int_0^r w(s) ds + r^p \int_r^\infty \frac{w(s)}{s^p} ds \right).$$

Now, taking $c = \exp(-(1+A)^{1/(1-p)})$ we get $(\log \frac{r}{x})^{1-p} - A \geq 1$ if $0 < x < rc$, thus

$$\begin{aligned} \int_0^{rc} w(x) dx &\leq \int_0^{rc} \left(\left(\log \frac{r}{x} \right)^{1-p} - A \right) w(x) dx \leq A \left(\int_{cr}^r w(s) ds + r^p \int_r^\infty \frac{w(s)}{s^p} ds \right) \\ &\preceq r^p \int_{cr}^\infty \frac{w(s)}{s^p} ds + r^p \int_r^\infty \frac{w(s)}{s^p} ds \preceq (cr)^p \int_{cr}^\infty \frac{w(s)}{s^p} ds, \end{aligned}$$

i.e. $w \in RB_p$. □

Proposition 3.2 *The space $S_p(1/t)$ does not have the lattice property if $0 < p < 1$.*

Proof: By Proposition 3.1, $S_p(1/t) \subset \Gamma^{p,\infty}(1) = L^\infty$; and if $S_p(1/t)$ had the lattice property, by Lemma 3.1 also $L^\infty \subset S_p(1/t)$.

Let $1 < \alpha < 1/p$, and consider the decreasing function

$$h(t) = \frac{1}{t(1 - \log t)^\alpha} \chi_{[0, e^{-1}]}$$

Then, the function

$$f(t) = \frac{1}{t} (Ph(\frac{1}{t}) - h(\frac{1}{t}))$$

is also decreasing and belongs to L^∞ , since

$$\begin{aligned} \|f\|_\infty &= \lim_{t \rightarrow 0^+} f(t) = \lim_{t \rightarrow \infty} t(Ph(t) - h(t)) = \lim_{t \rightarrow \infty} \int_{h(t)}^\infty \lambda_h(t) = \int_0^\infty \lambda_h(t) dt \\ &= \int_0^\infty h(t) dt < \infty. \end{aligned}$$

However, $f \notin S_p(1/t)$; since, by Lemma 2.1,

$$h(t) = \frac{1}{t} (Pf(\frac{1}{t}) - f(\frac{1}{t})).$$

Thus,

$$\|f\|_{S_p(1/t)}^p = \int_0^{e^{-1}} \left(\frac{1}{t(1 - \ln t)^\alpha} \right)^p t^{p-1} dt = \infty. \quad \square$$

3.1.2 Normability

Now, we consider the problem of characterizing when $S_p(w)$ is a normed space, i.e. when there is a norm $\|\cdot\|$ on $S_p(w)$ and positive constants c_1 and c_2 such that

$$c_1 \|f\| \leq \|f\|_{S_p(w)} \leq c_2 \|f\|, \quad f \in S_p(w).$$

Our main result states that, for $p \geq 1$, normability is equivalent to the lattice property and hence, Theorem 3.2 and 3.3 can be used to describe the spaces $S_p(w)$.

To this end, let us introduce some notation that we shall use later. Given $r \in \mathbb{R}$, we denote by τ_r the translation operator

$$\tau_r f(x) := f(x + r), \quad x \in \mathbb{R},$$

and the symmetric operator is defined by

$$\tilde{f}(x) := f(-x).$$

Notice that

$$(\tau_r f)^* = (\tilde{f})^* = f^*.$$

Lemma 3.2 *If $S_p(w)$ is a quasi-normed space, then $\varphi_{p,w}$ is quasi-increasing.*

Proof: Let $0 < y \leq x$ and consider the functions $\chi_{[0,x]}$ and $\chi_{[0,x+y]}$. Then,

$$\varphi_{p,w}(y) = \|\chi_{[0,x+y]} - \chi_{[0,x]}\|_{S_p(w)} \leq c(\varphi_{p,w}(x+y) + \varphi_{p,w}(x)),$$

and since

$$\varphi_{p,w}(x+y) = (x+y) \left(\int_{x+y}^{\infty} \frac{w(s)}{s^p} ds \right)^{1/p} \leq 2x \left(\int_x^{\infty} \frac{w(s)}{s^p} ds \right)^{1/p} = 2\varphi_{p,w}(x),$$

we have

$$\varphi_{p,w}(y) \leq c\varphi_{p,w}(x). \quad \square$$

Theorem 3.5 *Let $p \geq 1$, the following statements are equivalent:*

1) $S_p(w)$ is a normed space.

2) $S_p(w)$ is a quasi-normed space.

3) $S_p(w)$ has the lattice property, and then

(a) If $p = 1$, there is a decreasing weight v such that, $S_1(w) = \Lambda^1(v)$ if $\varphi_{1,w}(0^+) = 0$, and $S_1(w) = \Lambda^1(v) \cap L^\infty$ if $\varphi_{1,w}(0^+) \neq 0$.

(b) If $p > 1$

$$S_p(w) = \Gamma^p(w).$$

Hence, $S_p(w)$ is also complete.

Proof: Obviously 1) \Rightarrow 2) and 3) \Rightarrow 1).

2) \Rightarrow 3) If $p = 1$, by the previous lemma, $\varphi_{1,w}$ is quasi-increasing, and then Theorem 3.2 applies.

If $p > 1$, as claimed in Theorem 3.3, we need to prove the embedding

$$S_p(w) \subset \Gamma^\infty(\varphi_{p,w}).$$

Given $f \in S_p(w)$ and $r > 0$, let

$$f_r^*(t) = f^*(r)\chi_{[0,r]}(t) + f^*(t)\chi_{[r,\infty)}(t) \quad \text{and} \quad f^{*,r}(t) = (f^*(t) - f^*(r))\chi_{[0,r]}(t).$$

Since $(f_r^*)^*(t) = f^*(r)\chi_{[0,r]}(t) + f^*(t)\chi_{[r,\infty)}(t)$, a simple computation shows that

$$(f_r^*)^{**}(t) - (f_r^*)^*(t) = (f^{**}(t) - f^*(t) - \frac{r}{t}(f^{**}(r) - f^*(r)))\chi_{[r,\infty)}(t).$$

Thus,

$$\|f_r^*\|_{S_p(w)} \leq \left(\int_r^\infty (f^{**}(t) - f^*(t))^p w(t) dt \right)^{1/p} \leq \|f\|_{S_p(w)}.$$

Also, since $(f^{*,r})^*(t) = (f^*(t) - f^*(r))\chi_{[0,r]}(t)$, we get that

$$(f^{*,r})^{**}(t) - (f^{*,r})^*(t) = (f^{**}(t) - f^*(t))\chi_{[0,r]}(t) + \frac{r}{t}(f^{**}(r) - f^*(r))\chi_{[r,\infty)}(t).$$

And, using that $t(f^{**}(t) - f^*(t))$ is increasing,

$$\begin{aligned} \|f^{*,r}\|_{S_p(w)} &= \left(\int_0^r (f^{**}(s) - f^*(s))^p w(s) ds + ((f^{**}(r) - f^*(r))r)^p \int_t^\infty \frac{w(s)}{s^p} ds \right)^{1/p} \\ &\leq \left(\int_0^\infty (f^{**}(s) - f^*(s))^p w(s) ds \right)^{1/p} = \|f\|_{S_p(w)}. \end{aligned}$$

On the other hand, since $S_p(w)$ is a quasi-normed space, $\tilde{f}^* - \tau_r(f_r^*) \in S_p(w)$ with

$$\|\tilde{f}^* - \tau_r(f_r^*)\|_{S_p(w)} \leq 2c\|f\|_{S_p(w)},$$

and the function

$$H = (\tilde{f}^* - \tau_r(f_r^*) - \widetilde{f^{*,r}})$$

belongs to $S_p(w)$ with

$$\|H\|_{S_p(w)} \leq c(c+1)\|f\|_{S_p(w)}.$$

But, since

$$(\tilde{f}^* - \tau_r(f_r^*) - \widetilde{f^{*,r}})(t) = \begin{cases} -f^*(t+r) & \text{if } t \geq 0, \\ 0 & \text{if } -r < t < 0, \\ f^*(-t) & \text{if } t \leq -r, \end{cases}$$

we have $\lambda_H = 2\lambda_{f^*\chi_{[r,\infty)}}$, and using Lemma 2.2 and the fact that $\varphi_{p,w}$ is increasing, we deduce that $f^*\chi_{[r,\infty)} \in S_p(w)$. Therefore, $f^*\chi_{[0,r]} = f^* - f^*\chi_{[r,\infty)} \in S_p(w)$ with

$$\|f^*\chi_{[0,r]}\|_{S_p(w)} \leq c^2(c+1)\|f\|_{S_p(w)},$$

and we now finish the proof as in Proposition 3.1–2). \square

Proposition 3.3 *Let $p > 0$, and suppose that $S_p(w)$ is a Banach space. Then $p \geq 1$.*

Proof: Assume that $0 < p < 1$. As in [7], we are going to find a sequence of functions $\{f_k\}$, satisfying

$$\|f_k\|_{S_p(w)} = 1 \quad \text{but} \quad \frac{1}{N} \left\| \sum_{k=1}^N f_k \right\|_{S_p(w)} \xrightarrow{N \rightarrow \infty} \infty.$$

Choose a decreasing sequence $\{r_k\}_k$ such that

$$\int_{\frac{1}{r_k}}^\infty \frac{w(s)}{s^p} ds = 2^{-pk}.$$

Let $f_k = 2^k r_k \chi_{[0,1/r_k]}$. Then

$$\|f_k\|_{S_p(w)}^p = (2^k r_k)^p \frac{1}{r_k^p} \int_{\frac{1}{r_k}}^\infty \frac{w(s)}{s^p} ds = 1.$$

But, if for N fixed, we let $F_N = \frac{1}{N} \sum_{k=1}^N f_k$, an easy computation shows that

$$\begin{aligned}
\|F_N\|_{S_p(w)} &= \frac{1}{N} \left(\sum_{k=1}^{N-1} \left(\sum_{j=1}^k 2^j \right)^p \int_{\frac{1}{r_k}}^{\frac{1}{r_{k+1}}} \frac{w(s)}{s^p} ds + \left(\sum_{j=1}^N 2^j \right)^p \int_{\frac{1}{r_N}}^{\infty} \frac{w(s)}{s^p} ds \right)^{1/p} \\
&= \frac{1}{N} \left\{ \sum_{k=1}^{N-1} \left(\sum_{j=1}^k 2^j \right)^p \left(\int_{\frac{1}{r_k}}^{\infty} \frac{w(s)}{s^p} ds - \int_{\frac{1}{r_{k+1}}}^{\infty} \frac{w(s)}{s^p} ds \right) \right. \\
&\quad \left. + \left(\sum_{j=1}^N 2^j \right)^p \int_{\frac{1}{r_N}}^{\infty} \frac{w(s)}{s^p} ds \right\}^{1/p} \\
&\geq \frac{1}{N} \left(\sum_{k=1}^N \left(2^{k+1} - 2 \right)^p \left(2^{-kp} - 2^{-(k+1)p} \right) \right)^{1/p} \geq c_p \frac{1}{N} N^{1/p} \xrightarrow{N \rightarrow \infty} \infty. \quad \square
\end{aligned}$$

3.1.3 Linear property

In general, the linearity of $S_p(w)$ does not imply that $\varphi_{p,w}$ is quasi-increasing. Effectively if $w(t) = \frac{1}{t} \chi_{[0,1]}$ then

$$S_1(w) = L^\infty \quad \text{with} \quad \|f\|_{S_1(w)} = \|f\|_{L^\infty} - \|f^* \chi_{[0,1]}\|_{L^1} \quad \text{and} \quad \varphi_{1,w}(t) = 1 - t, \quad (0 \leq t \leq 1).$$

To prove this claim, given $\varepsilon > 0$, write

$$\begin{aligned}
\int_\varepsilon^1 \left(\frac{1}{t} \int_0^t f^*(s) ds - f^*(t) \right) \frac{dt}{t} &= \int_0^1 f^*(s) \left(\int_{\max(\varepsilon, s)}^1 \frac{dt}{t^2} \right) ds - \int_\varepsilon^1 f^*(t) \frac{dt}{t} \\
&= \frac{1}{\varepsilon} \int_0^\varepsilon f^*(s) ds - \int_\varepsilon^1 f^*(s) ds.
\end{aligned}$$

Hence

$$\|f\|_{S_1(w)} = \lim_{\varepsilon \rightarrow 0} \left(\frac{1}{\varepsilon} \int_0^\varepsilon f^*(s) ds - \int_\varepsilon^1 f^*(s) ds \right) = \|f\|_{L^\infty} - \|f^* \chi_{[0,1]}\|_{L^1}.$$

Theorem 3.6 *Let $0 < p < \infty$. If $S_p(w)$ is a linear space, then there exist positive constants α, β such that*

$$\varphi_{p,w}(r) \preceq \varphi_{p,w}(2r) \quad \text{for all } r \leq \alpha \text{ and } r \geq \beta. \quad (12)$$

Proof: Suppose that (12) is not satisfied, then, following Theorem 1.4 of [11], we can find a decreasing function $h \in A$ such that $h \in \Lambda^p(s^{p-2}w(1/s))$ and $h(t/2) \notin \Lambda^p(s^{p-2}w(1/s))$.

Let $f = Th$. Using (8), it follows readily that

$$\left(\int_0^\infty (f^{**}(s) - f^*(s))^p w(s) ds \right)^{1/p} = \left(\int_0^\infty h(s)^p s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p}$$

and hence, $f \in S_p(w)$. On the other hand,

$$\left(\int_0^\infty (f^{**}(2s) - f^*(2s))^p w(s) ds \right)^{1/p} = \left(\int_0^\infty h\left(\frac{s}{2}\right)^p s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p},$$

and consequently, $f(2t) \notin S_p(w)$.

Let $F(x) = \tilde{f}(2x) + f(x)$, then

$$\lambda_F(t) = \lambda_f(t)/2 + \lambda_f(t) = 3/2\lambda_f(t)$$

and by Lemma 2.2,

$$\begin{aligned} \|F\|_{S_p(w)}^p &\simeq \int_0^{\|F\|_\infty} \left(\frac{\int_u^\infty \lambda_F(s) ds}{\lambda_F(u)} \right)^{p-1} \varphi_{p,w}(\lambda_F(u))^p du \\ &= \int_0^{\|f\|_\infty} \left(\frac{\int_u^\infty \lambda_f(s) ds}{\lambda_f(u)} \right)^{p-1} \left(\frac{3\lambda_f(u)}{2} \right)^p \int_{\frac{3\lambda_f(u)}{2}}^\infty \frac{w(s)}{s^p} ds du \\ &\leq \left(\frac{3}{2} \right)^p \int_0^{\|f\|_\infty} \left(\frac{\int_u^\infty \lambda_f(s) ds}{\lambda_f(u)} \right)^{p-1} \lambda_f(u)^p \int_{\lambda_f(u)}^\infty \frac{w(s)}{s^p} ds du \\ &\simeq \|f\|_{S_p(w)}^p. \end{aligned}$$

Thus $F \in S_p(w)$, and since $S_p(w)$ is linear, $\tilde{f}(2t) \in S_p(w)$ or equivalently $f(2t) \in S_p(w)$ which is not possible. \square

Remark 3.2 *The converse in the above theorem is not true, since it is known (see [2] and [17, Theorem 3.1]) that if $p > 1$, $S_p(1/t)$ is not a linear space, while, however, $\varphi_{p,1/t}(r) = (1/p)^{1/p}$ satisfies condition (12).*

3.2 The space $S_{p,\infty}(w)$

In this section we describe those weights w , for which $S_{p,\infty}(w)$ has the lattice property and is a normed space.

The following lemma can be proved in the same way as Proposition 3.1.

Lemma 3.3 *Let $0 < p < \infty$. If $S_{p,\infty}(w)$ has the lattice property, then*

- 1) $\varphi_{p,w}$ is quasi-increasing.
- 2) $S_{p,\infty}(w) \subset \Gamma^\infty(\varphi_{p,w})$.

In particular, since the converse embedding always holds, we get that

$$S_{p,\infty}(w) = \Gamma^\infty(\varphi_{p,w}).$$

Theorem 3.7 *Let $0 < p < \infty$. The following are equivalent:*

- 1) $S_{p,\infty}(w)$ has the lattice property.
- 2) $S_{p,\infty}(w) \subset \Gamma^\infty(\varphi_{p,w})$.
- 3) $w \in RB_p$.
- 4) $S_{p,\infty}(w) = \Gamma^\infty(\varphi_{p,w})$.

Proof: 1) \Rightarrow 2) is Lemma 3.3.

2) \Rightarrow 3) By hypothesis,

$$\frac{\varphi_{p,w}(r)}{r} \sup_{f \in S_{p,\infty}(w)} \frac{\int_0^r f^*(s) ds}{\sup_{t>0} (f^{**}(t) - f^*(t)) \varphi_{p,w}(t)} \leq c.$$

Since

$$\begin{aligned} \sup_{t>0} (f^{**}(t) - f^*(t)) \varphi_{p,w}(t) &= \sup_{s>0} \frac{1}{s} (f^{**}(\frac{1}{s}) - f^*(\frac{1}{s})) \left(\int_0^s u^{p-2} w(\frac{1}{u}) du \right)^{1/p} \\ &= \sup_{s>0} T f^*(s) \left(\int_0^s u^{p-2} w(\frac{1}{u}) du \right)^{1/p}, \end{aligned}$$

we can write the above expression as

$$\frac{\varphi_{p,w}(r)}{r} \sup_{f \in S_{p,\infty}(w)} \frac{r \int_0^{r^{-1}} T f^*(s) ds}{\sup_{s>0} T f^*(s) \left(\int_0^s u^{p-2} w(\frac{1}{u}) du \right)^{1/p}} \leq c.$$

Since by Lemma 2.1 every $h \in A$ can be represented as Tf for some $f \in A$.

$$\varphi_{p,w}\left(\frac{1}{r}\right) \sup_{g \in A} \frac{\int_0^r g(s) ds}{\sup_{s>0} g(s) \left(\int_0^s u^{p-2} w(\frac{1}{u}) du \right)^{1/p}} \leq c. \quad (13)$$

Since if $g = c$ is a constant function, the above expression is less than or equal to one, and since every $h \searrow$ can be represented as a sum of $g \in A$ and a constant, we can take the supremum in (13) over all decreasing functions.

On the other hand, Theorem 3.3 of [9] states that

$$\sup_{h \searrow} \frac{\int_0^r h(s) ds}{\sup_{s>0} h(s) \left(\int_0^s u^{p-2} w(\frac{1}{u}) du \right)^{1/p}} = \int_0^r \frac{dx}{\left(\int_0^x u^{p-2} w(\frac{1}{u}) du \right)^{1/p}}.$$

Hence, we have proved that

$$\int_0^r \frac{dx}{\left(\int_0^x u^{p-2} w(\frac{1}{u}) du \right)^{1/p}} \leq \frac{c}{\varphi_{p,w}\left(\frac{1}{r}\right)^{1/p}} = \frac{cr}{\left(\int_0^r u^{p-2} w(\frac{1}{u}) du \right)^{1/p}},$$

which is equivalent to $u^{p-2} w(\frac{1}{u}) \in B_p$ (see [24] Theorem 2.8).

3) \Rightarrow 4) Since by hypothesis $s^{p-2} w(\frac{1}{s}) \in B_p$, Theorem, 3.1 of [24] states that the Hardy operator is bounded on $\Gamma^\infty(v)$, where $v(r) = \left(\int_0^r s^{p-2} w(\frac{1}{s}) ds \right)^{1/p}$, and now since $\frac{1}{s} (f^{**}(\frac{1}{s}) - f^*(\frac{1}{s}))$ is decreasing, the result follows in the same way as Theorem 3.3.

4) \Rightarrow 1) is evident. \square

The following result is the counterpart of Theorem 3.5 for $S_{p,\infty}(w)$ spaces and can be proved in the same way.

Theorem 3.8 *Let $0 < p < \infty$, the following statements are equivalent*

- 1) $S_{p,\infty}(w)$ is a quasi-normed space.

2) $S_{p,\infty}(w)$ is a normed space.

3) $w \in RB_p$.

4) $S_{p,\infty}(w) = \Gamma^\infty(\varphi_{p,w})$.

And thus, in any of these cases, $S_{p,\infty}(w)$ is also complete.

We end this section with an analogue of Theorem 3.6 whose proof is the same using now as starting point Theorem 1.6 of [11].

Theorem 3.9 *Let $0 < p < \infty$. If $S_{p,\infty}(w)$ is a linear space, then there exist positive constants α, β such that*

$$\varphi_{p,w}(r) \preceq \varphi_{p,w}(2r) \text{ for all } r \leq \alpha \text{ and } r \geq \beta.$$

4 The associate space

In this section we describe the associate space of $S_p(w)$ and $S_{p,\infty}(w)$.

Definition 4.1

$$S_p(w)' = \left\{ g : \sup_{f \in S_p(w)} \frac{\int_0^\infty f(t)g(t) dt}{\|f\|_{S_p(w)}} < \infty \right\}. \quad (14)$$

($S_{p,\infty}(w)'$ is defined in the same way considering in (14) $S_{p,\infty}(w)$ instead of $S_p(w)$).

Obviously, $S_p(w)'$ is a normed space, and

$$\|g\|_{S_p(w)'} = \sup_{f \in A} \frac{\int_0^\infty f(t)g^*(t) dt}{\|Tf\|_{\Lambda^p(s^{p-2}w(1/s))}} < \infty.$$

Theorem 4.1 *Let $0 < p < \infty$ and $1/p + 1/p' = 1$.*

1) *If $p > 1$ and $\int_0^\infty \frac{w(s)}{s^p} ds < +\infty$, then $S_p(w)' = \Gamma^{p'}(v) \cap L^\infty$, where*

$$v(t) = \frac{w(t)}{t^p \left(\int_t^\infty \frac{w(s)}{s^p} ds \right)^{p'}}.$$

and if $\int_0^\infty \frac{w(s)}{s^p} ds = +\infty$, then $S_p(w)' = \Gamma^{p'}(v)$, with v as before.

2) *If $p \leq 1$,*

$$\|g\|_{S_p(w)'} \simeq \sup_{t>0} \frac{g^{**}(t)}{\left(\int_t^\infty \frac{w(s)}{s^p} ds \right)^{1/p}}.$$

Proof: First of all, notice that if $g^*(\infty) \neq 0$ then $g \notin S_p(w)'$ since, on the contrary, for all $r > 0$,

$$\|g\|_{S_p(w)'} \geq \frac{\int_0^r g^*(t) dt}{\|T\chi_{[0,r]}\|_{\Lambda^p(s^{p-2}w(1/s))}} \geq \frac{rg^*(\infty)}{r\left(\int_r^\infty \frac{w(s)}{s^p} ds\right)^{1/p}} \xrightarrow{r \rightarrow \infty} \infty.$$

Then, by Lemma 2.1,

$$\begin{aligned} \|g\|_{S_p(w)'} &= \sup_{f \in A} \frac{\int_0^\infty f(t)g^*(t) dt}{\|Tf\|_{\Lambda^p(s^{p-2}w(1/s))}} = \sup_{f \in A} \frac{\int_0^\infty Tf(t)Tg^*(t) dt}{\|Tf\|_{\Lambda^p(s^{p-2}w(1/s))}} \\ &= \sup_{h \in A} \frac{\int_0^\infty h(t)Tg^*(t) dt}{\|h\|_{\Lambda^p(s^{p-2}w(1/s))}} \simeq \sup_{h \searrow} \frac{\int_0^\infty h(t)Tg^*(t) dt}{\|h\|_{\Lambda^p(s^{p-2}w(1/s))}}, \end{aligned}$$

where in the last equivalence we have used the fact that the supremum over all nonnegative and decreasing functions is attained for

$$h(x) = \left(\int_x^\infty \frac{Tg^*(t)}{\int_0^t s^{p-2}w(1/s) ds} dt \right)^{p'-1} \quad \text{if } 1 < p < \infty$$

(see [23]) and on characteristic decreasing functions $\chi_{[0,r]}$ if $0 < p \leq 1$ (see [10]).

Then, if $p > 1$ (see [23]),

$$\begin{aligned} \|g\|_{S_p(w)'} &\simeq \left(\int_0^\infty (Tg)^{**}(t)^{p'} \frac{t^{p'} t^{p-2} w(1/t)}{\left(\int_0^t s^{p-2} w(1/s) ds \right)^{p'}} \right)^{1/p'} \\ &\quad + \left(\int_0^\infty s^{p-2} w(1/s) ds \right)^{-1/p} \int_0^\infty Tg(t) dt, \end{aligned}$$

which, by (8), is equivalent to

$$\|g\|_{S_p(w)'} \simeq \left(\int_0^\infty g^{**}(t)^{p'} \frac{w(t)}{\left(\int_t^\infty \frac{w(s)}{s^p} ds \right)^{p'} t^p} dt \right)^{1/p'} + \left(\int_0^\infty \frac{w(s)}{s^p} ds \right)^{-1/p} \|g\|_\infty.$$

and 1) follows. For the case $p \leq 1$, we apply the Carro–Soria duality result (see [10]), to obtain

$$\|g\|_{S_p(w)'} \simeq \sup_{t>0} (Tg)^{**}(t) \frac{t}{\left(\int_0^t s^{p-2} w(1/s) ds \right)^{1/p}} \simeq \sup_{t>0} \frac{g^{**}(t)}{\left(\int_t^\infty \frac{w(s)}{s^p} ds \right)^{1/p}}. \quad \square$$

Theorem 4.2 *Let w be a weight, and let us define*

$$u(t) = \left(\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{-1/p}.$$

Then

1. If u is locally integrable and $\int_0^\infty \frac{w(s)}{s^p} ds < +\infty$, then $S_{p,\infty}(w)' = \Lambda^1(Tu) \cap L^\infty$.
2. If u is locally integrable and $\int_0^\infty \frac{w(s)}{s^p} ds = +\infty$, then $S_{p,\infty}(w)' = \Lambda^1(Tu)$.
3. If u is not locally integrable, then $S_{p,\infty}(w)' = \{0\}$.

Proof: First of all, notice that using the same argument as in the previous theorem, it is easy to see that if $g^*(\infty) \neq 0$ then $g \notin S_{p,\infty}(w)'$, and again, by Lemma 2.1,

$$\|g\|_{S_{p,\infty}(w)'} = \sup_{f \in A} \frac{\int_0^\infty T f(t) T g^*(t) dt}{\sup_{t>0} T f(t) \left(\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p}} = \sup_{h \in A} \frac{\int_0^\infty h(t) T g^*(t) dt}{\sup_{t>0} h(t) \left(\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p}}.$$

Obviously,

$$\|g\|_{S_{p,\infty}(w)'} \leq \sup_{h \searrow} \frac{\int_0^\infty h(t) T g^*(t) dt}{\sup_{t>0} h(t) \left(\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p}} \leq \int_0^\infty T g^*(t) u(t) dt.$$

Conversely, given $R > 0$, let

$$u_R(t) = u(t) \chi_{[0,R]}(t).$$

Now, if u is locally integrable, we get that $u_R \in A$, and then

$$\|g\|_{S_{p,\infty}(w)'} \geq \sup_{R>0} \frac{\int_0^\infty u_R(t) T g^*(t) dt}{\sup_{t>0} u_R(t) \left(\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p}} = \int_0^\infty T g^*(t) u(t) dt,$$

and, by Lemma 2.1

$$\begin{aligned} \int_0^\infty T g^*(t) u(t) dt &= \int_0^\infty T g^*(t) (u(t) - u(\infty)) dt + u(\infty) \int_0^\infty T g^*(t) dt \\ &= \int_0^\infty g^*(t) T(u - u(\infty))(t) dt + u(\infty) \|g\|_{S_1(1/t)} \\ &= \int_0^\infty g^*(t) T u(t) dt + u(\infty) \|g\|_{S_1(1/t)} \\ &= \|g\|_{\Lambda^1(w)} + u(\infty) \|g\|_\infty, \end{aligned}$$

which proves 1 and 2.

To see 3, given $0 < R < 1$, let us consider the function

$$u_R(t) = u(R) \chi_{[0,R]}(t) + u(t) \chi_{[R,1/R]}(t).$$

Since $u_R \in A$,

$$\begin{aligned} \|g\|_{S_{p,\infty}(w)'} &\geq \sup_{0 < R < 1} \frac{u(R) \int_0^R T g^*(t) dt + \int_R^{1/R} u(t) T g^*(t) dt}{\sup_{t>0} u_R(t) \left(\int_0^t s^{p-2} w\left(\frac{1}{s}\right) ds \right)^{1/p}} \\ &\geq \sup_{0 < R < 1} \int_R^{1/R} u(t) T g^*(t) dt = \int_0^\infty T g^*(t) u(t) dt, \end{aligned}$$

and, since we are assuming that u is not locally integrable, the last integral is finite if and only if $g = 0$. \square

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References

- [1] Ariño M., and Muckenhoupt B. *Maximal functions on classical Lorentz spaces and Hardy's inequality with weights for non increasing functions*, Trans. Amer. Math. Soc. **320** (1990), 727–735.
- [2] Bastero J., Milman M. and Ruiz F., *A note in $L(\infty, q)$ spaces and Sobolev embeddings*, Indiana Univ. Math. J. **52** (2003), 1215–1230.
- [3] Bagby R. J. and Kurtz D. S., *A rearrangement good- λ inequality*, Trans. Amer. Math. Soc. **293** (1986), 71–81.
- [4] Bennett C., De Vore R. and Sharpley R., *Weak L^∞ and BMO*, Ann. of Math. **113** (1981), 601–611.
- [5] Bennett C. and Sharpley R., *Interpolation of Operators*, Academic Press Inc., New York 1988.
- [6] Boza S. and Martín J., *Equivalent expressions for norms in classical Lorentz spaces*, Forum Math. (To appear).
- [7] Carro M., García del Amo A. and Soria J., *Weak type weights and normable Lorentz spaces*, Proc. Amer. Math. Soc. **124** (1996), 849–857.
- [8] Carro M., Gogatishvili A., Martín J. and Pick L., *Weighted inequalities involving two Hardy operator with applications to embeddings of function spaces*, (Preprint).
- [9] Carro M., Pick L., Soria J. and Stepanov V.D., *On embeddings between classical Lorentz spaces*, Math. Inequal. Appl. **4** (2001), 397–428.
- [10] Carro M. and Soria J., *Boundedness of some integral operators*, Canad. J. Math. **124** (1996) no. 6, 1155–1166.
- [11] Cwikel M., Kamińska L., Maligranda L. and Pick L., *Are generalized Lorentz “spaces” really spaces?*, Proc. Amer. Math. Soc. **132** (2004) no. 12, 3615–3625.
- [12] Edmunds D. E., Kerman R. and Pick L., *Optimal Sobolev imdeddings involving rearrangement-invariant quasinorms*, J. Funct. Anal. **170** (2000) no. 2, 307–355.
- [13] Franciosi M., *A condition implying boundedness and VMO for a function f* , Studia Math. **123** (1997) no. 2, 109–116.
- [14] Krugljak N., Maligranda L. and Persson L. E., *On an elementary approach to the fractional Hardy inequality*, Proc. Amer. Math. Soc. **128** (2000), 727–734.
- [15] Kreĭn S. G., Petunin Ju. I. and Semenov E. M., *Interpolation of Linear Operators*, Amer. Math. Soc., Providence, R.I., 1982.

- [16] Martín J., and Milman M., *Symmetrization inequalities and Sobolev embeddings*, Proc. Amer. Math. Soc. (To appear).
- [17] Malý J. and Pick L., *An elementary proof of sharp Sobolev embeddings*, Proc. Amer. Math. Soc. **130** (2002), no. 2, 555–563.
- [18] Milman M. and Pustylnik E., *On sharp higher order Sobolev embeddings*, Commun. Contemp. Math. **6** (2004) no. 3, 495–511.
- [19] Pustylnik E., *Sobolev Type inequalities in Ultrasymmetric spaces with applications to Orlicz–Sobolev embeddings*, (Preprint).
- [20] Sagher Y. and Shvartsman P., *Rearrangement–function inequalities and interpolation theory*, J. Approx. Theory **119** (2002), 214–251.
- [21] Sagher Y. and Shvartsman P., *An interpolation theorem with perturbed continuity*, J. Funct. Anal. **188** (2002) no. 1, 75–110.
- [22] Sagher Y. and Shvartsman P., *The approximation functional and interpolation with perturbed continuity*, J. Approx. Theory **110** (2001) no. 2, 236–260.
- [23] Sawyer E., *Boundedness of classical operators on classical Lorentz spaces*, Studia. Math. **96** (1990), 145–158.
- [24] Soria J., *Lorentz Spaces of Weak–Type*, Quart. J. Math. Oxford **49** (1998), 93–103.