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Entropy function spaces and interpolation*

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Abstract

We associate to every function space, and to every entropy function E, a scale of spaces $\Lambda^{p,q}(E)$ similar to the classical Lorentz spaces $L^{p,q}$. Necessary and sufficient conditions for they to be normed spaces are proved, their role in real interpolation theory is analyzed, and a number of applications to functional and interpolation properties of several variants of Lorentz spaces and entropy spaces are given.

1 Introduction

An analysis of classical rearrangement invariant spaces of functions, with their associated Lorentz and Marcinkiewicz classes, suggests that a common method could be developed in order to study functional and interpolation properties in the setting of general function spaces.

The point is that in many important cases the quantity which controls the norm involves certain "entropy functions" which quantify precisely the size of the level sets of their elements.

In order to explain with more detail what we want to do, let us introduce some definitions.

By a **quasi-Banach function space** on a given measure space (Ω, Σ, μ) we denote a linear subspace E of $L^0 = L^0(\mu)$, the space of all (equivalence classes of) measurable functions on Ω, endowed with a (quasi-)norm $\|\cdot\|_E$ with the following two properties:

(i) $g \in E$ and $\|g\|_E \leq \|f\|_E$, whenever $g \in L^0$, $f \in E$ and $|g| \leq |f|$ a.e. (lattice property).

(ii) If $0 \leq f_n \uparrow f$ a.e., then $\|f_n\|_E \uparrow \|f\|_E$ (Fatou property).

We say that E is a **Banach function space** if $\|\cdot\|_E$ is equivalent to a norm.

By defining $E(A) := \|\chi_A\|_E$, we obtain a **quasi-entropy function** on Ω, that we denote with the same symbol E, which is a set function on Σ such that

(a) $0 \leq E(A) \leq \infty$,
(b) $E(A) = 0$ if and only if $\mu(A) = 0$,
(c) $E(A) \leq E(B)$ if $A \subset B$,
(d) $\lim_k E(A_k) = E(A)$ if $A_k \uparrow A$, and
(e) $E(A \cup B) \leq c(E(A) + E(B))$.

If $c = 1$, we say that E is an **entropy function**.

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Another example of entropy function, first considered in [11], is **Shannon entropy** E_φ on $[0,1/e]^n \subset \mathbf{R}^n$. It is associated to $\varphi(x) := x\log(1/x)$ by defining

$$E_\varphi(A) := \inf\Big\{\sum_{k=1}^\infty \varphi(|I_k|);\ A\subset \bigcup_{k=1}^\infty I_k,\ I_k\subset[0,1/e]^n \text{ intervals}\Big\}.$$

Here $|I_k|$ is the volume of I_k and μ the Hausdorff measure relative to φ.

Given a quasi-entropy function E on Ω, we define two function spaces $\Lambda(E)$ and $M(E)$ by the conditions

$$\|f\|_{\Lambda(E)} := \int_0^\infty E(\{|f|>s\})\,ds < \infty \quad \text{and} \quad \|f\|_{M(E)} := \sup_{s>0} sE(\{|f|>s\}) < \infty.$$

Since $\{|f+g|>s\}\subset\{|f|>s/2\}\cup\{|g|>s/2\}$ we get that $\|\cdot\|_{\Lambda(E)}$ and $\|\cdot\|_{M(E)}$ are quasi-norms which are complete since our conditions on E ensure that both have the Fatou property. It follows from the definition that, for any $A\in\Sigma$, $\|\chi_A\|_{\Lambda(E)} = \|\chi_A\|_{M(E)} = E(A)$.

Observe that $E(\{|f|>t\})$ is a decreasing function for $0<t<\infty$ and it will play the role of the usual distribution function in the theory of Lebesgue spaces. In particular, for any quasi-entropy function E, we can define in the same vein **entropy Lorentz spaces** with two parameters $p,q\in(0,\infty)$ by

$$\Lambda^{p,q}(E) = \Big\{f\in L^0;\ \|f\|_{\Lambda^{p,q}(E)} := \Big(\int_0^\infty t^{q-1}E(\{|f|>t\})^{q/p}\,dt\Big)^{1/q} < \infty\Big\},$$

and also $\Lambda^{p,\infty}(E)$ with $\|f\|_{p,\infty} := \sup_{t>0} tE(\{|f|>t\})^{1/p}$, so that $\Lambda^{1,\infty}(E) = M(E)$. We denote $\Lambda^p(E) = \Lambda^{p,p}(E)$, thus $\Lambda(E) = \Lambda^1(E)$.

The significance of these spaces will appear very clearly in the setting of the real $K-$method of interpolation (see Section 3 bellow).

Example 1 *If E is a rearrangement invariant space on $(0,\infty)$ whose fundamental function $\phi_E(t) := \|\chi_{(0,t)}\|_E$ is concave (see [12] and [4]), then $\Lambda(E) = \Lambda(\phi_E)$ and $M(E) = M(\phi_E)$, where $\Lambda(\phi_E)$ and $M(\phi_E)$ are the classical Lorentz and Marcinkiewicz spaces associated to $\phi_E(t)$ defined by the conditions*

$$\|f\|_{\Lambda(\phi_E)} := \int_0^\infty \phi_E\big(\mu_f(s)\big)ds < \infty, \quad \|f\|_{M(\phi_E)} := \sup_{s>0} s\phi_E\big(\mu_f(s)\big) < \infty.$$

Example 2 *If E_φ is the Shannon entropy, the corresponding spaces $\Lambda^p(E_\varphi)$ and $M(E_\varphi)$ were considered by R. Fefferman in order to obtain entropic versions of the theorems of Hardy and Littlewood and of Calderón and Zygmund.*

Example 3 *If $0<p,q\le\infty$ and $E = L^{p,q}(w)$, a weighted Lorentz space, then $E = \Lambda^{p,q}(L^1(w))$, $\Lambda(E) = L^{p,1}(w)$, and $M(L^{p,q}(w)) = L^{p,\infty}(w)$.*

Obviously $E\hookrightarrow M(E)$. If E is a Banach function space, from $|f|\le\sum_{k=-\infty}^\infty 2^{k+1}\chi_{\{|f|>2^k\}}$ we also obtain

$$\|f\|_E \le \sum_{k=-\infty}^\infty 2^{k-1}\|\chi_{\{|f|>2^k\}}\|_E \le 4\int_0^\infty E(\{|f|>s\})\,ds = 4\|f\|_{\Lambda(E)},$$

and $\Lambda(E)$ and $M(E)$ are extremal in the sense that if X is another Banach function space on Ω such that $\|A\|_X = \|A\|_E$ for any measurable set $A \subset \Omega$, then

$$\Lambda(E) \hookrightarrow X \hookrightarrow M(E). \tag{1}$$

If $0 < p < 1$, then $\Lambda(L^p) = L^{p,1}$ is strictly larger than L^p, but we claim that, if E is a quasi-Banach function space, there exits a number $0 < u \leq 1$ such that

$$\Lambda^{1,u}(E) \subset E. \tag{2}$$

To prove this claim, take u defined by $(2c)^u = 2$, where c is the quasi-norm constant. Then by the Aoki-Rolewicz theorem (cf. [3]), we know that there is a $u-$norm $\|\cdot\|_E^*$ such that

$$\|f\|_E^* \leq \|f\|_E^u \leq 2\,\|f\|_E^*\,.$$

Then, if $0 \leq f \in E$, $f \leq \sum 2^{k+1}\chi_{\{f>2^k\}}$, hence

$$\begin{aligned}
\|f\|_E^u &\leq \left\|\sum 2^{k+1}\chi_{\{f>2^k\}}\right\|_E^u \leq 2\left\|\sum 2^{k+1}\chi_{\{f>2^k\}}\right\|_E^* \leq 2\sum 2^{(k+1)u}\|\chi_{\{f>2^k\}}\|_E^* \\
&\leq 2\sum 2^{(k+1)u}\|\chi_{\{f>2^k\}}\|_E^u = 2\sum 2^{(k+1)u}E(\{f>2^k\})^u \\
&\preceq \int_0^\infty y^{u-1}E(\{f>y\})^u dy = \|f\|_{\Lambda^{1,u}(E)}^u\,.
\end{aligned}$$

Let us now briefly summarize the contents of the paper. Section 2 deals with the basic question of whether $\|\cdot\|_{\Lambda^{p,q}(E)}$ is a norm, and as an application we analyze subadditivity properties for Shannon entropy spaces, and for classical Lorentz spaces and their multidimensional variants. Our aim in Section 3 is to analyze the role of $\Lambda^{p,q}(E)$ spaces in the setting of real interpolation theory. The main idea is that it will be enough to restrict arguments to characteristic functions. We start by considering interpolation with L^∞ and we prove that

$$(\Lambda(E), L^\infty)_{\theta,q} = (M(E), L^\infty)_{\theta,q} = \Lambda^{p,q}(E) \qquad (p = 1/(1-\theta)).$$

Then we describe $(\Lambda^{p_0,q_0}(E_0), \Lambda^{p_1,q_1}(E_1))_{\theta,q}$, and the main idea here is to consider the Sparr's interpolation method for triples. As an application we obtain the description of $(E, L^\infty)_{\theta,q}$ when E is a function space, and this allows to provide an identification of the interpolation space $(B_q, L^\infty)_{\theta,p}$ that completes the results of [14] concerning real interpolation between B_q and L^p when B_q is the block space introduced by M. Taibleson and G. Weiss to refine some aspects of the theory of entropy spaces. We also obtain an interpolation result about interpolation of classical Lorentz spaces and, finally, we characterize all the pairs (E, L^∞) that are universal right Calderón couples.

As usual, by $A \simeq B$ we mean that $c^{-1}A \leq B \leq cA$, and by $A \preceq B$ that $A \leq cB$, for some constant $c > 0$ independent of appropriate quantities.

2 Normed $\Lambda^{p,q}(E)$ spaces

We always assume that E is a quasi-entropy function on Ω. The basic problem considered here is to know when the quasinorm $\|\cdot\|_{\Lambda^{p,q}(E)}$ is subadditive.

In fact, a reduction to a single parameter can be performed through s-convexifications $E^{(s)}$. For any entropy function E and $0 < s < \infty$,

$$E^{(s)}(A) := E(A)^{1/s}.$$

If E is a Banach functions space, $\|f\|_{E^{(s)}} = \||f|^s\|_E^{1/s}$ (cf. [13]). It is readily checked that, for any $0 < p, q < \infty$,
(a) $\Lambda^p(E) = \Lambda(E)^{(p)}$.
(b) $\Lambda^{p,q}(E) = \Lambda(E^{(p/q)})^{(q)} = \Lambda^q(E^{(p/q)})$.

Let $1 \leq p < \infty$, if $\|\cdot\|_{\Lambda^p(E)}$ is a norm, then $\|f_\delta + g_\delta\|_{\Lambda^p(E)} \leq \|f_\delta\|_{\Lambda^p(E)} + \|g_\delta\|_{\Lambda^p(E)}$, where $f_\delta = (1+\delta)\chi_A + \chi_{B\setminus A}$ and $g_\delta = (1+\delta)\chi_B + \chi_{A\setminus B}$ $(A \cap B \neq \emptyset)$.

It is easily checked that

$$\|f_\delta\|_{\Lambda^p(E)} = \Big(\frac{1}{p}\Big)^{1/p}\Big(E(A\cup B) + ((1+\delta)^p - 1)E(A)\Big)^{1/p},$$

and the same for g_δ, with $E(B)$ instead of $E(A)$. Then

$$\|f_\delta + g_\delta\|_{\Lambda^p(E)} = \Big(\frac{1}{p}\Big)^{1/p}\Big(E(A\cup B)(2+\delta)^q + [(2+2\delta)^p - (2+\delta)^p]E(A\cap B)\Big)^{1/p}.$$

Now, from the estimates $(x+y)^{1/p} \leq x^{1/p} + y^{1/p} \leq 2^{1-1/p}(x+y)^{1/p}$, collecting terms and by letting $\delta \to 0$ we have that

$$E(A\cup B) + E(A\cap B) \leq E(A) + E(B). \qquad (3)$$

For the converse, it will be convenient to consider simple functions $s_N = \sum_{i=1}^N \alpha_i \chi_{A_i}$ with $\alpha_i \geq 0$ and $A_i \cap A_j = \emptyset$ if $i \neq j$, and

$$\pi(s_N) := (\alpha_1 - \alpha_2)E(A_1) + (\alpha_2 - \alpha_3)E(A_1 \cup A_2) + \cdots + \alpha_N E(A_1 \cup \cdots \cup A_N).$$

We will say that E is **strongly subadditive** if condition (3) holds.

Theorem 1 *Let $1 \leq p < \infty$. The quasi-norm $\|\cdot\|_{\Lambda^p(E)}$ is a norm if and only if E is strongly subadditive.*

Proof. We have seen that the condition is necessary. For the converse we may assume that $p = 1$, since $\Lambda^p(E) = \Lambda(E)^{(p)}$ is a normed space provided that $\Lambda(E)$ is normed.

Using Fatou property, we are allowed to consider nonnegative simple functions

$$s_N = \sum_{i=1}^N \alpha_i \chi_{A_i}, \quad t_N = \sum_{i=1}^N \beta_i \chi_{A_i}$$

and assume that $\alpha_1 + \beta_1 \geq \alpha_2 + \beta_2 \geq \cdots \geq \alpha_N + \beta_N$, and $A_i \cap A_j = \emptyset$ if $i \neq j$. Then, since

$$\begin{aligned}\|s_N\|_{\Lambda(E)} &= (\alpha_{\sigma(1)} - \alpha_{\sigma(2)})E(A_{\sigma(1)}) + (\alpha_{\sigma(2)} - \alpha_{\sigma(3)})E(A_{\sigma(1)} \cup A_{\sigma(2)}) \\ &\quad + \cdots + \\ &\quad + (\alpha_{\sigma(N-1)} - \alpha_{\sigma(N)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(N-1)}) + \alpha_{\sigma(N)}E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(N)})\end{aligned}$$

if σ is a permutation of $\{1, 2, \ldots, N\}$ such that $\alpha_{\sigma(1)} \geq \alpha_{\sigma(2)} \geq \cdots \geq \alpha_{\sigma(N)}$, we only need to check that $\pi(s_N) \leq \|s_N\|_{\Lambda(E)}$ to obtain the triangle property,

$$\|s_N + t_N\|_{\Lambda(E)} = \pi(s_N) + \pi(t_N) \leq \|s_N\|_{\Lambda(E)} + \|t_N\|_{\Lambda(E)}.$$

By induction, we assume that the estimate is true for s_N, with N terms. Let

$$s_{N+1} = s_N + \alpha_{N+1}\chi_{A_{N+1}}$$

and consider a permutation σ such that $\alpha_{\sigma(1)} \geq \alpha_{\sigma(2)} \geq \cdots \geq \alpha_{\sigma(N)}$.

Let us assume that $\alpha_{\sigma(1)} \geq \cdots \alpha_{\sigma(r)} \geq \alpha_{N+1} \geq \alpha_{\sigma(r+1)} \cdots \geq \alpha_{\sigma(N)}$, the extreme cases $\alpha_{\sigma(N)} > \alpha_{N+1}$ and $\alpha_{N+1} > \alpha_{\sigma(1)}$ being similar and simpler. Then

$$\begin{aligned}
\|s_{N+1}\|_{\Lambda(E)} \;=\;& (\alpha_{\sigma(1)} - \alpha_{\sigma(2)})E(A_{\sigma(1)}) + \cdots + (\alpha_{\sigma(r-1)} - \alpha_{\sigma(r)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r-1)}) \\
&+(\alpha_{\sigma(r)} - \alpha_{N+1})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)}) \\
&+(\alpha_{N+1} - \alpha_{\sigma(r+1)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)} \cup A_{N+1}) \\
&+(\alpha_{\sigma(r+1)} - \alpha_{\sigma(r+2)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)} \cup A_{N+1} \cup A_{\sigma(r+1)}) \\
&+ \cdots + \alpha_{\sigma(N)}E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(N)} \cup A_{N+1})
\end{aligned}$$

and

$$\begin{aligned}
\|s_N\|_{\Lambda(E)} \;=\;& (\alpha_{\sigma(1)} - \alpha_{\sigma(2)})E(A_{\sigma(1)}) + \cdots + (\alpha_{\sigma(r-1)} - \alpha_{\sigma(r)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r-1)}) \\
&(\alpha_{\sigma(r)} - \alpha_{\sigma(r+1)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)}) \\
&+(\alpha_{\sigma(r+1)} - \alpha_{\sigma(r+2)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)} \cup A_{\sigma(r+1)}) \\
&+ \cdots + \alpha_{\sigma(N)}E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(N)})
\end{aligned}$$

with $A_{\sigma(1)} \cup \cdots \cup A_{\sigma(N)} = A_1 \cup \cdots \cup A_N$ and $A_{\sigma(1)} \cup \cdots \cup A_{\sigma(N)} \cup A_{N+1} = A_1 \cup \cdots \cup A_N \cup A_{N+1}$.

We have $\pi(s_{N+1}) = \pi(s_N) - \alpha_{N+1}E(A_1 \cup \cdots \cup A_N) + \alpha_{N+1}E(A_1 \cup \cdots \cup A_{N+1})$ and we want to prove that

$$\|s_{N+1}\|_{\Lambda(E)} - \pi(s_{N+1}) = \|s_N\|_{\Lambda(E)} - \pi(s_N) + I \geq 0,$$

with $I = \|s_{N+1}\|_{\Lambda(E)} - \|s_N\|_{\Lambda(E)} + \alpha_{N+1}E(A_1 \cup \cdots \cup A_N) - \alpha_{N+1}E(A_1 \cup \cdots \cup A_{N+1})$.

By induction, $\|s_N\|_{\Lambda(E)} - \pi(s_N) \geq 0$. To show that also $I \geq 0$ we observe that

$$\begin{aligned}
I = & (\alpha_{\sigma(r)} - \alpha_{N+1})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)}) + (\alpha_{N+1} - \alpha_{\sigma(r+1)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)} \cup A_{N+1}) \\
& -(\alpha_{\sigma(r)} - \alpha_{\sigma(r+1)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)}) \\
& +(\alpha_{\sigma(r+1)} - \alpha_{\sigma(r+2)})[E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)} \cup A_{N+1} \cup A_{\sigma(r+1)}) \\
& \qquad -E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)} \cup A_{\sigma(r+1)})] \\
& + \cdots + \\
& (\alpha_{N+1} - \alpha_{\sigma(N)})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(N)}) \\
& \qquad +(\alpha_{\sigma(N)} - \alpha_{N+1})E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(N)} \cup A_{N+1}),
\end{aligned}$$

and make the substitutions

$$\alpha_{\sigma(r)} - \alpha_{\sigma(r+1)} = (\alpha_{\sigma(r)} - \alpha_{N+1}) + (\alpha_{N+1} - \alpha_{\sigma(r+1)})$$

and

$$\alpha_{N+1} - \alpha_{\sigma(N)} = (\alpha_{N+1} - \alpha_{\sigma(r+1)}) + (\alpha_{\sigma(r+1)} - \alpha_{\sigma(r+2)}) + \cdots + (\alpha_{\sigma(N-1)} - \alpha_{\sigma(N)}).$$

Now we may associate similar terms of I and check that everyone is nonnegative. We have that $\alpha_{N+1} - \alpha_{\sigma(r+1)} \geq 0$ multiplies

$$-E(A_{\sigma(1)}\cup\cdots\cup A_{\sigma(r)})+E(A_{\sigma(1)}\cup\cdots\cup A_{\sigma(r)}\cup A_{N+1})+E(A_1\cup\cdots\cup A_N)-E(A_1\cup\cdots\cup A_N\cup A_{N+1}),$$

which is nonnegative, by condition (3). Similarly, $\alpha_{\sigma(r+1)} - \alpha_{\sigma(r+2)} \geq 0$ is multiplied by

$$\begin{aligned}-E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r+1)}) \quad + \quad & E(A_{\sigma(1)} \cup \cdots \cup A_{\sigma(r)} \cup A_{N+1} \cup A_{\sigma(r+1)}) \\ & +E(A_1 \cup \cdots \cup A_N) - E(A_1 \cup \cdots \cup A_N \cup A_{N+1}),\end{aligned}$$

also nonnegative, and the same for the remaining terms.

Remark 1 *Not all Banach function norms are strongly subadditive. The mixed norm spaces* $L^p[L^q]$ $(1 \leq p, q < \infty)$*, with*

$$\|f\|_{(p,q)} := \Big(\int_{\mathbf{R}} \Big(\int_{\mathbf{R}} |f(x,y)|^q dy \Big)^{p/q} dx \Big)^{1/p} < \infty,$$

satisfy condition (3) if $p \leq q$*. If* $0 < a < 2^r - 2$ *with* $r := p/q > 1$*, then (3) fails for* $A = (0, a+1) \times (0,1)$ *and* $B = ((0,a) \times (1,2)) \cup ((1,2) \times (0,1))$*.*

As a simple application we may give a very simple proof of the subadditivity of the maximal function

$$f^{**}(t) := \frac{1}{t} \int_0^t f^*(s)\, ds, \quad (t > 0),$$

where $f^*(t) := \inf\{s : \mu_f(s) \leq t\}$ and μ_f is the distribution function of $|f|$.

Example 4 *The entropy function* $E(A) := \min(\mu(A), t)$ *is strongly subadditive and, if* f *and* g *are two measurable function on* Ω*, then* $(f+g)^{**} \leq f^{**} + g^{**}$*.*

Since $\mu(A \cup B) + \mu(A \cap B) = \mu(A) + \mu(B)$ it follows that E is strongly subadditive. Since

$$\int_0^t f^*(s)\, ds = \int_0^\infty \mu_{f^* \chi_{[0,t]}}(s)\, ds = \int_0^\infty \min(\mu_f(s), t)\, ds = ||f||_{\Lambda(E)},$$

it follows that $t(f+g)^{**}(t) = \|f+g\|_{\Lambda(E)} \leq \|f\|_{\Lambda(E)} + \|g\|_{\Lambda(E)} = tf^{**}(t) + tf^{**}(t)$.

Let us now consider the counterpart of Theorem 1 for $\Lambda^{p,\infty}(E)$ spaces.

Theorem 2 *Let* E *be a quasi-Banach function space and* $1 \leq p < \infty$*. The following properties are equivalent:*
(a) $\|\cdot\|_{\Lambda^{p,\infty}(E)}$ *is a norm.*
(b) If A *and* B *are two disjoint measurable sets, then* $E(A \cup B) = \max(E(A), E(B))$*.*
(c) $\Lambda^{p,\infty}(E)$ *is an AM-space, i.e.,*

$$\|f+g\|_{\Lambda^{p,\infty}(E)} = \max(\|f\|_{\Lambda^{p,\infty}(E))}, \|g\|_{\Lambda^{p,\infty}(E)}) \text{ if } \min(|f|,|g|) = 0.$$

Proof. Let us start with the case $p = 1$. Obviously, (c) implies (a).

To show that (a) implies (b), assume that there exist A and B such that $E(A \cup B) > \max(E(A), E(B))$ with $A \cap B = \emptyset$ and let $f = (1+\delta)\chi_A + \chi_B$, $g = (1+3\delta/2)\chi_A + (1+2\delta)\chi_B$. Then a simple computation shows that

$$\|f\|_{M(E)} = \max((1+\delta)E(A), E(A \cup B)), \quad \|g\|_{M(E)} = \max((1+2\delta)E(B), (1+3\delta/2)E(A \cup B))$$

and

$$\|f+g\|_{M(E)} = \max((2+5\delta/2)E(A), (2+2\delta)E(A \cup B)).$$

If we choose $\delta > 0$ such that $(1+\delta)E(A) < E(A \cup B)$, $(1+2\delta)E(B) < (1+3\delta/2)E(A \cup B)$ and $(2+5\delta/2)E(A) < (2+2\delta)E(A \cup B)$, it follows that $\| \cdot \|_{M(E)}$ is not a norm since

$$\|f\|_{M(E)} + \|g\|_{M(E)} < (2+2\delta)E(A \cup B) = \|f+g\|_{M(E)}.$$

Assume now that (b) holds true. For any $s_N = \sum_{i=1}^N \alpha_i \chi_{A_i} \geq 0$ $(A_i \cap A_j = \emptyset)$, where we may assume $\alpha_1 > \alpha_2 > \ldots > \alpha_N > 0$, let us check that

$$\|s_N\|_{M(E)} = \max(\alpha_1 E(A_1), \ldots, \alpha_N E(A_N)).$$

If

$$\|s_N\|_{M(E)} = \max(\alpha_1 E(A_1), \alpha_2 E(A_1 \cup A_2), \ldots, \alpha_N E(A_1 \cup \cdots \cup A_N)) = \alpha_j E(A_1 \cup \cdots \cup A_j),$$

then $E(A_1 \cup \cdots \cup A_j) > E(A_1 \cup \cdots \cup A_{j-1})$ since $\alpha_j < \alpha_{j-1}$, and it follows from (b) that

$$E(A_1 \cup \cdots \cup A_j) = E(A_j).$$

Now, if $s_N = \sum_{i=1}^N \alpha_i \chi_{A_i} \geq 0$, $t_N = \sum_{i=N+1}^M \alpha_i \chi_{A_i} \geq 0$ $(A_i \cap A_j = \emptyset)$ are two disjoint simple functions, we obtain

$$\|s_N + t_N\|_{M(E)} = \max(\alpha_1 E(A_1), \ldots, \alpha_M E(A_M)) = \max(\|s_N\|_{M(E)}, \|t_N\|_{M(E)}).$$

Moreover, for any couple $s_N = \sum_{i=1}^N \alpha_i \chi_{A_i}$, $t_N = \sum_{i=1}^N \beta_i \chi_{A_i}$ $(A_i \cap A_j = \emptyset)$ of simple functions, $\|s_N + t_N\|_{M(E)} \leq \|s_N\|_{M(E)} + \|t_N\|_{M(E)}$, since

$$\begin{aligned}
\|s_N + t_N\|_{M(E)} &\leq \max((\alpha_1+\beta_1)E(A_1), \ldots, (\alpha_N+\beta_N)E(A_N)) \\
&\leq \max(\alpha_1 E(A_1), \ldots, \alpha_N E(A_N)) + \max(\beta_1 E(A_1), \ldots, \beta_N E(A_N)) \\
&= \|s_N\|_{M(E)} + \|t_N\|_{M(E)}.
\end{aligned}$$

The case $p > 1$ reduces to the previous one, since if $M(E)$ is a normed space, $\Lambda^{p,\infty}(E) = M(E)^{(p)}$ is also a normed space. Conversely, if $\Lambda^{p,\infty}(E)$ is a normed space, then $\Lambda^{p,\infty}(E) = M(\Lambda^{p,\infty}(E))$ since $\|f\|_{M(\Lambda^{p,\infty}(E))} = \sup_{y>0} y\|\chi_{\{|f|>y\}}\|_{\Lambda^{p,\infty}(E)} = \sup_{y>0} yE(\{|f| > y\})^{1/p}$; thus $M(\Lambda^{p,\infty}(E))$ is normed space.

2.1 Applications

Very often, the proof of a subadditivity property requires some careful work. Theorem 1 can be useful for this purpose, as we have seen in Example 4. Let us also show how the subadditivity of the norm follows very easily from that theorem for Shannon entropy spaces, and for classical Lorentz spaces and their multidimensional variants.

2.1.1 Shannon entropy spaces

For $\varphi(x) := x\log(1/x)$ on $[0,1/e]$, in [11] it is proved that $\Lambda(E_\varphi)$ is a normed space by first checking the easy fact $E_\varphi(I\cup J)+E_\varphi(I\cap J)\le E_\varphi(I)+E_\varphi(J)$ for intervals of $[0,1/e]$. Starting also from this fact, we can provide a new short proof as an application of Theorem 1.

Proposition 1 *$\Lambda(E_\varphi)$ is a normed space.*

Proof. We prove that $\|\cdot\|_{\Lambda(E_\varphi)}$ is a norm by directly checking that, for all sets, $E_\varphi(A\cup B)+E_\varphi(A\cap B)\le E_\varphi(A)+E_\varphi(B)$ holds. Observe that $E_\varphi(I)=\varphi(|I|)$, if I is an interval in $[0,1/e]$.

If $\varepsilon>0$, let

$$A\subset\bigcup_k I_k,\qquad B\subset\bigcup_j J_j$$

such that $\sum_k E_\varphi(I_k)+\sum_j E_\varphi(J_j)\le(1+\varepsilon)(E_\varphi(A)+E_\varphi(B))$.

Denote $D_0:=\{(k,j);\, I_k\cap J_j\neq\emptyset\}$, $D_1:=\{k;\, I_k\cap J_j=\emptyset\ \forall j\}$, $D_2:=\{j;\, I_k\cap J_j=\emptyset\ \forall k\}$. Then

$$A\cap B\subset\bigcup_{D_0}(I_k\cap J_j),\quad A\cup B\subset\Big(\bigcup_{D_0}(I_k\cup J_j)\Big)\cup\Big(\bigcup_{D_1}I_k\Big)\cup\Big(\bigcup_{D_2}J_j\Big)$$

and

$$\begin{aligned}E_\varphi(A\cup B)+E_\varphi(A\cap B)&\le\sum_{D_0}((E_\varphi(I_k\cup J_j)+E_\varphi(I_k\cap J_j))+\sum_{D_1}((E_\varphi(I_k)+\sum_{D_2}E_\varphi(J_j))\\&\le\sum_{D_0}((E_\varphi(I_k)+E_\varphi(J_j))+\sum_{D_1}((E_\varphi(I_k)+\sum_{D_2}E_\varphi(J_j))\\&\le(1+\varepsilon)(E_\varphi(A)+E_\varphi(B)).\end{aligned}$$

As also observed in [11], the condition $E_\varphi(I\cup J)+E_\varphi(I\cap J)\le E_\varphi(I)+E_\varphi(J)$ is false for cubes of $[0,1/e]^n$ if $n>1$. But E_φ is equivalent to

$$E^d_\varphi(A):=\inf\Big\{\sum_{k=1}^\infty\varphi(|(Q_k|);\, A\subset\bigcup_{k=1}^\infty Q_k,\, Q_k\subset[0,1/e]^n\text{ dyadic cubes}\Big\},$$

and the property $E^d_\varphi(A\cup B)+E^d_\varphi(A\cap B)\le E^d_\varphi(A)+E^d_\varphi(B)$ is checked as above, since it is trivially true for dyadic cubes. Hence, $\Lambda(E_\varphi)$ is a Banach space.

2.1.2 Classical Lorentz Spaces

We shall start by giving a very short proof of a well known fact.

Proposition 2 *If $1\le p<\infty$, $\Lambda^p(w)$ is a normed space if and only if the weight w is nonincreasing.*

Proof. Recall that $\|f\|^p_{\Lambda^p(w)}=\int_0^\infty f^*(x)^pw(x)dx=\int_0^\infty W(\mu_{|f|^p}(t))dt$, where $W(s)=\int_0^s w(s)\,ds$.

With our notations, $\Lambda^p(w)=\Lambda^p(\Lambda^1(w))$, since

$$\|f\|^p_{\Lambda^p(w)}=p\int_0^\infty t^{p-1}W(\mu_f(t))\,dt=p\int_0^\infty t^{p-1}\|\chi_{\{|f|>t\}}\|_{\Lambda^1(w)}\,dt,$$

which yields $\|f\|_{\Lambda^p(w)} = p^{1/p}\|f\|_{\Lambda^p(\Lambda^1(w))}$. We have seen that this is a norm if and only if

$$\|\chi_{A\cup B}\|_{\Lambda^1(w)} + \|\chi_{A\cap B}\|_{\Lambda^1(w)} \le \|\chi_A\|_{\Lambda^1(w)} + \|\chi_B\|_{\Lambda^1(w)}. \qquad (4)$$

Let $0 < \varepsilon < a < b$. For $A = (0, a)$ and $B = (\varepsilon, b)$, (4) reads $W(b) - W(b-\varepsilon) \le W(a) - W(a-\varepsilon)$, thus $w(b) \le w(a)$ and w is nonincreasing.

Conversely, if W is concave, then $W(x+y-t) \le W(x)+W(y)-W(t)$ if $0 \le t \le \min(x, y)$, and $W(\mu(A \cup B)) \le W(\mu(A)) + W(\mu(B)) - W(\mu(A \cap B))$. Since $W(\mu(A)) = \|\chi_A\|_{\Lambda^1(w)}$, we obtain condition (4).

To consider multidimensional analogs of classical estimates, the corresponding variant of Lorentz spaces have been also introduced. Let us recall the definitions.

For every measurable set $E \subset \mathbf{R}^2$ let $\varphi_E(x) := \mu(\{y;\,(x, y) \in E\})$ the measure of the x-section of E,

$$E_2^* := \big\{(s, t) \in (0, \infty)^2;\, 0 < t \le \varphi_E^*(s)\big\},$$

where f_E^* is the decreasing rearrangement of f_E, and

$$f_2^*(x) := \int_0^\infty \chi_{\{|f|>y\}_2^*}(x)\,dy \qquad (x \in (0, \infty)^2)$$

the **2-dimensional decreasing rearrangement** of f.

Then, if $\omega > 0$ is a locally integrable weight on $(0, \infty)^2$ and $p \ge 1$, the **2-dimensional Lorentz space** $\Lambda_2^p(\omega)$ contains all functions $f \in L^0((0, \infty)^2)$ such that

$$\|f\|_{\Lambda_2^p(\omega)} := \Big(\int_{(0,\infty)^2} f_2^*(x)^p \omega(x)\,dx\Big)^{1/p} < \infty.$$

It is a result of [2] that $\|\cdot\|_{\Lambda_2^p(\omega)}$ is a norm if and only if an equivalent condition to (3) holds true. Let us see how this is again an easy consequence of the above results.

Proposition 3 $\Lambda_2^p(\omega) = \Lambda^p(\Lambda_2^1(\omega))$, *and it is a normed space if and only if*

$$\|\chi_{A\cup B}\|_{\Lambda_2^1(\omega)} + \|\chi_{A\cap B}\|_{\Lambda_2^1(\omega)} \le \|\chi_A\|_{\Lambda_2^1(\omega)} + \|\chi_B\|_{\Lambda_2^1(\omega)}.$$

Proof. Since $f_2^*(x)^p = (f_2^p)^*(x) = p\int_0^\infty t^{p-1}\chi_{\{|f|>t\}_2^*}(x)\,dt$ (cf. [2]), we obtain

$$\begin{aligned}\|f\|^p_{\Lambda_2^p(\omega)} &= \int_{(0,\infty)^2}\Big(p\int_0^\infty t^{p-1}\chi_{\{|f|>t\}_2^*}(x)\,dt\Big)\omega(x)\,dx\\ &= p\int_0^\infty t^{p-1}\|\chi_{\{|f|>t\}}\|_{\Lambda_2^1(\omega)}dt = p\|f\|^p_{\Lambda^p(\Lambda_2^1(\omega))}.\end{aligned}$$

The last part is a direct application of Theorem 1.

3 Interpolation

Recall that, if $\bar{A} = (A_0, A_1)$ is a couple of quasi-Banach spaces, $0 < \theta < 1$ and $0 < q \le \infty$, the interpolation space $\bar{A}_{\theta,q}$ is the quasi-Banach space of all $f \in A_0 + A_1$ such that

$$\|f\|_{\theta,q} := \Big(\int_0^\infty (t^{-\theta}K(t, f, \bar{A}))^q \frac{dt}{t}\Big)^{1/q} < \infty$$

where $K(t,f,\bar{A})$ is the K-functional,

$$K(t,f;\bar{A}) := \inf\left\{\|f_0\|_{A_0} + t\|f_1\|_{A_1};\ f = f_0 + f_1\right\}$$

($\|f\|_X := \infty$ if $f \notin X$). In the case of function spaces on Ω, there exists $\Omega_f(t) \subset \Omega$ such that

$$K(t,f;\bar{A}) \simeq \|f\chi_{\Omega_f(t)}\|_{A_0} + t\|f\chi_{\Omega\setminus\Omega_f(t)}\|_{A_1}. \tag{5}$$

This well known fact is easily checked: if $f = f_0 + f_1$ and $\|f_0\|_{A_0} + t\|f_1\|_{A_1} \le 2K(t,f;\bar{A})$, take $\Omega_f(t) := \{\omega \in \Omega;\ |f_0(\omega)| \ge |f_1(\omega)|\}$, and then $|f|\chi_{\Omega_f(t)} \le 2|f_0|$ and $|f|\chi_{\Omega\setminus\Omega_f(t)} \le 2|f_1|$. Also

$$K(t,\chi_B;\bar{A}) \simeq \inf\{A_0(B_0) + tA_1(B_1);\ B = B_0 \cup B_1,\ B_0 \cap B_1 = \emptyset\}. \tag{6}$$

if $f = \chi_A$. Here, $A_j(B) := \|\chi_B\|_{A_j}$.

We refer to [3], [4], [5] and [12] for undefined facts concerning interpolation and function spaces.

3.1 Interpolation with L^∞

Let now E be any quasi-entropy function, and $0 < r < \infty$. It follows from (6) that

$$K(t,\chi_A;\Lambda^{1,r}(E),L^\infty) \simeq K(t,\chi_A;M(E),L^\infty) \simeq \min(E(A),t) \tag{7}$$

since $\|\chi_A\|_{\Lambda^{1,r}(E)} \simeq E(A) = \|\chi_A\|_{M(E)}$. Moreover, if $A = A_0 \cup A_1$ with $A_0 \cap A_1 = \emptyset$ and $E(A) \le t$, then

$$E(A) \le E(A_0) + t = E(A_0) + t\,\|\chi_{A_1}\|_\infty\,,$$

and $t \le E(A_0) + t\,\|\chi_{A_1}\|_\infty$ if $E(A) > t$. In both cases, $\min(E(A),t) \le cK(t,\chi_A;M(E),L^\infty)$.

Theorem 3 *If E is a quasi-entropy function on Ω and $0 < p < \infty$, then*
(a) $K(t,f;\Lambda^p(E),L^\infty) \simeq \left(\int_0^\infty y^{p-1}\min(E(\{|f|>y\}),t^p)\,dy\right)^{1/p}$, and
(b) $K(t,f;M(E),L^\infty) \simeq \sup_{y>0} y\min(E(\{|f|>y\}),t)$.

Proof. (a) Let $0 \le f \in \Lambda^p(E) + L^\infty$. For a given $t > 0$, if we consider

$$y^* := \inf\{y > 0;\ E(\{f > y\}) \le t^p\}$$

and

$$f_0(x) := \int_{y^*}^\infty \chi_{\{f>y\}}(x)\,dy, \quad f_1(x) := \int_0^{y^*} \chi_{\{f>y\}}(x)\,dy, \tag{8}$$

then $f = f_0 + f_1$ and $\{f_0 > y\} = \{f > y + y^*\}$. Hence

$$\begin{aligned}
\|f_0\|^p_{\Lambda^p(E)} &= \int_0^\infty y^{p-1}E(\{f > y + y^*\})\,dy \\
&\le \int_0^{y^*} y^{p-1}E(\{f > y + y^*\})\,dy + \int_{y^*}^\infty y^{p-1}E(\{f > y\})\,dy \\
&\preceq E(\{f > y^*\})y^{*p} + \int_{y^*}^\infty y^{p-1}E(\{f > y\})\,dy \\
&\le t^p y^{*p} + \int_{y^*}^\infty y^{p-1}E(\{f > y\})\,dy,
\end{aligned}$$

and then

$$\begin{aligned}
K(t,f;\Lambda^p(E),L^\infty) &\leq \|f_0\|_{\Lambda^p(E)} + t\|f_1\|_\infty \\
&\leq \left(t^p y^{*p} + \int_{y^*}^\infty y^{p-1} E(\{f>y\})\,dy\right)^{1/p} + ty^* \\
&\preceq \left(t^p y^{*p} + \int_{y^*}^\infty y^{p-1} E(\{f>y\})\,dy\right)^{1/p} + \left(t^{\,p}\int_0^{y^*} y^{p-1}dy\right)^{1/p} \\
&\preceq \left(\int_{y^*}^\infty y^{p-1} E(\{f>y\})\,dy + t^{\,p}\int_0^{y^*} y^{p-1}dy\right)^{1/p} \\
&\preceq \left(\int_0^\infty y^{p-1}\min(E(\{|f|>y\}),t^p)\,dy\right)^{1/p}
\end{aligned}$$

For the converse we use (5) and set $f_0 := f\chi_{\Omega(t)}$, $f_1 := f\chi_{\Omega\setminus\Omega(t)}$. Then

$$K(t,f;\Lambda^p(E),L^\infty) \simeq \|f_0\|_{\Lambda^p(E)} + t\|f_1\|_\infty.$$

By (7), and since $\chi_{\{f>y\}} = \chi_{\{f_0>y\}} + \chi_{\{f_1>y\}}$ (f_0, f_1 are disjointly supported),

$$\min(E(\{f>y\}),t) \simeq K(t,\chi_{\{f>y\}};\Lambda(E),L^\infty) \leq E(\{f_0>y\}) + t\|\chi_{\{f_1>y\}}\|_\infty.$$

Using now that $\|f_1\|_\infty = \int_0^\infty \|\chi_{\{f_1>y\}}\|_\infty dy \simeq \left(\int_0^\infty y^{p-1}\|\chi_{\{f_1>y\}}\|_\infty dy\right)^{1/p}$, we obtain

$$\begin{aligned}
K(t,f;\Lambda^p(E),L^\infty) &\simeq \left(\int_0^\infty y^{p-1}E(\{f_0>y\})dy\right)^{1/p} + \left(t^p\int_0^\infty y^{p-1}\|\chi_{\{f_1>y\}}\|_\infty dy\right)^{1/p} \\
&\simeq \left(\int_0^\infty y^{p-1}\left(E(\{f_0>y\}) + t^p\|\chi_{\{f_1>y\}}\|_\infty\right)dy\right)^{1/p} \\
&\geq \left(\int_0^\infty y^{p-1}\left(K(t^p,\chi_{\{f>y\}};\Lambda(E),L^\infty)\right)dy\right)^{1/p} \\
&\simeq \left(\int_0^\infty y^{p-1}\min(E(\{f>y\}),t^p)\,dy\right)^{1/p}.
\end{aligned}$$

(b) We observe that $K(t,f;M(E),L^\infty) \geq \sup_y yK(t,\chi_{\{f>y\}};M(E),L^\infty)$, since $f \geq y\chi_{\{f>y\}}$, and that $K(t,\chi_{\{f>y\}};M(E),L^\infty) \simeq \min(E(\{f>y\}),t)$, by (7). Hence

$$K(t,f;M(E),L^\infty) \geq c\sup_y y\min(E(\{f>y\}),t).$$

Conversely, let f_0 and f_1 be as in (8), but with $p=1$ in the definition of y^*. Then

$$\begin{aligned}
K(t,f;M(E),L^\infty) &\leq \|f_0\|_{M(E)} + t\|f_1\|_\infty \leq \sup_{y\geq y^*} yE(\{f>y\}) + ty^* \\
&\preceq \max(\sup_{y\geq y^*} yE(\{f>y\}), \sup_{y\leq y^*} yt) \leq \sup_y y\min(E(\{f>y\}),t).
\end{aligned}$$

If f is a measurable function on Ω, f_E^* will be the distribution function of $E(\{|f|>\cdot\})$ with respect to the Lebesgue measure m on $(0,\infty)$, i.e.

$$f_E^*(y) := m_{E(\{|f|>\cdot\})}(y) = m\left(\{s>0 : E(\{|f|>s\})>y\}\right).$$

Then we have the estimates

$$\int_0^\infty y^{p-1}\min(E(\{|f|>y\}),t^p)\,dy \simeq \int_0^{t^p} f_E^*(y)^p dy$$

and

$$\|f\|_{\Lambda^{p,q}(E)} = \Big(\int_0^\infty y^{q-1}E(\{|f|>y\})^{q/p}\Big)^{1/q} \simeq \Big(\int_0^\infty \Big(y^{1/p}f_E^*(y)\Big)^q \frac{dy}{y}\Big)^{1/q},$$

that follow from the fact that $m_{f_E^*}(y) = E(\{|f|>y\})$ a.e. $y>0$.

Theorem 4 *Let E be a quasi-entropy function and $0<\theta<1$. Then*
(a) $(\Lambda^{p_0}(E), L^\infty)_{\theta,q} = \Lambda^{p,q}(E)$, with $1/p = (1-\theta)/p_0$, and $0<p_0<q\le\infty$ or $0<p_0\le q<\infty$
(b) $(\Lambda^{1,r}(E), L^\infty)_{\theta,q} = (M(E), L^\infty)_{\theta,q}$, $0<r<\infty$ and $0<q\le\infty$.

Proof. (a) Let $q<\infty$ and denote $\|\cdot\|_{\theta,q}$ be the norm of $(\Lambda^{p_0}(E), L^\infty)_{\theta,q}$. Then

$$\begin{aligned}\|f\|_{\theta,q}^{p_0} &\simeq \Big(\int_0^\infty \Big(t^{-\theta p_0}\int_0^{t^{p_0}} f_E^*(y)^{p_0}dy\Big)^{q/p_0}\frac{dt}{t}\Big)^{p_0/q}\\ &\simeq \Big(\int_0^\infty \Big(t^{-\theta p_0+p_0}\int_0^1 f_E^*(yt^{p_0})^{p_0}y\frac{dy}{y}\Big)^{q/p_0}\frac{dt}{t}\Big)^{p_0/q}\end{aligned}$$

and, by Minkowski inequality,

$$\|f\|_{\theta,q}^{p_0} \preceq \int_0^1 \Big(y^{q/p_0}\int_0^\infty t^{(1-\theta)q}(f_E^*(yt^{p_0}))^q\frac{dt}{t}\Big)^{p_0/q}\frac{dy}{y} \preceq \Big(\int_0^\infty \Big(z^{\frac{(1-\theta)}{p_0}}f_E^*(z)\Big)^q\frac{dz}{z}\Big)^{p_0/q}$$

with $(1-\theta)/p_0 = 1/p$, hence

$$\|f\|_{\theta,q} \preceq \|f\|_{\Lambda^{p,q}(E)}.$$

Conversely, f_E^* being decreasing,

$$\|f\|_{\Lambda^{p,q}(E)} = \Big(\int_0^\infty \Big(z^{\frac{(1-\theta)}{p_0}}f_E^*(z)\Big)^q\frac{dz}{z}\Big)^{1/q} \simeq \Big(\int_0^\infty (t^{-\theta p_0}t^{p_0}(f_E^*(t^{p_0}))^{p_0})^{q/p_0}\frac{dt}{t}\Big)^{1/q} \preceq \|f\|_{\theta,q}.$$

(b) If $q=\infty$, then $\Lambda^{1,r}(E) = \Lambda^r(E^{(1/r)})$, by (a),

$$(\Lambda^r(E^{(1/r)}), L^\infty)_{\theta,\infty} = \Lambda^{\frac{1-\theta}{r},\infty}(E^{(1/r)}); \tag{9}$$

moreover, since

$$\|\chi_{\{f>y\}}\|_{\Lambda^{\frac{1-\theta}{r},\infty}(E^{(1/r)})} = E^{(1/r)}(\{f>y\})^{\frac{1-\theta}{r}} = E(\{f>y\})^{1-\theta}, \tag{10}$$

we have that $\|f\|_{\Lambda^{\frac{1-\theta}{r},\infty}(E^{(1/r)})} = \sup_{y>0} yE(\{f>y\})^{1-\theta}$. Now, by Theorem 3, (10) and (9),

$$\begin{aligned}\|f\|_{(M(E),L^\infty)_{\theta,\infty}} &\simeq \sup_{t>0} t^{-\theta}\sup_{y>0} y\min(E(\{|f|>y\}),t) = \sup_{y>0} y\sup_{t>0} t^{-\theta}\min(E(\{|f|>y\}),t)\\ &= \sup_{y>0} y\|\chi_{\{f>y\}}\|_{\Lambda^{\frac{1-\theta}{r},\infty}(E^{(1/r)})} \simeq \sup_{y>0} yE(\{f>y\})^{1-\theta}\\ &= \|f\|_{\Lambda^{\frac{1-\theta}{r},\infty}(E^{(1/r)})} \simeq \|f\|_{(\Lambda^r(E^{(1/r)}),L^\infty)_{\theta,\infty}}.\end{aligned}$$

The case $0<q<\infty$ is proved by reiteration considering $0<\theta_0<\theta<\theta_1<1$. Observe that

$$(M(E), L^\infty)_{\theta,q} = ((M(E), L^\infty)_{\theta_0,\infty}, (M(E), L^\infty)_{\theta_1,\infty})_{\eta,q}$$

if $\theta = (1-\eta)\theta_0 + \eta\theta_1$, and

$$(\Lambda^{1,r}(E), L^\infty)_{\theta,q} = ((\Lambda^{1,r}(E), L^\infty)_{\theta_0,\infty}, (\Lambda^{1,r}(E), L^\infty)_{\theta_1,\infty})_{\eta,q}$$

with $(M(E), L^\infty)_{\theta_j,\infty} = (\Lambda^{1,r}(E), L^\infty)_{\theta_j,\infty}$ $(j = 0,1)$ as we have seen above.

Corollary 1 *If $\mathcal{M}$ is the Hardy-Littlewood maximal operator, $1 \le q \le \infty$ and $1 < p < \infty$, then*

$$\mathcal{M} : \Lambda^{p,q}(E_\varphi) \longrightarrow \Lambda^{p,q}(E_\varphi).$$

Here E_φ is the Shannon entropy.

Proof. In [11] it is seen that

$$E_\varphi(\{\mathcal{M}g > t\}) \le \frac{c}{t}\int_0^\infty E_\varphi(\{|g| > s\})\, ds,$$

which means that $\mathcal{M} : \Lambda(E_\varphi) \longrightarrow M(E_\varphi)$. By interpolation,

$$\mathcal{M} : (\Lambda(E_\varphi), L^\infty)_{\theta,q} \longrightarrow (\Lambda(E_\varphi), L^\infty)_{\theta,q}.$$

The following result follows from (1) and Theorem 4 (see also [9] for related results).

Corollary 2 *If E is a Banach function space, $0 < \theta < 1$ and $1 \le q \le \infty$, then $(\Lambda(E), L^\infty)_{\theta,q} = (M(E), L^\infty)_{\theta,q} = (E, L^\infty)_{\theta,q} = \Lambda^{p,q}(E)$, $1/p = 1-\theta$. In particular, $\Lambda^{p,q}(E)$ are Banach function spaces if $p > 1$ and $1 \le q \le \infty$.*

3.2 Interpolation of Lorentz spaces with two parameters

We say that $\bar{E} = (E_0, E_1)$ is an **entropy couple** if E_0 and E_1 are two entropy functions on the same measure space Ω. Then, if $\overline{\Lambda(E)} = (\Lambda(E_0), \Lambda(E_1))$ and $t > 0$,

$$[E_0 + tE_1](A) := K(t, \chi_A; \overline{\Lambda(E)})$$

is an entropy function. It follows from (6) that, for any couple $\bar{X} = (X_0, X_1)$ of quasi-Banach function spaces such that $\|\chi_A\|_{X_i} = E_i(A)$ $(i = 0,1)$,

$$[E_0 + tE_1](A) \simeq K(t, \chi_A; \bar{X}). \tag{11}$$

Proposition 4 *Let $\bar{E} = (E_0, E_1)$ be an entropy couple.*
(a) If f and g are two nonnegative measurable disjointly supported functions, then

$$K(t, f+g; \overline{\Lambda(E)}) \le K(t, f; \overline{\Lambda(E)}) + K(t, g; \overline{\Lambda(E)}).$$

(b)

$$K(t, f; \overline{\Lambda(E)}) \simeq \int_0^\infty [E_0 + tE_1](\{|f| > y\})\, dy.$$

Proof. (a) Since f and g are disjointly supported, $\chi_{\{f+g>y\}} = \chi_{\{f>y\}} + \chi_{\{g>y\}}$ and then

$$\|f+g\|_{\Lambda(E_i)} \leq \|f\|_{\Lambda(E_i)} + \|g\|_{\Lambda(E_i)} \quad (i=0,1).$$

But if $f = f_0 + f_1$ and $g = g_0 + g_1$ (with f_i and g_i nonnegative) then f_i and g_i are disjointly supported $(i = 0, 1)$. Hence

$$\begin{aligned} K(t, f+g; \overline{\Lambda(E)}) &\leq \inf_{\substack{f=f_0+f_1 \\ g=g_0+g_1}} \left\{ \|f_0+f_1\|_{\Lambda(E_0)} + t\,\|g_0+g_1\|_{\Lambda(E_1)} \right\} \\ &\leq \inf_{\substack{f=f_0+f_1 \\ g=g_0+g_1}} \left\{ \|f_0\|_{\Lambda(E_0)} + \|f_0\|_{\Lambda(E_0)} + t\,\|g_0\|_{\Lambda(E_1)} + t\,\|g_0\|_{\Lambda(E_1)} \right\} \\ &\leq K(t, f; \overline{\Lambda(E)}) + K(t, g; \overline{\Lambda(E)}). \end{aligned}$$

(b) We may assume that $f \geq 0$. Using (5) we have

$$K(t, f; \overline{\Lambda(E)}) \simeq \|f_0\|_{\Lambda(E_0)} + t\|f_1\|_{\Lambda(E_1)}$$

where f_0 , f_1 are disjointly supported, thus $\chi_{\{f>y\}} = \chi_{\{f_0>y\}} + \chi_{\{f_1>y\}}$ and

$$K(t, f; \overline{\Lambda(E)}) \simeq \int_0^\infty \Big(E_0(\{f_0 > y\}) + tE_1(\{f_1 > y\}) \Big)\, dy \geq \int_0^\infty [E_0 + tE_1](\{f > y\}) dy.$$

For the reverse estimate, since $f \leq \sum_{k\in\mathbf{Z}} 2^{k+1}\chi_{\{2^k < f \leq 2^{k+1}\}}$, from (a) we get

$$\begin{aligned} K(t, f; \overline{\Lambda(E)}) &\leq \sum_{k\in\mathbf{Z}} 2^{k+1}[E_0 + tE_1](\{2^k < f \leq 2^{k+1}\}) \leq 4 \sum_{k\in\mathbf{Z}} 2^{k-1}[E_0 + tE_1](\{f > 2^k\}) \\ &\leq 4 \int_0^\infty [E_0 + tE_1](\{f > y\})\, dy. \end{aligned}$$

If $0 < \theta < 1$ and $0 < q \leq \infty$, we denote

$$\bar{E}_{\theta,q}(A) := \Big(\int_0^\infty (t^{-\theta}\, [E_0 + tE_1]\, (A))^q \frac{dt}{t} \Big)^{1/q} \qquad (A \in \Sigma).$$

Corollary 3 *If* $0 < \theta < 1$, $\overline{\Lambda(E)}_{\theta,1} = \Lambda(\bar{E}_{\theta,1})$.

New interpolation results for couples of $\Lambda^{p,q}$-spaces can be obtained by combining Theorem 4 with Sparr's interpolation method for several spaces (cf. [17]).

Let $\bar{X} = (X_0, X_1, X_2)$ be a compatible quasi-Banach triple, and $(\alpha, \beta) \in (0, 1)^2$ such that $\alpha + \beta < 1$ and $1 \leq q \leq \infty$. The corresponding interpolated space $\bar{X}_{\alpha,\beta,q;K}$ is defined by the condition

$$\|x\|_{\bar{X}_{\alpha,\beta,q;K}} := \Big(\int_0^\infty \int_0^\infty \Big(t_1^{-\alpha} t_2^{-\beta} K(t_1, t_2, x; \bar{X}) \Big)^q \frac{dt_1}{t_1} \frac{dt_2}{t_2} \Big)^{1/q} < \infty \qquad (x \in X_0 + X_1 + X_2),$$

with

$$K(t_1, t_2, x; \bar{X}) := \inf_{x = x_0 + x_1 + x_2} \Big\{ \|x_0\|_{X_0} + t_1\|x_1\|_{X_1} + t_2\|x_2\|_{X_2} \Big\} \qquad (t_1, t_2 > 0).$$

In our setting of function spaces, as shown in [17] and [1], the reiteration formulae

$$((X_0, X_2)_{\theta,\infty}, (X_1, X_2)_{\theta,\infty})_{\eta,\infty} = (X_0, X_1, X_2)_{(1-\theta)\eta,\theta,\infty}. \tag{12}$$

and

$$((X_0,X_2)_{\alpha_0,q_0},(X_1,X_2)_{\alpha_1,q_1})_{\eta,q}=(X_0,X_1,X_2)_{\theta_1,\theta_2,q}. \tag{13}$$

hold if $1\le q_0,q_1,q<\infty$, $1/q=(1-\eta)/q_0+\eta/q_1$, $\theta_1=(1-\alpha_1)\eta$ and $\theta_2=\alpha_0(1-\eta)+\alpha_1\eta$.

Lemma 1 *Let $\bar E=(E_0,E_1)$ be an entropy couple, and $1\le r<\infty$. Then*

$$K(t_1,t_2,f;\Lambda^{r,1}(E_0),\Lambda^{r,1}(E_1),L^\infty)\simeq\int_0^\infty \min\Big([E_0^{(r)}+t_1E_1^{(r)}](\{|f|>y\}),t_2\Big)\,dy,$$

Proof. Obviously,

$$K(t_1,t_2,f;\Lambda^{r,1}(E_0),\Lambda^{r,1}(E_1),L^\infty)=K(t_2,f;\Lambda^{r,1}(E_0)+t_1\Lambda^{r,1}(E_1),L^\infty).$$

Moreover since $\Lambda(\Lambda^{r,1}(E_j))=\Lambda^{r,1}(E_j)=\Lambda(E_j^{(r)})$ $(j=0,1)$, and $(E_0^{(r)},E_1^{(r)})$ is an entropy couple (since $1\le r<\infty$) we have that

$$\Lambda^{r,1}(E_0)+t_1\Lambda^{r,1}(E_1)=\Lambda([E_0^{(r)}+t_1E_1^{(r)}])$$

since, by Proposition 4,

$$\begin{aligned}\|g\|_{\Lambda^{r,1}(E_0)+t_1\Lambda^{r,1}(E_1)} &\simeq \int_0^\infty K(t_1,\chi_{\{|g|>y\}};\Lambda^{r,1}(E_0),\Lambda^{r,1}(E_1))\,dy\\ &= \int_0^\infty [E_0^{(r)}+t_1E_1^{(r)}](\{|g|>y\})dy=\|g\|_{\Lambda([E_0^{(r)}+t_1E_1^{(r)}])}.\end{aligned}$$

Now the result follows from Theorem 4.

Theorem 5 *Let $\bar E=(E_0,E_1)$ be an entropy couple. Let $0<\eta<1$. Then*
(a) If $1<p_0,p_1<\infty$, $1\le q_0,q_1<\infty$, $1/p=(1-\eta)/p_0+\eta/p_1$ and $1/q=(1-\eta)/q_0+\eta/q_1$,

$$(\Lambda^{p_0,q_0}(E_0),\Lambda^{p_1,q_1}(E_1))_{\eta,q}=\Lambda^{p,q}(\bar E_{\eta p/p_1,q/p}).$$

(b) If $1<p<\infty$,

$$(\Lambda^{p,\infty}(E_0),\Lambda^{p,\infty}(E_1))_{\eta,\infty}=\Lambda^{p,\infty}(\bar E_{\eta,\infty}).$$

(c)

$$(M(E_0),M(E_1))_{\eta,\infty}=M(\bar E_{\eta,\infty}).$$

Proof. (a) Let $1<r<\min(p_0,p_1)$. Since

$$\Lambda^{p_i,q_i}(E_i)=(\Lambda^{r,1}(E_i),L^\infty)_{\alpha_i,q_i}\qquad(\alpha_i=(p_i-r)/p_i=\varepsilon/p;\ i=0,1),$$

we may use (13) to obtain

$$(\Lambda^{p_0,q_0}(E_0),\Lambda^{p_1,q_1}(E_1))_{\eta,q}=\bar X_{\theta_1,\theta_2,q},$$

with $\bar X=(\Lambda^{r,1}(E_0),\Lambda^{r,1}(E_1),L^\infty)$, $\theta_1=(1-\alpha_1)\eta$ and $\theta_2=\alpha_0(1-\eta)+\alpha_1\eta$. Moreover, by Lemma 1,

$$K(t_1,t_2,f;\bar X)\simeq\int_0^\infty \min([E_0^{(r)}+t_1E_1^{(r)}](\{|f|>y\}),t_2)\,dy$$

Hence,

$$\|f\|^q_{\bar X_{\theta_1,\theta_2,q}} \simeq \int_0^\infty \int_0^\infty \Big(t_1^{-\theta_1} t_2^{-\theta_2} \int_0^\infty \min([E_0^{(r)} + t_1 E_1^{(r)}](\{|f| > y\}), t_2)\, dy \Big)^q \frac{dt_2}{t_2} \frac{dt_1}{t_1},$$

where, by Theorem 4,

$$\begin{aligned} &\int_0^\infty \Big(t_2^{-\theta_2} \int_0^\infty \min([E_0^{(r)} + t_1 E_1^{(r)}](\{|f| > y\}), t_2) dy \Big)^q \frac{dt_2}{t_2} \\ &\qquad \simeq \|f\|_{(\Lambda([E_0^{(r)}+t_1E_1^{(r)}]),L^\infty)_{\theta_2,q}} = \|f\|_{\Lambda^{1/(1-\theta_2),q}([E_0^{(r)}+t_1E_1^{(r)}])} \\ &\qquad \simeq \int_0^\infty y^{q-1} ([E_0^{(r)} + t_1 E_1^{(r)}](\{|f| > y\}))^{(q(1-\theta_2))}\, dy. \end{aligned}$$

On the other hand, $E_i^{(r)}(A) = \|\chi_A\|_{\Lambda(E_i)^{(r)}}$ $(i = 0, 1)$. Then, by (11),

$$[E_0^{(r)} + t_1 E_1^{(r)}](A) \simeq K(t_1, \chi_A; \Lambda(E_0)^{(r)}, \Lambda(E_1)^{(r)})$$

and, since $K(|g|^r, t^r; X_0, X_1) \simeq K(g, t; X_0^{(r)}, X_1^{(r)})^r$, it follows that

$$\begin{aligned} &\int_0^\infty y^{q-1} ([E_0^{(r)} + t_1 E_1^{(r)}](\{|f| > y\}))^{q(1-\theta_2)}\, dy \\ &\qquad \simeq \int_0^\infty y^{q-1} K(t_1, \chi_{\{|f|>y\}}; \Lambda(E_0)^{(r)}, \Lambda(E_1)^{(r)})^{(1-\theta_2)q} dy \\ &\qquad \simeq \int_0^\infty y^{q-1} K(t_1^r, \chi_{\{|f|>y\}}; \Lambda(E_0), \Lambda(E_1))^{(1-\theta_2)q/r}\, dy \\ &\qquad = \int_0^\infty y^{q-1} ([E_0 + t_1^r E_1](\{|f| > y\}))^{(1-\theta_2)q/r}\, dy. \end{aligned}$$

Thus

$$\begin{aligned} \|f\|^q_{\bar X_{\theta_1,\theta_2,q}} &\simeq \int_0^\infty t_1^{-\theta_1 q} \int_0^\infty y^{q-1} ([E_0 + t_1^r E_1](\{|f| > y\}))^{(1-\theta_2)q/r} dy \frac{dt_1}{t_1} \\ &= \frac{1}{r} \int_0^\infty y^{q-1} \Big(\int_0^\infty (\tau^{-\theta_1} [E_0 + \tau E_1](\{|f| > y\}))^{(1-\theta_2)q/r} d\tau \Big)\, dy, \end{aligned}$$

where $\theta_1 = \frac{r\eta}{p_1}$ and $1 - \theta_2 = \frac{r}{p}$. Hence

$$\begin{aligned} \|f\|^q_{\bar X_{\theta_1,\theta_2,q}} &\simeq \int_0^\infty y^{q-1} \int_0^\infty (\tau^{-\eta p/p_1} [E_0 + \tau E_1](\{|f| > y\}))^{q/p} d\tau\, dy \\ &= \int_0^\infty y^{q-1} \|\chi_{\{|f|>y\}}\|^{q/p}_{\bar E_{\eta p/p_1, q/p}} dy = \|f\|^q_{\Lambda^{p,q}(\bar E_{\eta p/p_1, q/p})}. \end{aligned}$$

(b) Let $1 < r < p$. Since $\Lambda^{p,\infty}(E_i) = (\Lambda^{r,1}(E_i), L^\infty)_{\theta,\infty}$ when $\theta = (p - r)/p$ (by Theorem 4), we obtain from (12) that $(\Lambda^{p,\infty}(E_0), \Lambda^{p,\infty}(E_1))_{\eta,\infty} = \bar X_{(1-\theta)\eta,\theta,\infty}$ if $\bar X = (\Lambda^{r,1}(E_0), \Lambda^{r,1}(E_1), L^\infty)$, and

$$\|f\|_{\bar X_{(1-\theta)\eta,\theta,\infty}} \simeq \sup_{t_1>0} t_1^{-(1-\theta)\eta} \sup_{t_2>0} t_2^{-\theta} \int_0^\infty \min([E_0^{(r)} + t_1 E_1^{(r)}](\{|f| > y\}), t_2)\, dy$$

with

$$\sup_{t_2>0} t_2^{-\theta} \int_0^\infty \min([E_0^{(r)} + t_1 E_1^{(r)}](\{|f| > y\}), t_2)\, dy \simeq \sup_{y>0} y([E_0^{(r)} + t_1 E_1^{(r)}](\{|f| > y\}))^{1-\theta},$$

since, again by Theorem 4, $(\Lambda([E_0^{(r)} + t_1 E_1^{(r)}]), L^\infty)_{\theta,\infty} = \Lambda^{p,\infty}([E_0^{(r)} + t_1 E_1^{(r)}])$. Thus,

$$\begin{aligned}
\|f\|_{\bar{X}_{(1-\theta)\eta,\theta,\infty}} &\simeq \sup_{t_1>0} t_1^{-(1-\theta)\eta} \sup_{y>0} y \left([E_0^{(r)} + t_1 E_1^{(r)}](\{|f| > y\})\right)^{1-\theta} \\
&\simeq \sup_{t_1>0} t_1^{-(1-\theta)\eta} \sup_{y>0} K(t_1, \chi_{\{|f|>y\}}; \Lambda(E_0)^{(r)}, \Lambda(E_1)^{(r)})^{1-\theta} \\
&\simeq \sup_{y>0} \sup_{\tau>0} \tau^{-(1-\theta)\eta/r} \left([E_0 + \tau E_1](\{|f| > y\})\right)^{(1-\theta)r} \\
&= \sup_{y>0} y \|\chi_{\{|f|>y\}}\|_{\bar{E}_{\eta,\infty}}^{1/p} = \|f\|_{\Lambda^{p,\infty}(\bar{E}_{\eta,\infty})}
\end{aligned}$$

and also $\bar{X}_{(1-\theta)\eta,\theta,\infty} = \Lambda^{p,\infty}(\bar{E}_{\eta,\infty})$.
(c) Let $1 < p < \infty$. Since $(\Lambda^{p,\infty}(E_0), \Lambda^{p,\infty}(E_1))_{\eta,\infty} = \Lambda^{p,\infty}(\bar{E}_{\eta,\infty})$ (by (b)). And $\Lambda^{p,\infty}(E) = M(E)^{(p)}$, we have

$$(M(E_0)^{(p)}, M(E_1)^{(p)})_{\eta,\infty} = M(\bar{E}_{\eta,\infty})^{(p)}$$

and, using again that $K(t, f; M(E_0)^{(p)}, M(E_1)^{(p)}) \simeq K(t^p, |f|^p; \overline{M(E)})^{1/p}$, an easy computation shows that $(M(E_0)^{(p)}, M(E_1)^{(p)})_{\eta,\infty} = \overline{M(E)}_{\eta,\infty}^{(p)}$.

3.3 Applications

We shall finish the paper by showing how the previous work, that allow us to restrict arguments to characteristic functions, can be applied to study interpolation properties of some variants of Lorentz spaces, and of entropy spaces and the related block spaces.

Let us start by describing the real interpolated space $(E, L^\infty)_{\theta,q}$ when E is quasi-Banach function space

Theorem 6 *Let E be a quasi-Banach function space, $0 < \theta < 1$ and $1 \le q \le \infty$. Then*

$$(E, L^\infty)_{\theta,q} = \Lambda^{p,q}(E),$$

where $1/p = 1 - \theta$.

Proof. As in (2), we take $0 < u \le 1$ such that $\Lambda^{1,u}(E) \hookrightarrow E \hookrightarrow M(E)$. Then

$$\left(\Lambda^{1,u}(E), L^\infty\right)_{\theta,q} \hookrightarrow (E, L^\infty)_{\theta,q} \hookrightarrow (M(E), L^\infty)_{\theta,q}$$

and, by Theorem 4,

$$\left(\Lambda^{1,u}(E), L^\infty\right)_{\theta,q} = (M(E), L^\infty)_{\theta,q} = (\Lambda(E), L^\infty)_{\theta,q} = \Lambda^{p,q}(E).$$

3.3.1 Block spaces

Let X denote the Euclidean space $\mathbf{R}^n$ or the fundamental cube of $\mathbf{R}^n$. For $1 \le q \le \infty$, a measurable function $b(x)$ is said to be a *$q-$block* if there is a cube $Q \subset X$ containing the support of b such that

$$\left(\int |b(x)|^q \, dx\right)^{1/q} \le \frac{1}{|Q|^{1-1/q}}.$$

Classes of functions generated by $q-$*blocks* and numerical sequences $c = \{c_k\}$ satisfying

$$M(c) = \sum_{k=1}^{\infty} |c_k| \left(1 + \log^{+} \frac{1}{|c_k|}\right) < \infty \tag{14}$$

were introduced in [18] in connection with the convergence of Fourier series.

The block space $(B_q(X), N_q)$ is defined as the class of all functions f which can be written as

$$f(x) = \sum c_k b_k(x), \tag{15}$$

where the b_k are $q-$*blocks* and $\{c_k\}$ satisfies (14), and

$$N_q(f) := \inf \left\{ \sum_{k=1}^{\infty} |c_k| \left(1 + \log^{+} \frac{\sum_{k=1}^{\infty} |c_k|}{|c_k|}\right) \right\} < \infty,$$

where the infimum is taken over all representations (15) of f.

Its topology can also be define through the metric $d(f, g) = M_q(f - g)$, where

$$M_q(f) := \inf \left\{ \sum_{k=1}^{\infty} |c_k| \left(1 + \log^{+} \frac{1}{|c_k|}\right) \right\}.$$

We refer to [14], [16] and [18] for the details.

The entropy space $\Lambda(E_\varphi)$ is related with the block space B_∞ on $[0, 1/e]$, since, as shown in [15,Theorem 3.3 and Remark 3.6],

$$E_\varphi(A) \simeq \int_0^\infty \|\chi_{\{\chi_A > t\}}\|_{M_\infty}\, dt = M_\infty(A).$$

Using Theorem 6 we can give the following description of $(B_q, L^\infty)_{\theta,r}$.

Theorem 7 *Let B_q endowed with the quasi-norm N_q $(1 \le q \le \infty)$. If $0 < \theta < 1$ and $1 \le r \le \infty$ and $1/p = 1 - \theta$, then*

$$(B_q, L^\infty)_{\theta,r} = \Lambda^{p,r}(N_q) \quad \text{and} \quad \|f\|_{\Lambda^{p,r}(N_q)} = \left(\int_0^\infty y^{r-1} N_q\left(\{|f| > y\}\right)^{r/p} dy\right)^{1/r}.$$

Proof. The quasi-Banach space (B_q, N_q) has the lattice property, since, if $f \in B_q$ and $|g| \le |f|$, then $(g/f)b_k$ is a $q-$block,

$$g = f\frac{g}{f} = \sum_k c_k \left(\frac{g}{f} b_k\right)$$

and $N_q(g) \le N_q(f)$. Although it is not known whether Fatou property holds for B_q, still

$$\Lambda^{1,u}(B_q) \subset B_q \subset M(B_q)$$

and the spaces are complete. Thus, as in Theorem 6,

$$(B_q, L^\infty)_{\theta,r} = \Lambda^{p,r}(N_q)$$

The inclusion $\Lambda^{1,u}(B_q) \subset B_q$ and the completeness of both $\Lambda^{1,u}(B_q)$ and $M(B_q)$ follow from the completeness of B_q and the embedding in L^0.

Remark 2 *Real interpolation for the couple (B_∞, L^∞), with B_∞ endowed with M_∞, has been considered by M. Milman in [15] using the error functional.*

3.3.2 Interpolation of Lorentz spaces with change of measures

The above results are also useful for interpolation of classical Lorentz spaces $\Lambda^p(w)$, with

$$\|f\|^p_{\Lambda^p(w)} = \int_0^\infty f^*(x)^p w(x)\,dx = \int_0^\infty W(\mu_{|f|^p}(t))\,dt,$$

where $W(s) := \int_0^s w(s)\,ds$.

Remark that, since we will be dealing with rearrangement invariant spaces, it follows from [8] (see also [7]) that in this case

$$K(t,\chi_A;\Lambda^{p_0}(w_0),\Lambda^{p_1}(w_1)) \simeq \min(W_0(\mu(A))^{1/p_0}, tW_1(\mu(A))^{1/p_1}). \tag{16}$$

Weighted Lorentz spaces with two parameters $\Lambda^{p,q}(w)$ can be considered as in [6], with $\|f\|_{\Lambda^{p,q}(w)} = \|yW(\mu_{|f|}(y))^{1/p}\|_{L^q(dy/y)}$, and it follows from the definition that

$$\Lambda^{p,q}(w) = \Lambda^{p,q}(\Lambda^1(w)).$$

Theorem 8 *Let w_0 and w_1 be two decreasing weights, and $0<\eta<1$. Then*
(a) If $1<p_0,p_1<\infty$ and $1\le q_0,q_1<\infty$,

$$(\Lambda^{p_0,q_0}(w_0),\Lambda^{p_1,q_1}(w_1))_{\eta,q} = \Lambda^q(dW),$$

where $1/q=(1-\eta)/q_0+\eta/q_1$ and $W = W_0^{(1-\eta)p/p_0}W_1^{\eta p/p_1}$.
(b) If $1\le p_0\le p_1<\infty$ and $1\le q\le\infty$, then

$$(\Lambda^{p_0,1}(w_0),\Lambda^{p_1,1}(w_1))_{\eta,1} = \Lambda(W) \quad \text{and} \quad (\Lambda^{p_0,\infty}(w_0),\Lambda^{p_1,\infty}(w_1))_{\eta,\infty}) = M(W),$$

where $W = W_0^{(1-\eta)/p_0}W_1^{\eta/p_1}$, and $\Lambda(W)$ and $M(W)$ are the Lorentz and the Marcinkiewicz spaces associated to the concave function W, as in Example 1.

Proof. (a) By Theorem 5, if $1/p=(1-\eta)/p_0+\eta/p_1$ and $1/q=(1-\eta)/q_0+\eta/q_1$,

$$(\Lambda^{p_0,q_0}(w_0),\Lambda^{p_1,q_1}(w_1))_{\eta,q} = \Lambda^{p,q}((\Lambda^1(w_0),\Lambda^1(w_1))_{\eta p/p_1,q/p}).$$

By (16)

$$K(t,\chi_A;\Lambda^1(w_0),\Lambda^1(w_1)) \simeq \min(W_0(\mu(A)), tW_1(\mu(A)))$$

and, if we denote $\|f\| := \|f\|_{\Lambda^{p,q}((\Lambda^1(w_0),\Lambda^1(w_1))_{\eta p/p_1,q/p})}$, we have

$$\begin{aligned}
\|f\|^q &= \int_0^\infty y^{q-1}\Big(\int_0^\infty \big(t^{\eta p/p_1}\min(W_0(\mu_f(y)), tW_1(\mu_f(y)))\big)^{q/p}\frac{dt}{t}\Big)^{p/q} dy\\
&\simeq \int_0^\infty y^{q-1}W_0(\mu_f(y))^{(1-\eta)p/p_0}W_1(\mu_f(y))^{\eta p/p_1}\,dy\\
&\simeq \int_0^\infty W_0(\mu_{|f|^q}(y))^{(1-\eta)p/p_0}W_1(\mu_{|f|^q}(y))^{\eta p/p_1}\,dy = \int_0^\infty W(\mu_{|f|^p}(y))\,dy.
\end{aligned}$$

(b) Since $\Lambda^{p_1,1}(w_i) = \Lambda(\Lambda^{p_i}(w_i))$ $(i=0,1)$, by Theorem 5,

$$(\Lambda^{p_0,1}(w_0),\Lambda^{p_1,1}(w_1))_{\eta,1} = (\Lambda(\Lambda^{p_0}(w_0)),\Lambda(\Lambda^{p_1}(w_1)))_{\eta,1} = \Lambda\big((\Lambda^{p_0}(w_0),\Lambda^{p_1}(w_1))_{\eta,1}\big).$$

Then, using again (16), we obtain

$$\begin{aligned}\|f\|_{\Lambda\left((\Lambda^{p_0}(w_0),\Lambda^{p_1}(w_1))_{\eta,1}\right)} &= \int_0^\infty \int_0^\infty t^{-\eta}\min(W_0(\mu_f(y))^{1/p_0}, tW_1(\mu_f(y))^{1/p_1})\frac{dt}{t}\,dy \\ &\simeq \int_0^\infty W_0(\mu_f(y))^{(1-\eta)/p_0}W_1(\mu_f(y))^{\eta/p_1}\,dy = \int_0^\infty W(\mu_f(y))\,dy.\end{aligned}$$

The last identity $(\Lambda^{p_0,\infty}(w_0), \Lambda^{p_1,\infty}(w_1))_{\eta,\infty} = M(W)$ follows from $\Lambda^{p_i,\infty}(w_i) = M(\Lambda^{p_i}(w_i))$ and by applying Theorem 5 once again.

3.3.3 Universal right Calderón couples (E, L^∞)

Our last application is devoted to the problem of finding the relative interpolation spaces for the couple (E, L^∞).

Let X and Y be two intermediate Banach spaces for the Banach couples $\bar{X}$ and $\bar{Y}$, respectively. It is said that they are relatively K-monotone with respect to $\bar{X}$ and $\bar{Y}$ if the property

$$K(\cdot, y; \bar{Y}) \leq K(\cdot, x; \bar{X}), \qquad x \in X$$

implies $y \in Y$. In this case, X and Y are relative interpolation spaces with respect to $\bar{X}$ and $\bar{Y}$, i.e., $T(X) \subset Y$ for every $T \in \mathcal{L}(\bar{X}; \bar{Y})$.

We say that $\bar{Y}$ is a universal right Calderón couple (or, as in [10], that it has the universal right K property) if the converse is true for every Banach couple $\bar{X}$, i.e., if all relative interpolation spaces X and Y with respect to $\bar{X}$ and $\bar{Y}$ are relatively K-monotone.

If a couple of Banach function spaces $\bar{E}$ is a universal right Calderón couple, then

$$\bar{E}(\theta) = \bar{E}_{\theta,\infty} \tag{17}$$

for some (or for all) $\theta \in (0,1)$, and the converse is true when $\bar{E} = (E, L^\infty)$ (see [10]). Here $\bar{E}(\theta) := E_0^{1-\theta}E_1^\theta$ is the Calderón product, defined by the norm

$$\|f\|_{E(\theta)} := \inf\left\{t > 0;\ |f| \leq t|f_0|^{1-\theta}|f_1|^\theta, \quad \|f_0\|_{E_0} \leq 1, \quad \|f_1\|_{E_1} \leq 1\right\}.$$

The following theorem characterizes this property for (E, L^∞) in terms of the maximal space. We shall use that, for $1-\theta = 1/p$, $E^{1-\theta}(L^\infty)^\theta = E^{(p)}$, the p-convexification of E (cf. [10]).

Theorem 9 *If E is a Banach function space, then*
(a) For all $0 < \theta < 1$, $(E, L^\infty)_{\theta,\infty} = M(E)^{(p)}$ $(1-\theta = 1/p)$, and
(b) (E, L^∞) is a universal right Calderón couple if and only if $E = M(E)$.

Proof. If $1-\theta = 1/p$ and $f \geq 0$, by Theorem 4,

$$\|f\|_{M(E)^{(p)}} = \sup_{y>0} y^{1-\theta}E(\{f^p > y\})^{1-\theta} = \sup_{s>0} sE(\{f > s\})^{1-\theta} = \|f\|_{(M(E),L^\infty)_{\theta,\infty}}$$

and $(M(E), L^\infty)_{\theta,\infty} = (E, L^\infty)_{\theta,\infty}$.

To prove (b) assume first that (E, L^∞) is a universal right Calderón couple. It follows from (a) and (17) that $E^{(p)} = M(E)^{(p)}$, and then $E = M(E)$.

Conversely, if $E = M(E)$, then $M(E)$ is a Banach function space and, if $1-\theta = 1/p$, $M(E)^{1-\theta}(L^\infty)^\theta = M(E)^{(p)} = E^{(p)} = E^{1-\theta}(L^\infty)^\theta$. ¿From (a) we obtain $(E, L^\infty)_{\theta,\infty} = M(E)^{(p)} = E^{1-\theta}(L^\infty)^\theta$, and (E, L^∞) is a universal right Calderón couple, by (17).

Corollary 4 *If E_0 and E_1 are two Banach function spaces such that (E_i, L^∞) is a universal right Calderón couple ($i = 0, 1$) and $0 < \theta < 1$, then $(\bar{E}_{\theta,\infty}, L^\infty)$ is also a universal right Calderón couple.*

Proof. This follows from Theorem 9, since $(M(E_0), M(E_1))_{\theta,\infty} = M(\bar{E}_{\theta,\infty})$ by Theorem 5.

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