RECENT RESULTS ON LINEAR SYSTEMS ON GENERIC $K3$ SURFACES

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Abstract. In this note we relate about the problem of evaluate the dimension of linear systems through fat points defined on generic $K3$ surfaces.

1. Introduction and statement of the problem

In what follows we assume that the ground field is algebraically closed of characteristic 0. With S we always denote a smooth projective *generic* $K3$ surface, i.e. $\text{Pic}(S) = \langle H\rangle$ and let $n = H^2$. Consider r points in general position on S, to each one of them associate a natural number m_i called the *multiplicity* of the point. We will denote by $\mathcal{L} = \mathcal{L}^n(d, m_1, \ldots, m_n)$ the linear system $|dH|$ through the r points with the given multiplicities. Define the *virtual dimension* of the system as $v(\mathcal{L}) = d^2n/2+1-\sum m_i(m_i+1)/2$ and its *expected dimension* by $e = \max\{v, -1\}$. Observe that $e \leq \dim(\mathcal{L})$ and that the inequality may be strict if the conditions imposed by the points are dependent. In this case we say that the system is *special*. By S' we will denote the blow-up of S along the r points, given two curves A, B on S, the intersection AB will be defined as the intersection of their strict transforms on S'. The problem of classifying special systems has been largely studied for linear systems on the plane [2, 6, 11] and more generally for systems on rational surfaces [7, 8]. The main conjecture on the structure of such systems has been formulated in [8]. In this note we report about some recent results in the case of generic $K3$ surfaces. In [3] the authors proved that on the projective plane this conjecture is equivalent to an older one given by Segre in [11]. The advantage of Segre conjecture is that it can be formulated in the same way on any surface. Starting from this idea we proved in [4] the equivalence of Conjecture 2.1 with Conjecture 2.2 on a generic $K3$ surface. An attempt to prove Conjecture 2.2 has been done in [5] by using a degeneration technique inspired by [1]. The main result, by using this technique, is Theorem 3.1 which relates the speciality of some linear systems through points of the same multiplicity with the speciality of systems through just one point.

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2. The equivalence of the two conjectures

As stated in the introduction we consider here an extension, to any surface, of Segre conjecture about special linear systems.

Conjecture 2.1. *If $\mathcal{L}$ is non-empty and reduced linear system on a surface S, then it is non-special.*

By Bertini second theorem, this conjecture tell us that if $\mathcal{L}$ is special, then there exists an irreducible curve C such that $2C \subseteq \mathrm{Bs}\,(\mathcal{L})$. This means that, if Conjecture 2.1 is true, then in order to give a classification of special systems on a surface we should be able to classify the type of the curve C. In the case of generic $K3$ surfaces we proved the equivalence of the preceding conjecture with the following (see [4]).

Conjecture 2.2. *Let $\mathcal{L}$ and S be as above, then*

(i) *$\mathcal{L}$ is special if and only if $\mathcal{L} = \mathcal{L}^4(d, 2d)$ or $\mathcal{L} = \mathcal{L}^2(d, d^2)$ with $d \geq 2$;*
(ii) *if $\mathcal{L}$ is non-empty then its general divisor has exactly the imposed multiplicities in the points p_i;*
(iii) *if $\mathcal{L}$ is non-special and has a fixed irreducible component C then*
  a) *$\mathcal{L} = \mathcal{L}^2(m+1, m+1, m) = mC + \mathcal{L}^2(1, 1)$ with $C = \mathcal{L}^2(1, 1^2)$ or*
  b) *$\mathcal{L} = 2C$ with $C \in \{\mathcal{L}^4(1, 1^3), \mathcal{L}^6(1, 2, 1), \mathcal{L}^{10}(1, 3)\}$ or*
  c) *$\mathcal{L} = C$.*
(iv) *if $\mathcal{L}$ has no fixed components then either its general element is irreducible or $\mathcal{L} = \mathcal{L}^2(2, 2)$.*

The proof of this result proceeds by analyzing the base locus of the system $\mathcal{L}$. Assume that there exist distinct irreducible curves C_i and D_j such that

$$\mathcal{L} = \sum_{i=1}^{a} \mu_i C_i + \sum_{i=1}^{b} D_i + \mathcal{M},$$

where $\mu_i \geq 2$ and $\mathcal{M}$ has no fixed components. By putting A, B to be two of the irreducible curves into the fixed part of $\mathcal{L}$ and assuming conjecture 2.1 to be true, we have that $v(A) = v(B) = v(A+B) = 0$. Since $v(A+B) = v(A)+v(B)+AB-1$, this implies that $AB = 1$. Hence this gives that $C_iC_j = C_iD_j = D_iD_j = 1$ and $C_i^2 \leq 1$. Now, it is possible to prove (see [4]) that given two distinct irreducible curves A and B on S then either $AB \neq 1$ or $A = \mathcal{L}^2(1, 1^2)$ and B is an irreducible element of $\mathcal{L}^2(1, 1)$.

3. A degeneration of K3 surfaces

In this section we consider an attempt to prove conjecture 2.2 by using a degeneration of $K3$ surfaces to a union of planes and the blow-up of a $K3$ along points. Let Δ be an open disk and let X be the blow-up of $S \times \Delta$ along b general points of $S \times \{0\}$. The threefold X is equipped with two projections p_1, p_2 on Δ and S respectively and the general fiber X_t of p_1 is isomorphic to S, while X_0 is a reducible surface given by the union of b planes with a surface $\mathbb{S}$. The last surface is the blow-up of S along the b points. Each one of the b planes $\mathbb{P}_i$ cuts a curve R_i on $\mathbb{S}$ which is a line in $\mathbb{P}_i$ and a (-1)-curve in $\mathbb{S}$. Now given a line bundle L on

S it is possible to construct infinitely many line bundles (depending on the integer k) $\mathcal{O}_X(L,k) := p_2^*(L)\otimes\mathcal{O}_X(k\mathbb{S})$ on X such that each one restricted to X_t gives L. Defining $\mathcal{X}(L,k)$ as the restriction to X_0 we have that

$$\begin{array}{rcl}\mathcal{X}(L,k)_{|\mathbb{P}^i} & = & \mathcal{O}_{\mathbb{P}^2}(k)\\ \mathcal{X}(L,k)_{|\mathbb{S}} & = & \mathfrak{b}^*(L)\otimes\mathcal{O}_{\mathbb{S}}(-\sum_{i=1}^b kE_i),\end{array}$$

where $\mathfrak{b}:\mathbb{S}\to S$ is the blow-up map. This construction allows us to degenerate a system on S to a union of systems on the $\mathbb{P}_i$'s and S in the following way. Let $Z := m_1q_1+\cdots+m_rq_r$ be a subscheme of S with points in general position. Chosen $a_1,\ldots,a_b$ positive integers such that $a_1+\cdots+a_b\leq r$, let Z'_i be the specialization of a_i points of Z to points of $\mathbb{P}_i$ (with the same multiplicities). Let $Z'_{\mathbb{S}}$ be the residual subscheme, made of $r-\sum a_i$ general points of $\mathbb{S}$. Given $Z' := Z'_1+\ldots+Z'_b+Z'_{\mathbb{S}}$, one has that $\mathcal{X}(\mathcal{L},k)\otimes\mathcal{I}_{Z'}$, is a degeneration of $\mathcal{L}\otimes\mathcal{I}_Z$. In this way, the starting system $\mathcal{L}$ through r degenerate to the system $\mathcal{L}_0$ on X_0 made by the $\mathcal{L}^i$ on the $\mathbb{P}_i$ and by the $\mathcal{L}_{\mathbb{S}}$ on $\mathbb{S}$. Observe that the last system corresponds to a system on S through less than r points. In this way, by using the fact that the homogeneous planar systems $\mathcal{L}_2(d,m^4)$, $\mathcal{L}_2(d,m^9)$ are never special, it is possible to use the preceding degeneration in an inductive way. So, for example consider the system $\mathcal{L}^n(d,m^{4^h})$, take $b=4^{h-1}$ and put four general points on each of the $\mathbb{P}_i$. In this way the speciality of the starting system is related to that of $\mathcal{L}^n(d,m^{4^{h-1}})$ and so on. More generally we have the following (see [5]).

Theorem 3.1. *If $\mathcal{L}^n(d,m)$ is non-special for all non-negative integers (d,m) then $\mathcal{L}^n(d',m'^{4^h9^k})$ is non-special for all non-negative integers (d',m',h,k).*

Unfortunately it is an open problem to evaluate if a system through just one point is special or not. The only known example is $\mathcal{L}^4(d,2d)$ as stated in Conjecture 2.2.

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