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# Master-Slave Synchronization for Nonlinear Systems based on Reduced Observers

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**Abstract:** In this article a global method for master-slave synchronization of nonlinear systems is provided. Based on a diffeomorphic transformation of the original dynamics, a reduced observer is designed such that the nonlinear state feedback law is independent of the inertial frame. Generally, the method requires non trivial computations but leads to global convergence results for the invariant tracking error. We illustrate the design procedure in detail for a master-slave synchronization of a bacterial growth model, which may escape to infinity in finite time.

*Keywords:* synchronization, tracking, nonlinear observers, input-affine systems, invariance

## 1. INTRODUCTION

Within the last decade a growing community of researchers has been focusing on the cooperative control of networked dynamic systems, denoted as agents. A fundamental task in such cooperative control scenarios is the *synchronization* of the agents (Olfati-Saber et al., 2007). Key aspects to achieve synchronization are the exclusive use of only local knowledge in the control laws and hence, their distributed nature. To this end, only the states of the neighbors in the communication network are used in the controllers of each agent to achieve unison. As a result, this procedure is independent of a common inertial reference frame of the agents. General solutions for agents with nonlinear dynamics using passivity are reported in Arcak (2007); Listmann et al. (2009); Listmann and Woolsey (2009) and for systems on Lie groups in Sarlette et al. (2010).

In contrast to such new developments, the *tracking problem* is a classical problem of control design. Here, the goal is to let a dynamic system follow a given reference trajectory. Similar to the synchronization problem, to date solutions also use only local knowledge. In this case invariant tracking errors are defined, again independent of the chosen inertial reference frame. Very advanced techniques for general nonlinear systems are given in Martin et al. (2004) using exact linearization and for systems on Lie groups, Maithripala et al. (2006) provide appropriate solutions.

A combination of both problems, tracking and synchronization, is the *master-slave synchronization* considered in this article. Then, a collective of agents, the slaves, try to synchronize to or track the trajectory of a single master, all described by identical dynamics. This approach is often elaborated to study and achieve synchronization in nonlinear chaotic systems (Pecora and Carroll, 1990). Generally this problem is not fully distributed in nature. Although some computations may only be done by the master, all the slaves compute their local controller as well, contradicting a fully centralized approach. In this set-up, the only knowledge available to all agents, the master and the slaves, is the distance of each slave's trajectory to the trajectory of the master. This is again an invariant information referred to as *translational invariance* and is illustrated in Fig. 1. From a tracking perspective, master-slave synchronization achieves tracking (for the slave) for an unknown reference trajectory (of the master). A typical application would be to clone the behavior of an industrial plant to other ones.

In van de Wouw and Pavlov (2008) master-slave synchronization is considered for piecewise affine systems and the solution is based on full-order observers and the fact that the overall system is a convergent dynamics (Pavlov et al., 2004). Regardless of such particular system structures, the approach presented here is applicable to all nonlinear input affine systems. We present a general procedure to derive invariant tracking control laws for such systems as only the relative distance to the trajectory of the master is available to every slave and the master. Key to our solution is the application of a globally convergent reduced observer, recently defined in Karagiannis et al. (2008). The use of such observers is facilitated by transforming the dynamics so that only the master trajectory and the relative distances to the slaves are remaining. The observer then reconstructs the trajectory of the master based on the relative measurements available. Further, this estimated

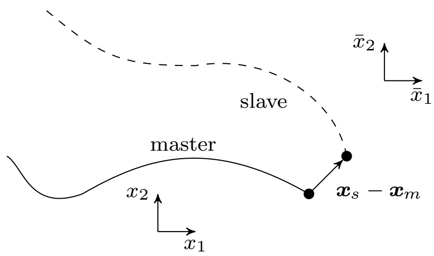


Fig. 1. Phase-plane trajectories of a master (−) and a slave (−−) indicating the invariance of $\boldsymbol{x}_s - \boldsymbol{x}_m$ w.r.t. translation of the reference frame from $\boldsymbol{x}_i$ to $\bar{\boldsymbol{x}}_i$.

value is used in the feedback design, of the master and of the slaves, to accomplish the goal.

Hence, the paper is structured as follows: In the next section the problem and the utilized reduced observer are introduced. Then in Section 3, the diffeomorphic transformation to the particular form considered is shown, and the general procedure for the control design is given. This procedure is illustrated for a bacterial growth model in Section 4. Finally, a conclusion is provided.

## 2. PRELIMINARIES

In this section a proper problem description is given and the utilized reduced observer is introduced briefly.

### *2.1 Problem Statement*

Generally, the goal is to synchronize a group of slaves to one single master. This problem is in spirit similar to the tracking of a reference trajectory, where the reference is given through the motion of an identical dynamic system, the master. To this end, we consider $N$ agents with dynamics

$$\dot{\boldsymbol{x}}_i = \boldsymbol{a}(\boldsymbol{x}_i) + \boldsymbol{b}(\boldsymbol{x}_i)\boldsymbol{u}_i \quad \forall i = 1, \dots, N, \tag{1}$$

where $\boldsymbol{x}_i \in \mathbb{R}^n$ and $\boldsymbol{u}_i \in \mathbb{R}^m$. The first agent is considered to be the *master*, all other $N-1$ agents are called *slaves*. Since the task is to synchronize the slaves with the master the tracking errors

$$\boldsymbol{y}_j = \boldsymbol{x}_j - \boldsymbol{x}_1 \quad \forall j = 2, \dots, N \tag{2}$$

are defined. Therefore, synchronization is equivalent to $\lim_{t\to\infty} \boldsymbol{y}_j(t) = \boldsymbol{0}$. Note that this implies full state synchronization and the only information available for control design are the tracking errors, i.e. relative information. The absolute position of the master agent, $\boldsymbol{x}_1$, is unknown, giving rise to an interpretation as an invariant tracking problem.

### *2.2 Reduced Nonlinear Observer*

Before introducing the concept of the reduced observer, some mandatory definitions are given. We consider the autonomous differential equation

$$\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}) \tag{3}$$

together with its solution at time $t$ denoted by $\boldsymbol{\phi}(t, \boldsymbol{x}_0, t_0)$, where $t_0$ and $\boldsymbol{x}_0$ are the initial time and state, respectively.

*Definition 1.* (Khalil (2001)). A set $\mathcal{M}$ is called *invariant* with respect to (3) if $\boldsymbol{x}_0 \in \mathcal{M} \Rightarrow \boldsymbol{x}(t) \in \mathcal{M}, \forall t \in \mathbb{R}$ and is said to be *positively invariant* if the same holds $\forall t \geq t_0$.

Hence, if a solution to (3) belongs to $\mathcal{M}$ at some time instant, then it will belong to $\mathcal{M}$ for all future times.

Assume that the equilibrium, i.e. $\boldsymbol{f}(\boldsymbol{x}_e) = \boldsymbol{0}$, is given by $\boldsymbol{x}_e = \boldsymbol{0}$. Further, let this equilibrium be isolated, i.e. its neighborhood contains no other point $\bar{\boldsymbol{x}}$ such that $\boldsymbol{f}(\bar{\boldsymbol{x}}) = \boldsymbol{0}$. This leads to

*Definition 2.* (Hahn (1968)). The equilibrium of the differential equation (3) is called *attractive* if there exists a number $\alpha > 0$ with the property

$$\lim_{t\to\infty} \boldsymbol{\phi}(t, \boldsymbol{x}_0, t_0) = \boldsymbol{0} \quad \text{whenever} \quad \|\boldsymbol{x}_0\| < \alpha.$$

Clearly, the key idea behind a design of a reduced observer is to create a positively invariant and attractive set. The distance to this set should be described by the observer error, to let the estimated value globally converge to the true state. This idea is elaborated in Karagiannis et al. (2008) and a globally convergent reduced order observer for nonlinear dynamical systems is presented. Assuming that the measurable states are $\boldsymbol{y} \in \mathbb{R}^q$ and the states $\boldsymbol{\eta} \in \mathbb{R}^n$ have to be estimated, the dynamical system is written as

$$\dot{\boldsymbol{y}} = \boldsymbol{f}_1(\boldsymbol{y}, \boldsymbol{\eta}), \tag{4a}$$
$$\dot{\boldsymbol{\eta}} = \boldsymbol{f}_2(\boldsymbol{y}, \boldsymbol{\eta}). \tag{4b}$$

The objective is to find a dynamical system

$$\dot{\boldsymbol{\xi}} = \boldsymbol{\alpha}(\boldsymbol{y}, \boldsymbol{\xi}) \tag{5}$$

with $\boldsymbol{\xi} \in \mathbb{R}^p$, $p \geq n$ and the mappings

$$\boldsymbol{\beta}(\boldsymbol{y}, \boldsymbol{\xi}) : \mathbb{R}^q \times \mathbb{R}^p \to \mathbb{R}^p, \tag{6a}$$
$$\boldsymbol{\Phi}_{\boldsymbol{y}}(\boldsymbol{\eta}) : \mathbb{R}^n \to \mathbb{R}^p \tag{6b}$$

such that the manifold

$$\mathcal{M} = \{(\boldsymbol{y}, \boldsymbol{\eta}, \boldsymbol{\xi}) : \boldsymbol{\beta}(\boldsymbol{y}, \boldsymbol{\xi}) = \boldsymbol{\Phi}_{\boldsymbol{y}}(\boldsymbol{\eta})\} \tag{7}$$

is positively invariant and attractive.

To design the observer the variable $\boldsymbol{z}$ measuring the distance to the manifold $\mathcal{M}$ is introduced,

$$\boldsymbol{z} = \boldsymbol{\beta}(\boldsymbol{y}, \boldsymbol{\xi}) - \boldsymbol{\Phi}_{\boldsymbol{y}}(\boldsymbol{\eta}). \tag{8}$$

Since we have $\boldsymbol{z} = \boldsymbol{0}$ on $\mathcal{M}$, for the estimation of $\boldsymbol{\eta}$

$$\tilde{\boldsymbol{\eta}} = \boldsymbol{\Phi}_{\boldsymbol{y}}^L(\boldsymbol{\beta}(\boldsymbol{y}, \boldsymbol{\xi})) \tag{9}$$

holds, where $\boldsymbol{\Phi}_{\boldsymbol{y}}^L(\cdot)$ is the left-inverse of $\boldsymbol{\Phi}_{\boldsymbol{y}}(\cdot)$, so that

$$\boldsymbol{\Phi}_{\boldsymbol{y}}^L(\boldsymbol{\Phi}_{\boldsymbol{y}}(\boldsymbol{\eta})) = \boldsymbol{\eta}. \tag{10}$$

The invariance of $\mathcal{M}$ is ensured by

$$\begin{aligned}\boldsymbol{\alpha}(\boldsymbol{y}, \boldsymbol{\xi}) = -\left(\frac{\partial \beta}{\partial \boldsymbol{\xi}}\right)^{-1} \Bigg[ &\frac{\partial \beta}{\partial \boldsymbol{y}} \boldsymbol{f}_1(\boldsymbol{y}, \tilde{\boldsymbol{\eta}}) \\ &- \left.\frac{\partial \boldsymbol{\Phi}_{\boldsymbol{y}}}{\partial \boldsymbol{y}}\right|_{\boldsymbol{\eta}=\tilde{\boldsymbol{\eta}}} \boldsymbol{f}_1(\boldsymbol{y}, \tilde{\boldsymbol{\eta}}) \\ &- \left.\frac{\partial \boldsymbol{\Phi}_{\boldsymbol{y}}}{\partial \boldsymbol{\eta}}\right|_{\boldsymbol{\eta}=\tilde{\boldsymbol{\eta}}} \boldsymbol{f}_2(\boldsymbol{y}, \tilde{\boldsymbol{\eta}}) \Bigg]\end{aligned} \tag{11}$$

if the inverse $(\partial\beta/\partial\boldsymbol{\xi})^{-1}$ exists. Using this, the time derivative of $\boldsymbol{z}$ is given by

$$\begin{aligned}\dot{\boldsymbol{z}} = &-\frac{\partial \beta}{\partial \boldsymbol{y}} (\boldsymbol{f}_1(\boldsymbol{y}, \tilde{\boldsymbol{\eta}}) - \boldsymbol{f}_1(\boldsymbol{y}, \boldsymbol{\eta})) \\ &+ \left.\frac{\partial \boldsymbol{\Phi}_{\boldsymbol{y}}}{\partial \boldsymbol{y}}\right|_{\boldsymbol{\eta}=\tilde{\boldsymbol{\eta}}} \boldsymbol{f}_1(\boldsymbol{y}, \tilde{\boldsymbol{\eta}}) - \frac{\partial \boldsymbol{\Phi}_{\boldsymbol{y}}}{\partial \boldsymbol{y}} \boldsymbol{f}_1(\boldsymbol{y}, \boldsymbol{\eta}) \\ &+ \left.\frac{\partial \boldsymbol{\Phi}_{\boldsymbol{y}}}{\partial \boldsymbol{\eta}}\right|_{\boldsymbol{\eta}=\tilde{\boldsymbol{\eta}}} \boldsymbol{f}_2(\boldsymbol{y}, \tilde{\boldsymbol{\eta}}) - \frac{\partial \boldsymbol{\Phi}_{\boldsymbol{y}}}{\partial \boldsymbol{\eta}} \boldsymbol{f}_2(\boldsymbol{y}, \boldsymbol{\eta}).\end{aligned} \tag{12}$$

The task of constructing a globally convergent observer is therefore reduced to finding the mappings $\boldsymbol{\beta}(\cdot)$ and $\boldsymbol{\Phi}_{\boldsymbol{y}}(\cdot)$ guaranteeing invariance and attractiveness of $\mathcal{M}$.

## 3. DESIGN PROCEDURE

The idea of the design procedure presented in the current paper can be split into four aspects described in this section. Fig. 2 shows a summary of the design steps.

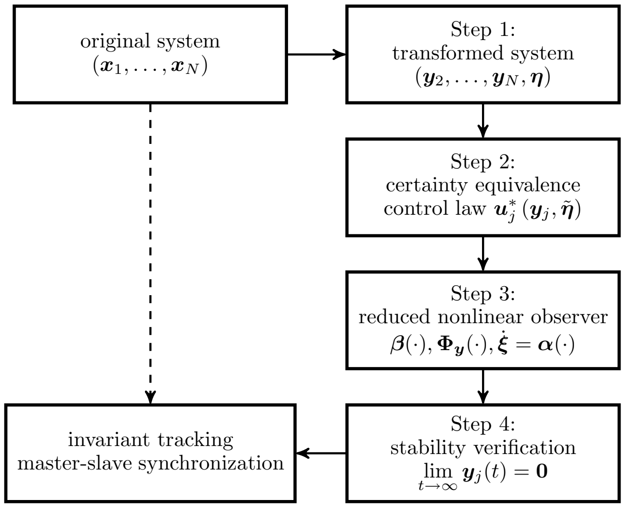


Fig. 2. Design procedure

In Step 1, the system dynamics are formulated in terms of the tracking errors. To achieve this, the system is transformed by

$$\begin{bmatrix} \boldsymbol{y}_2 \\ \vdots \\ \vdots \\ \boldsymbol{y}_N \\ \boldsymbol{\eta} \end{bmatrix} = \begin{bmatrix} -\boldsymbol{I} & \boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} \\ \vdots & \boldsymbol{0} & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \boldsymbol{0} \\ -\boldsymbol{I} & \boldsymbol{0} & \cdots & \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{I} & \boldsymbol{0} & \cdots & \cdots & \boldsymbol{0} \end{bmatrix} \boldsymbol{x} \tag{12}$$

to describe the unknown reference state $\boldsymbol{\eta} = \boldsymbol{x}_1$ and the tracking errors $\boldsymbol{y}_j$. In this, $\boldsymbol{I}$ denotes the $n \times n$-dimensional identity matrix. The transformation results in

$$\dot{\boldsymbol{y}} = \boldsymbol{f}_1(\boldsymbol{y}, \boldsymbol{\eta}, \boldsymbol{u}) \tag{13a}$$
$$= \begin{bmatrix} \boldsymbol{a}(\boldsymbol{y}_2 + \boldsymbol{\eta}) - \boldsymbol{a}(\boldsymbol{\eta}) + \boldsymbol{b}(\boldsymbol{y}_2 + \boldsymbol{\eta})\boldsymbol{u}_2 - \boldsymbol{b}(\boldsymbol{\eta})\boldsymbol{u}_1 \\ \vdots \\ \boldsymbol{a}(\boldsymbol{y}_N + \boldsymbol{\eta}) - \boldsymbol{a}(\boldsymbol{\eta}) + \boldsymbol{b}(\boldsymbol{y}_N + \boldsymbol{\eta})\boldsymbol{u}_N - \boldsymbol{b}(\boldsymbol{\eta})\boldsymbol{u}_1 \end{bmatrix},$$
$$\dot{\boldsymbol{\eta}} = \boldsymbol{f}_2(\boldsymbol{\eta}, \boldsymbol{u}_1) = \boldsymbol{a}(\boldsymbol{\eta}) + \boldsymbol{b}(\boldsymbol{\eta})\boldsymbol{u}_1. \tag{13b}$$

Obviously, $\lim_{t\to\infty} \boldsymbol{x}_j(t) = \lim_{t\to\infty} \boldsymbol{x}_1(t)$ in the original system is equivalent to $\lim_{t\to\infty} \boldsymbol{y}_j(t) = \boldsymbol{0}$ in the transformed version. Therefore, given the dynamics of the tracking errors, a control law stabilizing $\boldsymbol{y}_j$ has to be found in Step 2. To this end, we first assume that $\boldsymbol{u}_1$ as well as $\boldsymbol{\eta}$ are known for the design of $\boldsymbol{u}_j(\boldsymbol{y}_j, \boldsymbol{\eta})$. As we are dealing with input affine systems, exact linearization is a common approach but we emphasize that other design procedures such as Lyapunov-based controllers may be employed. Following the certainty equivalence heuristic (Rusnak et al., 1993), the control law uses the estimated absolute position $\tilde{\boldsymbol{\eta}}$ in the remainder of this paper, i.e. $\boldsymbol{u}_j(\boldsymbol{y}_j, \tilde{\boldsymbol{\eta}})$. Of course this requires observability of the system (13) for $\boldsymbol{y} \neq \boldsymbol{0}$.

The main ingredient in the concept proposed in this paper is an estimator for the unknown reference state $\boldsymbol{\eta}$ designed in Step 3. We are looking for a globally convergent estimator contrary to local approaches such as high-gain observers or Extended Kalman Filters. These would significantly simplify the design while closed loop stability would be hard to proof. Moreover, the well-known nonlinear full order observer designs (e.g. Marino and Tomei, 1995) are not applicable for the problem discussed in this paper. Due to the special system structure one is not able to find a transformation into a linear system with output injection terms. However, one possible solution for our task is given by the reduced order invariant-manifold based observer developed in Karagiannis et al. (2008). By using the dynamical equations of the closed loop with the certainty equivalence based controller we specifically design such an observer for the control law $\boldsymbol{u}_j(\boldsymbol{y}_j, \tilde{\boldsymbol{\eta}})$.

As a consequence, the global convergence of the estimator is assured. However, it is well known that closed-loop stability is generally not guaranteed when a stabilizing control law is used in combination with a convergent observer (Andrieu and Praly, 2009). To this end, we verify stability of the closed loop system

$$\dot{\boldsymbol{y}}_j = \boldsymbol{a}(\boldsymbol{y}_j + \boldsymbol{\eta}) - \boldsymbol{a}(\boldsymbol{\eta}) + \boldsymbol{b}(\boldsymbol{y}_j + \boldsymbol{\eta})\boldsymbol{u}_j(\boldsymbol{y}, \tilde{\boldsymbol{\eta}}) - \boldsymbol{b}(\boldsymbol{\eta})\boldsymbol{u}_1 \tag{14}$$

for $j = 2, \ldots, N$ in Step 4. In order to guarantee closed-loop stability globally, we require the bounds on the initial observer error to be known *a priori*. However, the initial deviation will always be finite so that without loss of generality the observer error may be enclosed within a ball of appropriate radius.

In view of the master-slave synchronization problem, the information flow in the system is visualized in Fig. 3.

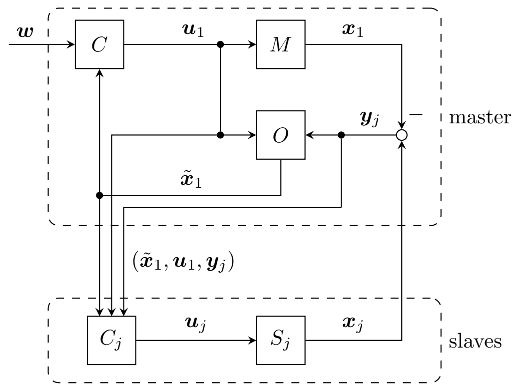


Fig. 3. Information flow in master-slave synchronization

The master with dynamics $M$ is able to measure the state differences $\boldsymbol{x}_j - \boldsymbol{x}_1$. Based on this information the master agent estimates its absolute position by means of the reduced order observer $O$ designed in Step 3 which in turn is based on the control law of Step 2. The estimated position as well as the control input of the master agent and the measured differences are communicated to the slaves. The slaves with dynamics $S_j$ use this information in their controller $C_j$ to track the master. Additionally, it is possible for the master to track an exogenous reference signal $\boldsymbol{w}$ based on the estimated absolute position using the controller $C$.

We would like to point out again that the tracking of the master as well as the tracking of the exogenous reference signal are based solely on relative information, i.e. the state differences between the master and the slaves. No absolute state information is incorporated in the proposed design.

## 4. EXAMPLE

For illustration purposes we choose the population dynamics of bacteria cultures. The so called logistic differential equation is a first order nonlinear differential equation and is shown to give a good approximation of the population growth dynamics (Britton, 2003). Here, it is augmented by an affine control input $u_i$ and thus given by

$$\dot{x}_i = r x_i \left(1 - \frac{x_i}{K}\right) + u_i = r x_i - \frac{c_1}{2} x_i^2 + u_i \tag{15}$$

where $r$ describes the growth rate, $K$ is the capacity of the autonomous system and $c_1 = \frac{2r}{K}$ is introduced for notational convenience.

### 4.1 Transformed system dynamics

Following the proposed design procedure we evaluate (13) for a system of $N$ bacterial populations. This results in

$$\dot{\boldsymbol{y}} = \boldsymbol{f}_1(\boldsymbol{y}, \eta, \boldsymbol{u}) = \begin{bmatrix} r y_2 - c_1 \eta y_2 - \frac{c_1}{2} y_2^2 + u_2^* \\ \vdots \\ r y_N - c_1 \eta y_N - \frac{c_1}{2} y_N^2 + u_N^* \end{bmatrix}, \tag{16a}$$

$$\dot{\eta} = f_2(\boldsymbol{y}, \eta, \boldsymbol{u}) = r\eta - \frac{c_1}{2}\eta^2 + u_1 \tag{16b}$$

where $u_j^* = u_j - u_1$ is used for notational purposes.

To check the observability of the system, we calculate the observability mapping (Kou et al., 1973)

$$\boldsymbol{q}(\eta, \boldsymbol{y}, \boldsymbol{u}) = \begin{bmatrix} y_2 \\ \dot{y}_2 \\ \vdots \\ y_N \\ \dot{y}_N \end{bmatrix} = \begin{bmatrix} y_2 \\ r y_2 - c_1 \eta y_2 - \frac{c_1}{2} y_2^2 + u_2^* \\ \vdots \\ y_N \\ r y_N - c_1 \eta y_N - \frac{c_1}{2} y_N^2 + u_N^* \end{bmatrix}. \tag{17}$$

Then, for the observability matrix $\boldsymbol{Q}(\boldsymbol{y}, \eta, \boldsymbol{u}) \in \mathbb{R}^{2(N-1)\times N}$

$$\boldsymbol{Q}(\boldsymbol{y}, \eta, \boldsymbol{u}) = \frac{\partial \boldsymbol{q}(\boldsymbol{y}, \eta, \boldsymbol{u})}{\partial \left[\boldsymbol{y}^\top \ \eta\right]^\top} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \chi_2 & 0 & 0 & \cdots & -c_1 y_2 \\ 0 & 1 & 0 & & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \\ 0 & \cdots & 0 & \chi_N & -c_1 y_N \end{bmatrix} \tag{18}$$

holds with $\chi_j = r - c_1(\eta + y_j)$. Therefore, $y_j \neq 0$ for at least one measurable output implies

$$\operatorname{rank}(\boldsymbol{Q}(\boldsymbol{y}, \eta, \boldsymbol{u})) = N \tag{19}$$

and the system is globally observable.

### 4.2 Certainty Equivalence Based Control Law

Assuming the absolute position of the master is known, an exactly linearizing control law resulting in the closed loop dynamics $\dot{y}_j = -a_0 y_j$ can easily be found. This motivates the use of the certainty equivalence based controller

$$u_j^* = -\left(r y_j - c_1 \tilde{\eta} y_j - \frac{c_1}{2} y_j^2 + a_0 y_j\right) \tag{20}$$

which leads to

$$\dot{y}_j = -c_1(\eta - \tilde{\eta}) y_j - a_0 y_j. \tag{21}$$

The aim then is to ensure global convergence of $y_j$ for all slaves.

### 4.3 Observer design

Setting $\Phi_{\boldsymbol{y}}(\eta) = \varepsilon(\boldsymbol{y})\eta$ and evaluating (12) results in

$$\begin{aligned} \dot{z} = \Bigg( & -\frac{\partial \beta}{\partial \boldsymbol{y}} \begin{bmatrix} -c_1 y_2 \\ \vdots \\ -c_1 y_N \end{bmatrix} + \varepsilon r \\ & + \frac{\partial \varepsilon}{\partial \boldsymbol{y}} \begin{bmatrix} r y_2 - \frac{c_1}{2} y_2^2 + u_2^* \\ \vdots \\ r y_N - \frac{c_1}{2} y_N^2 + u_N^* \end{bmatrix} \Bigg) (\tilde{\eta} - \eta) \\ & + \left( -\varepsilon \frac{c_1}{2} + \frac{\partial \varepsilon}{\partial \boldsymbol{y}} \begin{bmatrix} -c_1 y_2 \\ \vdots \\ -c_1 y_N \end{bmatrix} \right) (\tilde{\eta}^2 - \eta^2) \end{aligned} \tag{22}$$

after some simplifications. The arguments of $\beta(\cdot)$ and $\varepsilon(\cdot)$ were omitted for notational convenience. To eliminate the quadratic terms in $\eta$ and $\tilde{\eta}$,

$$-\varepsilon \frac{c_1}{2} + \frac{\partial \varepsilon}{\partial \boldsymbol{y}} \begin{bmatrix} -c_1 y_2 \\ \vdots \\ -c_1 y_N \end{bmatrix} = 0 \quad \forall t > 0 \tag{23}$$

has to be fulfilled. In this example it is straightforward to verify that

$$\varepsilon(\boldsymbol{y}) = \sum_{j=2}^{N} |y_j|^{-\frac{1}{2}} \quad \forall y_j \neq 0 \tag{24}$$

solves the differential equation (23). Substituting (24), the control law (20), $\tilde{\eta} = \varepsilon^{-1}(\boldsymbol{y})\beta$ and $\tilde{\eta} - \eta = \varepsilon^{-1}(\boldsymbol{y})z$ into (22) gives

$$\begin{aligned} \dot{z} &= \left( \frac{\partial \beta}{\partial \boldsymbol{y}} \begin{bmatrix} c_1 y_2 \\ \vdots \\ c_1 y_N \end{bmatrix} + \varepsilon r + \frac{\partial \varepsilon}{\partial \boldsymbol{y}} \begin{bmatrix} c_1 \varepsilon \beta y_2 - a_0 y_2 \\ \vdots \\ c_1 \varepsilon \beta y_N - a_0 y_N \end{bmatrix} \right) \varepsilon^{-1} z \\ &= \Bigg( \frac{\partial \beta}{\partial \boldsymbol{y}} \begin{bmatrix} c_1 y_2 \\ \vdots \\ c_1 y_N \end{bmatrix} \frac{1}{\sum_{j=2}^{N} |y_j|^{-\frac{1}{2}}} + r \\ &\quad + \begin{bmatrix} -\frac{1}{2}\operatorname{sgn}(y_2)|y_2|^{-\frac{3}{2}} \\ \vdots \\ -\frac{1}{2}\operatorname{sgn}(y_N)|y_N|^{-\frac{3}{2}} \end{bmatrix}^\top \\ &\quad \begin{bmatrix} c_1 \frac{1}{\sum_{j=2}^{N} |y_j|^{-\frac{1}{2}}} \beta y_2 - a_0 y_2 \\ \vdots \\ c_1 \frac{1}{\sum_{j=2}^{N} |y_j|^{-\frac{1}{2}}} \beta y_N - a_0 y_N \end{bmatrix} \frac{1}{\sum_{j=2}^{N} |y_j|^{-\frac{1}{2}}} \Bigg) z \end{aligned} \tag{25}$$

which is linear in $z$. It remains to find a mapping $\beta(\cdot)$ such that $z \to 0$ for $t \to \infty$, making $\mathcal{M}$ invariant and attractive. Despite the relatively simple nature of the example this task requires non trivial computations. To the authors' knowledge there exists no constructive method to find such a mapping. It can however be verified that

$$\beta(\boldsymbol{y}, \xi) = \sum_{j=2}^{N} (1 + \xi)|y_j|^{\frac{1}{2}} + \frac{a_0}{c_1}|y_j|^{-\frac{1}{2}} \tag{26}$$

leads to

$$\dot{z} = \left(-\frac{1}{2}a_0 + r\right) z = -c_2 z. \tag{27}$$

Thus $a_0 > 2r$ implies $c_2 = \frac{1}{2}a_0 - r > 0$ and guarantees global asymptotic convergence of the observer.

The estimated state is calculated from the observer state $\xi$ and the measured output $\boldsymbol{y}$ by means of (24) and (26):

$$\tilde{\eta} = \frac{1}{\varepsilon(\boldsymbol{y})}\beta(\boldsymbol{y}, \xi). \tag{28}$$

For the implementation, we compute and insert into (11)

$$\frac{\partial \beta}{\partial \xi} = \sum_{j=2}^{N} |y_j|^{\frac{1}{2}}, \tag{29a}$$

$$\frac{\partial \beta}{\partial \boldsymbol{y}}^{\top} = \begin{bmatrix} \frac{1}{2}\mathrm{sgn}(y_2)\left((1+\xi)|y_2|^{-\frac{1}{2}} - \frac{a_0}{c_1}|y_2|^{-\frac{3}{2}}\right) \\ \vdots \\ \frac{1}{2}\mathrm{sgn}(y_N)\left((1+\xi)|y_N|^{-\frac{1}{2}} - \frac{a_0}{c_1}|y_N|^{-\frac{3}{2}}\right) \end{bmatrix}, \tag{29b}$$

$$\left.\frac{\partial \Phi_{\boldsymbol{y}}}{\partial \boldsymbol{y}}\right|_{\eta=\tilde{\eta}}^{\top} = \begin{bmatrix} -\frac{1}{2}\mathrm{sgn}(y_2)|y_2|^{-\frac{3}{2}} \\ \vdots \\ -\frac{1}{2}\mathrm{sgn}(y_N)|y_N|^{-\frac{3}{2}} \end{bmatrix}, \tag{29c}$$

$$\left.\frac{\partial \Phi_{\boldsymbol{y}}}{\partial \eta}\right|_{\eta=\tilde{\eta}} = \sum_{j=2}^{N} |y_j|^{-\frac{1}{2}}, \tag{29d}$$

$$\boldsymbol{f}_1(\boldsymbol{y}, \tilde{\eta}) = -a_0 [y_2 \cdots y_N]^{\top}, \tag{29e}$$

$$f_2(\boldsymbol{y}, \tilde{\eta}, u_1) = r\tilde{\eta} - \frac{c_1}{2}\tilde{\eta}^2 + u_1. \tag{29f}$$

We recall that the designed observer depends on the certainty equivalence based control law developed in (20). It is therefore possible that the observer design might be simplified if a different control law was chosen. Generally the design is difficult in the case of agents with higher order dynamics.

### *4.4 Tracking error stability*

In the previous section a globally convergent observer is designed. Since the estimated absolute position $\tilde{\eta}$ is used to control the agents, it remains to verify the stability of the closed loop dynamics. Again using $\tilde{\eta} - \eta = \varepsilon(\boldsymbol{y})^{-1} z$, the tracking error dynamics (21) can be rewritten in terms of $\boldsymbol{y}$ and $z$ as

$$\dot{y}_j = c_1 \frac{1}{\varepsilon(\boldsymbol{y})} z y_j - a_0 y_j. \tag{30}$$

The time solution of the observer error obviously is $z(t) = z_0 e^{-c_2 t}$. To show the convergence of the tracking errors we thus have to ensure asymptotic stability of

$$\dot{y}_j = c_1 \frac{1}{\sum_{k=2}^{N} |y_k|^{-\frac{1}{2}}} y_j z_0 e^{-c_2 t} - a_0 y_j. \tag{31}$$

In the special case $N = 2$ we can explicitly solve the differential equation for $y > 0$ and get

$$y(t) = y_2(t) = \frac{e^{2c_2 t}(a_0 + 2c_2)^2}{\left(c_1 z_0 + e^{\left(\frac{a_0}{2}+c_2\right)t}\left(\frac{a_0+2c_2}{\sqrt{y_0}} - c_1 z_0\right)\right)^2}.$$

Obviously, $y(t)$ escapes to infinity in finite time if the denominator of $y(t)$ becomes zero. Otherwise we have $y(t) \to 0$ because

$$\lim_{t\to\infty} \frac{e^{2c_2 t}}{e^{2\left(\frac{a_0}{2}+c_2\right)t}} = \lim_{t\to\infty} e^{-a_0 t} = 0. \tag{32}$$

However, we can derive a condition for the tracking errors to converge to zero by regarding

$$e^{\left(\frac{a_0}{2}+c_2\right)t_{\mathrm{esc}}}\left(c_1 z_0 - \frac{a_0 + 2c_2}{\sqrt{y_0}}\right) = c_1 z_0, \tag{33}$$

where $t_{\mathrm{esc}}$ denotes the finite escape time. The above equation has no solution if $z_0 < 0$. Therefore, asymptotic tracking is achieved if $a_0 > 2r$ is satisfied. The case of $z_0 > 0$ requires further examination as then, (33) has no solution if

$$\frac{c_1 z_0}{c_1 z_0 - \frac{a_0+2c_2}{\sqrt{y_0}}} \leq 0. \tag{34}$$

A similar result can be obtained for $y < 0$ which finally gives the condition

$$a_0 \geq c_1 z_0 \sqrt{|y_0|} - 2c_2. \tag{35}$$

Resubstituting $c_1$ and $c_2$ we can deduce a condition for the controller parameter $a_0$ to guarantee the convergence of the tracking error as

$$a_0 \geq \frac{r}{K} z_0 \sqrt{|y_0|} + r. \tag{36}$$

In this equation $z_0$ is unknown but since $z_0 < z_{0,\max}$ we can use the upper bound on the initial estimation error $z_{0,\max}$ to calculate a stabilizing $a_0$.

When multiple slaves are considered, i.e. $N > 2$, it becomes far more difficult to solve the differential equations of $y_j$ because of couplings between the tracking error dynamics. We can overcome this problem at the cost of some conservatism if we require the tracking error to be monotonically decreasing for $y > 0$. This is obvious if we write

$$\dot{y} = y\left(-a_0 + c_1\sqrt{|y|} z_0 e^{-c_2 t}\right). \tag{37}$$

Since $e^{-c_2 t}$ is positive and monotonically decreasing, $y$ is alike if $\dot{y}(0)y(0) < 0$ is satisfied. Therefore, the tracking error will converge to zero as $t \to \infty$. Similar reasoning for $y < 0$ leads to

$$a_0 \geq c_1 \sqrt{|y_0|} z_0 \tag{38}$$

which is a conservative estimate of the condition given beforehand in (35), since $c_2 > 0$. From this, we can derive a sufficient condition for the tracking errors to converge asymptotically in the case of $N > 2$ as

$$a_0 \geq \frac{c_1 z_0}{\sum_{j=2}^{N} |y_{j,0}|^{-\frac{1}{2}}}. \tag{39}$$

In addition to the slaves tracking the master, the estimated absolute position $\tilde{\eta}$ allows to design a control law for the master agent. If we choose

$$u_1(w, \dot{w}, \tilde{\eta}) = \dot{w} + \gamma w - (r + \gamma)\tilde{\eta} + \frac{c_1}{2}\tilde{\eta}^2 \tag{40}$$

the master agent tracks the exogenous differentiable signal $w(t)$ asymptotically. We emphasize that this is possible despite the unmeasurable absolute position $\eta$. Introducing the master tracking error $e_t = w - \eta$ we calculate

$$\dot{e}_t = -\gamma e_t \underbrace{- (r + \gamma)(\eta - \tilde{\eta}) + \frac{c_1}{2}\left(\eta^2 - \tilde{\eta}^2\right)}_{v}. \tag{41}$$

As shown before, $z(t) \to 0$ as $t \to \infty$ implies $\eta(t) \to \tilde{\eta}(t)$ if condition (39) is satisfied. Therefore, we can conclude that $v(t) \to 0$ as well guaranteeing $e_t(t) \to 0$ for $\gamma > 0$.

### *4.5 Simulation results*

To illustrate the obtained results, a system with two slaves is simulated with parameter values $r = 0.8$ and $K = 10$. Following (27) the choice of $a_0 = 2$ ensures convergence of the observer. With $y_2(0) = 8$, $y_3(0) = -3$, $\eta(0) = 5$ and $\tilde{\eta}(0) = 1$ the condition (39) for the stability of the tracking errors is also satisfied. Fig. 4 shows the absolute agent states, the control inputs and the tracking errors over time. Therein, the master is controlled by $u_1 = \dot{w} + \gamma w - (r+\gamma)\,\tilde{\eta} + \frac{c_1}{2}\tilde{\eta}^2$. In this $w(t) = 5 + 3\sin(2\pi t)$ is a given reference signal and $\gamma = 1$. In addition to showing the applicability of the proposed invariant tracking controller the simulation shows that the estimated absolute state $\tilde{\eta}$ can also be used to control the unmeasurable reference state $\eta$.

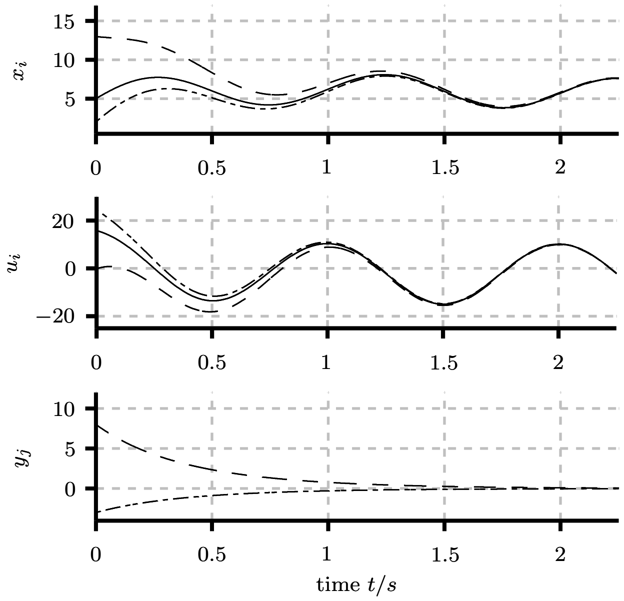


Fig. 4. Tracking of bacteria population size (master $-$, 1st slave $--$, 2nd slave $-\cdot-$)

## 5. CONCLUSION

In this article a solution to the master-slave synchronization of nonlinear input affine systems was proposed. The methodology is based on an appropriate transformation of the complete system dynamics so that only locally available information is used to reconstruct the global state of the trajectory of the master. Despite the challenging problem of finding a stabilizing mapping for the observer error, the method provides a globally convergent result. If in contrast, local estimators would be used, more general dynamic systems could be studied but the stability analysis gets slightly more involved.

For the future, it would be interesting if the same could be achieved if the state difference to the master is only known partially. Moreover, a full distribution of this concept would be able to solve the synchronization problem for a very general class of dynamic systems.

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