GENERALIZED MULTIDIMENSIONAL HILBERT TRANSFORMS IN CLIFFORD ANALYSIS

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Two specific generalizations of the multidimensional Hilbert transform in Clifford analysis are constructed. It is shown that though in each of these generalizations some traditional properties of the Hilbert transform are inevitably lost, new bounded singular operators emerge on Hilbert or Sobolev spaces of L_2-functions

1. Introduction

During the last fifty years, Clifford analysis has gradually developed to a comprehensive theory offering a direct, elegant, and powerful generalization to higher dimension of the theory of holomorphic functions in the complex plane. In its most simple but still useful setting, flat m-dimensional Euclidean space, Clifford analysis focusses on the so-called *monogenic functions*, that is, null solutions of the Clifford vector-valued Dirac operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}, \tag{1.1}$$

where $(e_1,\ldots,e_m)$ forms an orthogonal basis for the quadratic space $\mathbb{R}^m$ underlying the construction of the Clifford algebra $\mathbb{R}_{0,m}$. Monogenic functions have a special relationship with harmonic functions of several variables in that they are refining their properties. Note for instance that each harmonic function can be split into a so-called *inner* and an *outer* monogenic function, and that a real harmonic function is always the real part of a monogenic one, which does not need to be the case for a harmonic function of several complex variables. The reason is that, as does the Cauchy-Riemann operator in the complex plane, the rotation-invariant Dirac operator factorizes the m-dimensional Laplace operator. This has, a.o., allowed for a nice study of Hardy spaces of monogenic functions and the related multidimensional Cauchy and Hilbert transform, see [1, 11–14, 23].

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The Hilbert transform on the real line, given for an appropriate function or distribution f by

$$\mathscr{H}[f](x) = \frac{1}{\pi} Pv \int_{-\infty}^{+\infty} \frac{f(y)}{x-y} dy \tag{1.2}$$

was first generalized to m-dimensional Euclidean space by means of the Riesz transforms R_j, given by

$$R_j[f](\underline{x}) = \lim_{\varepsilon \to 0+} \frac{2}{a_{m+1}} \int_{\mathbb{R}^m \setminus B(\underline{x},\varepsilon)} \frac{x_j - y_j}{|\underline{x} - \underline{y}|^{m+1}} f(\underline{y}) dV(\underline{y}), \quad j = 1, \ldots, m, \tag{1.3}$$

where $a_{m+1} = 2\pi^{(m+1)/2}/\Gamma((m+1)/2)$ denotes the area of the unit sphere S^m in $\mathbb{R}^{m+1}$.

It was Horváth who, already in his paper [20], introduced the Clifford vector-valued Hilbert operator:

$$\mathscr{H} = \sum_{j=1}^{m} e_j R_j. \tag{1.4}$$

The m-dimensional Hilbert transform in the Clifford analysis setting was taken up again in the 1980's and further studied in, for example, [15, 16, 18, 22, 27]. We recall its alternative definition and main properties in Section 4.

In the early 2000's, four broad families of specific distributions in Clifford analysis were introduced and thoroughly studied (see [2, 9, 10]) and it was shown that the Hilbert kernel is one of those distributions acting as a convolution operator (see also [5, 7]). The definition of the normalizations of those distributions and a study of their convolvability are given in Section 3.

In this paper, we treat two possible generalizations of the Hilbert transform in $\mathbb{R}^m$, making use of these families of Clifford distributions, and aiming at preserving in these approaches as much traditional properties of the Hilbert transform as possible. It is shown that in each case some of the properties—different ones—are inevitably lost. Nevertheless, we twice obtain a new bounded singular integral operator on L_2 or on appropriate Sobolev spaces.

In the first approach, the Hilbert transform is generalized by using other kernels for the convolution, stemming from the families of distributions mentioned above. They constitute a refinement of the generalized Hilbert kernels introduced by Horváth in [21], who considered convolution kernels, homogeneous of degree $(-m)$, of the form

$$K = Pv \frac{S(\underline{\omega})}{r^m}, \quad r = |\underline{x}|, \tag{1.5}$$

where $S(\underline{\omega})$, $\underline{\omega} \in S^{m-1}$ is a surface spherical harmonic. We investigate generalized Hilbert convolution kernels, which are homogeneous of degree $(-m)$ as well, however, involving inner and outer spherical monogenics, that is, restrictions to the unit sphere S^{m-1} of monogenic homogeneous polynomials in $\mathbb{R}^m$, respectively, monogenic homogeneous functions in the complement of the origin. The resulting generalized Hilbert transform will no longer be a unitary operator, yet it remains a bounded singular operator on $L_2(\mathbb{R}^m; \mathbb{R}_{0,m})$.

The second approach is based on the intimate relationship between the Hilbert and the Cauchy transform and starts with the construction of a generalized Cauchy transform in $\mathbb{R}^{m+1}$ involving a distribution from one of the above-mentioned families as a generalized Cauchy kernel. A new generalized Hilbert transform in $\mathbb{R}^m$ is then defined as part of the L_2 or distributional boundary values of the generalized Cauchy transform considered, and it is shown to be a bounded operator on Sobolev spaces W_2^n.

Finally a connection is established between both generalizations, through the action of a higher-order derivative of the Dirac operator.

In order to keep the paper self-contained, the necessary definitions and results of Clifford analysis are given in the next section.

2. Clifford analysis

In this section, we briefly present the basic definitions and some results of Clifford analysis which are necessary for our purpose. For an in-depth study of this higher-dimensional function theory and its applications, we refer to, for example, [8, 17–19, 24–26].

Let $\mathbb{R}^{0,m}$ be the real vector space $\mathbb{R}^m$, endowed with a nondegenerate quadratic form of signature $(0,m)$, let $(e_1,\dots,e_m)$ be an orthonormal basis for $\mathbb{R}^{0,m}$, and let $\mathbb{R}_{0,m}$ be the universal Clifford algebra constructed over $\mathbb{R}^{0,m}$.

The noncommutative multiplication in $\mathbb{R}_{0,m}$ is governed by the rules

$$e_i e_j + e_j e_i = -2\delta_{i,j}, \quad i,j \in \{1,\dots,m\}. \tag{2.1}$$

For a set $A = \{i_1,\dots,i_h\} \subset \{1,\dots,m\}$ with $1 \le i_1 < i_2 < \cdots < i_h \le m$, let $e_A = e_{i_1} e_{i_2} \cdots e_{i_h}$. Moreover, we put $e_\varnothing = 1$, the latter being the identity element. Then $(e_A : A \subset \{1,\dots,m\})$ is a basis for the Clifford algebra $\mathbb{R}_{0,m}$. Any $a \in \mathbb{R}_{0,m}$ may thus be written as $a = \sum_A a_A e_A$ with $a_A \in \mathbb{R}$ or still as $a = \sum_{k=0}^m [a]_k$, where $[a]_k = \sum_{|A|=k} a_A e_A$ is a so-called k-vector $(k = 0,1,\dots,m)$. If we denote the space of k-vectors by $\mathbb{R}_{0,m}^k$, then $\mathbb{R}_{0,m} = \bigoplus_{k=0}^m \mathbb{R}_{0,m}^k$.

We will also identify an element $\underline{x} = (x_1,\dots,x_m) \in \mathbb{R}^m$ with the one-vector (or vector for short) $\underline{x} = \sum_{j=1}^m x_j e_j$. The multiplication of any two vectors $\underline{x}$ and $\underline{y}$ is given by

$$\underline{x}\,\underline{y} = -\langle \underline{x}, \underline{y} \rangle + \underline{x} \wedge \underline{y} \tag{2.2}$$

with

$$\begin{aligned} \langle \underline{x}, \underline{y} \rangle &= \sum_{j=1}^m x_j y_j = -\frac{1}{2}(\underline{x}\,\underline{y} + \underline{y}\,\underline{x}), \\ \underline{x} \wedge \underline{y} &= \sum_{i<j} e_{ij}(x_i y_j - x_j y_i) = \frac{1}{2}(\underline{x}\,\underline{y} - \underline{y}\,\underline{x}) \end{aligned} \tag{2.3}$$

being a scalar and a 2-vector (also called bivector), respectively. In particular, $\underline{x}^2 = -\langle \underline{x}, \underline{x} \rangle = -|\underline{x}|^2 = -\sum_{j=1}^m x_j^2$.

Conjugation in $\mathbb{R}_{0,m}$ is defined as the anti-involution for which $\overline{e}_j = -e_j$, $j = 1,\ldots,m$. In particular for a vector $\underline{x}$, we have $\overline{\underline{x}} = -\underline{x}$.

The Dirac operator in $\mathbb{R}^m$ is the first-order vector-valued differential operator

$$\underline{\partial} = \sum_{j=1}^{m} e_j \partial_{x_j}, \tag{2.4}$$

its fundamental solution being given by

$$E(\underline{x}) = \frac{1}{a_m} \frac{\overline{\underline{x}}}{|\underline{x}|^m}. \tag{2.5}$$

Considering functions defined in $\mathbb{R}^m$ and taking values in $\mathbb{R}_{0,m}$, we say that the function f is left monogenic in the open region Ω of $\mathbb{R}^m$ if and only if f is continuously differentiable in Ω and satisfies in Ω the equation $\underline{\partial} f = 0$. As $\overline{\underline{\partial} f} = \overline{f}\,\overline{\underline{\partial}} = -\overline{f}\underline{\partial}$, a function f is left monogenic in Ω if and only if $\overline{f}$ is right monogenic in Ω. As, moreover, the Dirac operator factorizes the Laplace operator Δ, $-\underline{\partial}^2 = \underline{\partial}\overline{\underline{\partial}} = \overline{\underline{\partial}}\underline{\partial} = \Delta$, a monogenic function in Ω (as well as its components) is harmonic and hence C_∞ in Ω.

Introducing spherical coordinates $\underline{x} = r\underline{\omega}$, $r = |\underline{x}|$, $\underline{\omega} \in S^{m-1}$, the Dirac operator $\underline{\partial}$ may be written as

$$\underline{\partial} = \underline{\omega}\partial_r + \frac{1}{r}\partial_{\underline{\omega}} = \underline{\omega}\left(\partial_r - \frac{1}{r}\underline{\omega}\partial_{\underline{\omega}}\right), \tag{2.6}$$

while the Laplace operator takes the form

$$\Delta = \partial_r^2 + \frac{m-1}{r}\partial_r + \frac{1}{r^2}\Delta^*, \tag{2.7}$$

Δ^* being the Laplace-Beltrami operator.

In this paper, a fundamental role is played by the homogeneous polynomials $P_p(\underline{x})$ of degree $p \in \mathbb{N}$ which we take to be vector-valued and left (and hence also right) monogenic. Note that such kind of polynomials is easily obtained by considering $P_p(\underline{x}) = \underline{\partial} S_{p+1}(\underline{x})$, where $S_{p+1}(\underline{x})$ is a scalar-valued harmonic polynomial of degree $(p+1)$. By spherical inversion, the functions $Q_p(\underline{x}) = (\overline{\underline{x}}/|\underline{x}|^{m+2p})P_p(\underline{x})$ are left monogenic homogeneous functions of degree $(-m+1-p)$ in the complement of the origin. By taking restrictions to the unit sphere S^{m-1}, we obtain the so-called inner spherical monogenics $P_p(\underline{\omega})$ and the so-called outer spherical monogenics $Q_p(\underline{\omega}) = \underline{\omega}P_p(\underline{\omega})$. For $p = 0$, we put $P_0(\underline{x}) = 1$.

Finally, in this paper we adopt the following definition of the Fourier transform:

$$\mathcal{F}[f(\underline{x})](\underline{y}) = \int_{\mathbb{R}^m} f(\underline{x}) \exp\left(-2\pi i\langle \underline{x}, \underline{y}\rangle\right) dV(\underline{x}) \tag{2.8}$$

for which some well-known basic formulae hold:

$$\begin{aligned}
\mathcal{F}[\underline{\partial} f](\underline{y}) &= 2\pi i \underline{y}\mathcal{F}[f](\underline{y}),\\
2\pi i\, \mathcal{F}[\underline{x} f](\underline{y}) &= -\underline{\partial}\mathcal{F}[f](\underline{y}),\\
2\pi i \mathcal{F}[f\underline{x}](\underline{y}) &= -\mathcal{F}[f](\underline{y})\underline{\partial},\\
\mathcal{F}[\delta(\underline{x})](\underline{y}) &= 1,\\
\mathcal{F}[1](\underline{y}) &= \delta(\underline{y}).
\end{aligned} \tag{2.9}$$

3. Four families of distributions

In [9, 10], four families of distributions in Euclidean space: $T_{\lambda,p}$, $U_{\lambda,p}$, $V_{\lambda,p}$, and $W_{\lambda,p}$, depending on parameters $\lambda \in \mathbb{C}$ and $p \in \mathbb{N}_0 = \{0,1,2,\dots\}$ were studied in the framework of Clifford analysis. In this section, we first recall the definition of their normalizations $T^*_{\lambda,p}$, $U^*_{\lambda,p}$, $V^*_{\lambda,p}$, and $W^*_{\lambda,p}$, which is then followed by a convolvability study inside as well as in between the $T^*_{\lambda,p}$- and the $U^*_{\lambda,p}$-families of distributions. We have (see, e.g., [5])

$$\begin{aligned}
T^*_{\lambda,p} &= \pi^{(\lambda+m)/2+p}\frac{T_{\lambda,p}}{\Gamma((\lambda+m)/2+p)}, \quad \lambda \neq -m-2p-2l,\\
T^*_{-m-2p-2l,p} &= \frac{(-1)^p l!\pi^{m/2-l}}{2^{2p+2l}(p+l)!\Gamma(m/2+p+l)}P_p(\underline{x})\underline{\partial}^{2p+2l}\delta(\underline{x}), \quad l \in \mathbb{N}_0,
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
U^*_{\lambda,p} &= \pi^{(\lambda+m+1)/2+p}\frac{U_{\lambda,p}}{\Gamma((\lambda+m+1)/2+p)}, \quad \lambda \neq -m-2p-2l-1,\\
U^*_{-m-2p-2l-1,p} &= \frac{(-1)^{p+1} l!\pi^{m/2-l}}{2^{2p+2l+1}(p+l)!\Gamma(m/2+p+l+1)}(\underline{\partial}^{2p+2l+1}\delta(\underline{x}))P_p(\underline{x}), \quad l \in \mathbb{N}_0,
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
V^*_{\lambda,p} &= \pi^{(\lambda+m+1)/2+p}\frac{V_{\lambda,p}}{\Gamma((\lambda+m+1)/2+p)}, \quad \lambda \neq -m-2p-2l-1,\\
V^*_{-m-2p-2l-1,p} &= \frac{(-1)^{p+1} l!\pi^{m/2-l}}{2^{2p+2l+1}(p+l)!\Gamma(m/2+p+l+1)}P_p(\underline{x})(\underline{\partial}^{2p+2l+1}\delta(\underline{x})),
\end{aligned} \tag{3.3}$$

$$\begin{aligned}
W^*_{\lambda,p} &= \pi^{(\lambda+m)/2+p}\frac{W_{\lambda,p}}{\Gamma((\lambda+m)/2+p)}, \quad \lambda \neq -m-2p-2l,\\
W^*_{-m-2p-2l,p} &= \frac{(-1)^{p+1} l!\pi^{m/2-l}}{2^{2p+2l+2}(p+l+1)!\Gamma(m/2+p+l+1)}\underline{x}P_p(\underline{x})\underline{x}\underline{\partial}^{2p+2l+2}\delta(\underline{x}),
\end{aligned} \tag{3.4}$$

the action of the original distributions $T_{\lambda,p}$, $U_{\lambda,p}$, $V_{\lambda,p}$ and $W_{\lambda,p}$ on a testing function ϕ being given by

$$\begin{aligned}
\langle T_{\lambda,p},\phi\rangle &= a_m\left\langle Fpr_+^{\mu+p_e},\Sigma_p^{(0)}[\phi]\right\rangle,\\
\langle U_{\lambda,p},\phi\rangle &= a_m\left\langle Fpr_+^{\mu+p_e},\Sigma_p^{(1)}[\phi]\right\rangle,\\
\langle V_{\lambda,p},\phi\rangle &= a_m\left\langle Fpr_+^{\mu+p_e},\Sigma_p^{(3)}[\phi]\right\rangle,\\
\langle W_{\lambda,p},\phi\rangle &= a_m\left\langle Fpr_+^{\mu+p_e},\Sigma_p^{(2)}[\phi]\right\rangle.
\end{aligned} \tag{3.5}$$

We explain the notations in the above expressions. First, the symbol Fp stands for the well-known distribution "finite parts" on the real line, furthermore $\mu = \lambda + m - 1$ and p_e denotes the "even part of p," defined by $p_e = p$ if p is even and $p_e = p - 1$ if p is odd. Finally, $\Sigma_p^{(0)}$, $\Sigma_p^{(1)}$, $\Sigma_p^{(2)}$, and $\Sigma_p^{(3)}$ are the generalized spherical mean operators defined on scalar-valued testing functions ϕ by

$$\begin{aligned}
\Sigma_p^{(0)}[\phi] &= r^{p-p_e}\Sigma^{(0)}\left[P_p(\underline{\omega})\phi(\underline{x})\right] = \frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} P_p(\underline{\omega})\phi(\underline{x})dS(\underline{\omega}),\\
\Sigma_p^{(1)}[\phi] &= r^{p-p_e}\Sigma^{(0)}\left[\underline{\omega}P_p(\underline{\omega})\phi(\underline{x})\right] = \frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} \underline{\omega}P_p(\underline{\omega})\phi(\underline{x})dS(\underline{\omega}),\\
\Sigma_p^{(2)}[\phi] &= r^{p-p_e}\Sigma^{(0)}\left[\underline{\omega}P_p(\underline{\omega})\underline{\omega}\phi(\underline{x})\right] = \frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} \underline{\omega}P_p(\underline{\omega})\underline{\omega}\phi(\underline{x})dS(\underline{\omega}),\\
\Sigma_p^{(3)}[\phi] &= r^{p-p_e}\Sigma^{(0)}\left[P_p(\underline{\omega})\underline{\omega}\phi(\underline{x})\right] = \frac{r^{p-p_e}}{a_m}\int_{S^{m-1}} P_p(\underline{\omega})\underline{\omega}\phi(\underline{x})dS(\underline{\omega}),
\end{aligned} \tag{3.6}$$

where $P_p(\underline{\omega})$ is an inner spherical monogenic of degree p as defined in the previous section.

For a detailed study of the intra- and interrelationships between these families of distributions, we refer to—in chronological order—[3, 5, 9, 10].

In [4], the convolvability of the distributions $T^*_{\lambda,0}$ and $U^*_{\lambda,0}$ has been studied. Here we proceed with this study by considering the convolution of arbitrary members of the $T^*_{\lambda,p}$- and/or the $U^*_{\lambda,p}$-family. First of all, we recall the most important results of [4, Section 4] in the following lemma.

Lemma 3.1. *For each couple* $(\lambda,\mu) \in \mathbb{C}\times\mathbb{C}$ *such that*

(i) $\lambda \neq 2j$, $\mu \neq 2k$, *and* $\lambda+\mu \neq -m+2l$, $j,k,l \in \mathbb{N}_0$, *one has*

$$T^*_{\lambda,0} * T^*_{\mu,0} = c_m(\lambda,\mu)T^*_{\lambda+\mu+m,0}; \tag{3.7}$$

(ii) $\lambda \neq 2j$, $\mu \neq 2k+1$, *and* $\lambda+\mu \neq -m+2l+1$, $j,k,l \in \mathbb{N}_0$, *one has*

$$T^*_{\lambda,0} * U^*_{\mu,0} = U^*_{\mu,0} * T^*_{\lambda,0} = c_m(\lambda,\mu-1)U^*_{\lambda+\mu+m,0}; \tag{3.8}$$

(iii) $\lambda \neq 2j+1$, $\mu \neq 2k+1$, *and* $\lambda+\mu \neq -m+2l$, $j,k,l \in \mathbb{N}_0$, *one has*

$$U^*_{\lambda,0} * U^*_{\mu,0} = \frac{-2\pi}{\lambda+\mu+m} c_m(\lambda-1,\mu-1) T^*_{\lambda+\mu+m,0}, \tag{3.9}$$

where the constants $c_m(\lambda,\mu)$ *are given by*

$$c_m(\lambda,\mu) = \pi^{m/2} \frac{\Gamma(-(\lambda+\mu+m)/2)}{\Gamma(-(\lambda)/2)\Gamma(-\mu/2)}. \tag{3.10}$$

The formulae above, along with the respective conditions restraining their validity, have to be elucidated through some additional comments, since they should be interpreted with care. Consider for instance the formula in (i) for the "convolution" of two T^*-distributions, which apparently holds in the region

$$\widetilde{\Omega} = \{(\lambda,\mu) \in \mathbb{C}\times\mathbb{C} \mid \lambda \neq 2j,\ \mu \neq 2k,\ \lambda+\mu \neq -m+2l,\ j,k,l \in \mathbb{N}_0\} \tag{3.11}$$

in $\mathbb{C}\times\mathbb{C}$. However, only in a subset of $\widetilde{\Omega}$, the left-hand side exists as a genuine convolution. In the complementary subset, one defines the left-hand side by analytic continuation as $c_m(\lambda,\mu)T^*_{\lambda+\mu+m,0}$, leading to a $*$-operator which, although not being the genuine convolution operator, still preserves its basic properties. Finally, notice that for the excluded values of the couple (λ,μ) a simple pole occurs in at least one of the Γ-functions constituting the coefficient $c_m(\lambda,\mu)$. Hence the formula in (i) cannot be given any meaning in those cases. Similar remarks apply to (ii) and (iii); for more details, we refer to [4, Section 4].

Next the convolvability problem is tackled stepwise. First, in Lemma 3.2, a specific relation between $T^*_{\lambda,p}$ and $T^*_{\lambda+2p,0}$, respectively, between $U^*_{\lambda,p}$ and $U^*_{\lambda+2p,0}$, is established, by means of which we will be able to convert new convolutions into already known ones. This lemma is then used to deal with convolutions within or in between the $T^*_{\lambda,p}$- and $U^*_{\lambda,p}$-families where, for one of the involved distributions, we still have $p=0$. Finally the main results are given in Proposition 3.4, completing the picture in the most general case, where $p \neq 0$ for both distributions involved.

Lemma 3.2. *For each couple* $(\lambda,p) \in \mathbb{C}\times\mathbb{N}_0$, *one has*

$$(-2)^p \frac{\Gamma(-\lambda/2)}{\Gamma(-\lambda/2-p)} T^*_{\lambda,p} = T^*_{\lambda+2p,0} P_p(\underline{\partial}), \tag{3.12}$$

$$(-2)^p \frac{\Gamma(-(\lambda-1)/2)}{\Gamma(-(\lambda-1)/2-p)} U^*_{\lambda,p} = U^*_{\lambda+2p,0} P_p(\underline{\partial}). \tag{3.13}$$

Proof. We only prove the first equality, the proof of the second one runs along similar lines. From (3.1), one can derive that

$$T^*_{\lambda,p} = \pi^p \frac{\Gamma((\lambda+m)/2)}{\Gamma((\lambda+m)/2+p)} T^*_{\lambda,0} P_p(\underline{x}) \tag{3.14}$$

for $\lambda \neq -m-2l$, $l = 0, 1, \ldots, p-1$. Invoking

$$\mathscr{F}[T^*_{\lambda,p}] = i^{-p} T^*_{-\lambda-m-2p,p} \quad \forall (\lambda, p) \in \mathbb{C} \times \mathbb{N}_0, \tag{3.15}$$

see [3, Theorem 2], and some of the basic properties (2.9), we convert (3.14) to frequency space, which leads to

$$i^{-p} T^*_{-\lambda-m-2p,p} = \pi^p \frac{\Gamma((\lambda+m)/2)}{\Gamma((\lambda+m)/2+p)} \frac{i^p}{(2\pi)^p} T^*_{-\lambda-m,0} P_p(\underline{\partial}). \tag{3.16}$$

Replacing λ by $-\lambda-m-2p$, we obtain

$$(-2)^p T^*_{\lambda,p} = \frac{\Gamma(-\lambda/2-p)}{\Gamma(-\lambda/2)} T^*_{\lambda+2p,0} P_p(\underline{\partial}) \tag{3.17}$$

for $\lambda \neq -2p+2l$, $l = 0, \ldots, p-1$. Finally, rewriting this equality in the form of (3.12) reveals its validity for all couples (λ, p), since both sides reduce to 0 whenever λ takes one of the values excluded above. □

As announced, the previous lemma gives rise to a first generalization of Lemma 3.1.

Lemma 3.3. *For each triplet $(\lambda, \mu, p) \in \mathbb{C} \times \mathbb{C} \times \mathbb{N}$ such that*

(i) *$\lambda \neq 2j$, $\mu \neq 2k$, and $\lambda + \mu \neq -m+2l$, $j, k, l \in \mathbb{N}_0$, one has*

$$T^*_{\lambda,p} * T^*_{\mu,0} = T^*_{\mu,0} * T^*_{\lambda,p} = c_m(\lambda, \mu) T^*_{\lambda+\mu+m,p}; \tag{3.18}$$

(ii) *$\lambda \neq 2j$, $\mu \neq 2k+1$, and $\lambda + \mu \neq -m+2l+1$, $j, k, l \in \mathbb{N}_0$, one has*

$$\begin{aligned} T^*_{\lambda,p} * U^*_{\mu,0} &= c_m(\lambda, \mu-1) V^*_{\lambda+\mu+m,p}, \\ U^*_{\mu,0} * T^*_{\lambda,p} &= c_m(\lambda, \mu-1) U^*_{\lambda+\mu+m,p}; \end{aligned} \tag{3.19}$$

(iii) *$\lambda \neq 2j+1$, $\mu \neq 2k$, and $\lambda + \mu \neq -m+2l+1$, $j, k, l \in \mathbb{N}_0$, one has*

$$U^*_{\lambda,p} * T^*_{\mu,0} = T^*_{\mu,0} * U^*_{\lambda,p} = c_m(\lambda-1, \mu) U^*_{\lambda+\mu+m,p}; \tag{3.20}$$

(iv) *$\lambda \neq 2j+1$, $\mu \neq 2k+1$, and $\lambda + \mu \neq -m+2l$, $j, k, l \in \mathbb{N}_0$, one has*

$$\begin{aligned} U^*_{\lambda,p} * U^*_{\mu,0} = {} & \frac{2\pi}{(\lambda+\mu+2m+2p)(\lambda+\mu+m)} c_m(\lambda-1, \mu-1) \\ & \times \left[(m-2) T^*_{\lambda+\mu+m,p} + (\lambda+\mu+m) W^*_{\lambda+\mu+m,p}\right] \quad \text{if } \lambda+\mu \neq -2m-2p, \\ U^*_{\mu,0} * U^*_{\lambda,p} = {} & \frac{-2\pi}{\lambda+\mu+m} c_m(\lambda-1, \mu-1) T^*_{\lambda+\mu+m,p}. \end{aligned} \tag{3.21}$$

Proof. We only treat the case of $T^*_{\lambda,p} * T^*_{\mu,0}$, the other cases being similar.

First, take $\lambda \neq -2p+2j$, $j = 0, 1, \ldots, p-1$. In that case, (3.12) can be rewritten as

$$T^*_{\lambda,p} = \frac{(-1)^p}{2^p} \frac{\Gamma(-\lambda/2-p)}{\Gamma(-\lambda/2)} T^*_{\lambda+2p,0} P_p(\underline{\partial}). \tag{3.22}$$

Then, from (3.22), it follows that

$$\begin{aligned} T^*_{\lambda,p} * T^*_{\mu,0} &= \frac{(-1)^p}{2^p}\frac{\Gamma(-\lambda/2-p)}{\Gamma(-\lambda/2)}\left(T^*_{\lambda+2p,0}P_p(\underline{\partial}) * T^*_{\mu,0}\right) \\ &= \frac{(-1)^p}{2^p}\frac{\Gamma(-\lambda/2-p)}{\Gamma(-\lambda/2)}P_p(\underline{\partial})\left(T^*_{\lambda+2p,0} * T^*_{\mu,0}\right). \end{aligned} \tag{3.23}$$

In order for Lemma 3.1 to be applicable to the last expression, we need to assume, in addition to the premised conditions of (i), that $\lambda+\mu \neq -m-2p+2l$, $l=0,1,\ldots,p-1$. We are then lead to

$$T^*_{\lambda,p} * T^*_{\mu,0} = \frac{(-1)^p}{2^p}\frac{\Gamma(-\lambda/2-p)}{\Gamma(-\lambda/2)}c_m(\lambda+2p,\mu)P_p(\underline{\partial})T^*_{\lambda+\mu+m+2p,0} \tag{3.24}$$

from which the desired formula is easily obtained again exploiting (3.22):

$$T^*_{\lambda,p} * T^*_{\mu,0} = \frac{\Gamma(-\lambda/2-p)\Gamma(-(\lambda+\mu+m)/2)}{\Gamma(-\lambda/2)\Gamma(-(\lambda+\mu+m)/2-p)}c_m(\lambda+2p,\mu)T^*_{\lambda+\mu+m,p} = c_m(\lambda,\mu)T^*_{\lambda+\mu+m,p}. \tag{3.25}$$

We now further examine the values $\lambda=-2p+2j$ and $\lambda+\mu=-m-2p+2l$, $j,l=0,1,\ldots,p-1$, which had to be excluded temporarily in the course of the proof. For these values, we may write $T^*_{-2p+2j,p}=\lim_{\lambda\to -2p+2j}T^*_{\lambda,p}$, respectively, $T^*_{-\mu-m-2p+2l,p}=\lim_{\lambda\to-\mu-m-2p+2l}T^*_{\lambda,p}$, allowing us to repeat the procedure above, where we only effectuate the limit at the end of the calculations. □

The previous lemma now leads, in a second step, to more general results for the convolution of arbitrary $T^*_{\lambda,p}$- and/or $U^*_{\lambda,p}$-distributions, apart from some exceptional values for the involved parameters which remain excluded.

Proposition 3.4. *For each 4-tuple* $(\lambda,\mu,p,q)\in\mathbb{C}\times\mathbb{C}\times\mathbb{N}\times\mathbb{N}$ *such that*

(i) $\lambda\neq 2j$ *and* $\mu\neq 2k$, $j,k\in\mathbb{N}_0$, *one has*

$$T^*_{\lambda,p} * T^*_{\mu,q} = \begin{cases} c_{m,q}(\lambda,\mu)T^*_{\lambda+\mu+m+2q,p}P_q(\underline{\partial}) & \text{if } \lambda+\mu\neq -m-2q+2l,\ l\in\mathbb{N}_0, \\ c_{m,p}(\lambda,\mu)P_p(\underline{\partial})T^*_{\lambda+\mu+m+2p,q} & \text{if } \lambda+\mu\neq -m-2p+2l,\ l\in\mathbb{N}_0; \end{cases} \tag{3.26}$$

(ii) $\lambda\neq 2j+1$, $\mu\neq 2k$, *and* $\lambda+\mu\neq -m-2q+2l+1$, $j,k,l\in\mathbb{N}_0$, *one has*

$$U^*_{\lambda,p} * T^*_{\mu,q} = c_{m,q}(\lambda-1,\mu)U^*_{\lambda+\mu+m+2q,p}P_q(\underline{\partial}); \tag{3.27}$$

(iii) $\lambda \neq 2j$, $\mu \neq 2k+1$, *and* $\lambda+\mu \neq -m-2q+2l+1$, $j,k,l \in \mathbb{N}_0$, *one has*

$$T^*_{\lambda,p} * U^*_{\mu,q} = c_{m,q}(\lambda,\mu-1)V^*_{\lambda+\mu+m+2q,p}P_q(\underline{\partial}); \tag{3.28}$$

(iv) $\lambda \neq 2j+1$, $\mu \neq 2k+1$, *and* $\lambda+\mu \neq -m-2q+2l$, $j,k,l \in \mathbb{N}_0$, *one has*

$$\begin{aligned} U^*_{\lambda,p} * U^*_{\mu,q} = {} & \frac{2\pi}{(\lambda+\mu+2m+2p+2q)(\lambda+\mu+m+2q)}c_{m,q}(\lambda-1,\mu-1) \\ & \times\left[(m-2)T^*_{\lambda+\mu+m+2q,p} + (\lambda+\mu+m+2q)W^*_{\lambda+\mu+m+2q,p}\right]P_q(\underline{\partial}) \end{aligned} \tag{3.29}$$

if moreover $\lambda+\mu \neq -2m-2p-2q$,
where the constants $c_{m,p}(\lambda,\mu)$ *are given by*

$$c_{m,p}(\lambda,\mu) = \frac{(-1)^p}{2^p}\pi^{m/2}\frac{\Gamma\big(-(\lambda+\mu+m)/2-p\big)}{\Gamma(-\lambda/2)\Gamma(-\mu/2)} \tag{3.30}$$

with $c_{m,0}(\lambda,\mu) \equiv c_m(\lambda,\mu)$.

Proof. The proof directly follows from Lemmas 3.2 and 3.3. □

4. The classical Hilbert transform in Clifford analysis

In this section, we recall the definition and some important properties of the Hilbert transform in $\mathbb{R}^m$ in the framework of Clifford analysis.

First we pass to $(m+1)$-dimensional space by introducing an additional basis vector e_0 which follows the usual multiplication rules, that is, $e_0^2 = -1$ and it anticommutes with the other basis vectors, viz $e_0e_j + e_je_0 = 0$, $j = 1,\ldots,m$. The variable $x = (x_0,x_1,\ldots,x_m) \in \mathbb{R}^{m+1}$ is then identified with the vector $x = \sum_{j=0}^m e_jx_j$ in the Clifford algebra $\mathbb{R}_{0,m+1}$. Furthermore, the Dirac operator in $\mathbb{R}^{m+1}$ reads $\partial = e_0\partial_{x_0} + \underline{\partial}$.

For a suitable function or distribution f, its Hilbert transform in $\mathbb{R}^m$ is defined as

$$\mathcal{H}[f](\underline{x}) = \bar{e}_0H(\cdot) * f(\cdot)(\underline{x}) \tag{4.1}$$

with H the convolution kernel given by

$$H(\underline{x}) = \frac{2}{a_{m+1}}\mathrm{Pv}\frac{\bar{\underline{\omega}}}{r^m} = -\frac{2}{a_{m+1}}U^*_{-m,0}, \tag{4.2}$$

the last equality being shown in [10].

The corresponding Cauchy transform in $\mathbb{R}^{m+1}$ is defined by the convolution

$$\mathcal{C}[f](x_0,\underline{x}) = C(x_0,\cdot) * f(\cdot)(\underline{x}) \tag{4.3}$$

with the Cauchy kernel

$$C(x) = C(x_0,\underline{x}) = \frac{1}{a_{m+1}}\frac{\bar{x}e_0}{|x|^{m+1}} = \frac{1}{a_{m+1}}\frac{x_0+e_0\underline{x}}{|x_0+e_0\underline{x}|^{m+1}} \tag{4.4}$$

which is the fundamental solution of the Cauchy-Riemann operator $D_x = \bar{e}_0\partial$ in $\mathbb{R}^{m+1}$.

Some important properties of the Hilbert transform (4.1) are

P(1) $\mathcal{H}$ is translation invariant, that is, $\mathcal{H}[f(\underline{y}-\underline{t})](\underline{x}) = \mathcal{H}[f](\underline{x}-\underline{t})$ for all $\underline{t} \in \mathbb{R}^m$;

P(2) H is a homogeneous distribution of degree $(-m)$, which, for a convolution operator, is equivalent with its dilation invariance, that is, $\mathcal{H}[f(a\underline{y})](\underline{x}) = \mathcal{H}[f](a\underline{x})$ for all $a > 0$;

P(3) the Fourier symbol $\mathcal{F}[H](\underline{x}) = (2i/a_{m+1})\underline{\omega}$ is a bounded function, which is equivalent with $\mathcal{H}$ being a bounded linear operator on $L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$;

P(4) $\mathcal{H}^2 = 1$;

P(5) $\mathcal{H}$ is a unitary operator;

P(6) the Hilbert transform arises in a natural way by considering boundary values (in L_2 or in distributional sense) of the Cauchy transform in $\mathbb{R}^{m+1}$ of an appropriate function or distribution in $\mathbb{R}^m$.

Clearly, property P(6) requires some more detailed explanation. Taking nontangential limits for $x_0 \to 0$ and identifying $\mathbb{R}^m$ with the hyperplane $\{x_0 = 0\}$ in $\mathbb{R}^{m+1}$, the following distributions in $\mathbb{R}^m$ are obtained:

$$C(0+,\underline{x}) = \lim_{x_0 \to 0+} C(x_0,\underline{x}), \qquad C(0-,\underline{x}) = \lim_{x_0 \to 0-} C(x_0,\underline{x}). \tag{4.5}$$

They satisfy the relations

$$\begin{aligned} C(0+,\underline{x}) &= \frac{1}{2}\delta(\underline{x}) + \bar{e}_0\frac{1}{2}H(\underline{x}), \\ C(0-,\underline{x}) &= -\frac{1}{2}\delta(\underline{x}) + \bar{e}_0\frac{1}{2}H(\underline{x}), \end{aligned} \tag{4.6}$$

which are equivalent with the well-known distributional limits

$$\begin{aligned} \lim_{x_0 \to 0\pm} \frac{1}{a_{m+1}} \frac{2x_0}{|x_0 + \underline{x}|^{m+1}} &= \pm\delta(\underline{x}), \\ \lim_{x_0 \to 0\pm} \frac{1}{a_{m+1}} \frac{2\overline{\underline{x}}}{|x_0 + \underline{x}|^{m+1}} &= \frac{2}{a_{m+1}} Pv \frac{\overline{\underline{\omega}}}{r^m} = H(\underline{x}). \end{aligned} \tag{4.7}$$

If in particular $f \in L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$, then $\mathcal{C}[f]$ belongs to the Hardy spaces $H^2(\mathbb{R}^{m+1}_{\pm}; \mathbb{R}_{0,m+1})$, and its nontangential limits $\mathcal{C}^{\pm}[f]$ for $x_0 \to 0\pm$ satisfy the so-called Plemelj-Sokhotzki formulae

$$\mathcal{C}^{\pm}[f](\underline{x}) = \lim_{x_0 \to 0\pm} \mathcal{C}[f](x_0,\underline{x}) = \pm\frac{1}{2}f(\underline{x}) + \frac{1}{2}\mathcal{H}[f](\underline{x}), \quad \text{for a.e. } \underline{x} \in \mathbb{R}^m. \tag{4.8}$$

5. Generalizations of the Hilbert transform

5.1. First generalization. We consider the following specific distributions:

$$\begin{aligned}
T_{-m-p,p} &= Fp\frac{1}{r^m}P_p(\underline{\omega}) = Pv\frac{P_p(\underline{\omega})}{r^m},\\
U_{-m-p,p} &= Fp\frac{1}{r^m}\underline{\omega}P_p(\underline{\omega}) = Pv\frac{\underline{\omega}P_p(\underline{\omega})}{r^m},\\
V_{-m-p,p} &= Fp\frac{1}{r^m}P_p(\underline{\omega})\underline{\omega} = Pv\frac{P_p(\underline{\omega})\underline{\omega}}{r^m},\\
W_{-m-p,p} &= Fp\frac{1}{r^m}\underline{\omega}P_p(\underline{\omega})\underline{\omega} = Pv\frac{\underline{\omega}P_p(\underline{\omega})\underline{\omega}}{r^m},\\
Pv\frac{S_{p+1}(\underline{\omega})}{r^m} &= -\frac{1}{2(p+1)}\left(U_{-m-p,p}+V_{-m-p,p}\right),\\
Pv\frac{\underline{\omega}S_{p+1}(\underline{\omega})}{r^m} &= -\frac{1}{2(p+1)}\left(W_{-m-p,p}-T_{-m-p,p}\right),
\end{aligned} \tag{5.1}$$

where $P_p(\underline{x}) = \partial S_{p+1}(\underline{x})$, $S_{p+1}(\underline{x})$ being a scalar-valued solid spherical harmonic and hence, $P_p(\underline{x})$ being a vector-valued solid spherical monogenic. These distributions are homogeneous of degree $(-m)$ and the functions occurring in the numerator satisfy the cancellation condition

$$\int_{S^{m-1}} \Omega(\underline{\omega})\,d\underline{\omega} = 0, \tag{5.2}$$

$\Omega(\underline{\omega})$ being either of $P_p(\underline{\omega})$, $\underline{\omega}P_p(\underline{\omega})$, $P_p(\underline{\omega})\underline{\omega}$, or $\underline{\omega}P_p(\underline{\omega})\underline{\omega}$.

Their Fourier symbols, given by (see [5])

$$\begin{aligned}
\mathcal{F}\left[T_{-m-p}\right] &= i^{-p}\pi^{m/2}\frac{\Gamma(p/2)}{\Gamma((m+p)/2)}P_p(\underline{\omega}),\\
\mathcal{F}\left[U_{-m-p}\right] &= i^{-p-1}\pi^{m/2}\frac{\Gamma((p+1)/2)}{\Gamma((m+p+1)/2)}\underline{\omega}P_p(\underline{\omega}),\\
\mathcal{F}\left[V_{-m-p}\right] &= i^{-p-1}\pi^{m/2}\frac{\Gamma((p+1)/2)}{\Gamma((m+p+1)/2)}P_p(\underline{\omega})\underline{\omega},\\
\mathcal{F}\left[W_{-m-p}\right] &= i^{-p-2}\pi^{m/2}\frac{p\Gamma(p/2)}{(m+p)\Gamma((m+p)/2)}\left(\underline{\omega}P_p(\underline{\omega})\underline{\omega}-\frac{m-2}{p}P_p(\underline{\omega})\right)
\end{aligned} \tag{5.3}$$

are homogeneous of degree 0 and moreover are bounded functions, whence

$$T_{-m-p,p} * f,\qquad U_{-m-p,p} * f,\qquad V_{-m-p,p} * f,\qquad W_{-m-p,p} * f \tag{5.4}$$

are bounded singular integral operators on $L_2(\mathbb{R}^m;\mathbb{R}_{0,m+1})$ which are direct generalizations of the Hilbert transform $\mathcal{H}$, preserving (properly adapted analogues of the) properties P(1)–P(3).

We now investigate whether these new operators will fulfil some appropriate analogues of the remaining properties P(4)–P(6) as well. To this end we closely examine the kernel $T_{-m-p,p}$.

First, from Proposition 3.4 it follows that

$$T_{-m-p,p} * T_{-m-p,p} = \frac{(-1)^p}{2^p}\pi^{m/2}\frac{\Gamma(m/2)}{\Gamma(p)}\left[\frac{\Gamma(p/2)}{\Gamma((m+p)/2)}\right]^2 T_{-m,p}P_p(\underline{\partial}) \tag{5.5}$$

which directly implies that the generalized Hilbert transform $T_{-m-p,p} * f$ does not satisfy an analogue of property P(4).

Next, as it can easily be shown that the considered operator coincides with its adjoint—up to a minus sign when p is even, we may also conclude, in view of (5.5), that it will not be unitary.

Finally, we fail to establish an analogue of property P(6) as well, since it is not possible to find a generalized Cauchy kernel in $\mathbb{R}^{m+1} \setminus \{0\}$, for which a part of the boundary values is precisely the generalized Hilbert kernel $T_{-m-p,p}$. Similar conclusions hold for the other generalized kernels used in (5.1).

5.2. Second generalization. Subsequent to the observations above, we now want to find a type of generalized Hilbert kernel which actually preserves property P(6). To that end, we define the function

$$C_p(x) = C_p(x_0,\underline{x}) = \frac{1}{a_{m+1,p}}\frac{\bar{x}e_0}{|x|^{m+1+2p}}P_p(\underline{x}) = \frac{1}{a_{m+1,p}}\frac{x_0+e_0\underline{x}}{\left|x_0+\underline{x}\right|^{m+1+2p}}P_p(\underline{x}), \tag{5.6}$$

where

$$a_{m+1,p} = \frac{(-1)^p}{2^p}\frac{2\pi^{(m+1)/2}}{\Gamma((m+1)/2+p)} \tag{5.7}$$

involving a homogeneous polynomial $P_p(\underline{x})$ of degree p which we take to be vector valued and monogenic (as defined in Section 2). In the next proposition, we show that these functions C_p are good candidates for generalized Cauchy kernels.

Proposition 5.1. *The function C_p satisfies the following properties:*

(i) $C_p \in L_1^{\mathrm{loc}}(\mathbb{R}^{m+1};\mathbb{R}_{0,m+1})$ *and* $\lim_{|x|\to\infty} C_p(x) = 0$ *for all* $p \in \mathbb{N}$*;*

(ii) $D_x C_p(x) = P_p(\underline{\partial})\delta(x)$ *in distributional sense for all* $p \in \mathbb{N}$*;*

(iii) *for* $p = 0$*,* C_0 *coincides with the traditional Cauchy kernel* C*.*

Proof. The proof of (i) being straightforward, we focus on the proofs of (ii) and (iii). First recall that in $\mathbb{R}^m$ the following formula holds for each couple $(\lambda,p) \in \mathbb{C}\times\mathbb{N}$ (see, e.g., [5]):

$$\underline{\partial}U^*_{\lambda,p} = -2\pi T^*_{\lambda-1,p}. \tag{5.8}$$

Hence, passing to $\mathbb{R}^{m+1}$, and using the tilde-notation for the corresponding families of distributions there, we still have $\partial\widetilde{U}^*_{\lambda,p} = -2\pi\widetilde{T}^*_{\lambda-1,p}$. Applying this formula in the specific case where $\lambda = -m-2p$, $p \in \mathbb{N}$, we get

$$\partial \widetilde{U}^*_{-m-2p,p} = -2\pi \widetilde{T}^*_{-m-2p-1,p} \tag{5.9}$$

which, invoking (3.1) and (3.2)—however with m being replaced by $m+1$, can be rewritten as

$$\partial\left(\frac{\pi}{\Gamma(1)}\widetilde{U}_{-m-2p,p}\right) = -2\pi\left(\frac{\pi^{(m+1)/2}}{2^{2p}p!\Gamma((m+1)/2+p)}\widetilde{P}_p(x)\Delta^p\delta(x)\right) \tag{5.10}$$

or as

$$\partial\left(\frac{\Omega}{|x|^{m+2p}}\widetilde{P}_p(x)\right) = -\frac{1}{2^p p!}a_{m+1,p}\widetilde{P}_p(x)\partial^{2p}\delta(x) \tag{5.11}$$

with $x = |x|\Omega$, $\Omega \in S^m$. In particular, substituting $\widetilde{P}_p(x) = e_0 P_p(\underline{x})$ in (5.11) yields

$$D_x\left(\frac{\bar{x}e_0}{|x|^{m+1+2p}}P_p(\underline{x})\right) = \frac{1}{2^p p!}a_{m+1,p}P_p(\underline{x})\partial^{2p}\delta(x). \tag{5.12}$$

On the other hand, we know from [3, Proposition 2] that in $\mathbb{R}^m$

$$P_p(\underline{x})\underline{\partial}^{2p}\delta(\underline{x}) = 2^p p! P_p(\underline{\partial})\delta(\underline{x}) \tag{5.13}$$

which can be rewritten in $\mathbb{R}^{m+1}$ as $\widetilde{P}_p(x)\partial^{2p}\delta(x) = 2^p p!\widetilde{P}_p(\partial)\delta(x)$. Again taking $\widetilde{P}_p(x) = e_0 P_p(\underline{x})$ then gives

$$P_p(\underline{x})\partial^{2p}\delta(x) = 2^p p! P_p(\underline{\partial})\delta(x). \tag{5.14}$$

Finally, substitution of (5.14) in the right-hand side of (5.12) yields $D_x C_p(x) = P_p(\underline{\partial})\delta(x)$. Now, as for $p = 0$, we have $P_0(\underline{x}) = 1$ and $a_{m+1,0} = a_{m+1}$, this implies

$$C_0(x) = \frac{1}{a_{m+1}}\frac{\bar{x}e_0}{|x|^{m+1}} \tag{5.15}$$

which is precisely the standard Cauchy kernel in Clifford analysis. □

As a nice additional result, using a similar method as in the previous proof, one also can construct a generalized fundamental solution for the Dirac operator $\underline{\partial}$ in $\mathbb{R}^m$, viz

$$E_p(\underline{x}) = \frac{1}{a_{m,p}}\frac{\bar{\underline{x}}P_p(\underline{x})}{|\underline{x}|^{m+2p}} = -\frac{1}{\pi a_{m,p}}U^*_{-m-2p+1,p} \tag{5.16}$$

for which $\underline{\partial}E_p(\underline{x}) = P_p(\underline{\partial})\delta(\underline{x})$ and $E_0 = E$, the standard fundamental solution of the Dirac operator (see Section 2).

In the next proposition, we calculate the nontangential distributional boundary values for $x_0 \to 0\pm$ of the generalized Cauchy kernels $C_p(x_0,\underline{x})$, $p \in \mathbb{N}_0$. To this end, we first formulate an auxiliary result in the following lemma.

LEMMA 5.2. *For $p \in \mathbb{N}_0$, one has*

$$\lim_{x_0 \to 0+} \frac{x_0}{|x_0 + \underline{x}|^{m+1+2p}} = \frac{1}{2^{p+1} p!} a_{m+1,p} \underline{\partial}^{2p} \delta(\underline{x}). \tag{5.17}$$

Proof. We will prove (5.17) by induction on p.

Clearly, for $p = 0$, (5.17) yields the following well-known distributional limit:

$$\lim_{x_0 \to 0+} \frac{x_0}{|x_0 + \underline{x}|^{m+1}} = \frac{1}{2} a_{m+1} \delta(\underline{x}). \tag{5.18}$$

Next, assume (5.17) to be valid for $(p-1)$, that is, we have

$$\lim_{x_0 \to 0+} \frac{x_0}{|x_0 + \underline{x}|^{m-1+2p}} = \frac{1}{2^p (p-1)!} a_{m+1,p-1} \underline{\partial}^{2p-2} \delta(\underline{x}). \tag{5.19}$$

From the action of the Dirac operator on both sides of this equality, we obtain

$$\lim_{x_0 \to 0+} \underline{\partial} \left(\frac{x_0}{|x_0 + \underline{x}|^{m-1+2p}} \right) = \frac{1}{2^p (p-1)!} a_{m+1,p-1} \underline{\partial}^{2p-1} \delta(\underline{x}). \tag{5.20}$$

On the other hand, one can directly calculate that

$$\lim_{x_0 \to 0+} \underline{\partial} \left(\frac{x_0}{|x_0 + \underline{x}|^{m-1+2p}} \right) = -(m-1+2p) \lim_{x_0 \to 0+} \frac{x_0 \underline{x}}{|x_0 + \underline{x}|^{m+1+2p}}. \tag{5.21}$$

Comparison between (5.20) and (5.21) leads to

$$\frac{1}{2^p (p-1)!} a_{m+1,p-1} \underline{\partial}^{2p-1} \delta(\underline{x}) = -(m-1+2p) \underline{x} \lim_{x_0 \to 0+} \frac{x_0}{|x_0 + \underline{x}|^{m+1+2p}}. \tag{5.22}$$

From [6, Lemma 3.1], we have $\underline{x}\underline{\partial}^{2p} \delta(\underline{x}) = 2p \underline{\partial}^{2p-1} \delta(\underline{x})$. Thus (5.22) can be rewritten as

$$\frac{1}{2^{p+1} p!} a_{m+1,p-1} \underline{x}\underline{\partial}^{2p} \delta(\underline{x}) = -(m-1+2p) \underline{x} \lim_{x_0 \to 0+} \frac{x_0}{|x_0 + \underline{x}|^{m+1+2p}}, \tag{5.23}$$

leading to the desired result

$$\lim_{x_0 \to 0+} \frac{x_0}{|x_0 + \underline{x}|^{m+1+2p}} = \frac{1}{2^{p+1} p!} a_{m+1,p} \underline{\partial}^{2p} \delta(\underline{x}) \tag{5.24}$$

when we invoke the definition (5.7) of $a_{m+1,p}$. □

PROPOSITION 5.3. *For each $p \in \mathbb{N}_0$, one has*

$$\begin{aligned} C_p(0+, \underline{x}) &= \lim_{x_0 \to 0+} C_p(x_0, \underline{x}) = \frac{1}{2} P_p(\underline{\partial}) \delta(\underline{x}) + \bar{e}_0 \frac{1}{2} H_p(\underline{x}), \\ C_p(0-, \underline{x}) &= \lim_{x_0 \to 0-} C_p(x_0, \underline{x}) = -\frac{1}{2} P_p(\underline{\partial}) \delta(\underline{x}) + \bar{e}_0 \frac{1}{2} H_p(\underline{x}), \end{aligned} \tag{5.25}$$

where

$$H_p(\underline{x}) = \frac{2}{a_{m+1,p}} Fp \frac{\bar{\underline{\omega}} P_p(\underline{\omega})}{r^{m+p}} = -\frac{2}{a_{m+1,p}} U^*_{-m-2p,p}. \tag{5.26}$$

Proof. We only calculate $C_p(0+,\underline{x})$, the computation for $C_p(0-,\underline{x})$ runs along similar lines. Multiplying both sides of (5.17) with $P_p(\underline{x})$ and applying (5.13), already yields

$$\lim_{x_0 \to 0+} \frac{x_0 P_p(\underline{x})}{|x_0 + \underline{x}|^{m+1+2p}} = \frac{1}{2} a_{m+1,p} P_p(\underline{\partial}) \delta(\underline{x}). \tag{5.27}$$

Next one can show that in distributional sense,

$$\lim_{x_0 \to 0+} e_0 \frac{\underline{x} P_p(\underline{x})}{|x_0 + \underline{x}|^{m+1+2p}} = e_0 Fp \frac{\underline{\omega} P_p(\underline{\omega})}{r^{m+p}}. \tag{5.28}$$

Expressions (5.27) and (5.28) then result into the following distributional limit:

$$\begin{aligned} C_p(0+,\underline{x}) &= \lim_{x_0 \to 0+} \frac{1}{a_{m+1,p}} \frac{x_0 P_p(\underline{x})}{|x_0 + \underline{x}|^{m+1+2p}} + \lim_{x_0 \to 0+} \frac{1}{a_{m+1,p}} \frac{e_0 \underline{x} P_p(\underline{x})}{|x_0 + \underline{x}|^{m+1+2p}} \\ &= \frac{1}{2} P_p(\underline{\partial}) \delta(\underline{x}) + \frac{1}{a_{m+1,p}} e_0 Fp \frac{\underline{\omega} P_p(\underline{\omega})}{r^{m+p}}, \end{aligned} \tag{5.29}$$

which had to be proved. $\square$

The distribution H_p arising in the previous proposition allows for the definition of a generalized Hilbert transform $\mathcal{H}_p$, given by

$$\mathcal{H}_p[f] = \bar{e}_0 H_p * f. \tag{5.30}$$

The Fourier symbol

$$\mathcal{F}[H_p] = -\frac{2}{a_{m+1,p}} i^{-p-1} U^*_{0,p} \tag{5.31}$$

of the kernel H_p not being a bounded function, the operator $\mathcal{H}_p$ will not be bounded on $L_2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$. However, the Fourier symbol is polynomial of degree p, implying that $\mathcal{H}_p$ is a bounded operator between the Sobolev spaces $W_2^n(\mathbb{R}^m; \mathbb{R}_{0,m+1}) \to W_2^{n-p}(\mathbb{R}^m; \mathbb{R}_{0,m+1})$, for $n \geq p$. This is also confirmed in Corollary 5.5.

Proposition 5.4. *The generalized Cauchy transform $\mathcal{C}_p$ maps the Sobolev space $W_2^n(\mathbb{R}^m; \mathbb{R}_{0,m+1})$ into the Hardy space $H^2(\mathbb{R}_+^{m+1}; \mathbb{R}_{0,m+1})$, for each natural number $n \geq p$.*

Proof. First of all, we notice that the Hardy spaces $H^2(\mathbb{R}_+^{m+1}; \mathbb{R}_{0,m+1})$ and $H^2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$ are isomorphic; each element of the latter space can be identified with the nontangential limit $\lim_{x_0 \to 0+} F(x_0, \underline{x})$, with $F \in H^2(\mathbb{R}_+^{m+1}; \mathbb{R}_{0,m+1})$. Moreover, $H^2(\mathbb{R}^m; \mathbb{R}_{0,m+1})$ can be characterized as follows:

$$g \in H^2(\mathbb{R}^m;\mathbb{R}_{0,m+1}) \Longleftrightarrow \begin{cases} (C1) & g \in L_2(\mathbb{R}^m;\mathbb{R}_{0,m+1}), \\ (C2) & \mathcal{H}[g] = g. \end{cases} \tag{5.32}$$

So, it is necessary and sufficient to prove that

$$\lim_{x_0 \to 0+} \mathscr{C}_p[f](x_0,\underline{x}) = \frac{1}{2} P_p(\underline{\partial}) f + \frac{1}{2} \mathcal{H}_p[f] \tag{5.33}$$

satisfies conditions (C1) and (C2) for each $f \in W_2^n$, $n \geq p$.

For such a function f we immediately have that $P_p(\underline{\partial}) f \in W_2^{n-p} \subset L_2$. For the second term on the right-hand side of (5.33) we apply Lebesgue's dominated convergence theorem, which yields $H_p \in L_1$. Then, from Young's inequality, it follows that $\mathcal{H}_p[f] \in L_2$, fulfilling condition (C1).

Now we examine whether condition (C2) is satisfied as well, that is, we check if

$$\mathcal{H}\Big[\lim_{x_0 \to 0+} \mathscr{C}_p[f](x_0,\underline{x})\Big] = \lim_{x_0 \to 0+} \mathscr{C}_p[f](x_0,\underline{x}). \tag{5.34}$$

Successively invoking Lemmas 3.1 and 3.2, we find

$$\mathcal{H}[P_p(\underline{\partial}) f] = \bar{e}_0 \frac{-2}{a_{m+1}} U^*_{-m} * P_p(\underline{\partial}) f = \bar{e}_0 \frac{-2}{a_{m+1,p}} U^*_{-m-2p,p} * f = \mathcal{H}_p[f], \tag{5.35}$$

$$\begin{aligned} \mathcal{H}[\mathcal{H}_p[f]] &= \frac{2}{a_{m+1}} \frac{2}{a_{m+1,p}} U^*_{-m,0} * (U^*_{-m-2p,p} * f) \\ &= \frac{2}{a_{m+1}} \frac{\Gamma((m+1)/2)}{\pi^{(m+1)/2}} (U^*_{-m,0} * U^*_{-m,0}) * P_p(\underline{\partial}) f = P_p(\underline{\partial}) f \end{aligned} \tag{5.36}$$

which completes the proof. □

Corollary 5.5. *The generalized Hilbert transform $\mathcal{H}_p$ is a bounded linear operator between the Sobolev spaces $W_2^n(\mathbb{R}^m;\mathbb{R}_{0,m+1})$ and $W_2^{n-p}(\mathbb{R}^m;\mathbb{R}_{0,m+1})$ for each natural number $n \geq p$.*

Proof. The previous proposition learns that $\mathcal{H}_p[f] = \mathcal{H}[P_p(\underline{\partial}) f]$, for each function $f \in W_2^n$, with $n \geq p$. As $\mathcal{H}$ is a bounded operator on L_2 and $P_p(\underline{\partial}) f \in W_2^{n-p} \subset L_2$, this ensures that $\mathcal{H}_p[f] \in L_2$. Moreover, relying on $\mathcal{H}[\partial_{x_i} f] = \partial_{y_i} \mathcal{H}[f]$, $i = 1,\dots,m$, we have a fortiori that $\mathcal{H}_p[f] \in W_2^{n-p}$. □

Comparing further the properties of $\mathcal{H}_p$ with those of the standard Hilbert transform $\mathcal{H}$ in Clifford analysis learns that the main objective for this second generalization is fulfilled on account of Proposition 5.3: H_p pops up as a part of the boundary values of a generalized Cauchy kernel C_p, an analogue of the "classical" property P(6). However, the kernel H_p is a homogeneous distribution of degree $(-m-p)$, meaning that $\mathcal{H}_p$ is not dilation invariant. Finally, a link with the first type of generalized Hilbert transforms is established below.

Remark 5.6. H_p can be written as a higher-order Dirac derivative, say $\underline{\partial}^p$, of the generalized Hilbert kernels of the first kind $T_{-m-p,p}$ and $U_{-m-p,p}$, depending on the parity of p. More specifically, for a suitable function f and a natural number p, one has

$$\begin{aligned}\mathcal{H}_p[f] &= \bar{e}_0 H_p * f \\ &= \bar{e}_0 \begin{cases} \dfrac{-1}{2^{(p+1)/2}(p-2)!!}\dfrac{\Gamma((m+p)/2)}{\Gamma((m+2p+1)/2)}\underline{\partial}^p T_{-m-p,p} * f & \text{if } p \text{ is odd,} \\[2ex] \dfrac{1}{2^{p/2}(p-1)!!}\dfrac{\Gamma((m+p+1)/2)}{\Gamma((m+2p+1)/2)}\underline{\partial}^p U_{-m-p,p} * f & \text{if } p \text{ is even.} \end{cases}\end{aligned} \tag{5.37}$$

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