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Bonetto, Riccardo; Kojakhmetov, Hildeberto Jardón

Nonlinear Laplacian Dynamics: Symmetries, Perturbations, and Consensus

Riccardo Bonetto† and Hildeberto Jardón Kojakhmetov†

†University of Groningen — Bernoulli Institute for Mathematics, Computer Science and Artificial Intelligence; Nijenborgh 9, 9747AG, Groningen, The Netherlands

June 10, 2022

Abstract

In this paper, we study a class of dynamic networks called Absolute Laplacian Flows under small perturbations. Absolute Laplacian Flows are a type of nonlinear generalisation of classical linear Laplacian dynamics. Our main goal is to describe the behaviour of the system near the consensus space. The nonlinearity of the studied system gives rise to potentially intricate structures of equilibria that can intersect the consensus space, creating singularities. For the unperturbed case, we characterise the sets of equilibria by exploiting the symmetries under group transformations of the nonlinear vector field. Under perturbations, Absolute Laplacian Flows behave as a slow-fast system. Thus, we analyse the slow-fast dynamics near the singularities on the consensus space. In particular, we prove a theorem that provides existence conditions for a maximal canard, that coincides with the consensus subspace, by using the symmetry properties of the network. Furthermore, we provide a linear approximation of the intersecting branches of equilibria at the singular points; as a consequence, we show that, generically, the singularities on the consensus space turn out to be *transcritical*.

Keywords: Laplacian dynamics, symmetries, consensus, singular perturbations.

1 Introduction

Laplacian dynamical systems are among the most renowned subjects in dynamic networks science. Indeed, given a graph structure, a linear Laplacian system, $\dot{\boldsymbol{x}} = -L\boldsymbol{x}$, where L is the graph Laplacian, describes a (linear) diffusion process on a network. In fact, the Laplacian matrix plays the role of discrete counterpart of the Laplacian differential operator, Δ, present for example in the heat equation, $\dot{u} = \Delta u$. Unfolding the equations for a linear Laplacian system, we obtain a set of ordinary differential equations that componentwise read as $\dot{x}_i = \sum_j a_{ij}(x_j - x_i)$, where x_i represents the state of the ith-agent, or node of the network, and a_{ij} encodes the graph structure and the 'strength' of interaction. A lot of effort has been made to precisely analyse the behaviour of Laplacian systems, especially for the linear case [44, 43, 36]. More recently, also nonlinear extensions of the Laplacian dynamics are being explored [42, 3, 1, 24, 25]. The potential extensions from a linear Laplacian system to a nonlinear one are various; in the present paper we aim to look at a class of nonlinear Laplacian systems called *Absolute Laplacian Flows* (ALFs) [42, 3], defined by

$$\dot{\boldsymbol{x}} = -LF(\boldsymbol{x}) + \epsilon H(\boldsymbol{x}, \Lambda), \tag{1}$$

where $F(\boldsymbol{x})$ is a nonlinear vector field, $0 < \epsilon \ll 1$, and $H(\boldsymbol{x}, \Lambda)$ is a perturbation; more precise definitions follow in the next sections.

An ALF can be regarded as a discrete analogue of an extended system's diffusion process, namely the *Fokker-Planck Equation* (FPE) [37]. Let us recall that for an extended system on the real line, the (one-dimensional) FPE reads

$$\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}\left(D(x,t)u(x,t)\right) - \frac{\partial}{\partial x}\left(\mu(x,t)u(x,t)\right), \tag{2}$$

where $u : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$, $D(x,t)$ is called the diffusion coefficient, and $\mu(x,t)$ the drift coefficient. We notice that the term $F(\boldsymbol{x})$ in (1) can be written as $F(\boldsymbol{x}) =: K(\boldsymbol{x})\boldsymbol{x}$, where $K(\boldsymbol{x})$ is a diagonal matrix, and the precise structure follows from the definition of the components of the vector field $F(\boldsymbol{x})$, see (4) below. Then, one can recognise a correspondence between FPE and a perturbed ALF, i.e.,

$$\frac{\partial^2}{\partial x^2}\left(D(x,t)u(x,t)\right) \mapsto -LF(\boldsymbol{x}), \qquad -\frac{\partial}{\partial x}\left(\mu(x,t)u(x,t)\right) \mapsto \epsilon H(\boldsymbol{x},\Lambda). \tag{3}$$

So, the first term of (1), $-LF(\boldsymbol{x})$, can be interpreted as a nonlinear diffusion process on a network, while the perturbation $\epsilon H(\boldsymbol{x},\Lambda)$ describes a small drift.

The above is just one possible interpretation of a perturbed Absolute Laplacian Flow. From another perspective, it is possible to recognise a connection with directed graphs with state-dependent weights. If we consider the rewriting $F(\boldsymbol{x}) =: K(\boldsymbol{x})\boldsymbol{x}$, we can define a state-dependent Laplacian, $L_{\mathrm{K}}(\boldsymbol{x}) := LK(\boldsymbol{x})$. The matrix $L_{\mathrm{K}}(\boldsymbol{x})$ can be seen as the Laplacian of a directed graph under the definition of out-degree Laplacian [2, 35]. Then, the diffusion term in (1) is given by $-L_{\mathrm{K}}(\boldsymbol{x})\boldsymbol{x}$, retrieving a form similar to the linear Laplacian dynamics but for directed graphs with state-dependent weights. Let us notice that the form of the state-dependent weights is constrained by the structure of the matrix $K(\boldsymbol{x})$, and so in turn by the definition of $F(\boldsymbol{x})$.

Within the context of Laplacian dynamics the diffusion of 'information' among the agents can lead the system to reach a common state, referred to as *consensus*. As such, Laplacian systems find a remarkable application in consensus problems [36, 39, 12], which have been extensively studied for their importance in several branches of science, e.g., synchronisation, opinion dynamics, coordination of robots, rendezvous problems, among many others.

Recapping, the models we are going to study (1), called ALFs, are relevant in the study of, for instance, nonlinear diffusion processes on networks, or state-dependent directed graphs. We consider ALFs subject to a small perturbation that, as we have argued, acts as a small drift in the undergoing nonlinear diffusion.

Let us now delineate the main contributions of the paper. Notice that it is quite natural to consider graphs with positive weights, since, for example, they can represent the strength of the interaction among the agents of the system. So, first we prove topological equivalence near consensus between (unperturbed) ALFs with arbitrary positive weights and with unit weights (proposition 5). Next, recall that linear Laplacian systems have well-known, simple, sets of equilibria; but that is not the case for the ALFs we consider. Thus, we then characterise new equilibria emerging from the invariance of $F(\boldsymbol{x})$ under group transformations (proposition 6).

Later, we consider the perturbation problem (1), which turns out to be a *singular perturbation problem* [46]. Due to the more complicated structure of the equilibria, and the presence of multiple timescales, we study the existence of canard solutions (theorem 2); and we characterize the local geometry of the consensus space at the singular points (proposition 17). In particular, such results show that under certain conditions related to the preservation of symmetry, *ALFs can only have transcritical singularities on the consensus space*, and thus the possible canard solutions are induced by such singularities. Along this paper, we make a strong connection between dynamic behaviour and symmetry properties of the networked system (1). In that regard, the cell-network formalism [19, 21] is a prominent relevant framework. Indeed, in sections 3 and 7, we discuss the connections between ALFs and the cell-network formalism.

The paper is structured as follows: in section 2 we define the (unperturbed) ALF, and we exploit some general properties. Section 3 is dedicated to understanding the role of symmetries in ALFs. Later, in section 4 we introduce a perturbation term leading to (1). As we will see, the perturbations, in general, give rise to a slow-fast behaviour of the system. A brief summary of the main ideas of singular perturbation theory useful for the main analysis can be found in appendix A; while the slow-fast consensus dynamics of ALFs with complete graph structure is studied in section 5. In order to highlight our results, in section 6, we show some numerical examples. A brief discussion and the conclusions are drawn in section 7.

2 Definitions and preliminaries

From a mathematical point of view, a network structure can be described by a graph, $\mathcal{G} = \{\mathcal{V}, \mathcal{E}\}$, which consists of a finite set of nodes (or vertices) $\mathcal{V} = \{1, ..., n\}$, and a set of edges $e_{ij} \in \mathcal{E}$, where each e_{ij} represents a connection between the node j and i [18]. Furthermore, if we assign a set of weights $\mathcal{W}$ to the graph $\mathcal{G}$, then we obtain a weighted graph, $\mathcal{G}_{\mathrm{w}} = \{\mathcal{V}, \mathcal{E}, \mathcal{W}\}$. We assume (unless otherwise stated) that the weights, $w_{ij} \in \mathcal{W}$, are positive reals. Such a choice is natural both for the algebraic properties that follow from it [4], and for the modelling perspective, since often the weights represent some measure of distance, or strengths of the coupling [33]. In other contexts negative weights are also considered [41], for example in models involving inhibition, or antagonism; in the present paper we do not consider such situations.

The edges and the weights are in one-to-one correspondence, in fact, we assign to each edge e_{ij} a weight w_{ij}. If the network is *directed* then in general $w_{ij} \neq w_{ji}$, otherwise for an *undirected* network we have $w_{ij} = w_{ji}$.

Definition 1. *A* simple graph *is an undirected graph without self-loops or multiple edges.*

Unless otherwise stated, we consider simple graphs. For weighted graphs, we assume a simple unweighted structure, i.e., $\mathcal{G}_{\mathrm{w}} \setminus \mathcal{W}$ is a simple graph. Moreover, we make no nomenclature distinction between graphs and weighted graphs with unit weights, because the algebraic structures in such cases are exactly the same.

We now recall a few well-known concepts of graph theory [33, 6, 18]. A *path* is a sequence of edges that joins a sequence of distinct vertices. A *connected component* of a graph is a sub-graph such that every pair of nodes is connected by a path. A relevant class of graphs is given by the complete graphs.

Definition 2. *A* complete graph *with n nodes, denoted by K_n, is a graph such that for all pair of nodes there is an edge connecting them, i.e., $\forall i, j \in \mathcal{V}\ \exists\ e_{ij} \in \mathcal{E}$.*

Given a graph it is possible to define some algebraic structures on it, we recall here the ones that are useful for the purpose of the paper.

- The adjacency matrix, A, of a graph is the $n \times n$ matrix with components $A_{ij} = w_{ij}$ if $e_{ij} \in \mathcal{E}$, and $A_{ij} = 0$ if $e_{ij} \notin \mathcal{E}$.
- The degree matrix, Δ, of a graph is the $n \times n$ diagonal matrix with (diagonal) components $\Delta_{ii} = \sum_j w_{ij}$.
- The Laplacian matrix, L, of a graph is the $n \times n$ matrix defined by $L := \Delta - A$.

For simple graphs, let us recall that the Laplacian matrix, L, is symmetric, degenerate and semi-positive definite; moreover, the following proposition holds [4].

Proposition 1. *The algebraic multiplicity of the* 0 *eigenvalue of L, $\mu_a(0)$, is equal to the number of connected components of the graph. Furthermore, the vector $\mathbf{1} = (1, ..., 1)^\intercal$ is an eigenvector of L with eigenvalue* 0*, i.e., $\mathbf{1} \in \ker(L)$.*

We now introduce the class of systems we are going to investigate in the present paper. Let us assign to each node of a graph, $i \in \mathcal{V}$, a state $x_i \in \mathbb{R}$ and a function

$$\begin{aligned} f_i : \mathbb{R} &\to \mathbb{R} \\ x_i &\mapsto f_i(x_i), \end{aligned} \tag{4}$$

which we call *response function.* So, given the set of nodes $\mathcal{V}$, we obtain a state vector $\boldsymbol{x} = (x_1, ..., x_n)^\intercal$, and a response vector field $F(\boldsymbol{x}) = (f_1(x_1), ..., f_n(x_n))^\intercal$. An Absolute Laplacian Flow [42, 3] is defined by the equation

$$\dot{\boldsymbol{x}} = -LF(\boldsymbol{x}). \tag{5}$$

Moreover, we say that an ALF is *homogeneous* if the response functions satisfy the equalities

$$f_1 = \cdots = f_n. \tag{6}$$

Remark 1. *Due to proposition 1, the Laplacian decomposes in a direct sum $L = L^{(1)} \oplus \cdots \oplus L^{(\mu_a(0))}$, and therefore an ALF decomposes in $\mu_a(0)$ indipendent systems. So, in general, we consider connected graphs.*

Note that if we choose the response vector field to be the identity, i.e., $F(\boldsymbol{x}) = \boldsymbol{x}$, then (5) becomes the widely studied (linear) Laplacian dynamics $\dot{\boldsymbol{x}} = -L\boldsymbol{x}$, [44, 43, 36]. Indeed, the ALF is a direct nonlinear generalisation of the linear Laplacian dynamics [42, 3, 1, 24]. ALFs have received considerably less attention than the linear Laplacian dynamics, although they still have strong relations with consensus problems. For example, ALFs have been considered as models for nonlinear communications protocols in [47, 34].

Our objective is to exploit the symmetry properties of ALFs, and characterise the near consensus behaviour for such systems under small perturbations.

2.1 General properties

ALFs have some generic properties deriving from the symmetric and degenerate structure of the Laplacian.

Proposition 2. *ALFs are invariant under the transformation of the response vector field*

$$F(\boldsymbol{x}) \mapsto F(\boldsymbol{x}) + h(\boldsymbol{x})\mathbf{1}, \tag{7}$$

where $h(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ *is a* k*-differentiable function,* $k \geq 1$*.*

Proof. Let $F^{(h)}(\boldsymbol{x}) = F(\boldsymbol{x}) + h(\boldsymbol{x})\mathbf{1}$. Since $\mathbf{1} \in \ker(L)$, we have $LF^{(h)}(\boldsymbol{x}) = LF(\boldsymbol{x})$. □

As a consequence of proposition 2 we have that the response vector field is not unique, in fact there exist an infinite set of vector fields leading to the same dynamics. This freedom can be used in practice to choose the "simplest" response field. Such property reminds in some way a gauge freedom [30], where one can choose among a class of response fields $F^{(h)}(\boldsymbol{x}) = F(\boldsymbol{x}) + h(\boldsymbol{x})\mathbf{1}$ without affecting the evolution equations (5).

Another important property of ALFs is the existence of a constant of motion.

Proposition 3. *Given an ALF* (5)*, there exists a constant of motion* k*,*

$$k = \langle \mathbf{1}, \boldsymbol{x} \rangle, \tag{8}$$

where $\langle \cdot, \cdot \rangle$ *is the usual Euclidean scalar product in* $\mathbb{R}^n$*.*

Proof. The time evolution of k is given by $\dot{k} = \langle \mathbf{1}, \dot{\boldsymbol{x}} \rangle$. Then we have $\langle \mathbf{1}, \dot{\boldsymbol{x}} \rangle = -\mathbf{1}^\intercal LF(\boldsymbol{x}) = -(L\mathbf{1})^\intercal F(\boldsymbol{x}) = 0$. So, k is a constant (in time) value. □

Let us notice that the constant of motion k is related to the arithmetic mean of the states, $k = n\langle \boldsymbol{x} \rangle$, where $\langle \boldsymbol{x} \rangle := \langle \mathbf{1}, \boldsymbol{x} \rangle / n$. So, we can interpret the presence of the constant of motion k as a conservation law for $\langle \boldsymbol{x} \rangle$. More generally, every quantity of the form ck, with c a constant, is a constant of motion for an ALF.

Proposition 4. *The set of equilibria of an ALF is given by*

$$E = \{\boldsymbol{x} \in \mathbb{R}^n \mid f_1(x_1) = \cdots = f_n(x_n)\}. \tag{9}$$

Proof. In order to have an equilibrium for (5) we need $F(\boldsymbol{x}) \in \ker(L)$, which means $f_1(x_1) = \cdots = f_n(x_n)$, implying in turn that $\boldsymbol{x}$ is an equilibrium if and only if $\boldsymbol{x} \in E$. □

Corollary 1. *Given a homogeneous ALF the set*

$$C = \{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 = \cdots = x_n\} \tag{10}$$

is a subset of equilibria of the system, i.e., $C \subseteq E$.

The set C is the so-called *consensus* space [25, 36, 39]. Notice that for heterogeneous ALFs we can expect clustering, for example, if $f_1 = f_2 = \cdots = f_{n/2} \neq f_{n/2+1} = \cdots = f_n$, then two clusters of equilibria arise. The present work is mainly concerned with homogeneous ALF, for which we have a global consensus space. Although this assumption might seem restrictive, we will see that the nonlinearity of the response function can lead to nontrivial structures of equilibria.

As we already mentioned, one of our goals is to characterise the behaviour near the consensus space (10). Our characterisation aims to be qualitative, in the sense of *topological equivalence* [22, 48].

Definition 3. *Two vector fields, X, Y, are said to be topologically equivalent if there exists a homeomorphism which takes orbits of X to orbits of Y, preserving directions but not necessarily time parametrisation.*

It is known that for linear Laplacian dynamics, systems associated with weighted graphs are topologically equivalent to the corresponding unweighted ones [7]. We show, for the nonlinear case, that weighted homogeneous ALFs are topologically equivalent to simple homogeneous ALFs near the consensus space.

Proposition 5. *Let the response function f be fixed. Then, in a neighbourhood of the consensus space, a homogeneous ALF with graph structure $\mathcal{G}_\mathrm{w}$ is topologically equivalent to an ALF with simple graph structure $\mathcal{G}$.*

Proof. Let L_w be the Laplacian of a weighted graph $\mathcal{G}_\mathrm{w}$. Then the Jacobian of the associated homogeneous ALF on the consensus space is given by $-L_\mathrm{w}\mathrm{D}F(\boldsymbol{x}^*)$, where $\mathrm{D}F(\boldsymbol{x}^*)$ is the Jacobian of $F(\boldsymbol{x})$ evaluated at C. By setting all the weights to one on $\mathcal{G}_\mathrm{w}$ we obtain the simple graph $\mathcal{G}$. So, for the homogeneous ALF with unweighted graph $\mathcal{G}$ the Jacobian is $-L\mathrm{D}F(\boldsymbol{x}^*)$, where L is the Laplacian matrix associated to $\mathcal{G}$.

Let us recall that the Laplacian matrix is a semi-positive definite matrix [4], which means that its eigenvalues are all greater than or equal to zero, moreover such property holds for both simple graphs and positive-weighted graphs. Then, the signs of the eigenvalues of L_w and L are the same, compactly we write $\mathrm{sgn}(\mathrm{spec}(L_\mathrm{w})) = \mathrm{sgn}(\mathrm{spec}(L)) \geq 0$. Now, since for $\boldsymbol{x}^* \in C$ it holds that $x_1^* = \cdots = x_n^* := x^*$, we have $\mathrm{D}F(\boldsymbol{x}^*) = \mathrm{Id}\,\mathrm{d}_x f(x^*)$, where $\mathrm{d}_x f$ is a short notation for the derivative of f with respect to x, and Id is the identity matrix. So, the eigenvalues of the ALFs weighted and unweighted, at the consensus space C, are given by $\mathrm{d}_x f(x^*)\,\mathrm{spec}(L_\mathrm{w})$ and $\mathrm{d}_x f(x^*)\mathrm{spec}(L)$ respectively. Let us first consider $\mathrm{d}_x f(x^*) \neq 0$. Since we are assuming connectedness, the Laplacians have one zero eigenvalue associated with the eigenvector $\mathbf{1}$ (independently of the weights), see proposition 1. The eigendirection associated with the zero eigenvalue points along the consensus space, which is a line of equilibria. So, along the nonhyperbolic eigendirection $\mathbf{1}$ the two systems are equivalent. Therefore considering a Lyapunov–Schmidt reduction, along the other directions the topological equivalence follows form Hartman-Grobman theorem [22]. It is clear that when $\mathrm{d}_x f(x^*) = 0$ the nonhyperbolic nature is entirely determined by $\mathrm{d}_x f(x^*)$, which is a common factor of the two systems. □

Having in mind proposition 5, from now on we assume that homogeneous ALFs have all weights equal to one, i.e., they are defined by a simple (unweighted) graph $\mathcal{G}$. In general, we also assume homogeneity of the response vector field without referring to it, but mentioning only the graph structure.

The results we stated until this point rely mainly on the algebraic properties of the Laplacian. For simple graphs it is particularly useful to exploit the symmetry properties induced by the graph, which in turn can provide insightful information on the ALF.

3 Symmetries

Before introducing the role of symmetries in ALFs, it is worth recalling some basic notions about group theory [38, 17].

Definition 4. *Let Γ be a set and $\star : \Gamma \to \Gamma$ a binary operation. A group is a couple $(\Gamma, \star)$ satisfying the following properties*

1. *Associativity:* $(\gamma_i \star \gamma_j) \star \gamma_k = \gamma_i \star (\gamma_j \star \gamma_k)$, $\forall \gamma_i, \gamma_j, \gamma_k \in \Gamma$.

2. Identity: $\exists\ e \in \Gamma$ *such that* $e \star \gamma = \gamma \star e = \gamma$, $\forall \gamma \in \Gamma$.

3. Inverse: $\forall \gamma \in \Gamma$, $\exists\ \gamma^{-1} \in \Gamma$ *such that* $\gamma^{-1} \star \gamma = \gamma \star \gamma^{-1} = e$.

The operation $\star$ is called group operation. Note that, closure under the group operation follows from the definition of the operation itself. Unless strictly necessary, we refer to a group just by the set, e.g., Γ, implicitly considering it equipped with a group operation; we also omit the group operation symbol, e.g., $\gamma_i \gamma_j = \gamma_i \star \gamma_j$.

Definition 5. *A group is said to be finite if the set* Γ *has a finite number of elements. The order of a finite group,* $\mathrm{Ord}(\Gamma)$*, is the number of elements in the group.*

Given a vector space, a representation, ψ, is a homomorphism from the group to the general linear group of the vector space. Since the states, $\boldsymbol{x}$, of an ALF belong to $\mathbb{R}^n$, we are interested on how Γ acts on $\mathbb{R}^n$, therefore we consider representations $\psi : \Gamma \to GL(\mathbb{R}^n)$. To avoid new unnecessary notation, we omit to explicitly write the representation. It is implicit that when we write a group element applied to a vector we are considering the representation on the appropriate vector space, e.g., $\gamma \boldsymbol{x} = \psi(\gamma)\boldsymbol{x}$.

Definition 6. *Given two groups* $\Gamma^{(1)}$*,* $\Gamma^{(2)}$*, the direct product* $\Gamma^{(1)} \times \Gamma^{(2)}$ *is a group with set of elements given by the Cartesian product*

$$\Gamma^{(1)} \times \Gamma^{(2)} = \left\{ (\gamma^{(1)}, \gamma^{(2)}) \mid \gamma^{(1)} \in \Gamma^{(1)}, \gamma^{(2)} \in \Gamma^{(2)} \right\}, \tag{11}$$

and operation defined as follows

$$(\gamma_i^{(1)}, \gamma_j^{(2)})(\gamma_k^{(1)}, \gamma_l^{(2)}) = (\gamma_i^{(1)}\gamma_k^{(1)}, \gamma_j^{(2)}\gamma_l^{(2)}), \tag{12}$$

$\forall \gamma_{i,k}^{(1)} \in \Gamma^{(1)}$ *and* $\forall \gamma_{j,l}^{(2)} \in \Gamma^{(2)}$.

It follows that the direct product can be extended to an arbitrary number of groups, $\Gamma^{(1)} \times \Gamma^{(2)} \times \cdots \times \Gamma^{(n)}$. Another important concept is the one of *fixed-point space*, i.e., the set of points that are kept fixed by the action of the group.

Definition 7. *Let* $\Sigma \subseteq \Gamma$ *be a subgroup of* Γ*. The fixed-point space (under the action in* $\mathbb{R}^n$*) of the group* Σ *is denoted by* $\mathrm{Fix}(\Sigma)$ *and it is given by*

$$\mathrm{Fix}(\Sigma) = \{ \boldsymbol{x} \in \mathbb{R}^n \mid \sigma \boldsymbol{x} = \boldsymbol{x},\ \forall \sigma \in \Sigma \}. \tag{13}$$

A direct interplay between graph theory and group theory is given by the graph's automorphism group. For graphs the automorphism group is a subgroup of the (finite dimensional) symmetric group $\mathfrak{S}_n$ [40], i.e., the group of permutations of n symbols.

Definition 8. *Given a (simple) graph $\mathcal{G}$, the automorphism group,* $\mathrm{Aut}(\mathcal{G})$*, is the group of permutations which preserves the adjacency structure of the vertices of $\mathcal{G}$.*

Remark 2. *From definition 8 follows that a graph $\mathcal{G}$ has* $\mathrm{Aut}(\mathcal{G}) = \Gamma$*, with $\Gamma \subseteq \mathfrak{S}_n$, if and only if $[A, \sigma]$, $\forall \sigma \in \Gamma$, where $[\cdot, \cdot]$ represent the commutator of matrices, i.e., $[A, \sigma] := A\sigma - \sigma A$. Consequently, the same statement holds if we substitute the adjacency matrix with the Laplacian matrix.*

In the following sections, we are going to look at how the group properties of response functions and graphs affect the equilibria, and the dynamics of ALFs.

3.1 Equilibria of homogeneous ALFs

We have proven that the consensus space is always a subset of equilibria for a homogeneous ALF. We now show that the group transformations leaving the response function invariant induce new equilibria from the consensus space.

Definition 9. *Let Γ be a group*[1]*. A function f is called* Γ*-invariant if*

$$f \circ \Gamma = f, \tag{14}$$

where $\circ$ denotes the composition of maps.

Proposition 6. *Let f be a Γ-invariant response function. Then, the group $\Gamma_n := \underbrace{\Gamma \times \cdots \times \Gamma}_{n\text{-times}}$ maps the consensus space C to equilibria, i.e.,*

$$\Gamma_n(C) \subseteq E. \tag{15}$$

Proof. Let us consider $-LF(\Gamma_n \boldsymbol{x}^*)$, where $\boldsymbol{x}^* \in C$. Notice that $\Gamma \boldsymbol{x}^* \neq \boldsymbol{x}^*$ in general, but thanks to homogeneity of the ALF and invariance of the response function, we have

[1]Notice that this is not related to $\mathrm{Aut}(\mathcal{G})$, we use the same symbol Γ for different groups in different context.

$-LF(\Gamma_n \boldsymbol{x}^*) = -LF(\boldsymbol{x}^*) = 0$, where in the last equality we used the fact that the consensus space is an equilibrium. Then we have that $\Gamma_n(C)$ is also a set of equilibria for the system. $\square$

Remark 3. *From Proposition 6 it follows that any subgroup of* Γ_n *maps* C *to equilibria.*

The invariance of the response function leads to a richer structure of the equilibria, similarly to what happens for equivariant bifurcations [20, 9]. Such a richer structure can give rise, for example, to *bipartite consensus* [49, 3, 31], where the agents converge to two separate clusters of consensus. A simple exemplification of appearance of bipartite consensus in ALFs is given by considering the response function $f(x) = x^2$. It is clear that the symmetry group for such a function is $\mathbb{Z}_2$, in particular we are considering the representation $\{1, -1\}$. From Proposition 6 we have that the set of equilibria consists of all the possible combinations $\pm x_1 = \pm x_2 = \cdots = \pm x_n$, and therefore we have bipartite consensus among the potential equilibria for the system. Let us notice that the linear set-up for bipartite consensus requires negative weights on the network structure in order to obtain such result, while for ALFs bipartite consensus follows form nonlinearity, and symmetry properties of the response function. It is interesting to notice that the lines of bipartite consensus, $\pm x_1 = \pm x_2 = \cdots = \pm x_n$, arising from generic even response functions are saddles; this property follows by straightforward computations of the Jacobian.

3.2 Equivariant ALFs

The concept of symmetry, in dynamical systems theory, refers to the fact that there exists a group mapping solutions to solutions. When a vector field is given, such a concept can be translated in terms of equivariance of the vector field.

Definition 10. *Let* Γ *be a group, and* X *a vector field.* X *is called* Γ*-equivariant if*

$$X \circ \Gamma = \Gamma \circ X, \tag{16}$$

and $\gamma \in \Gamma$ *is called a symmetry of the vector field.*

Remark 4. *It is straightforward to see that if a system is* Γ*-equivariant, then* Γ *maps solutions to solutions.*

A solution of a dynamical system is a simple example of an invariant set under the flow. Invariant sets, in general, are extremely relevant in the study of ODEs, since they allow to reduce the analysis to potentially simpler equations. We recall here the general definition of invariant set under the flow induced by a vector field. Later we state the interplay between symmetries and invariant sets.

Definition 11. *Let ϕ be the flow generated by a vector field on $\mathbb{R}^n$. A set $\mathcal{M} \subset \mathbb{R}^n$ is said to be invariant (respectively positively/negatively invariant) under the flow if for every $x \in \mathcal{M}$, $\phi_t(x) \in \mathcal{M}$, $\forall t \in \mathbb{R}$ (respectively $\forall t \in \mathbb{R}_+/\mathbb{R}_-$).*

In order to keep a compact notation, we call a set invariant under the flow a ϕ-invariant set. From equivariance of the vector field we can immediately infer the existence of a ϕ-invariant set.

Proposition 7. *The fixed-point space,* $\mathrm{Fix}(\Gamma)$*, is a ϕ-invariant set for Γ-equivariant systems [20].*

Since we are considering systems with a network structure, the symmetry group associated to the equivariance properties will be a subgroup of the symmetric group. Moreover, if an ALF is $\mathfrak{S}_n \supseteq \Gamma$-equivariant it follows that it is also equivariant under any subgroup $\Sigma \subset \Gamma$. Therefore, also $\mathrm{Fix}(\Sigma)$ is a ϕ-invariant set, and $\mathrm{Fix}(\Gamma) \subseteq \mathrm{Fix}(\Sigma)$. From such considerations, if we are considering a Γ-equivariant ALF, the analysis of the dynamics can be reduced to the fixed-point spaces $\mathrm{Fix}(\Sigma)$, for all Σ subgroups of Γ.

Let us now consider the particular case of homogeneous ALFs with a complete graph structure, we shortly call them K_n-ALFs. We notice that $\mathrm{Aut}(K_n) = \mathfrak{S}_n$. As a consequence, we show that K_n-ALFs are $\mathfrak{S}_n$-equivariant. Considering that the phase space of a K_n-ALF is $\mathbb{R}^n$, when acting on a state of a K_n-ALF we use the permutation representation of $\mathfrak{S}_n$ in $\mathbb{R}^n$. An element of the permutation representation is called permutation matrix.

Definition 12. *A permutation matrix $\sigma \in \mathfrak{S}_n$ is an $n \times n$ matrix with components*

$$\sigma_{ij} = \begin{cases} 1 & \text{if } \sigma \text{ permutes the } i\text{-th element with the } j\text{-th element}, \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Proposition 8. *Let L be the Laplacian matrix of a K_n graph. Then $[L, \sigma] = 0$ for all $\sigma \in \mathfrak{S}_n$, where $[\cdot, \cdot]$ is the commutator.*

Proof. The statement follows from the graph completeness and edges indistinguishability. □

Equivalently, thanks to orthogonality of permutations, proposition 8 can be written as $\sigma^{\intercal} L \sigma = L$; in order words, the Laplacian is invariant under conjugacy by permutations.

Proposition 9. *K_n-ALFs are $\mathfrak{S}_n$-equivariant.*

Proof. We need to prove that $LF(\sigma \boldsymbol{x}) = \sigma LF(\boldsymbol{x})$, for all $\sigma \in \mathfrak{S}_n$. Let us start by considering $\sigma LF(\boldsymbol{x})$. Thanks to proposition 8 we have that $\sigma LF(\boldsymbol{x}) = L\sigma F(\boldsymbol{x})$. Next, the permutation applied to the vector field $F(\boldsymbol{x}) = (f(x_1), \dots, f(x_n))^{\intercal}$ exchanges the order of the components; since the functions are indistinguishable, we have that $\sigma F(\boldsymbol{x}) = F(\sigma \boldsymbol{x})$. Therefore, $\sigma LF(\boldsymbol{x}) = LF(\sigma \boldsymbol{x})$, which proves the statement. □

Remark 5. *In virtue of homogeneity of the response functions, the action of $\mathfrak{S}_n$ on the (non-linear) vector field $F(\boldsymbol{x})$ is linear.*

Proposition 10. *The fixed-point space of the symmetric group is the consensus space, i.e.,* $\mathrm{Fix}(\mathfrak{S}_n) = C$.

Proof. We proceed by contradiction. Suppose that $\boldsymbol{x} \notin C$, and $\boldsymbol{x} \in \mathrm{Fix}(\mathfrak{S}_n)$. From definition 7, we have $x_{\sigma(i)} = x_i$, $i = 1, \dots, n$, $\forall \sigma \in \mathfrak{S}_n$. From the fact that $\mathrm{Ord}(\mathfrak{S}_n) = n! \geq n$, we obtain $x_1 = \cdots = x_n$, i.e., $\boldsymbol{x} \in C$ which is a contradiction. □

We established a link between K_n-ALFs and equivariant systems. Let us notice that, if we have a dynamical system equivariant under a finite group, such dynamical system can be seen as a *cell-network* [19]. A cell-network is a system of differential equations equivariant under a group(oid), where the group(oid) is the automorphism group(oid) of a (multi)graph. So, there is a strict correspondence between cell-networks and the ALF framework. For example, a K_n-ALF is a cell-network with K_n graph structure. This alternative point of view, provides an extra tool in the analysis of ALFs. Indeed, from proposition 7 follows that $\mathrm{Fix}(\mathfrak{S}_n)$ is ϕ-invariant, and so, in turn, the consensus space, is an invariant space for the K_n-ALFs, or in the language of cell-networks it is a pattern of synchrony, which is already known from corollary 1. Actually, corollary 1, not only holds for all homogeneous ALFs, but also tells us that the consensus space is an equilibrium for the system, not just an invariant set. However, proposition 10 does not rely on the algebraic structure of the Laplacian, it is a general result for $\mathfrak{S}_n$-equivariant systems. Let us notice that, the results following from equivariance are, in general, more robust than the algebraic ones. Therefore, the symmetry perspective could be very useful when some transformation is applied to the system.

In general, a simple (connected) graph with n nodes can be regarded as a subgraph of K_n. As such, a generic simple graph has a subgroup of $\mathfrak{S}_n$ as automorphism group. So, a question that we can ask is: what are the subgroups of $\mathfrak{S}_n$ having C as a fixed-point space?

Proposition 11. *Let* Γ *be a subgroup of* $\mathfrak{S}_n$*, if* $\mathrm{Ord}(\Gamma) \geq n$ *then* $\mathrm{Fix}(\Gamma) = C$*.*

Proof. If $\mathrm{Ord}(\Gamma) \geq n$, then by considering the equations $\Gamma \boldsymbol{x} = \boldsymbol{x}$, we have at least n independent equalities that constrain all the components of $\boldsymbol{x}$ to be equal. Therefore $\mathrm{Fix}(\Gamma) = C$. □

When $\mathrm{Ord}(\Gamma) < n$ then $\dim \mathrm{Fix}(\Gamma) > 1$, so the fixed-point space is related to clustering, and $C \subset \mathrm{Fix}(\Gamma)$; we illustrate such a case in the following second example. Let us first consider the cycle graph C_n, for which $\mathrm{Aut}(C_n) = \mathbb{Z}_n$, where $\mathbb{Z}_n$ is the cyclic group. Since $\mathrm{Ord}(\mathbb{Z}_n) = n$, then $\mathrm{Fix}(\mathbb{Z}_n) = C$; this can be practically seen by applying a generator of $\mathbb{Z}_n$ to $\boldsymbol{x}$, e.g.,

$$x_1 = x_n \tag{18}$$

$$x_2 = x_1 \tag{19}$$

$$\vdots \tag{20}$$

$$x_n = x_{n-1}, \tag{21}$$

implying $x_1 = \cdots = x_n$. As a second example, we consider the path graph P_n ($n \geq 2$) which has $\mathbb{Z}_2$ as group of symmetry. In this case $\mathrm{Ord}(\mathbb{Z}_2) = 2$, therefore P_n has the consensus as fixed-point space only if $n = 2$. In fact, if we consider the generic case we can see that the equalities imposed by $\mathbb{Z}_2 \boldsymbol{x} = \boldsymbol{x}$ are

$$x_1 = x_n \tag{22}$$

$$x_2 = x_{n-1} \tag{23}$$

$$\vdots \tag{24}$$

$$x_{n-1} = x_2 \tag{25}$$

$$x_n = x_1, \tag{26}$$

which do not lead to consensus in general, but to clustering $\{\boldsymbol{x} \in \mathbb{R}^n \mid x_1 = x_n, \ldots, x_n = x_1\}$.

In summary, in this section we outlined a different perspective on the consensus space. Let us recall that a homogeneous ALF, near the consensus space, is equivalent to an ALF with simple graph structure (proposition 5). So, the ALF acquires the property of equivariance under the automorphism group of the (simple) graph; the symmetry group is a subgroup of the symmetric group. Therefore, an ALF possesses also a cell-network structure. At this point, forgetting momentarily the specific properties of ALFs, we can study the connections between the consensus space and the symmetry properties. It turns out that the consensus space arises as an invariant space for a class of subgroups of $\mathfrak{S}_n$ (proposition 11).

We conclude this section recalling a theorem highlighting the fundamental role of the graph structure in the definition of a dynamic network. Although, as we have seen, graphs and groups are intimately related, if we consider the logic flow of definitions for dynamic networks the graph structure should be defined first.

Theorem 1 (Frucht [16])**.** *For any finite group Γ there exists a finite graph $\mathcal{G}$ such that* $\mathrm{Aut}(\mathcal{G}) \cong \Gamma$.

Theorem 1, ensures that for any given finite group we can construct a graph which is invariant under the group. However, uniqueness is not provided, au contraire, in general there are infinitely many graphs with such property [6]. For this reason, the definition of the graph structure of a dynamic network precedes the group properties.

4 Perturbations

A perturbation is a small 'change' of the differential equations governing the system. The order of magnitude can be quantified by a parameter ϵ, $0 < \epsilon \ll 1$. A perturbation is called *singular* if the solutions of the perturbed ODEs have a different qualitative behaviour with respect to the solutions of the unperturbed system [46]. On the other hand, *regular* perturbations are the ones that preserve the qualitative structure. For this reason, singular perturbations are the most interesting if we aim to characterise qualitatively a system. In appendix A, we briefly review some of the standard terminology of singular perturbations.

4.1 Perturbed ALFs

As already mentioned in the introduction, we now consider perturbations of ALFs given by:

$$\dot{\boldsymbol{x}} = -LF(\boldsymbol{x}) + \epsilon H(\boldsymbol{x}, \Lambda), \tag{27}$$

where $0 < \epsilon \ll 1$, $H(\boldsymbol{x}, \Lambda) = (h_1(\boldsymbol{x}, \Lambda), \ldots, h_n(\boldsymbol{x}, \Lambda))^\intercal$, and Λ is a set of parameters.

Since we are considering homogeneous ALF, the consensus space C is always an set of equilibrium points for $\epsilon = 0$ (corollary 1). Moreover, the consensus set is a one-dimensional space, then generic perturbations are singular [46]. The system as written in (27) is in a non-standard form. In many cases, it is useful to transform a singularly perturbed system into a standard form, one reason among all is that a large amount of results in singular perturbation theory are stated for standard forms.

Proposition 12. *Given a perturbed ALF in the non-standard form* (27)*, then the coordinate transformation*

$$x_l\left(k, x_{j\neq l}\right) = k - \sum_{j\neq l} x_j, \tag{28}$$

where $l \in \{1, \ldots, n\}$ *is arbitrary, and* k *is the constant of motion* (8)*, transforms the system into the standard form*

$$\begin{aligned} \dot{x}_i &= -\sum_{j\neq l} L_{ij} f(x_j) - L_{il} f\left(x_l\left(k, x_{j\neq l}\right)\right) + \epsilon g_i(x_{j\neq l}, k, \Lambda) \quad i \neq l \\ \dot{k} &= \epsilon \sum_j g_j(x_{j\neq l}, k, \Lambda), \end{aligned} \tag{29}$$

where $g_i(x_{j\neq l}, k, \Lambda) := h_i\left(x_1, \ldots, x_l\left(k, x_{j\neq l}\right), \ldots, x_n, \Lambda\right)$.

Proof. Performing the coordinate change (28) into (27) we obtain the set of equations for x_i, $i \neq l$. Then, the equation for k is retrieved evaluating the scalar product $\langle \mathbf{1}, \dot{\boldsymbol{x}} \rangle$. □

Let us notice that the perturbation transforms the constant of motion into a slow variable of the system. Since the constant k is related to the arithmetic mean of the states, $\langle \boldsymbol{x} \rangle$, perturbing the system, in general, means that the states' density is changing, e.g., the concentration of the system is increasing/decreasing. In other words, the perturbation acts as a drift/dissipation in the undergoing diffusive evolution.

Lemma 1. *If* $\langle \mathbf{1}, H(\boldsymbol{x}, \Lambda) \rangle = 0$*, then equation* (27) *is a regular perturbation problem.*

Proof. If $\langle \mathbf{1}, H(\boldsymbol{x}, \Lambda) \rangle = 0$ then $\dot{k} = 0$. So, the standard form (29) is no more a slow-fast system, and therefore the perturbation is regular. □

Remark 6. *Notice that lemma 1 holds also for heterogeneous ALFs.*

As we already mentioned, we are interested in the singular perturbations because they are the ones that could lead to a different qualitative behaviour of the system. In the following section, we look at the near consensus behaviour under a singular perturbation, and we characterise the dynamics for a selected class of systems.

5 Consensus dynamics

In this section, we further specialise our analysis by studying the consensus dynamics of (homogeneous) ALFs with a K_n graph structure, already introduced in section 3.2 for their symmetry properties.

5.1 The layer problem

Let us consider a perturbed K_n-ALF in the standard form. The layer problem, obtained from (29) by setting ϵ to zero, reads

$$\dot{x}_i = -\left[\sum_{j\neq l} L_{ij} f(x_j) + L_{il} f\left(x_l\left(k, x_{j\neq l}\right)\right)\right] \quad i \neq l. \tag{30}$$

The system described by (30) is $(n-1)$-dimensional with one parameter, k.

It is worth noting that although the graph structure in (30) seems to be lost, some information is still readily available. On the one hand, the term $\sum_{j\neq l} L_{ij} f(x_j)$ still corresponds to a K_{n-1}-ALF, i.e., on a complete graph with $n-1$ nodes. On the other hand, the scalar term $L_{il} f(x_l)$, where in fact x_l is a function of all other nodes and the constant k, can be interpreted as a higher-order interaction [5] naturally arising from the reduction performed.

Furthermore, and for our specific purposes, there is a more relevant property that survives the transformation: the symmetry. In fact, starting with a $\mathfrak{S}_n$-equivariant unperturbed system, after the transformation into the standard form we get a layer problem which is $\mathfrak{S}_{n-1}$-equivariant. This is a first example of robustness of the symmetries. Since we are interested in the consensus dynamics, we consider equation (30) on the invariant space given by $\mathrm{Fix}(\mathfrak{S}_{n-1})$. Let us notice that, in general, the invariant space $\mathrm{Fix}(\mathfrak{S}_{n-1})$ is not globally stable. For example, in a neighbourhood of an attracting branch of the consensus space it is stable. Instead, in a neighbourhood of a repelling part of the consensus space it is unstable. This means that the restriction to $\mathrm{Fix}(\mathfrak{S}_{n-1})$ provides a robust description of the system only in the basin of attraction of a stable region of $\mathrm{Fix}(\mathfrak{S}_{n-1})$; in the unstable regions the restriction works only if the initial conditions lie exactly in the invariant space and if the perturbation preserves $\mathrm{Fix}(\mathfrak{S}_{n-1})$. Then, restricting equation (30) on $\mathrm{Fix}(\mathfrak{S}_{n-1})$ we obtain

$$\dot{x} = -\left[f(x) - f(k-(n-1)x)\right], \tag{31}$$

where we used the fact that for a K_n-graph $L_{ii} = n-1$, and $L_{i\neq j} = -1$. So, in the invariant space the system reduces to a one dimensional system (with one parameter). The former consensus space, $C = \mathrm{Fix}(\mathfrak{S}_n)$, gives the equilibria of (31). This is a consequence of the fact that the layer problem is a coordinate transformation of the unperturbed non-standard form (5), for which we know the consensus space is a set of equilibria.

Proposition 13. *The consensus states of the form*

$$x_k^* = \frac{k}{n} \tag{32}$$

are equilibria of (31).

Proof. Inserting x_k^* into (31) we obtain $\dot{x} = 0$, which proves that x_k^* is an equilibrium. To prove that it correspond to consensus, we recall that, since we are in the invariant space $\mathrm{Fix}(\mathfrak{S}_{n-1})$, then $n-1$ states are already equal to each other. The only state left to check is the one given by (28), and by inserting $x_{j\neq l} = k/n$, we obtain $x_l = k/n$, which proves the statement. □

Remark 7. *Recalling that $k = n\langle \boldsymbol{x} \rangle$, we have that the equilibrium is exactly the mean value of the states of the system, $\langle \boldsymbol{x} \rangle$.*

Since the equilibria of the layer equation constitute the critical manifold for the slow-fast system, then the consensus space is a (part of the) critical manifold of the system. We use the notation $\boldsymbol{x}_k^* := (x_k^*, \ldots, x_k^*)$ for the points in C having the form (32).

Proposition 14. *The stability of the consensus* (32) *is determined by the derivative of the response function. In particular, we can decompose the consensus into three components,*

$$\begin{aligned} C^a &= \{\boldsymbol{x}_k^* \in C \mid \mathrm{d}_x f(x_k^*) > 0\}, \\ C^r &= \{\boldsymbol{x}_k^* \in C \mid \mathrm{d}_x f(x_k^*) < 0\}, \\ C^s &= \{\boldsymbol{x}_k^* \in C \mid \mathrm{d}_x f(x_k^*) = 0\}, \end{aligned} \tag{33}$$

where C^a is attracting (stable), C^r is repelling (unstable), and C^s is singular.

Proof. By straightforward computations we obtain the Jacobian of the system evaluated at the consensus, $-\mathrm{d}_x f\left(\frac{k}{n}\right) n$, which implies the proposition. □

Thanks to theorem 3 (Fenichel), the regions $C^{a/r}$ are diffeomorphic to invariant regions of the singularly perturbed system (29). For the singularities C^s we need further analyses. We are going to consider the case of *transcritical* singularities which, as we will show, turn out to be generic due to the geometric properties of the problem.

5.2 The slow-fast dynamics around the consensus space

We start by considering some symmetry condition on the perturbation, in order to preserve the restriction to the invariant space.

Lemma 2. *Let $H(\boldsymbol{x}, \Lambda)$ be a perturbation such that $h_i = h_j := h$, $\forall i, j \neq l$, and $h_l =: \tilde{h}$. Then,* $\mathrm{Fix}(\mathfrak{S}_{n-1})$ *is an invariant space for the perturbed system* (29).

Assuming a perturbation satisfying lemma 2, we are allowed to continue our analysis on the invariant space $\mathrm{Fix}(\mathfrak{S}_{n-1})$. Indeed, under the chosen perturbation the perturbed system (29) becomes $\mathfrak{S}_{n-1}$-equivariant, preserving the invariant space $\mathrm{Fix}(\mathfrak{S}_{n-1})$. We remark once again that, in a neighbourhood of a stable regions of $\mathrm{Fix}(\mathfrak{S}_{n-1})$, the reduction is robust; while in unstable regions the reduction is preserved only by fine tuning of initial conditions and perturbation. For the perturbed system the invariant space reads $\mathrm{Fix}(\mathfrak{S}_{n-1}) = \{(x,k) \in \mathbb{R}^2 \mid x := x_i = x_j,\ \forall i,j \neq l\}$, which we call (x,k)-plane. Then, the perturbed equations (29) evaluated on the (x,k)-plane, take the form

$$\begin{aligned} \dot{x} &= -\left[f(x) - f(k-(n-1)x)\right] + \epsilon g(x,k,\Lambda) \\ \dot{k} &= \epsilon(n-1)g(x,k,\Lambda) + \epsilon\tilde{g}(x,k,\Lambda), \end{aligned} \tag{34}$$

where $g(x,k,\Lambda) := h(x,\ldots,k-(n-1)x,\ldots,x,\Lambda)$, and $\tilde{g}(x,k,\Lambda) := \tilde{h}(x,\ldots,k-(n-1)x,\ldots,x,\Lambda)$.

We now state the conditions for (nondegenerate) transcritical singularities.

Proposition 15. *Let $\boldsymbol{x}_k^s \in C^s$ be a singular consensus point, and assume that lemma 2 holds. If*

$$\mathrm{d}_x^2 f(x_k^s) \neq 0, \tag{35}$$

$$n \neq 2, \tag{36}$$

$$\langle \mathbf{1}, H(\boldsymbol{x}_k^s,\Lambda)\rangle \neq 0, \tag{37}$$

then $\boldsymbol{x}_k^s$ is a transcritical singularity of (34)

Proof. We check the conditions for transcritical singularities on planar systems provided by Krupa and Szmolyan [27]. Given a planar system of the form (67) then the origin is a transcritical singularity if

$$f(0,0,0)=0, \tag{38}$$

$$\partial_x f(0,0,0)=0, \tag{39}$$

$$\partial_y f(0,0,0)=0, \tag{40}$$

$$\begin{vmatrix} \partial_x^2 f(0,0,0) & \partial_{xy}^2 f(0,0,0) \\ \partial_{yx}^2 f(0,0,0) & \partial_y^2 f(0,0,0) \end{vmatrix} < 0, \tag{41}$$

$$\partial_x^2 f(0,0,0) \neq 0, \tag{42}$$

$$g_0 := g(0,0,0) \neq 0. \tag{43}$$

We now prove that these conditions applied to our system lead to the conditions of the statement. Conditions (38), (39), (40) are satisfied because x_k^s is a singular consensus point. For the non-degeneracy condition (41), we obtain

$$\mathrm{d}_x^2 f(x_k^s) \neq 0. \tag{44}$$

The transversality condition (42) gives,

$$n(n-2)\mathrm{d}_x^2 f(x_k^s) \neq 0, \tag{45}$$

which implies $n \neq 2$. Finally, the last condition (43) leads to $\langle \mathbf{1}, H(\boldsymbol{x}_k^s, \Lambda)\rangle \neq 0$. □

As previously mentioned, we consider the case where all the singularities on the consensus space are transcritical, i.e., the conditions of proposition 15 are met on C^s. Notice, however, that at least the first two conditions of proposition 15 are rather mild, we in fact argue about genericity later. The first says that the singularity is non-degenerate, while the second refers to networks of at least three nodes. So, in our current setting, the consensus space consists of a cascade of transcritical singularities changing the stability from attracting to repelling, and vice-versa. If at x_k^s the stability transition is from attracting to repelling, we call x_k^s: *type-1* *transcritical point.* Type-1 transcritical points are characterised by the sign ratio

$$\rho := \frac{\operatorname{sgn}\left(\mathrm{d}_x^2 f(x_k^s)\right)}{\operatorname{sgn}\left(\langle \mathbf{1}, H(\boldsymbol{x}_k^s, \Lambda)\rangle\right)} = -1. \tag{46}$$

If at x_k^s the stability transition is from repelling to attracting, we call x_k^s: *type-2 transcritical point.* Type-2 transcritical points are characterised by the sign ratio $\rho = 1$.

At this point, we are going to study the continuation of the consensus space after a transcritical point. A trajectory reaching a transcritical point could proceed in three different manners:

1. Being attracted to the stable branch of the critical manifold (a curve of equilibria crossing the consensus),
2. Being repelled along the fast flow,
3. Continuing along the consensus space (even if it is unstable).

The third characteristic behaviour is called *canard*, and acts as a separating case between the other two. Usually, the presence of a canard in a planar system is associated to a critical value for the parameters of the system. We are going to show that for ALFs the condition for canards is geometrical in nature, and can be stated as an algebraic condition on the perturbation, without specific requirements on the system's parameters.

Proposition 16. *Let $\boldsymbol{x}_k^s \in C^s$, if the perturbation satisfies the property*

$$H(\boldsymbol{x}_k^s, \Lambda) \propto \mathbf{1}, \tag{47}$$

then, for ϵ sufficiently small, the system (29) *admits canard solutions.*

We call *critical*, $H^{\mathrm{crit}}(\boldsymbol{x}, \Lambda)$, a perturbation satisfying such a property.

Proof. We use again the results from [27] to check that our statement holds. Considering a system of the form (67), satisfying conditions (38)-(43), then there exist a parameter λ controlling the behaviour at the transcritical singularity (which we remind to be at the origin for this benchmark system). The parameter λ is given by

$$\lambda = \frac{1}{|g_0|\sqrt{\beta^2 - \gamma\alpha}}(\delta\alpha + g_0\beta), \tag{48}$$

where

$$\alpha = \frac{1}{2}\partial_x^2 f(0,0,0), \qquad \beta = \frac{1}{2}\partial_{xy}^2 f(0,0,0),$$
$$\gamma = \frac{1}{2}\partial_y^2 f(0,0,0), \qquad \delta = \partial_\epsilon f(0,0,0).$$

So, the parameter λ for the system (34), at a transcritical point x_k^s, reads

$$\lambda = -\rho\frac{h(\boldsymbol{x}_k^s,\Lambda) + (n-1)\tilde{h}(\boldsymbol{x}_k^s,\Lambda)}{\tilde{h}(\boldsymbol{x}_k^s,\Lambda) + (n-1)h(\boldsymbol{x}_k^s,\Lambda)}. \tag{49}$$

For type-1 transcritical points, we have $\rho = -1$, and the condition for canards is $\lambda = \lambda^{\text{crit}} := 1$. So, we obtain the equation

$$(n-2)\tilde{h}(\boldsymbol{x}_k^s,\Lambda) = (n-2)h(\boldsymbol{x}_k^s,\Lambda). \tag{50}$$

Since the case $n = 2$ is excluded by proposition 15, we have $\tilde{h}(\boldsymbol{x}_k^s,\Lambda) = h(\boldsymbol{x}_k^s,\Lambda)$. Therefore the critical perturbation $H^{\text{crit}}(\boldsymbol{x},\Lambda)$ at $\boldsymbol{x}_k^s$ reads

$$\begin{aligned} H^{\text{crit}}(\boldsymbol{x}_k^s,\Lambda) &= (h(\boldsymbol{x}_k^s,\Lambda),\dots,h(\boldsymbol{x}_k^s,\Lambda),\dots,h(\boldsymbol{x}_k^s,\Lambda))^\intercal \\ &= h(\boldsymbol{x}_k^s,\Lambda)\mathbf{1}, \end{aligned}$$

which agrees with the statement. In order to complete the proof, we show that for a type-2 transcritical point a critical perturbation admits faux-canards. Let us recall that faux-canards for planar systems are generic, and appear for $\lambda < \lambda^{\text{crit}}$ [27]. By using the fact that for a type-2 transcritical point $\rho = 1$, we get $\lambda = -1$. □

In figure 1 we sketch the blow-up of the transcritical points under a critical perturbation [27, 32, 26]. The blow-up picture makes clear the different behaviour of the type-1/2 transcritical points and illustrates how the continuation of the consensus space occurs.

Let us notice that the critical perturbation preserves the full symmetry of the system on the consensus space, i.e., $H^{\text{crit}}(\boldsymbol{x}_k^s,\Lambda)$ is $\mathfrak{S}_n$-equivariant. Indeed, a similar result can be stated in terms of equivariant dynamical systems theory.

Theorem 2. *Let $\mathcal{G}$ be a graph with $\text{Aut}(\mathcal{G}) = \Gamma \subseteq \mathfrak{S}_n$, where $\text{Ord}(\Gamma) \geq n$. Let $H_\Gamma(\boldsymbol{x},\Lambda)$ be*

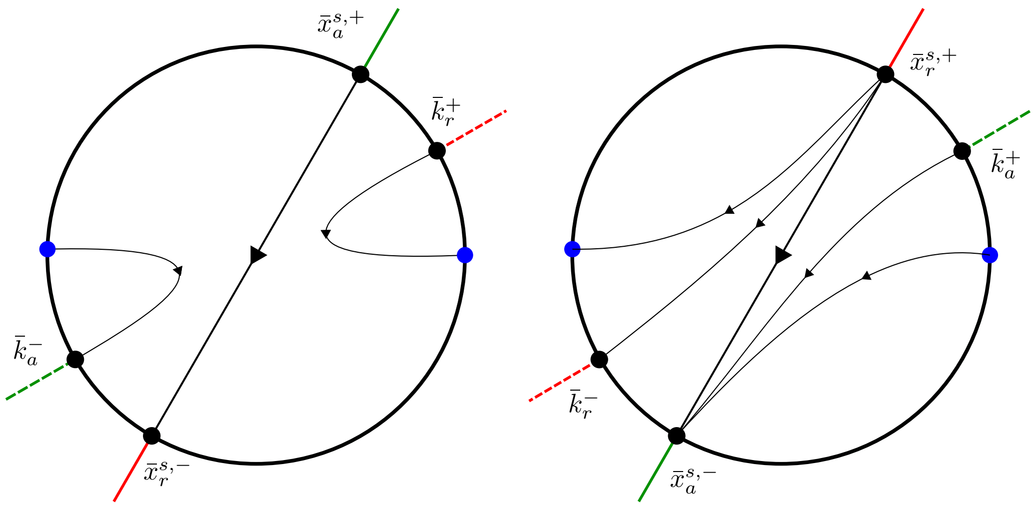


Figure 1: Sketch of blown-up transcritical points on the consensus space.
The picture displays typical blow-up phase portraits for a type-1 transcritical point (left), and a type-2 transcritical point (right); in both cases we are considering a critical perturbation. For a detailed exposition on the blow-up phase portraits for transcritical singularities see [27, 32]. The points $\bar{x}_{a/r}^{s,-/+}$ correspond to the singularity on the consensus space (solid lines), while the points $\bar{k}_{a/r}^{-/+}$ correspond to the singularity on the branch of the critical manifold crossing the consensus space (dashed lines). The labels $-/+$ indicate if we are below, or above, the transcritical point, and the labels a/r stand for attracting, and repelling, respectively. The attracting curves of equilibria are shown in green, while the repelling ones in red. The blue points are entering or exit points depending on the direction of the curves drawn.

a Γ-equivariant perturbation of a $\mathcal{G}$-ALF, such that $H_\Gamma(\mathrm{Fix}(\Gamma), \Lambda) \neq 0$. Then the consensus space is a trajectory of the perturbed system.

Proof. From proposition 11 we have that $\mathrm{Fix}(\Gamma) = C$. So, projecting equation (27) onto the invariant space C we have

$$\dot{\boldsymbol{x}}^* = \epsilon H_\Gamma(\boldsymbol{x}^*, \Lambda), \tag{51}$$

$\boldsymbol{x}^* \in C$. Since $H_\Gamma(\boldsymbol{x}^*, \Lambda) \neq 0$, $\forall \boldsymbol{x}^* \in C$, and $\dim(C) = 1$, then C is a trajectory for the system. □

In essence, theorem 2 gives some conditions under which a perturbed equivariant ALF (27) exhibits a maximal canard. Let us also notice that theorem 2 holds for a class of Γ-equivariant ALFs where $\mathrm{Ord}(\Gamma) \geq n$, and it does not make any particular assumption on the singularities of the consensus space. This means that in possibly more complicated cases where the singularities along the consensus space are not transcritical, the consensus space is still a maximal canard under the appropriate perturbation.

5.3 Polynomial response functions

Until this point, we considered a generic response function $f(x)$. Let us now consider a more specific class of functions: polynomials. A polynomial response function can be written as

$$f(x) = \sum_{j=0}^{N} a_j x^j, \tag{52}$$

where a_j are real constant coefficients, and $N \geq 2$. We know that the consensus space, for a homogeneous ALF, does not depend on the particular choice of the response function. On the other hand, the remaining equilibrium sets, that together with C give rise to the full critical manifold, depend on the form of the response function. By choosing a polynomial form for f, we are able to study in further detail the nature of the equilibria crossing the consensus space. We focus our analysis on a neighbourhood of the consensus space. So, in order to have a local picture, it is sufficient to understand the shape of the linear approximation of the intersecting branches of equilibria.

Proposition 17. *For a polynomial response function* (52)*, in the* (x,k)*-plane* (34)*, the branches of the critical manifold intersecting the consensus space locally have the form*

$$\kappa(x) = 2x_k^s + (n-2)x. \tag{53}$$

Proof. Let us consider the function $\tilde{\phi}(x,y) := -[f(x) - f(y+x)]$ and the auxiliary system

$$\begin{aligned} \dot{x} &= \tilde{\phi}(x,y) \\ \dot{y} &= 0, \end{aligned} \tag{54}$$

where y acts as a parameter. We are going to evaluate the slope of the tangents of the curves of zeros of $\tilde{\phi}(x,y)$ intersecting the line $y=0$. Considering a polynomial response function (52), we expand $f(x+y)$ by using the binomial theorem,

$$\begin{aligned} f(y+x) &= \sum_{j=0}^{N} a_j (y+x)^j \\ &= \sum_{j=0}^{N} a_j x^j + \sum_{j=0}^{N} \sum_{k=1}^{j} \binom{j}{k} a_j y^k x^{j-k}. \end{aligned}$$

So, we can rewrite $\tilde{\phi}(x,y)$ as

$$\tilde{\phi}(x,y) = y \sum_{j=0}^{N} \sum_{l=0}^{j-1} \binom{j}{l+1} a_j y^l x^{j-l-1}, \tag{55}$$

where we factored out y, and set $l = k-1$. We regularise $\tilde{\phi}(x,y)$ by removing the line $y=0$ from the set of zeros, i.e. $\tilde{\phi}(x,y) =: y\tilde{\phi}^{\text{reg}}(x,y)$, where

$$\tilde{\phi}^{\text{reg}}(x,y) = \sum_{j=0}^{N} \sum_{l=0}^{j-1} \binom{j}{l+1} a_j y^l x^{j-l-1}. \tag{56}$$

Then, the regularized auxiliary system reads

$$\begin{aligned} \dot{x} &= \tilde{\phi}^{\text{reg}}(x,y) \\ \dot{y} &= 0. \end{aligned} \tag{57}$$

Let us compute the Jacobian, J, for the regularised auxiliary system,

$$J = \begin{pmatrix} \frac{\partial \tilde{\phi}^{\text{reg}}}{\partial x} & \frac{\partial \tilde{\phi}^{\text{reg}}}{\partial y} \\ 0 & 0 \end{pmatrix}, \tag{58}$$

where

$$\frac{\partial \tilde{\phi}^{\text{reg}}}{\partial x} = \sum_{j=0}^{N} \sum_{l=0}^{j-1} \binom{j}{l+1} a_j (j-l-1) y^l x^{j-l-2}, \tag{59}$$

$$\frac{\partial \tilde{\phi}^{\text{reg}}}{\partial y} = \sum_{j=0}^{N} \sum_{l=0}^{j-1} \binom{j}{l+1} a_j l y^{l-1} x^{j-l-1}. \tag{60}$$

Since we are interested on what happens at the intersection with $y = 0$, we evaluate the Jacobian on that line, namely

$$J_0 := J\big|_{y=0} = \sum_{j=0}^{N} j(j-1) a_j x^{j-2} \begin{pmatrix} 1 & \frac{1}{2} \\ 0 & 0 \end{pmatrix}. \tag{61}$$

The eigenvalues of J_0 are determined by the equation $\det(J_0 - \mu\,\text{Id}) = 0$, which in our case reads

$$-\mu \left[\sum_{j=0}^{N} j(j-1) a_j x^{j-2} - \mu \right] = 0, \tag{62}$$

and so the eigenvalues are

$$\mu_1 = 0, \tag{63}$$

$$\mu_2 = \sum_{j=0}^{N} j(j-1) a_j x^{j-2}. \tag{64}$$

Let us notice that, in general, for the regularised auxiliary system the Jacobian J_0 cannot be interpreted as the linearisation of the system, because $y = 0$ is no more an equilibrium. However, there could be other equilibria on the line $y = 0$ at the intersections with the other curves of equilibria. At such points the eigenvector with zero eigenvalue gives the direction of the tangent of the intersecting curve of zeros. The eigenvector $\tilde{v}^{(\mu_1)} = (\tilde{v}_x^{(\mu_1)}, \tilde{v}_y^{(\mu_1)})^\intercal$ must satisfy the equation $J_0\tilde{v}^{(\mu_1)} = 0$, that is

$$\sum_{j=0}^{N} j(j-1)a_j x^{j-2}(\tilde{v}_x^{(\mu_1)} + \frac{1}{2}\tilde{v}_y^{(\mu_1)}) = 0.$$

Therefore $\tilde{v}_x^{(\mu_1)} = -\frac{1}{2}\tilde{v}_y^{(\mu_1)}$, and so $\tilde{v}^{(\mu_1)} \propto (1,-2)^\intercal$. Notice that $\tilde{v}^{(\mu_1)}$ does not depend on x, that means wherever the intersection takes place, then the tangent will have the direction of $\tilde{v}^{(\mu_1)}$. At this point, we need to obtain the equivalent vector for the system of our interest (31). Note that the equation $y = k - nx$ puts in relation the auxiliary system with the layer problem (31). Thanks to this relation we can construct the transformation connecting the tangent vectors of the two systems, in particular we have

$$\Pi := \begin{pmatrix} \frac{\partial x(x,k)}{\partial x} & \frac{\partial x(x,k)}{\partial k} \\ \frac{\partial y(x,k)}{\partial x} & \frac{\partial y(x,k)}{\partial k} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -n & 1 \end{pmatrix},$$

and then the inverse transformation reads

$$\Pi^{-1} = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}. \tag{65}$$

So, the tangents of the critical manifold at the intersections with the consensus space, in the original system, have direction given by $v_{\mu_1} = \Pi^{-1}\tilde{v}_{\mu_1} \propto (1, n-2)$, which proves that the slope of the tangent in the (x,k)-plane is $n-2$. Therefore, the tangent lines have an equation of the form $\kappa(x) = \tilde{\kappa} + (n-2)x$. In order to find $\tilde{\kappa}$, we use the fact that $\kappa(x_k^s) = nx_k^s$, obtaining $\tilde{\kappa} = 2x_k^s$; the statement is proven. A sketch of the result of this proposition is shown in figure 2. □

Proposition 17 tells us that, up to linear approximation, all other branches of equilibria of the critical manifold crossing the consensus space are parallel to each other and are in fact transverse to the consensus space. Moreover, we notice that proposition 17 is in agreement with the transversality condition, $n \neq 2$, for transcritical points. Indeed, for $n = 2$ we have that the tangents (53), of the branches of the critical manifold intersecting the consensus, are parallel to the fast foliation. As a consequence of proposition 17 we have that pitchfork singularities are not possible on the consensus space for the class of ALFs under examination,

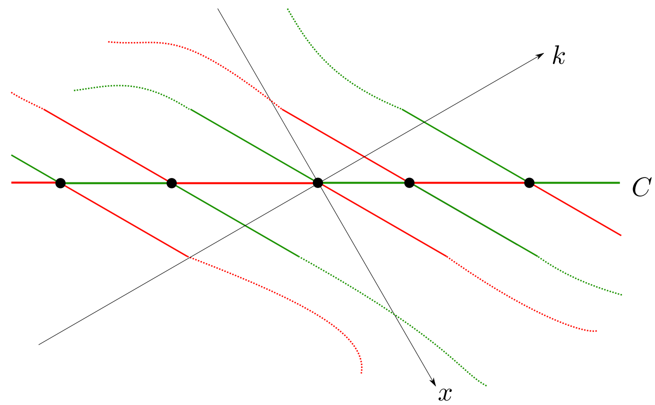


Figure 2: Local structure of the critical manifold near the consensus space in the (x,k)-plane.
In the figure, the (x,k)-plane is rigidly rotated in order to visualise the consensus space as a horizontal line. The black dots represent singular points, the coloured lines crossing the consensus space at the singular points are other branches of the critical manifold. We draw in red the unstable sections of the critical manifold, while the stable ones are drawn in green. We emphasise that, in general, the branches of equilibria that cross the consensus manifold are not straight lines, but due to proposition 17 they are all parallel to each other.

this statement is also confirmed by checking the conditions for pitchfork singularities [27]. By a similar reasoning, fold singularities on the consensus space are also excluded. Finally, Hopf singularities do not appear either by the fact that the layer system has always real eigenvalues. So, one can conclude that for K_n-ALFs transcritical singularities are generic on the consensus space, in the sense of codimension one bifurcations; while on other branches of the critical manifold it is possible to have other singularities, see some examples in section 6.

6 Examples and simulations

In this section, we present explicit examples of ALFs, and show numerical simulations for such concrete systems.

6.1 Example 1

For the first example, we choose the response function $f(x) = (x-1)^2(x+1)^2$, see figure 3. Let us notice that $f(x)$ is an even function, and so it is invariant under the symmetry group $\mathbb{Z}_2$; this implies that we expect a nontrivial critical manifold for the ALF. In fact, if we set the number of nodes $n = 3$ we can visualise the critical manifold in $\mathbb{R}^3$, see figure 4.

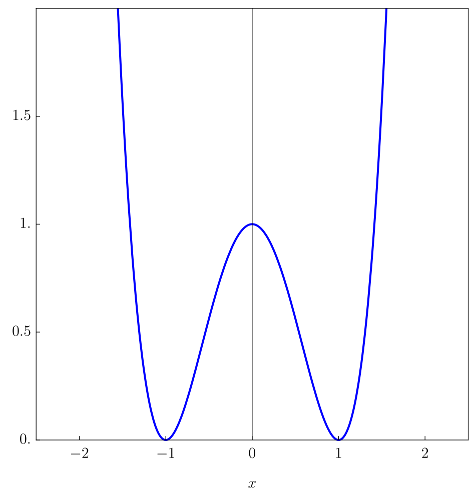


Figure 3: Response function $f(x) = (x-1)^2(x+1)^2$.

Following the analysis of section 5.2, we consider a symmetry preserving perturbation and we look at the behaviour of the system in the (x, k)-plane. Considering (34), we fix the perturbation parameter $\epsilon = 1/10$, and a critical perturbation $g(x, k, \Lambda) = \tilde{g}(x, k, \Lambda) = -1$, that is, the slow-flow is directed downwards in figure 5. We numerically simulate the system by tracking some characteristic canard solutions, see figure 5. It is worth recalling now a few issues with numerical simulations of transcritical points, see more details in [14]. Let us notice that, although the simulations are performed with a working precision of 50 digits they still suffer of numerical problems. In particular we refer to the fact that a canard solution, under an appropriate perturbation, should spend the same time 'close' to the attracting part as to the repelling part. In this examples, such a time symmetry should be visible as a spatial symmetry in correspondence of a transcritical point on the consensus manifold. As one can see in figure 5, the symmetry described is not very well reproduced by the numerical simulation. There are several factors that induce such problem, first, we have that the symmetry is exact only in the limit for $\epsilon \to 0$, so we expect that a numerical simulation will just approximate such symmetry. On the more numerical side, we have that the ϵ parameter is quite large to clearly see this effect, however lowering ϵ would imply to further increase the working precision, leading to problems with the time performance of the CPU. Since our present work has a theoretical

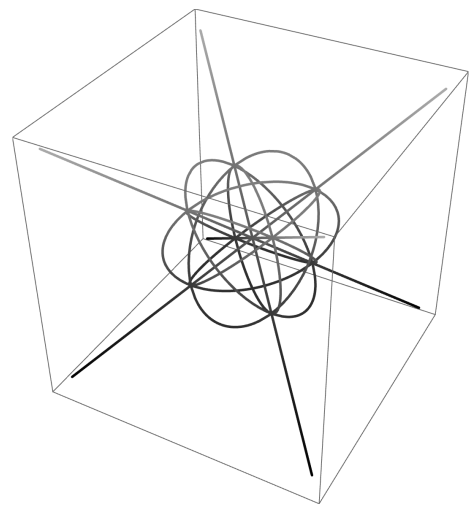
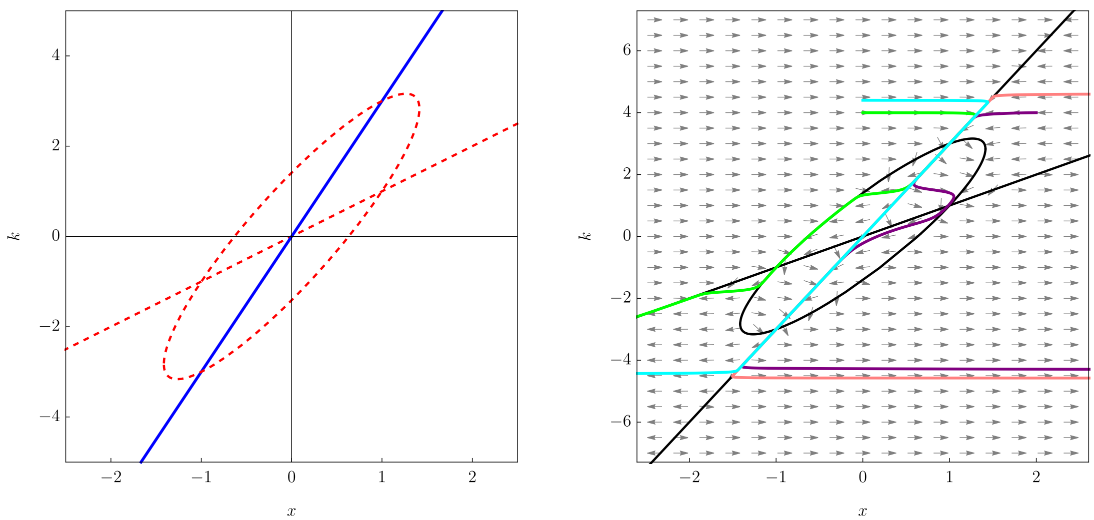


Figure 4: Critical manifold for n=3 and $f(x) = (x-1)^2(x+1)^2$

focus, we do not dive deeper in the numerical analysis of these delicate systems, leaving the problem of highly accurate simulations for future works. However, there is another detail to take into account: in figure 5 we have canard solutions that cross three transcritical points on the consensus space without leaving the repelling intermediate section (light blue and pink curves); here the question is: where should we expect the canards to leave the consensus manifold? As we just discussed, we expect some symmetry, but here we need to be more careful stating with respect to which transcritical point. We argue that, for these particular solutions, the intermediate section has a null effect, and therefore the solution should leave the consensus after the third transcritical point (counting top to bottom) after a distance (approximately) equal to the distance travelled close to the attracting section before the first transcritical point. In order to prove the aforementioned statement, we use the slow-divergence integral [13, 11, 10]. So, let us consider the slow-divergence integral along the consensus manifold for the system under analysis. Since the perturbation is constant the divergence of the vector field in the (x, k)-plane, evaluated on the consensus space, is simply $-n\mathrm{d}_x f(k/n)$. Therefore the slow-divergence integral becomes

$$-n \int \mathrm{d}_x f(k/n) dk. \tag{66}$$

We now notice two facts: first the intermediate section is divided in two spatially equal sections by a transcritical point. Second, on the first subsection $\mathrm{d}_x f(k/n) > 0$ and on the second subsection $d_x f(k/n) < 0$. Therefore the slow-divergence integral on the intermediate section of the consensus space is null, so on the intermediate section the attracting and repelling effect compensate each other. The considerations we have done hold for the dynamics near consensus under a constant critical perturbation. It is worth mentioning that the nonlocal interaction of transcritical singularities via hysteresis processes could give rise to an enhanced delay for the canard [15, 45].



(a) Critical manifold in the (x, k)-plane

(b) Canards solutions

Figure 5: On the left, figure 5a, we can see in blue the consensus space, and in red dashed the other branches of the critical manifold. This critical manifold is obtained from the critical manifold of figure 4. On the right, figure 5b, we computed numerically four canard solutions for a critical perturbation. Namely, in green we have a canard crossing two transcritical points (only one on the consensus space), after the first transcritical point it leaves the consensus space; in light blue and pink we have two canard solutions that cross three transcritical points without leaving the intermediate unstable section of the consensus space; in purple we have a solution that after the first transcritical point leaves the consensus space, after crossing a second transcritical point (not on consensus) due to the particular shape of the critical manifold it is attracted again to consensus and finally crosses a third transcritical point on the consensus. Notice that all the solutions, when crossing transcritical points on the consensus space, show the typical canard behaviour. We remark that not all the solutions displayed are robust, in particular this depends on the stability properties of the (x, k)-plane. We can expect that solutions starting in a neighbourhood of attracting branches of the consensus are robust, and so the reduction is representative of the dynamics. In the unstable regions the reduction still applies if we consider ad-hoc perturbations and initial conditions.

6.2 Example 2

Let us consider the larger network, K_{10}, in figure 6. We keep the same response function $f(x) = (x-1)^2(x+1)^2$ as in example 1, but now the perturbation is randomly generated, $H = (r_1, \ldots, r_n)$, where $r_1, \ldots, r_n$ are random reals between 0 and 1. For this example, we consider the equations in the non-standard form (27), and we compute the time-series for each node state with initial conditions randomly chosen between -1 and 0, see figure 7. Moreover, we also compute the timeseries for a weighted counterpart of a K_{10}-ALF in order to highlight the equivalence between the weighted and unweighted ALFs near consensus.

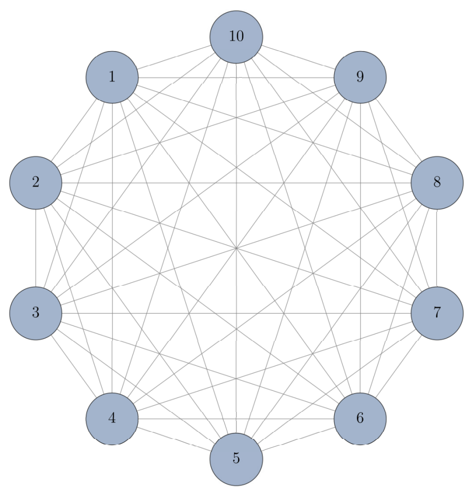


Figure 6: K_{10}

The timeseries in figure 7 show a characteristic two-timescale behaviour, alternating long periods of slow drift and short fast transitions of regime. In particular after an initial fast convergence to consensus, then the system slowly reaches the first transcritical point, there we can see that the system starts a sequence of fast transitions that bring the system out of consensus. Finally, due to the geometry of the critical manifold the system converges to consensus again.

6.3 Example 3

As a final example, we consider the phenomenon of symmetry breaking. We briefly recall what we mean by (spontaneous) symmetry breaking. Let us consider a dynamical system

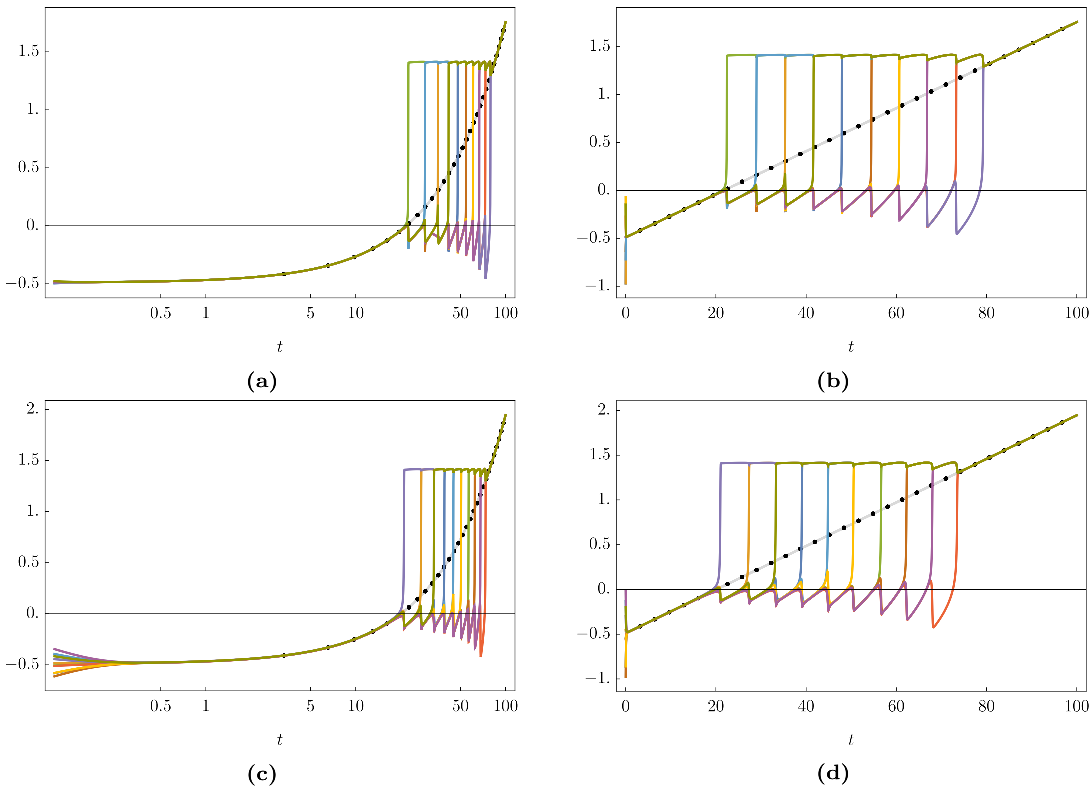
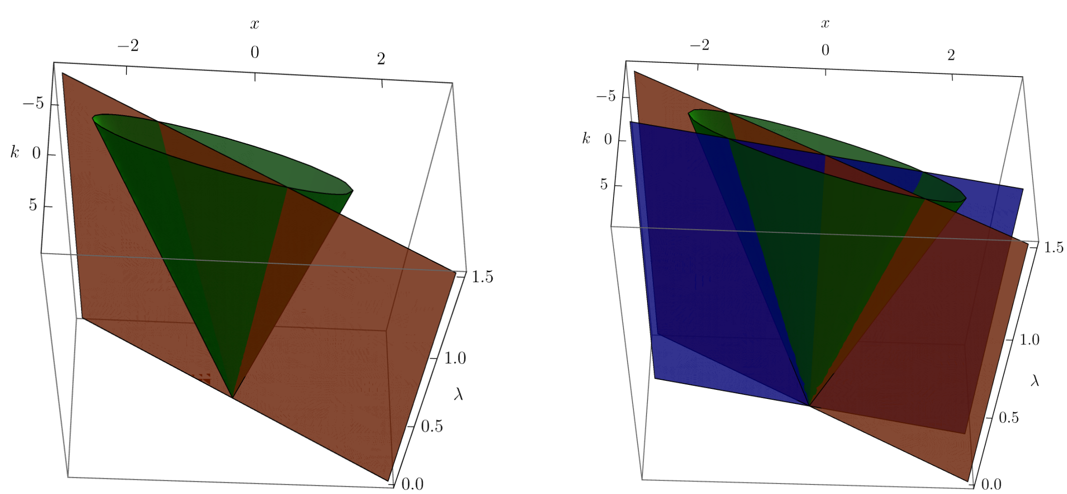


Figure 7: Timeseries for a K_{10}-ALF under a random constant and positive perturbation. Figures 7a, 7c display time in a logarithmic scale, while in figures 7b, 7d time is linear. Different colours represent different node's states. Figures 7a and 7b represent a simulation for a weighted K_{10}-ALF, where the weights are randomly chosen between 1 and 5; on the other hand, in figures 7c and 7d one can see the behaviour for a (unweighted) K_{10}-ALF. The plots confirm the topological equivalence proved in proposition 5. The black dots represent the simulation for the same system under a critical perturbation, where the expected canard solution takes place. Let us remark that the initial conditions for the weighted and for the unweighted case are not the same but they are chosen randomly in the same interval, between -1 and 0. Such random initial conditions are chosen nearby an attracting branch of the consensus space, implying the robustness of the solutions. Indeed, the behaviour of the two simulations is similar, and well predicted by the reduced system.

equivariant under a group Γ. Then, if there are equilibria that are not fixed under Γ we say that the symmetry is broken. As we have seen in section 3.1, for ALFs the symmetries of the response function induce, in general, a set of equilibria larger than the consensus space, which is the fixed-point space for groups of permutations of order greater than or equal to n. So, when we have a response function with nontrivial symmetry we should expect symmetry breaking. What we just mentioned works for unperturbed ALFs. When we perturb the system, in general, we destroy the symmetry of the unperturbed system and the behaviour changes dramatically. However, if we consider the particular case of a symmetry preserving perturbation, then the maximal canard becomes the solution fixed under the symmetry group replacing, in some sense, the role of the consensus space in the unperturbed system.

Let us consider the response function $f(x,\lambda) = (x-\lambda)(x+\lambda)^2$. We can construct a bifurcation diagram by plotting the critical manifold on the (x,k)-plane for a range of values of λ, i.e., $\{(x,k,\lambda) \in \mathbb{R}^3 \mid -[f(x,\lambda) - f(k-(n-1)x,\lambda)] = 0\}$, see figure 8a.



(a) **(b)**

Figure 8: Bifurcation diagram for the response function $f(x,\lambda) = (x-\lambda)(x+\lambda)^2$, figure 8a, and bifurcation diagram for the response function $f(x,\lambda) = (x-\lambda)^2(x+\lambda)^2$, figure 8b; in both cases we fixed $n = 3$. We show in orange the consensus manifold, in green the branches of the critical manifold arising by varying λ common to both response functions. Notice that, in figure 8b, there is an extra branch of the critical manifold (shown in blue), this is due to the $\mathbb{Z}_2$ invariance of the response function, or, in other words, it is due to an extra degeneracy of the response function.

We notice, by looking at figure 8a, that for $\lambda = 0$ the critical manifold consists of the consensus space only. Then, for $\lambda > 0$, another manifold appears breaking the symmetry of the system. If we think for a moment at the response function alone, we could consider another bifurcation diagram: the one given by $\{(x, \lambda) \in \mathbb{R}^2 \mid f(x, \lambda) = 0\}$. This reasoning allows us to compare what the bifurcation diagram of the critical manifold on the (x, k)-plane is, and the bifurcation described by an hypothetical one dimensional system, $\dot{x} = f(x, \lambda)$, which we call *response bifurcation*. For the function $f(x, \lambda) = (x - \lambda)(x + \lambda)^2$ the response bifurcation diagram is the one of a degenerate transcritical singularity. On the other hand, if we consider the response function $f(x, \lambda) = (x - \lambda)^2(x + \lambda)^2$, it is also representative of a degenerate transcritical singularity; in particular for $\lambda = 1$ it becomes the response function considered in the first and second examples. It is interesting to notice that, if we compare the response bifurcation diagrams for the function $f(x, \lambda) = (x-\lambda)^2(x+\lambda)^2$ and $f(x, \lambda) = (x-\lambda)(x+\lambda)^2$ the branches of equilibria have the same structure of degenerate transcritical bifurcations. However, if we consider the bifurcation diagrams of the critical manifold on the (x, k)-plane induced by those functions, we notice that the degeneracy induces different critical manifolds, see figure 8. Thus, with this example, we have argued that the network structure plays a highly non-trivial part when qualitatively studying networked dynamics and their bifurcations.

7 Conclusions and discussion

This paper has been dedicated to the analysis of a class of nonlinear Laplacian dynamical systems, ALFs, and their perturbations. The analysis starts by looking at (unperturbed) ALFs on graphs with positive weights. We emphasise the importance of the topological equivalence proved in proposition 5. Indeed, a weighted graph, in general, loses the symmetry properties that its own unweighted counterpart has. The fact that a positive-weighted graph induces dynamics of the ALF that are topological equivalent to the one induced by the unweighted counterpart, allows us to consider unweighted simple graphs. In turn, an ALF with simple graph structure inherits the symmetry properties of the underling graph, enabling us to use the tools of equivariant dynamical systems theory. In such a context, the cell-network formalism was developed with the purpose of exploiting network symmetries in dynamical systems, actually going even beyond the group symmetries. In the language of cell-networks, the linear invariant spaces arising from the symmetry properties of the underlying network are referred to as pattern of synchrony. So, in fact, the result of theorem 2 is a particular case of a robust pattern of synchrony; indeed, the existence of a maximal canard is obtained by using the symmetry properties of the network. We note that theorem 2 brings more stringent conditions for the consensus than being just an invariant space. Specifically, when the system is unperturbed, given the specific form of the ALF's equation, we know that the the consensus space is a set of equilibria. When a perturbation is switched on, we require the perturbation not only to preserve the symmetry, but also to be non-zero along the consensus space. So, asking the consensus space to be a robust pattern of synchrony is a necessary, but not a sufficient, condition to have a maximal canard. It is also worth remarking that, theorem 2 does not rely on specific tools of slow-fast systems, which are the classical techniques used to determine the existence of canards.

Several new directions arise from the results of our paper. For example, one could consider response functions ($F(\boldsymbol{x})$) depending also on "neighbouring" nodes; in such a case the network structure appears also inside the response vector field, and not only through the Laplacian matrix. Moreover, one may be interested in studying the case of heterogeneous response functions, i.e., different nodes might contribute differently to the mutual interaction. Regarding the graph structure, one may take into account negative weights, which even for the linear case give rise to much richer dynamics than the positive-weights case; directed graphs could be considered as well. Another possibility is to study Laplacian systems where the states have dimension greater than one, for example, one might consider Laplacian systems where the states evolve in the real plane, or even in a two-dimensional manifold. The dynamics of planar systems entail interesting phenomena, such as periodic orbits, that could potentially give rise to new phenomena unseen in the one-dimensional framework. For similar reasons, states of dimension three might be relevant as well, since it is well-known that it is for three-dimensional systems that chaos, at the level of the nodes, may appear. In all aforementioned cases, a natural question is to investigate, and if possible classify, the possible singularities that arise and the influence of the network on them (for example in this paper we showed that complete ALFs can only have transcritical singularities along the consensus manifold). Moreover, one would then be interested in the behaviour under small perturbations, which as seen here, is also closely related to the network structure.

A Singular perturbations

In general, the symbols used in this section are not related to those in the main text.

A singularly perturbed system of ODEs, in the standard form, is also called *slow-fast system* and it has the form

$$\begin{aligned} \dot{x} &= f(x, y, \epsilon) \\ \dot{y} &= \epsilon g(x, y, \epsilon), \end{aligned} \tag{67}$$

where $0 < \epsilon \ll 1$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$, and the time-parametrization t is called *fast-time*. Equation (67) is said to be in the standard form because the time separation between the fast-variables, x, and the slow-variables, y, is explicit. In the non-standard form a singularly perturbed system has the form

$$\dot{z} = F(z, \epsilon), \tag{68}$$

where $z \in \mathbb{R}^{n+m}$, and the vector field $F(z, 0)$ has at least one equilibrium of dimension greater than zero [46].

Defining the *slow-time*, $\tau := \epsilon t$, equation (67) can be rewritten as

$$\begin{aligned} \epsilon x' &= f(x, y, \epsilon) \\ y' &= g(x, y, \epsilon). \end{aligned} \tag{69}$$

Setting ϵ to zero in equation (69), we obtain the *reduced system*

$$\begin{aligned} 0 &= f(x, y, 0) \\ y' &= g(x, y, 0), \end{aligned} \tag{70}$$

which is a set of constrained differential equations. Instead, setting ϵ to zero in equation (67) we obtain the *layer problem*

$$\begin{aligned} \dot{x} &= f(x, y, 0) \\ \dot{y} &= 0, \end{aligned} \tag{71}$$

where the slow-variables, y, play the role of parameters.

Definition 13. *The critical manifold is defined as the set*

$$\mathcal{C}_0 = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m \mid f(x, y, 0) = 0\}. \tag{72}$$

Remark 8. *Although the commonly accepted name, the critical manifold it is not necessarily a manifold.*

The critical manifold is the constraint space for the reduced system (70), or equivalently, it is the set of equilibria of the layer problem (71).

Definition 14. *A point $x^* \in \mathcal{C}_0$ is called hyperbolic if the Jacobian matrix at x^* has no eigenvalues with zero real part.*

Lemma 3. *A subset $\mathcal{M}_0 \subseteq \mathcal{C}_0$ such that $x^* \in \mathcal{M}_0$ is hyperbolic, $\forall x^* \in \mathcal{M}_0$, is a normally hyperbolic invariant manifold (NHIM) [23].*

Theorem 3 (Fenichel)**.** *For ϵ sufficiently small, compact subsets of NHIMs of the layer system* (71) *are diffeomorphic to compact subsets of NHIMs of the singularly perturbed system* (67) *(or, equivalently* (69)*).*

Remark 9. *Theorem 3 is restricted to the results relevant for the present paper. An exhaustive version can be found in [28].*

Another way to state theorem 3 is that normally hyperbolic critical manifolds persist as slow manifolds, and inherit all stability properties of the critical manifold. For nonhyperbolic points, $x^* \in \mathcal{C}_0$, also known as *singularities*, Fenichel's theorem does not provide information about the perturbed system in a neighbourhood of x^*. The process of analysis of the system in a neighbourhood a singularity is called desingularization. We briefly introduce the *blow-up* (desingularization) for *nilpotent* singularities [26].

Definition 15. *A point $x^* \in \mathcal{C}_0$ is a nilpotent singularity if all the eigenvalues of the Jacobian, at x^*, are zero.*

When a singular point has a mixture of zero eigenvalues, and non-zero (real part) eigenvalues it is possible to reduce the problem to a nilpotent singularity via a *center manifold* reduction [8].

Definition 16. *Let x^* be a nilpotent singularity. The spherical blow-up is a (local) coordinate transformation*

$$\begin{aligned} \beta : \mathbb{R}^{n+m+1} &\to \mathbb{S}^{n+m}_{x^*} \times I \\ (x, y, \epsilon) &\mapsto (\bar{x}, \bar{y}, \bar{\epsilon}, r), \end{aligned} \tag{73}$$

where $\mathbb{S}^{n+m}_{x^*}$ *is the* $n+m$*-dimensional sphere (embedded in* $\mathbb{R}^{n+m+1}$*) centred at* x^**,* $(\bar{x},\bar{y},\bar{\epsilon}) \in \mathbb{S}^{n+m}_{x^*}$*, and* $r \in I \subseteq \mathbb{R}_{\geq 0}$*,* $0 \in I$*.*

Let us notice that, in a neighbourhood of the singularity the transformation (73) is a diffeomorphism. Instead at the singular point the blow-up transforms the point x^* into a $n+m$-dimensional sphere. The core idea of the blow-up is to analyse the induced flow on the sphere and deduct the behaviour of the singularity. However, the blow-up transformation by itself does not desingularize the system. After the blow-up it is necessary to perform a conformal transformation (time reparametrization) in order to regularise the system, i.e., retrieve hyperbolicity. We described the spherical blow-up, but several other blow-up transformations are possible, e.g., quasihomogeneous, directional, or even to other manifolds, not necessarily spheres [29]. The choice of the appropriate blow-up should be done by considering the regularisation, and feasibility of the analysis. Since this work is not concerned with a direct application of the blow-up, but rather with the implementation of results involving the blow-up technique, we presented the spherical set-up which provides the most clear geometrical visualisation.

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