CAYLEY CHARTS AND PFAFFIANS

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Abstract

Especially for the Euclidean (i.e. positive definite case) the notions of Cayley chart is well known. With it the notion of complete Pfaffian can be introduced. In this article we consider the link between Cayley charts, complete Pfaffians, and outer exponentials both for the Euclidean and the non-definite case.

Notations

We assume here that a standard basis for $\mathbb{R}^{0n}$ (or $\mathbb{R}^{pq}$) is given, with the inner product $(\cdot,\cdot)$:

$$(e_i, e_j) = \begin{cases} -\delta_{ij} & \text{if } i \le p \\ \delta_{ij} & \text{if } i > p \end{cases}$$

Notice that notation is chosen such that $\mathbb{R}^{0n}$ (also written $\mathbb{R}^n$ for short) is Euclidean space with positive inner product, a notation which is a lot more practical than the opposite sign convention. The generators of the Clifford algebra then satisfy

$$e_i^2 = -(e_i, e_i),$$

and the dot product is defined by $\vec{x} \cdot \vec{y} = (1/2)(\vec{x}\vec{y} + \vec{y}\vec{x})$. The k-vector part of a Clifford number a is written as $[a]_k$. $[a]_0$ is called the real or scalar part of a. Following Lounesto and Ahlfors we define the wedge and dot products of an r-vector a and an s-vector b by

$$a \cdot b = [ab]_{|r-s|} \qquad a \wedge b = [ab]_{r+s},$$

unless r or s is zero, in which case $a \cdot b = 0$. Also the notation $\wedge^k b = \left(\wedge^{k-1} b\right) \wedge b$, where $\wedge^0 b = 1$, comes from [2].

The main antiautomorphism on the algebra is defined by the transformation of the generators: $\overline{e_i} = -e_i$, and of course $\overline{ab} = \bar{b}\,\bar{a}$.

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Any transformation of $\mathbb{R}^{pq}$ can be expressed by a matrix using the basis $e_1, \ldots, e_n$. Vectors in $\mathbb{R}^{pq}$ other than these basis vectors are written with a vector arrow: $\vec{x}$, $\vec{y}$, ... $\mathbf{n}$ is the set $\{1, \ldots, n\}$, $\sigma(\mathbf{n})$ is its set of permutations, and likewise for $\mathbf{m}$. As usual $n = p + q$. If g is in the Clifford group, $\mathrm{Pin}(g)$ is the orthogonal transformation of $\mathbb{R}^{pq}$ mapping $\vec{x}$ to $g\vec{x}g'^{-1}$.

$\mathrm{End}_{-}(\mathbb{R}^{pq})$ is the set of transformations s such that $(s\vec{x}, \vec{y}) = -(\vec{x}, s\vec{y})$ for all vectors, i.e. the set of all antisymmetric transformations.

1. The Cayley Chart

The Cayley chart is a mapping from $\mathrm{End}_{-}(\mathbb{R}^{pq})$ to $SO(p,q)$ defined by

$$\mathrm{Cay}s = (1+s)(1-s)^{-1}$$

for those s for which $1 - s$ is invertible. Originally it was considered as a conformal mapping in $\mathbb{C}$, which maps the imaginary axis to the unit circle (the point ∞ is mapped to -1). However, multiplication with an imaginary number is antisymmetric, and so the imaginary axis can be identified with $\mathrm{End}_{-}(\mathbb{R}^2)$. Likewise the unit circle can be identified with $SO(\mathbb{R}^2)$. This way the Cayley chart can be generalised into two directions: the one given above, and as a conformal mapping in n-dimensional space mapping the unit sphere to a hyperplane.

THEOREM 1.1 The image of the Cayley map consists of those elements A of $SO(n)$ such that $1 + A$ is invertible.

Proof.
Suppose first that $A = (1-s)^{-1}(1+s)$ for a certain s. Then for any $\vec{x}$ such that $A\vec{x} = -\vec{x}$ we have that $(1+s)\vec{x} = (1-s)(-\vec{x})$ or $2\vec{x} = 0$. This proves that $1 + A$ is invertible. For the proof that $A \in SO(p,q)$ we sketch the reasoning given in [4]. First notice that the adjoint of $(1-s)$ is $(1+s)$. It follows that

$$\begin{aligned}(Au, Av) &= ((1-s)^{-1}(1+s)u, (1-s)^{-1}(1+s)v) \\ &= ((1+s)^{-1}(1+s)u, (1+s)^{-1}(1+s)v) \\ &= (u, v),\end{aligned}$$

so $A \in O(p,q)$. Moreover, since the determinant of the adjoint is the determinant of the transformation, $\det(1-s) = \det(1+s)$. It follows that

$$\det A = (\det(1-s))^{-1} \det(1+s) = (\det(1+s))^{-1} \det(1+s) = 1.$$

Suppose now that $1 + A$ is invertible. Put then

$$s = (A - 1)(1 + A)^{-1}.$$

One sees immediately that $A = (1 - s)^{-1}(1 + s)$, but we still have to prove that s is in $\mathrm{End}_-(\mathbb{R}^{pq})$, i.e. that for arbitrary $\vec{u}$ and $\vec{v}$, $(s\vec{u}, \vec{v}) + (\vec{u}, s\vec{v}) = 0$. Put $\vec{t} = (1 + A)^{-1}\vec{u}$, $\vec{w} = (1 + A)^{-1}\vec{v}$, and the expression becomes

$$((A - 1)\vec{t}, (A + 1)\vec{w}) + ((A + 1)\vec{t}, (A - 1)\vec{w})$$

which is zero, as an elementary calculation shows. ∎

Remark
The Cayley map is one to one. Moreover, for $p = 0$, $(1 - s)$ is always invertible, and the condition can be omitted.

There is a bijective relation between the mappings in $\mathrm{End}_-(\mathbb{R}^{pq})$ and the bivectors of $\mathbb{R}_{pq}$. Indeed, let b be a bivector, and let B be the mapping $\vec{x} \to b \cdot \vec{x}$. Then $B \in \mathrm{End}_-(\mathbb{R}^{pq})$. Indeed $(B\vec{x}, \vec{y}) = [\overline{b \cdot \vec{x}}\vec{y}]_0 = [\overline{b\vec{x}}\vec{y}]_0 = -[\vec{x}\bar{b}\vec{y}]_0$. On the other hand $(\vec{x}, B\vec{y}) = -[\vec{x}b\vec{y}]_0$. since $b = -\bar{b}$, this means that $B \in \mathrm{End}_-(\mathbb{R}^{pq})$. On the other hand, for $B \in \mathrm{End}_-(\mathbb{R}_{pq})$, we can put

$$b = \frac{1}{2} \sum_{i,j=1}^{n} (Be_i, e_j) e_i^{-1} e_j^{-1}.$$

Hence we shall use also the notation $\mathrm{Cay}(b)$ where b is a bivector.

2. Wedge Products and Outer Exponentials

The definitions are taken from [2]. The results proved here also are inspired by this article, where similar results were proved for the positive definite case.

Lemma 2.1. *for any bivector b, any vector $\vec{x}$, and any $k > 0$*

$$\left(\wedge^k b\right) \cdot \vec{x} = k \left(\wedge^{k-1} b\right) \wedge (b \cdot \vec{x}).$$

Proof.
Take first an arbitrary r-vector a and an s-vector b. Then

$$(a \wedge b) \cdot \vec{x} = (-1)^s (a \cdot \vec{x}) \wedge b + a \wedge (b \cdot \vec{x}).$$

This is easy to prove if $a = e_I$, $b = e_J$, $\vec{x} = e_i$, since the four cases (i in I or not, i in J or not) can be treated separately; the general case follows by linearity. From this it then easily follows by induction on k that for a set of elements of the even subalgebra a_i

$$\left(\wedge_{i=1}^k a_i\right) \cdot \vec{x} = (a_1 \cdot \vec{x}) \wedge a_2 \wedge \ldots \wedge a_k + a_1 \wedge (a_2 \cdot \vec{x}) \wedge \ldots \wedge a_k + \\ \ldots + a_1 \wedge \ldots \wedge (a_k \cdot \vec{x}) \,.$$

Putting all $a_i = b$, and noticing that the vector $b \cdot \vec{x}$ commutes (for the wedge product) with any even element, gives the result. ∎

We now define the outer exponential function by

$$\text{oexp}(a) = \sum_{k=0}^{\infty} \frac{1}{k!} \wedge^k a.$$

Lemma 2.2. *Let $b = [b]_2$ be a bivector, $\beta = \text{oexp}(b)$, and let B be the linear operator mapping $\vec{x} \to b \cdot \vec{x}$. If $1 - B$ is invertible, then, for any $\vec{x}$ there exists a unique $\vec{y}$ such that $\beta\vec{x} = \vec{y}\beta$. This $\vec{y}$ is given by $\text{Cay}(B)\vec{x} = \vec{y}$. Moreover $\beta\overline{\beta}$ is real.* ∎

Proof.
Take x arbitrary. Since $1-B$ is invertible, it is possible to find $\vec{y}$ with $(1+B)\vec{x} = (1-B)\vec{y}$, and so $\vec{y} = \text{Cay}(\vec{x})$. Notice that $\vec{y} \cdot b = -B(\vec{y})$ and $b \cdot \vec{x} = B(\vec{x})$, so the identity can be written as

$$\vec{x} + b \cdot \vec{x} = \vec{y} + \vec{y} \cdot b.$$

It must be shown that $\beta\vec{x} = \vec{y}\beta$. From the definition of oexp and the previous lemma it follows that $(\beta - 1) \cdot \vec{x} = \beta \wedge (b \cdot \vec{x})$. Indeed

$$\begin{aligned} (\beta - 1) \cdot \vec{x} = & \sum_{k=1}^{\infty} \frac{1}{k!} \left(\wedge^k b \cdot \vec{x}\right) \\ = & \sum_{k=1}^{\infty} \frac{k}{k!} \left(\wedge^{k-1} b \wedge (b \cdot \vec{x})\right) \\ = & \; \beta \wedge (b \cdot \vec{x}) \end{aligned}$$

Since $1 \cdot \vec{x} = 0$,

$$\beta \cdot \vec{x} = \beta \wedge (b \cdot \vec{x}).$$

On the other hand, since $\vec{x}$ is a vector, $\beta \cdot \vec{x} = \beta\vec{x} - \beta \wedge \vec{x}$ which results in

$$\beta\vec{x} = \beta \wedge (\vec{x} + b \cdot \vec{x}) = (\vec{x} + b \cdot \vec{x}) \wedge \beta = (\vec{y} + \vec{y} \cdot b) \wedge \beta.$$

In a similar way $\vec{y}\beta = (\vec{y} + \vec{y} \cdot b) \wedge \beta$, and this proves that $\beta\vec{x} = \vec{y}\beta$.

We still must prove there is only one $\vec{y}$ satisfying this equation, i.e. that if $\beta\vec{x} = \vec{z}\beta$, then $\vec{z} = \vec{y}$. Since $[\vec{z}\beta]_1 = (1-B)\vec{z} = [\vec{y}\beta]_1 = (1-B)\vec{y}$, and $1-B$ is invertible, this is however immediately clear.

To prove that $\beta\overline{\beta}$ is real we start from the fact that for arbitrary $\vec{x}$, $\beta\vec{x} = \vec{y}\beta$ for certain $\vec{y}$. Multiplying with $\overline{\beta}$ gives $\overline{\beta}\beta\vec{x} = \overline{\beta}\vec{y}\beta$, taking the adjoint gives $\vec{x}\overline{\beta}\beta = \overline{\beta}\vec{y}\beta$. In other words, $\overline{\beta}\beta$ commutes with all vectors, and must be in the centre of the algebra. On the other hand $\overline{\beta}\beta$ is even, so it must be real.

Remark

In the positive definite case it now follows easily that β is in the Clifford group. Since $\beta\bar{\beta} \neq 0$, β is invertible, $\beta'^{-1} = \beta^{-1} = \bar{\beta}/(\beta\bar{\beta})$. With $\beta\vec{x} = \vec{y}\beta$ this gives $\beta\vec{x}\beta'^{-1} = \vec{y}$. In the general case we still have to prove that $\beta\bar{\beta}$ is different from zero. To prove this the so called complete Pfaffian will be used, which is an alternative expression for the outer exponent of a bivector.

3. The Pfaffian in the Definite Case

$\mathrm{End}_-(\mathbb{R}^{0n})$ is given by the matrices for which $s = -s^\tau$ (τ indicates the transpose). The Pfaffian of s is defined to be 0 if n is odd and

$$\mathrm{pf}s = \sum_{\pi \in P} \mathrm{sgn}\pi \prod_{i \in \mathbf{m}} s_{\pi(2i),\pi(2i+1)}$$

if n is even, where $m = n/2$ and P is the set of permutations of n satisfying

(i) $\pi(2i) < \pi(2i+2)$ for all $i < m$.
(ii) $\pi(2i) < \pi(2i+1)$ for all $i \leq m$.

An alternative formula for the Pfaffian is

$$\mathrm{pf}se_{\mathbf{n}} = \frac{1}{2^m m!} \sum_{\pi \in \sigma(\mathbf{n})} \prod_{i \in \mathbf{m}} s_{\pi(2i),\pi(2i+1)} e_{\pi(2i)} e_{\pi(2i+1)}$$

This expression is given e.g. in [3] and [2] if one takes into account that $\prod_{i \in \mathbf{m}} e_{\pi(2i)} e_{\pi(2i+1)} = \mathrm{sgn}\pi e_{\mathbf{n}}$, which is easily checked.

The complete Pfaffian here is given by

$$\mathrm{Pf}s = \sum_{I \subset \mathbf{n}} \mathrm{pf}s_I e_I,$$

where s_I is the matrix obtained retaining only the elements s_{ij} such that $\{i,j\} \subset I$. We introduce some notations:

$S(i_1,\dots,i_k)$ is the matrix obtained from s deleting the i_1th, ... and i_kth rows and columns.

$I(i_1,\dots,i_k) = I \setminus \{i_1,\dots,i_k\}$.

The following expression for the Pfaffian of s (given in [3]) will be used:

$$\mathrm{pf}s = \sum_{i=1}^{n} s_{ik}(-1)^{i+k-1}\mathrm{pf}S(i,k). \tag{1}$$

This expression is valid for any k.

THEOREM 3.1 The complete Pfaffian is in the Clifford group.

Proof.
It is sufficient to prove that, for arbitrary n and s, Pfs can be written as

$$\mathrm{Pf}s = (1+\vec{v}e_n)\mathrm{Pf}S(n)$$

for some vector $\vec{v}$, since then the theorem follows by induction on n ($1+\vec{v}e_n$ is in the Clifford group). Since Pf$s \in \mathbb{R}_{0n}$, it has the form

$$\mathrm{Pf}s = a + e_n b,$$

where $a, b \in \mathbb{R}_{0,n-1}$, and it is easily checked that $a = \mathrm{Pf}S(n)$, while $e_n b$ is linear in the s_{in}:

$$e_n b = \sum_i s_{in} e_n b_i,$$

and it is sufficient to prove that each of the coefficients $e_n b_i$ is of the form $\vec{v}_i e_n$. Explicitly however

$$\begin{aligned} s_{in}e_n b_i &= \sum_{\{i,n\}\subset I\subset \mathbf{n}} s_{in}\mathrm{pf}s_{I(i,n)}e_i e_{I(i,n)}e_n \\ &= s_{in}e_i\mathrm{Pf}S(i,n)e_n \end{aligned}$$

(it is easy to check that if i is the tth element in I, then $e_I = (-1)^{1+t}e_i e_{I(i,n)}e_n$. Then use (1) for pf$S_{I(i,n)}$). From the induction hypothesis there exists a vector

$\vec{w}_i$ such that $\mathrm{Pf}S(n) = (1+\vec{w}_i e_i)\mathrm{Pf}S(i,n)$ and so there exist λ and $\vec{u}$ such that $(\lambda + \vec{u}_i e_i)\mathrm{Pf}S(n) = \mathrm{Pf}S(i,n)$, which results in

$$s_{in}e_n b_i = s_{in}e_i(\lambda + \vec{u}_i e_i)\mathrm{Pf}S(n)e_n.$$

Since $\mathrm{Pf}S(n)$ commutes with e_n, putting $\vec{v} = \lambda e_i + e_i\vec{u}_i e_i$ proves the theorem. ∎

COROLLARY 3.2 Let b be the bivector

$$b = -\frac{1}{2}\sum_{i,j=1}^{n} s_{ij}e_{ij}.$$

Then $\mathrm{Pf}s = \mathrm{oexp}(b)$.

Proof.
Both $\mathrm{Pf}s$ and $\mathrm{oexp}(b)$ are in the Clifford group, have real part equal to one, and have the same bivector part. Hence they are equal. ∎

Remark
It is also possible to prove the equality of the complete Pfaffian and the outer exponent of b on purely algebraic grounds. A sketch of a possible method is the following: write the Pfaffian in r-vectors:

$$\mathrm{Pf}s = \sum_{k=0}^{\infty}[\mathrm{Pf}s]_{2k}.$$

If we write $P_k = [\mathrm{Pf}s]_{2k}$ for short, we have to prove that $P_{k+1} = (1/(k+1))P_k \wedge b$. This is obvious for $2k+2 > n$, where both terms are zero. As an example we prove this for $2k+2 = n$, where n is supposed to be even. In this case we develop $\mathrm{pf}s$ w.r.t. every row, and take the average:

$$\begin{aligned} P_{k+1} = \quad & \mathrm{pf}se_{\mathbf{n}} \\ = \quad & \frac{1}{n}\sum_{i,j}\mathrm{pf}S(i,j)s_{ij}e_{\mathbf{n}\setminus\{i,j\}}e_{ij} \\ = \quad & \frac{1}{2(k+1)}\left(\sum_{|J|=n-2}\mathrm{pf}s_J\right) \wedge \sum_{i,j} s_{ij}e_{ij} \\ = \quad & \frac{1}{k+1}P_k \wedge b. \end{aligned}$$

For the general case each factor $s_I e_I$ where $|I| = 2k+2$ has to be developed in a similar way.

COROLLARY 3.3 $1 + \mathrm{Pin}(g)$ *is invertible if and only if the real part of* g *is different from zero.*

Proof.
If $1 + \mathrm{Pin}(g)$ is invertible, there exists an s such that $\mathrm{Cay} s = \mathrm{Pin}(g)$. Hence $g = \lambda \mathrm{Pf} s$ for real λ, and the real part of g is not zero.

Conversely, assume that $[g]_0 \neq 0$. We can assume it to be 1 after rescaling, and then $g = \mathrm{Pf} s$, where the s is easily determined from the bivector part of g. Hence $\mathrm{Cay}\ s = \mathrm{Pin}(g)$, and so $1 + \mathrm{Pin}(g)$ is invertible. ∎

4. The Indefinite Case

Let I_{pq} be the matrix

$$I_{pq} = \begin{pmatrix} -I_p & 0 \\ 0 & I_q \end{pmatrix}.$$

$\mathrm{End}_-(\mathbb{R}^{pq})$ is given by the matrices s for which $s_{ij} = e_{ij}^2 s_{ji}$, or alternatively by the matrices s such that $I_{pq}s$ is antisymmetric. To find out anything about the Pfaffian we look for the bivector $a = \sum_{i<j} a_{ij} e_{ij}$ such that $a \cdot \vec{x} = s\vec{x}$. One easily finds that

$a_{ij} = -s_{ij}$ if $j \leq p$,
$a_{ij} = s_{ij}$ if $i > p$,
$a_{ij} = -s_{ij}$ if $i \leq p$ and $j > p$,

or simply $a_{ij} = -e_i^2 s_{ij}$. If we define the antisymmetric matrix t by $t = I_{pq}s$, then $a_{ij} = t_{ij}$. We define the complete Pfaffian for s in this case as being formally equal to the one of t in the positive definite case, that is

$$\mathrm{Pf} s = \sum_{I \subset \mathbf{n}} \mathrm{pf} t_I e_I.$$

Also in this case we have

Lemma 4.1. *Let* b *be the bivector*

$$b = -\frac{1}{2} \sum_{i,j=1}^{n} t_{ij} e_{ij}.$$

Then $\mathrm{Pf} s = \mathrm{oexp}(b)$.

Proof.
The inner product is used neither in the definition of the outer exponent, nor in the definition of the Pfaffian of t (in the positive definite case of course $t = s$).
. ∎

Of course Pfs is defined even in the case where $1 - s$ is not invertible. We prove that Pfs is invertible (and hence in the Clifford group) if and only if $1-s$ is invertible. In order to prove this the following development of a determinant w.r.t. the diagonal will be used: let $a = (a_{ij})$. Then

$$\det a = \sum_{I \subset \mathbf{n}} \det a_I \prod_{i \notin I} a_{ii},$$

where a_I is obtained keeping only the a_{ij} with i and j in I from a, and replacing the diagonal elements by zero. To prove this expression, let for $I \subset \mathbf{n}$, $P(I) = \{\pi \in \sigma(\mathbf{n}) : i \in I \Leftrightarrow \pi(i) \neq i\}$. Then

$$\begin{aligned}
\det a &= \sum_{\pi \in \sigma(\mathbf{n})} \operatorname{sgn}\pi \prod_{i=1}^{n} a_{i\pi(i)} \\
&= \sum_{I \subset \mathbf{n}} \sum_{\pi \in P(I)} \operatorname{sgn}\pi \prod_{i=1}^{n} a_{i\pi(i)} \\
&= \sum_{I \subset \mathbf{n}} \left(\prod_{i \notin I} a_{ii} \sum_{\pi \in P(I)} \operatorname{sgn}\pi \prod_{i \in I} a_{i\pi(i)} \right) \\
&= \sum_{I \subset \mathbf{n}} \det a_I \prod_{i \notin I} a_{ii}.
\end{aligned}$$

THEOREM 4.2

$$\mathrm{Pf}s\overline{\mathrm{Pf}s} = \det(1 - s).$$

Proof.
Assume first that $1 - s$ is invertible. Then, according to lemma 2.2 already $\mathrm{Pf}s\overline{\mathrm{Pf}s}$ is real. Developing the determinant of $I_{pq} - t = I_{pq}(1 - s)$ with respect to the elements of the diagonal gives

$$(-1)^p \det(1 - s) = \det(I_{pq} - t) = \sum_{I \subset \mathbf{n}} \det t_I F(I)$$

where $F(I)$ is the product of diagonal elements with indices not in I. This is clearly $(-1)^{p-k}$ where k is the number of elements of I smaller than or equal to p, in other words: $F(I) = (-1)^p e_I \overline{e_I}$. On the other hand $\mathrm{Pf}\, s \overline{\mathrm{Pf}\, s}$ is equal to its real part

$$\sum_{I \subset \mathbf{n}} (\mathrm{pf}\, t_I)^2 e_I \overline{e_I} = (-1)^p \sum_{I \subset \mathbf{n}} \det t_I F(I).$$

This proves the formula in the case $1-s$ is invertible. Both sides of the equation however are polynomials in the s_{ij}. By continuity the relation must hold even if $1 - s$ is not invertible. ∎

COROLLARY 4.3 $\mathrm{Pf}\, s \in \Gamma(p, q)$ *if and only if* $1 - s$ *is invertible.*

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