Union and intersection of Level-2 fuzzy regions

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***Abstract*—In many applications spatial data are considered, yet this data quite often are prone to uncertainty and imprecision. For this purpose, fuzzy regions have been developed. Our initial model, a fuzzy set over a two dimensional domain, allowed for both fuzzy regions and fuzzy points to be modelled. The model still had some shortcomings: all points where treated independently, and it was not possible to group points together. In some situations this makes the model more imprecise and uncertain than it should have been. Furthermore, while the model allowed for the representations of both fuzzy regions and fuzzy locations, simply by changing the interpretation of the fuzzy set, this interpretation needed to be specified as meta information. The model was extended to a level-2 fuzzy region to overcome these limitations, but this has an impact on operations. In this contribution, intersection and union will be discussed.**

I. Introduction

The concept of fuzzy regions, essentially fuzzy sets over a two dimensional domain, was introduced to overcome the limitations of spatial information systems in modelling uncertain or imprecise spatial features. The model allows for the representation of fuzzy regions (i.e. regions with undetermined boundaries), or fuzzy points (i.e. points at an imprecise or uncertain location). While the model improved the modelling of such features, some shortcomings prevent a true modelling of real world problems. The first of these shortcomings is the fact that all points in a fuzzy region are considered independently, making it impossible for a user to specify that some points belong together. The second shortcoming of is more subtle and concerns interpretations. Consider a fuzzy region: the membership grades carry a veristic interpretation: all the points belong to some extent to this region. However, it is possible to have additional knowledge on the boundary of the region and possible candidates. An example of this would be a lake with changing water levels: it will yield a fuzzy region for different water levels. As such, there is a possibilistic interpretation for some aspects: there is only one boundary at a time, but it is just unknown to us for some reason. It is not possible to give the fuzzy region the possibilistic interpretation, as this would imply that all the elements are candidates for a single point, thus not representing a region any more but more a fuzzy point (and a fuzzy set of candidate locations). The last shortcoming concerns the fact that a possibilistic interpretation can be adopted to represent fuzzy points, but the interpretation needs to be known by the system as it impacts some of the operations. This dependency of metadata can be a drawback, especially when data with different interpretations is considered. To overcome these limitations, a new extension using level-2 fuzzy sets has been developed and presented in [13]; in this contribution we consider this new extension and elaborate on the union and intersection of these new fuzzy regions.

II. Preliminaries

A. Fuzzy regions

The concept of the original fuzzy regions is very simple. A region can be seen as a set of points belonging together; from this point of view, it is a small step to augment the definition to a fuzzy set ([14], [15]) of points. In [8], the fuzzy region was defined over $\mathbb{R}^2$, thus with each element (point) a membership grade was associated.

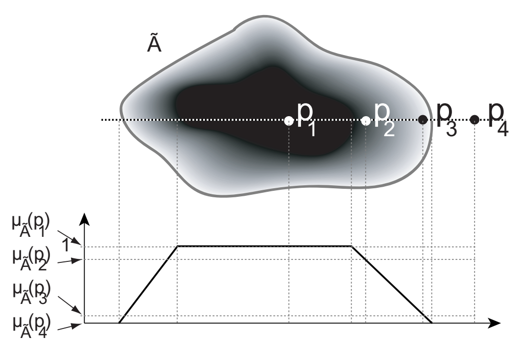

Fig. 1. The concept of fuzzy regions

1) Definitions: A fuzzy region essentially is a fuzzy set defined over a two dimensional domain; the concept is illustrated in Fig. 1. The definitions require a different view on the concept of a region: rather than defining the edge (by means of a polygon), the region is defined as a set of points, augmented to a fuzzy set of points. This yields the definition:

$$\tilde{R} = \{(p, \mu_{\tilde{R}}(p)) | p \in \mathbb{R}^2\} \quad (1)$$

This definition was extended to allow for grouping of points with the same membership grade [12]. In the fuzzy region above, the basic elements of the region are points. To group points together, we can consider groups of points to be regions;

these subregions are then assigned membership grades just like the points were. To obtain this, the concept of the powerset[1] is employed. By considering the powerset of $\mathbb{R}^2$ as the domain, the fuzzy region effectively becomes a fuzzy set of regions.

$$\tilde{R} = \{(P, \mu_{\tilde{R}}(P)) | P \in \wp(\mathbb{R}^2) \wedge \forall P_1, P_2 \in \tilde{R} : P_1 \cap P_2 = \emptyset\} \quad (2)$$

Note that the intersection between any two elements should be empty: it is required that no two elements of the fuzzy region share points. A point can only be considered to belong to the region once, even if it is to a membership grade less than 1. The concept is an extension of the above fuzzy region: if all subregions are singleton sets, the definition reverts to 1. The concept is illustrated in Fig. 2. The region $\tilde{R}$ consists of three non-overlapping subregions $\tilde{R}'_1$, $\tilde{R}'_2$, $\tilde{R}_3$, each of which carries a membership grade (the greyshade is the only indication of this). When a fuzzy region is defined by means of a limited number of subregions, the concept bears resemblance to the concept of plateau regions [3].

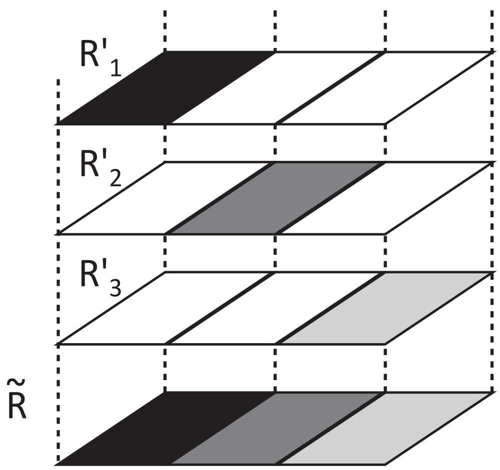

Fig. 2. The concept of fuzzy regions using the powerset. The region $\tilde{R}$ consists of three non-overlapping subregions $\tilde{R}'_1$, $\tilde{R}'_2$, $\tilde{R}_3$ that all carry a membership grade (not shown); these subregions can be as small as single points.

This extension was presented in [12]; and solves the first shortcoming by allowing elements of the fuzzy region to be grouped together. It does however imply that all the points that are grouped together must have the same membership grade.

2) Interpretations: The fuzzy regions are fuzzy sets, and as such the membership grades can carry one of three interpretations ([1]): versitic, possibilistic and degrees of truth. The veristic interpretation means that all points belong to the set to some specific extent; for fuzzy regions this interpretation is used to represent regions in which elements can have a partial membership. As such, the fuzzy region concept forms an extension to the traditional regions, commonly represented by polygons. When giving the fuzzy set a possibilistic interpretation, it means that one element of the set is considered to be valid, but there is uncertainty. The membership grades reflect this uncertainty. In the fuzzy region concept, this implies that all points are possible candidates for a specific crisp point. In this interpretation, we are modelling a point or location, but are not certain where the point or location is; the fuzzy set is used to model all the candidate locations for the point. The last interpretation, as degrees of truth, is currently not adopted in the fuzzy region model.

The interpretation of the fuzzy region is important, as it has an impact on the definition of several of the operators.

3) Operations: For fuzzy regions, a number of operations have been defined. This ranges from set operations, geometrical operations (bounding rectangle, convex hull; [8]), numerical operations (distance, surface area; [11]) to topology ([9]). In this contribution, we are only considering the set operations; because of this we repeat the set operations below. As the regions are fuzzy sets, the traditional fuzzy operations for intersection and union (t-norms and t-conorms) are immediately applicable. The intersection is defined as:

$$\tilde{R}'_1 \tilde{\cap} \tilde{R}'_2 = \{(x, \mu_{\tilde{R}'_1 \tilde{\cap} \tilde{R}'_2}(x))\} \quad (3)$$

with

$$\begin{aligned} \mu_{\tilde{R}'_1 \tilde{\cap} \tilde{R}'_2} : \mathbb{R}^2 &\mapsto [0,1] \\ x &\to T((\mu_{\tilde{R}'_1}(x), \mu_{\tilde{R}'_2}(x))) \end{aligned}$$

Here, T is a chosen T-norm for the intersection, e.g. the maximum. The union is defined as

$$\tilde{R}'_1 \tilde{\cup} \tilde{R}'_2 = \{(x, \mu_{\tilde{R}'_1 \tilde{\cup} \tilde{R}'_2}(x))\} \quad (4)$$

with

$$\begin{aligned} \mu_{\tilde{R}'_1 \tilde{\cap} \tilde{R}'_2} : \mathbb{R}^2 &\mapsto [0,1] \\ x &\to S((\mu_{\tilde{R}'_1}(x), \mu_{\tilde{R}'_2}(x))) \end{aligned}$$

Where S is an appropriate T-conorm, e.g. the maximum.

The set operations are independent of the interpretation; but it is assumed that both arguments carry the same interpretation, and that the end result will also carry the same interpretation. The set operations to suit the powerset extension mentioned above in equation (2) are similar and straight forward.

4) Limitations: In many situations a user has additional data concerning the distribution of membership grades, for instance knowledge that some points either belong to the region at the same time or don't belong to the region. The extension using the powerset only compensates to a limited extent, as only points with the same membership grade can be grouped together.

The second limitation concerns the interpretations; to illustrate consider the representation of a lake with varying water level: points at the same altitude around the lake will be flooded at the same time. The current model only allows for the lake to be modelled, with points (or groups of points) that belong to some extent to it. But every water level of the lake can give rise to a different fuzzy region, implying an additional level of uncertainty if the water level is not known. A veristic interpretation is needed as it is a region, so a possibilistic

[1]The powerset of a region A, denoted $\wp(\mathbb{A})$ is the set of all subsets of that region, including the emptyset and the region itself. For example: $\wp(\{0,1,2\}) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{0,1\}, \{0,2\}, \{1,2\}, \{0,1,2\}\}$.

interpretation of the set is not possible. The addition of a second level of uncertainty to the model would allow the fuzzy region in different circumstances to be modelled.

For fuzzy regions, a number of operations have been defined in the past, including distance and surface area [10]. Especially for the distance operation, the interpretation of the fuzzy set determines which definition of the distance is applicable. This dependency on meta data (the interpretation) makes the model less transparent and may give rise to confusion. A unified representation of both fuzzy points and fuzzy regions would be more elegant, and allow for a proper definition of operators that work on features with both interpretations.

B. Level-2 fuzzy regions

The level-2 fuzzy region is an extension to the traditional fuzzy regions. It is a further refinement of the extension introduced in [12] and defined above. In the previous extension, a fuzzy regions was defined as a fuzzy set of non-overlapping crisp regions. While it solved some initial issues, it had limitations among which the fact that there still was the need for additional metadata to carry the interpretation. The concept of the level-2 fuzzy region takes the *fuzzy set of regions* one step further by considering a *fuzzy set of fuzzy regions*. The elements are no longer considered subregions, but rather as possibilities for the region. The concept is illustrated in Fig. 3. The level-2 fuzzy region $\tilde{R}$ contains three possibilities: the fuzzy regions $\tilde{R}'_1$, $\tilde{R}'_2$ and $\tilde{R}'_3$. Each of this is a possible candidate for the feature we are modelling and thus is assigned a possibility (the membership grades are to be interpreted possibilistically) - these are not indicated. Every possible region is a fuzzy region and as such carries a veristic interpretation. This concept was first introduced in [13].

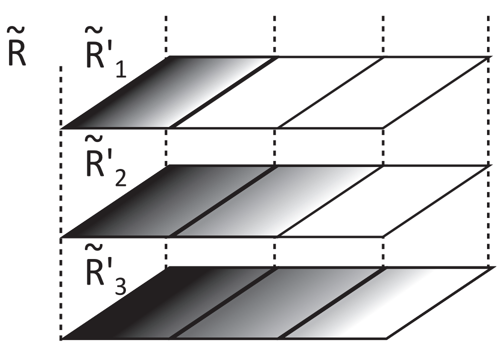

Fig. 3. The concept of level-2 fuzzy regions. The level-2 fuzzy region is a fuzzy set (with a possibilistic interpretation) that holds fuzzy regions. Each fuzzy region $\tilde{R}'_i$ can be seen as a candidate for the region being modelled (possibilistic interpretation) and as such the regions can overlap.

To create the definition, we will make use of the concept of the fuzzy powerset, denoted $\tilde{\wp}$. The fuzzy powerset of a set A is defined as the set of all fuzzy subsets of the set A. By considering $\tilde{\wp}(\mathbb{R}^2)$ as the domain, it is possible to define a level-2 fuzzy region similarly as has been done with the powerset.

$$\tilde{R} = \{(\tilde{R}', \mu_{\tilde{R}}(\tilde{R}')) | \tilde{R}' \in \tilde{\wp}(\mathbb{R}^2)\} \tag{5}$$

The membership function is defined as:

$$\begin{aligned} \mu_{\tilde{R}} : \tilde{\wp}(\mathbb{R}^2) &\mapsto [0,1] \\ \tilde{R}' &\rightarrow \mu_{\tilde{R}}(\tilde{R}') \end{aligned}$$

The elements of the fuzzy region are fuzzy regions as in the first definition (1) or the second definition (2). An important difference with the previous definition is that we now allow different elements to share points: the regions are not subregions, but candidate regions. This new definition is what is referred to as a level-2 fuzzy set: a fuzzy set defined over a fuzzy domain ([2],[4]). This concept is not to be confused with a type-2 fuzzy set ([5]), which is a fuzzy set defined over a crisp domain but with fuzzy membership grades. The type-2 fuzzy set would be more difficult to use as a model for candidate regions, as it would make it very complicated to still have the concept of a candidate region. The model now allows for a region to be represented by a number of fuzzy regions, each a candidate with a possibility degree.

III. Set operations on level-2 fuzzy regions

The level-2 fuzzy regions are level-2 fuzzy sets with an underlying two dimensional domain. The set operations are obtained by applying Zadeh's Extension principle [14] on both levels [2]. It should be noted that the set operations discussed here are pure set operations, intended to result in new level-2 fuzzy regions that comply with the concepts of union and intersection. For level-2 sets, it is also possible to consider alternative set operations that increase or decrease the number of possibilities (e.g. by only considering the common possible regions of two level-2 fuzzy regions). These are not considered to be set operations, but rather a higher level-reasoning with level-2 fuzzy sets, and will not considered in this paper.

A. Intersection of level-2 fuzzy regions

The intersection of two level-2 fuzzy regions is defined by means of a double application of the extension principle. The objective of the intersection is to provide a new region, which has the following interpretation. The two regions are obtained as the intersection of all possible fuzzy regions. The intersection of all possible regions is the set of all possible combinations of fuzzy regions in both sets.

$$\tilde{R}_1 \cap \tilde{R}_2 = \bigcup_{\substack{\tilde{R}'_1 \in \tilde{R}_1 \\ \tilde{R}'_2 \in \tilde{R}_2}} \{(\tilde{R}'_1 \cap \tilde{R}'_2, \mu_{\tilde{R}_1 \cap \tilde{R}_2}(\tilde{R}'_1 \cap \tilde{R}'_2))\} \tag{6}$$

The elements are fuzzy regions; the definition of their intersection is given in the previous section. The membership function is defined as:

$$\begin{aligned} \mu_{\tilde{R}} : \tilde{\wp}(\mathbb{R}^2) &\mapsto [0,1] \\ \mu_{\tilde{R}_1 \cap \tilde{R}_2}(\tilde{R}'_1 \cap \tilde{R}'_2) &\rightarrow T(\mu_{\tilde{R}_1}(\tilde{R}'_1), \mu_{\tilde{R}_2}(\tilde{R}'_2)) \end{aligned}$$

Where T also is a T-norm. It is not necessary to chose the same T-norm as for the intersection of the regions $\tilde{R}'_1$ and $\tilde{R}'_1$. The union operator in the definition (6) basically combines all possibilities of fuzzy regions at the possibilistic level of the fuzzy set.

B. Union of level-2 fuzzy regions

The union of two level-2 fuzzy regions is defined similarly to the intersection by means of a double application of the extension principle. The result is a new level-2 fuzzy region that holds all the possible combinations of unions between elements of both level-2 fuzzy regions.

$$\tilde{R}_1 \cup \tilde{R}_2 = \bigcup_{\substack{R'_1 \in \tilde{R}_1 \\ R'_2 \in \tilde{R}_2}} \{(\tilde{R}'_1 \cup \tilde{R}'_2, \mu_{\tilde{R}_1 \cup \tilde{R}_2}(\tilde{R}'_1 \cup \tilde{R}'_2))\} \quad (7)$$

As with the intersection, the definition of the union of the elements is given in the previous section. The membership function is defined as:

$$\begin{aligned} \mu_{\tilde{R}} : \tilde{\wp}(\mathbb{R}^2) &\mapsto [0,1] \\ \mu_{\tilde{R}_1 \cup \tilde{R}_2}(\tilde{R}'_1 \cup \tilde{R}'_2) &\rightarrow S(\mu_{\tilde{R}_1}(\tilde{R}'_1), \mu_{\tilde{R}_2}(\tilde{R}'_2)) \end{aligned}$$

Here, S is a T-conorm. Just as for the intersection; it is not required the conorm has to be the same as the one used to combine the subregions (but commonly this will be the case).

IV. Compatibility with earlier definitions

A. Impact on membership of points

In [13], the *membership* of points was considered. While this is not a true membership function (the region $\tilde{R}$ is a set that contains fuzzy regions), it is still possible to provide information on individual points. For every point, both its membership to every candidate fuzzy region $\tilde{R}'_i$ and the membership of the fuzzy region $\tilde{R}'_i$ in which it is contained are considered. This results in a type-2 fuzzy set as below:

$$\begin{aligned} \mu'_{\tilde{R}}(p) : \mathbb{R}^2 &\mapsto \widetilde{[0,1]} \\ p &\rightarrow \mu'_{\tilde{R}}(p) = \bigcup_{\tilde{R}' \in \tilde{R}} \{(\mu'_{\tilde{R}'}(p), \mu_{\tilde{R}}(\tilde{R}'))\} \end{aligned}$$

The membership grades of the point in the different candidate regions is combined with the memership grade of the candidate region in the level-2 fuzzy region. The type-2 fuzzy set expresses uncertainty of the point belonging to the region. When we consider two overlapping level-2 fuzzy regions $\tilde{R}_1$ and $\tilde{R}_2$, a point p will have type-2 fuzzy sets to indicate its membership in either: $\mu'_{\tilde{R}_1}(p)$ and $\mu'_{\tilde{R}_2}(p)$. In the following theorem, the relation between the membership of the point to the intersection of two level-2 fuzzy regions and the combination of the memberships of the points in the two level-2 fuzzy regions is investigated.

Theorem 1: $\mu'_{\tilde{R}_1 \cap \tilde{R}_2}(p) = T(\mu'_{\tilde{R}_1}(p), \mu'_{\tilde{R}_2}(p))$

Proof:

$$\begin{aligned} \mu'_{\tilde{R}_1 \cap \tilde{R}_2}(p) &= \bigcup_{\tilde{R}' \in \tilde{R}_1 \cap \tilde{R}_2} \{(\mu'_{\tilde{R}'}(p), \mu_{\tilde{R}_1 \cap \tilde{R}_2}(\tilde{R}'))\} \\ &= \bigcup_{\substack{\tilde{R}'_1 \in \tilde{R}_1 \\ \tilde{R}'_2 \in \tilde{R}_2}} \{(\mu'_{\tilde{R}'_1 \cap \tilde{R}'_2}(p), T(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(\tilde{R}'_2))\} \\ &= \bigcup_{\substack{\tilde{R}'_1 \in \tilde{R}_1 \\ \tilde{R}'_2 \in \tilde{R}_2}} \{T(\mu'_{\tilde{R}'_1}(p), \mu'_{\tilde{R}'_2}(p)), T(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(\tilde{R}'_2))\} \\ &= \bigcup_{\substack{\tilde{R}'_1 \in \tilde{R}_1 \\ \tilde{R}'_2 \in \tilde{R}_2}} \{\{(\mu'_{\tilde{R}'_1}(p), \mu_{\tilde{R}_1}(R'_1))\} \cap \{(\mu'_{\tilde{R}'_2}(p), \mu_{\tilde{R}_2}(\tilde{R}'_2))\} \\ &= \bigcup_{\tilde{R}'_1 \in \tilde{R}_1} \{(\mu'_{\tilde{R}'_1}(p), \mu_{\tilde{R}_1}(R'_1))\} \cap \bigcup_{\tilde{R}'_2 \in \tilde{R}_2} \{(\mu'_{\tilde{R}'_2}(p), \mu_{\tilde{R}_2}(\tilde{R}'_2))\} \\ &= \mu'_{\tilde{R}_1}(p) \cap \mu'_{\tilde{R}_2}(p) \end{aligned}$$

□

B. Compatibility with original fuzzy region model

The level-2 fuzzy regions unifies both the representation of known fuzzy regions (there is just one candidate region, but its elements are fuzzy) and the representation of fuzzy points (there are multiple candidates for a single point). For both of these, union and intersection has been considered; the definitions for level-2 fuzzy regions should still be compatible with them. The original fuzzy regions could be used to represent regions, in which case membershipgrades carried a veristic interpretation (all points of the fuzzy set belong to some extent to the set); or they could be used to represent a fuzzy point by using a possibilistic interpretation (all points in the set represent possible candidates for the point that is modelled. The interpretation was considered to be metadata; the user had to be aware of the interpretation as it impacted some operators. In the subsequent paragraphs, we will verify that the unified level-2 fuzzy region model complies with the intersection and union definitions, and as such is a straight forward and compatible extension of the original fuzzy region.

Theorem 2: The set operations on level-2 fuzzy regions are compatible with the set operations on fuzzy regions in a veristic interprestation.

Proof: When only considering one possibility in the level-2 fuzzy region (and assigning this the membership grade 1), the model can represent the same fuzzy region as before. Consider two level-2 fuzzy regions, each with only one candidate fuzzy region:

$$\begin{aligned} \tilde{R}_1 &= \{(\tilde{R}'_1, 1)\} \\ \tilde{R}_2 &= \{(\tilde{R}'_2, 1)\} \end{aligned}$$

Using the definition of the intersection of level-2 fuzzy regions, the intersection of both regions is:

$$\begin{aligned}\tilde{R}_1 \cup \tilde{R}_2 &= \bigcup_{\substack{R'_1 \in \tilde{R}_1 \\ R'_2 \in \tilde{R}_2}} \{(\tilde{R}'_1 \cap \tilde{R}'_2, \mu_{\tilde{R}_1 \cap \tilde{R}_2}(\tilde{R}'_1 \cap \tilde{R}'_2))\} \\ &= \{(\tilde{R}'_1 \cap \tilde{R}'_2, \mu_{\tilde{R}_1 \cap \tilde{R}_2}(\tilde{R}'_1 \cap \tilde{R}'_2))\}\end{aligned}$$

The associated membership grades will be

$$\begin{aligned}\mu_{\tilde{R}_1 \cap \tilde{R}_2}(\tilde{R}'_1 \cap \tilde{R}'_2) &= T(\mu_{\tilde{R}_1}(\tilde{R}'_1), \mu_{\tilde{R}_2}(\tilde{R}'_2)) \\ &= T(1,1) = 1\end{aligned}$$

This yields the same result. □

The proof for the union operator is completely similar.

Theorem 3: The set operations on level-2 fuzzy regions are compatible with the set operations on fuzzy regions in a possibilistic interprestation.

Proof: The representation of points was obtained by considering a fuzzy region, but employing a possibilistic interpretation. While the interpretation of the fuzzy set is of no importance to the set operations, the representation in the level-2 fuzzy region differs from the above representation, making it necessary to also verify this situation. The set operations should be seen as operations which increase or decrease the number of possible candidate points (or alter the membership grade). Fuzzy regions with a possibilistic interpretation can be represented by level-2 fuzzy regions by limiting the subregions to singleton sets in which the element has membership grade 1. Another limitation that needs to be imposed is that the singleton sets are mutually disjoint.

Consider two level-2 fuzzy regions $\tilde{R}_1$ and $\tilde{R}_2$ with only (non overlapping) singleton elements; their union is given by the definition:

$$\tilde{R}_1 \cup \tilde{R}_2 = \bigcup_{\substack{R'_1 \in \tilde{R}_1 \\ R'_2 \in \tilde{R}_2}} \{(\tilde{R}'_1 \cap \tilde{R}'_2, \mu_{\tilde{R}_1 \cap \tilde{R}_2}(\tilde{R}'_1 \cap \tilde{R}'_2))\}$$

Any two singleton fuzzy sets $\tilde{R}'_1$ and $\tilde{R}'_2$ either contain the same element, in which case it should be retained in the intersection (and with membership grade $T(\mu_{\tilde{R}_1}(\tilde{R}'_1), \mu_{\tilde{R}_2}(\tilde{R}'_2))$, but as the membership grades in all singleton sets is 1, the resulting membership grade will also be 1), or they contain a different element in which case it should not be and will not be in the resulting set (i.e. have membership grade 0). The application of the extension principle at the higher (possibilistic) level of the level-2 fuzzy set is thus in line with the original definition of the intersection of fuzzy regions in a possibilistic interpretation. □

V. Examples

To illustrate union and intersection of two level-2 fuzzy regions, consider two regions as defined below.

$$\tilde{R}_1 = \{(\tilde{R}'_1, 1), (\tilde{R}'_2, 0.6), (\tilde{R}'_3, 0.4)\}$$
$$\text{where} \begin{cases} \tilde{R}'_1 = \{(p_1, 1), (p_2, 0.7)\} \\ \tilde{R}'_2 = \{(p_1, 0.8), (p_2, 0.6)\} \\ \tilde{R}'_3 = \{(p_1, 0.4), (p_3, 0.8)\} \end{cases}$$

and

$$\tilde{R}_2 = \{(\tilde{R}'_4, 1), (\tilde{R}'_5, 0.6), (\tilde{R}'_6, 0.4)\}$$
$$\text{where} \begin{cases} \tilde{R}'_4 = \{(p_1, 1), (p_2, 0.6)\} \\ \tilde{R}'_5 = \{(p_1, 0.6), (p_4, 0.7)\} \\ \tilde{R}'_6 = \{(p_4, 0.4), (p_5, 0.6)\} \end{cases}$$

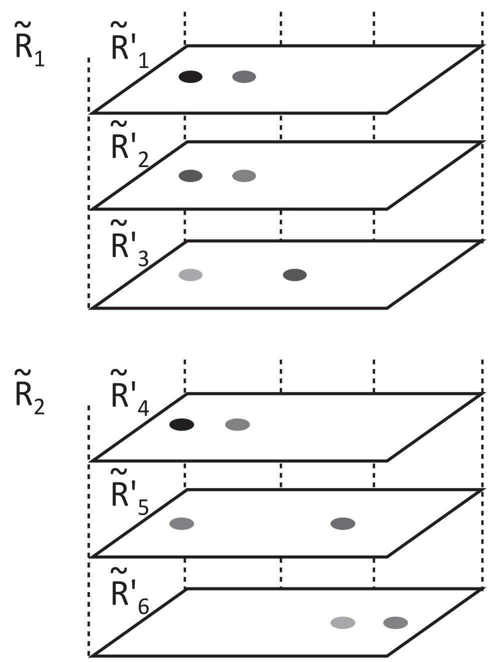

Fig. 4. The two regions used in the examples. In total, only 5 points are used (numbered p_1 to p_5 from left to right). The greyscale of the points reflects their membership grades.

Following the definition, the intersection of both regions will be given by:

$$\begin{aligned}\tilde{R}_1 \cap \tilde{R}_2 = \;& \{(\tilde{R}'_7, \min(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(R'_4)))\} \\ & \cup\{(R'_8, \min(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(R'_5)))\} \\ & \cup\{(R'_9, \min(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(R'_6)))\} \\ & \cup\{(R'_{10}, \min(\mu_{\tilde{R}_1}(R'_2), \mu_{\tilde{R}_2}(R'_4)))\} \\ & \cup\{(R'_{11}, \min(\mu_{\tilde{R}_1}(R'_2), \mu_{\tilde{R}_2}(R'_5)))\} \\ & \cup\{(R'_{12}, \min(\mu_{\tilde{R}_1}(R'_2), \mu_{\tilde{R}_2}(R'_6)))\} \\ & \cup\{(R'_{13}, \min(\mu_{\tilde{R}_1}(R'_3), \mu_{\tilde{R}_2}(R'_4)))\} \\ & \cup\{(R'_{14}, \min(\mu_{\tilde{R}_1}(R'_3), \mu_{\tilde{R}_2}(R'_5)))\} \\ & \cup\{(R'_{15}, \min(\mu_{\tilde{R}_1}(R'_3), \mu_{\tilde{R}_2}(R'_6)))\}\end{aligned} \tag{8}$$

with

$$
\begin{array}{rcl}
\tilde{R}'_7 & = & \tilde{R}'_1 \cap \tilde{R}'_4 \\
 & = & \{(p_1, \min(1,1)), (p_2, \min(0.7, 0.6))\} \\
 & = & \{(p_1, 1), (p_2, 0.6)\} \\
\tilde{R}'_8 & = & \tilde{R}'_1 \cap \tilde{R}'_5 \\
 & = & \{(p_1, \min(1, 0.6)), (p_2, \min(0.7, 0)), \\
 & & (p_4, \min(0, 0.7))\} \\
 & = & \{(p_1, 0.6)\} \\
\tilde{R}'_9 & = & \tilde{R}'_1 \cap \tilde{R}'_6 \\
 & = & \{(p_1, \min(1, 0)), (p_2, \min(0.7, 0)), \\
 & & (p_4, \min(0, 0.4)), (p_5, \min(0, 0.6))\} \\
 & = & \emptyset \\
\tilde{R}'_{10} & = & \tilde{R}'_2 \cap \tilde{R}'_4 \\
 & = & \{(p_1, \min(0.8, 1)), (p_2, \min(0.7, 0.6))\} \\
 & = & \{(p_1, 0.8), (p_2, 0.6)\} \\
\tilde{R}'_{11} & = & \tilde{R}'_2 \cap \tilde{R}'_5 \\
 & = & \{(p_1, \min(0.8, 0.6)), (p_2, \min(0.6, 0)), \\
 & & (p_4, \min(0, 0.7))\} \\
 & = & \{(p_1, 0.6)\} \\
\tilde{R}'_{12} & = & \tilde{R}'_2 \cap \tilde{R}'_6 \\
 & = & \{(p_1, \min(0.8, 0)), (p_2, \min(0.6, 0)), \\
 & & (p_4, \min(0, 0.7)), (p_5, \min(0, 0.6))\} \\
 & = & \emptyset \\
\tilde{R}'_{13} & = & \tilde{R}'_3 \cap \tilde{R}'_4 \\
 & = & \{(p_1, \min(0.4, 1)), (p_2, \min(0, 0.6)), \\
 & & (p_3, \min(0.8, 0))\} \\
 & = & \{(p_1, 0.4)\} \\
\tilde{R}'_{14} & = & \tilde{R}'_3 \cap \tilde{R}'_5 \\
 & = & \{(p_1, \min(0.4, 0.6)), (p_3, \min(0.8, 0)), \\
 & & (p_4, \min(0, 0.7))\} \\
 & = & \{(p_1, 0.4)\} \\
\tilde{R}'_{15} & = & \tilde{R}'_3 \cap \tilde{R}'_6 \\
 & = & \{(p_1, \min(0.4, 0)), (p_3, \min(0.8, 0)), \\
 & & (p_4, \min(0, 0.4)), (p_5, \min(0, 0.6))\} \\
 & = & \emptyset
\end{array}
\qquad (9)
$$

As $R'_8 = R'_{11}$ and $R'_{13} = R'_{14}$ and some R' would only carry elements with membership grade 0 , this simplifies to

$$
\tilde{R}_1 \cap \tilde{R}_2 = \{(\tilde{R}'_7, 1)\} \cup \{(R'_8, 0.6)\} \cup \{(R'_{10}, 0.6)\} \cup \{(R'_{13}, 0.4)\} \qquad (10)
$$

with the elements defined as above. Note that R_7 and R_{10} contain the same points, but as their elements carry different membership grades, they are different - albeit fully overlapping - regions. The result is illustrated in Fig. 5.

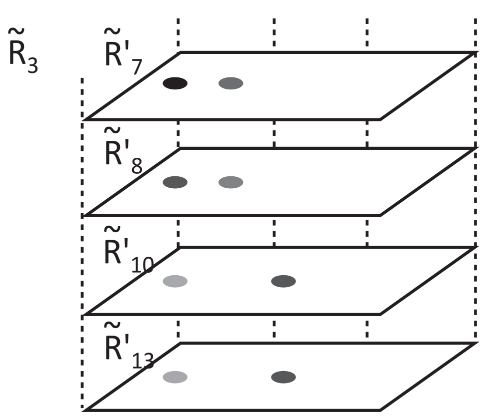

Fig. 5. The intersection of two regions $\tilde{R}'_1$ and $\tilde{R}'_2$ using the minimum as t-norm

The union of both regions will be obtained similarly:

$$
\begin{array}{rl}
\tilde{R}_1 \cup \tilde{R}_2 = & \{(\tilde{R}'_7, \max(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(R'_4)))\} \\
 & \cup\{(R'_8, \max(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(R'_5)))\} \\
 & \cup\{(R'_9, \max(\mu_{\tilde{R}_1}(R'_1), \mu_{\tilde{R}_2}(R'_6)))\} \\
 & \cup\{(R'_{10}, \max(\mu_{\tilde{R}_1}(R'_2), \mu_{\tilde{R}_2}(R'_4)))\} \\
 & \cup\{(R'_{11}, \max(\mu_{\tilde{R}_1}(R'_2), \mu_{\tilde{R}_2}(R'_5)))\} \\
 & \cup\{(R'_{12}, \max(\mu_{\tilde{R}_1}(R'_2), \mu_{\tilde{R}_2}(R'_6)))\} \\
 & \cup\{(R'_{13}, \max(\mu_{\tilde{R}_1}(R'_3), \mu_{\tilde{R}_2}(R'_4)))\} \\
 & \cup\{(R'_{14}, \max(\mu_{\tilde{R}_1}(R'_3), \mu_{\tilde{R}_2}(R'_5)))\} \\
 & \cup\{(R'_{15}, \max(\mu_{\tilde{R}_1}(R'_3), \mu_{\tilde{R}_2}(R'_6)))\}
\end{array}
\qquad (11)
$$

with

$$
\begin{array}{rcl}
\tilde{R}'_7 & = & \tilde{R}'_1 \cup \tilde{R}'_4 \\
 & = & \{(p_1, \max(1, 1)), (p_2, \max(0.7, 0.6))\} \\
 & = & \{(p_1, 1), (p_2, 0.7)\} \\
\tilde{R}'_8 & = & \tilde{R}'_1 \cup \tilde{R}'_5 \\
 & = & \{(p_1, \max(1, 0.6)), (p_2, \max(0.7, 0)), \\
 & & (p_4, \max(0, 0.7))\} \\
 & = & \{(p_1, 1), (p_2, 0.7), (p_4, 0.7)\} \\
\tilde{R}'_9 & = & \tilde{R}'_1 \cup \tilde{R}'_6 \\
 & = & \{(p_1, \max(1, 0)), (p_2, \max(0.7, 0)), \\
 & & (p_4, \max(0, 0.4))), (p_5, \max(0, 0.6))\} \\
 & = & \{(p_1, 1), (p_2, 0.7), (p_4, 0.4), (p_5, 0.6)\} \\
\tilde{R}'_{10} & = & \tilde{R}'_2 \cup \tilde{R}'_5 \\
 & = & \{(p_1, \max(0.8, 0.6)), (p_2, \max(0.7, 0)), \\
 & & (p_4, \max(0, 0.7))\} \\
 & = & \{(p_1, 0.8), (p_2, 0.7), (p_4, 0.7)\} \\
\tilde{R}'_{11} & = & \tilde{R}'_2 \cup \tilde{R}'_6 \\
 & = & \{(p_1, \max(0.8, 0)), (p_2, \max(0.6, 0)), \\
 & & (p_4, \max(0, 0.4)), (p_5, \max(0, 0.6))\} \\
 & = & \{(p_1, 0.8), (p_2, 0.6), (p_4, 0.4), (p_5, 0.6)\} \\
\tilde{R}'_{12} & = & \tilde{R}'_3 \cup \tilde{R}'_4 \\
 & = & \{(p_1, \max(0.4, 1)), (p_2, \max(0, 0.6)), \\
 & & (p_3, \max(0.8, 0))\} \\
 & = & \{(p_1, 1), (p_2, 0.6), (p_3, 0.8)\} \\
\tilde{R}'_{13} & = & \tilde{R}'_3 \cup \tilde{R}'_5 \\
 & = & \{(p_1, \max(0.4, 0.6)), (p_3, \max(0.8, 0)), \\
 & & (p_4, \max(0, 0.7))\} \\
 & = & \{(p_1, 0.6), (p_3, 0.8), (p_4, 0.7)\} \\
\tilde{R}'_{14} & = & \tilde{R}'_3 \cup \tilde{R}'_6 \\
 & = & \{(p_1, \max(0.4, 0)), (p_3, \max(0.8, 0)), \\
 & & (p_4, \max(0, 0.4)), (p_5, \max(0, 0.6))\} \\
 & = & \{(p_1, 0.4)), (p_3, 0.8), (p_4, 0.4), (p_5, 0.6)\}
\end{array}
\qquad (12)
$$

This result is illustrated in Fig. 6.

VI. Conclusion

In this contribution, we presented the union and intersection operators of level-2 fuzzy regions. Level-2 fuzzy regions allow for the representation of fuzzy regions under uncertainty. They also unify the possibilistic and veristic interpretations of the original fuzzy regions and can thus be used to represent a point or a (fuzzy) region of which there exists uncertainty. The set operations are compatible with the original fuzzy region and fuzzy point definitions. The fuzzy region concept has some parallels or applications in the use of suitability maps. At the moment, fuzzy regions are a theoretical concept with some success in prototype applications. The union and intersection were considered purely from the point of view where the fuzzy regions are considered as an extension of regions (e.g. in a

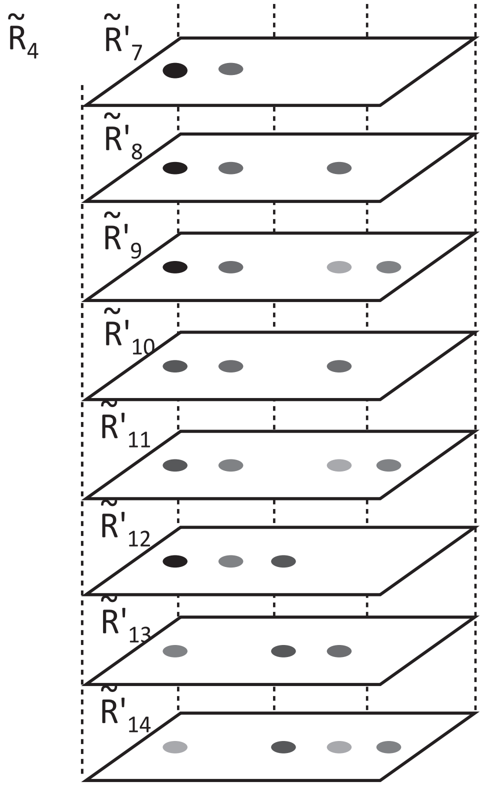

Fig. 6. The union of two regions $\tilde{R}_1'$ and $\tilde{R}_1'$ using the maximum as t-conorm

GIS), and are defined from that point of view. Due to the possible use in suitability maps, further research is required to bring the concepts closer and define adequate operations for the use of the level-2 fuzzy regions in this context; this includes weighted aggregation.

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