International Conference on Fractional Differentiation and its Applications,
Novi Sad, Serbia, July 18 - 20, 2016 ICFDA16

An indirect discretization method for fractional order PID controllers

Cristina I. Muresan[1], **Robin De Keyser**[2], **Clara Ionescu**[2]

[1] Department of Automation
Technical University of Cluj-Napoca, Memorandumului 28, Cluj-Napoca, Romania
e-mail: Cristina.Muresan@aut.utcluj.ro
[2] Department of Electrical energy, Systems and Automation
Ghent University, , Technologiepark 914, Gent, Belgium
e-mail: Robain.DeKeyser@UGent.be, ClaraMihaela.Ionescu@UGent.be

Abstract: The well-established advantages of fractional order PID controllers are only shadowed by the problems arising from their digital implementations. In this paper a new *s* to *z* transform is presented to produce digital realizations of fractional order PID controllers. The approach consists in obtaining first a continuous-time approximation of the ideal fractional order PID controller to which the new transform is applied to obtain the discrete-time rational transfer function. The mapping of the poles and zeros of the analog transfer function to the corresponding poles and zeros of the digital transfer function is discussed. It is shown that the discrete-time approximation based on the proposed *s* to *z* transform ensures a good fit of the ideal fractional order PID controller. An experimental study is also given.

Key words: fractional order PID controller, digital realization, indirect discretization, new *s* to *z* transform, experimental results

1. Introduction

The PID (proportional-integrative-derivative) controllers are amongst the most widely used control strategies in numerous industrial applications [1], accounting for more than 90% of the control loops [2]. The simplicity of the design and implementation of these controllers, coupled with their good performance, leads to many studies to improve the closed loop results, both in terms of new structures, as well as new tuning methods [1,3-5]. Such an attempt also leads to the generalization of the classical PID controller to the fractional order one [6]. The new controller is based on the use of a fractional integrator of order μ and a fractional differentiator of order λ, instead of the classical integer order integrator and differentiator [7]. The main advantage of the new generalized PID controller consists in a better shaping of the closed loop responses, mainly due to the two supplementary tuning parameters involved, the fractional orders μ and λ [8-13].

Unfortunately, this major advantage of fractional order $PI^{\mu}D^{\lambda}$ (FO-PID) controllers is shadowed by the problems that occur in terms of realization. An important aspect of fractional order derivatives and integrators is their infinite memory. Because of their infinite memory, the most difficult problem to be solved consists in the implementation of controllers that use fractional order differentiators and integrators. Quite often, the continuous-time FO-PID controllers are designed according to a set of performance specifications and then for practical implementation issues, a discrete-time approximation is computed.

The digital realizations of fractional order operators may be performed in two ways: either using indirect discretization or direct discretization methods. The indirect method presumes that first a continuous-time integer order transfer function is obtained that approximates the FO-PID controller and then the fitted continuous time transfer function is discretized. The direct methods, on the other hand, are based on power series expansion or continued fraction expansion, to name just a few, combined with a suitable generating function that maps the continuous time operator, s, into its discrete equivalent $w(z^{-1})$, where w is the generating function and z^{-1} is the backward shift operator. The focus of this paper is on indirect discretization methods.

Perhaps the continuous-time approximation most widely used is based on Oustaloup' s Recursive Approximation (ORA) method [8,14,15], as the fitting quality is much superior to those obtained with continued fraction based approaches [8]. The method implies that the fractional order operator s^{α} may be estimated as a generalized filter designed as:

$$F(s) = K\prod_{k=1}^{N}\frac{s+\omega_k'}{s+\omega_k} \quad (1)$$

where the poles, zeros and the gain are computed as $\omega_k' = \omega_b\omega_u^{(2k-1-\mu)/N}$, $\omega_k = \omega_b\omega_u^{(2k-1+\mu)/N}$ and $K = \omega_h^{\mu}$, with $\omega_u = \sqrt{\omega_h/\omega_b}$ and (ω_b, ω_h) is the frequency range where the filter in (1) must fit the original fractional order operator. The major drawback of Oustaloup' s Recursive Approximation method consists in the high order of the continuous-time (1) required to approximate the initial fractional order operator. This leads to an even higher order for the continuous-time transfer function that approximates the FO-PID controller, described as:

$$C_{FO-PID}(s) = k_p + \frac{k_i}{s^{\mu}} + k_d s^{\lambda} \quad (2)$$

where μ, $\lambda\epsilon(0\div2)$ and k_p, k_i and k_d are the proportional, integral and derivative gains.

Once the continuous-time approximation of the FO-PID controller has been obtained, its discrete-time equivalent is computed by using appropriate s to z transforms. The most efficient of these transforms are the Al-Alaoui and the bilinear (Tustin) transforms [16,17,18,19]. In [8], the *c2d* Matlab function is offered as a solution to discretize the continuous-time approximation of the FO-PID controller, using the Tustin transform with frequency prewarping. However, as indicated in [15], the *c2d* function might cause problems when used in the discretization of a high order continuous-time approximation of a FO-PID controller. Some of the unwanted effects that may occur when the continuous-time controller is replaced with an approximate discrete-time controller, and which should be avoided, refer to a possible decrease of the phase margin because of the sample-and-hold or even instability, due to truncation errors.

In this paper, a new s to z transform is proposed. The mapping of the s plane poles and zeros into the z plane is also given, along with the computation of the equivalent discrete-time gain. It is shown that using the proposed s to z transform, a discrete-time approximation is obtained that ensures a good fit of the initial FO-PID controller.

2. The digital implementation of the fractional order $PI^{\lambda}D^{\mu}$ controller

It is probably well established that one of the most common approaches in producing a continuous time rational approximation of a FO-PID controller is based on the Oustaloup Recursive Approximation methods [14]. However, even though the ORA method produces excellent continuous-time approximations of the FO-PID controller, a discrete-time realization is required for its final digital implementation. Therefore, the discretization of the fitted continuous-time rational transfer function is necessary. A great disadvantage here consists in the higher order of the continuous-time rational transfer function that needs to be discretized. And problems may arise [15] when using Matlab built-in functions such as the continuous-to-discrete-time function, *c2d*. To exemplify this, consider the following FO-PID controller transfer function:

$$C_{FO-PID}(s) = 16 + \frac{25}{s^{0.5}} + 2.5s^{0.5} \tag{3}$$

To obtain the continuous-time rational approximation of (3), the ORA method is used for the fractional order integrator and differentiator, with N=2, in a frequency range bounded by a high frequency end equal to ω_H=1000 rad/s and a low frequency end equal to $\omega_L=0.0001\omega_H$ rad/s. A final 10^{th} order continuous-time rational transfer function is obtained to approximate the FO-PID controller in (3). The discrete-time equivalent of the 10^{th} order fitted continuous-time rational is computed according to Tustin method, with a sampling period of 0.00314 seconds and using Matlab *c2d* function. To achieve this, the Laplace operator *s* in the rational continuous-time transfer function is replaced with:

$$s = \frac{2}{T_s}\frac{1 - z^{-1}}{1 + z^{-1}} \tag{4}$$

where T_s is the sampling time and z^{-1} is the backward shift operator. Figure 1 shows the continuous-time ideal FO-PID, as well as its discrete-time approximation. Although the high frequency approximation is acceptable, problems occur at low frequencies.

3. A new s to z transform

An alternative indirect discretization method is proposed in what follows, which is based on using a different generating function, as an interpolation of the Euler and Tustin discretization rules:

$$s = \frac{1+\alpha}{T_s}\frac{1 - z^{-1}}{1 + \alpha z^{-1}} \quad \text{or} \quad s = \frac{1+\alpha}{T_s}\frac{z - 1}{z + \alpha} \tag{5}$$

with $\alpha \epsilon [0 \div 1]$ – a weighting parameter. For α=0, the Euler discretization rule is obtained, while α=1 leads to the Tustin discretization rule. The choice of the parameter α has effect on the frequency response, weighing the magnitude error versus the phase error. The inverse relation is obtained based on (5) as:

$$z = \frac{1+\alpha+\alpha s T_s}{1+\alpha - sT_s} \tag{6}$$

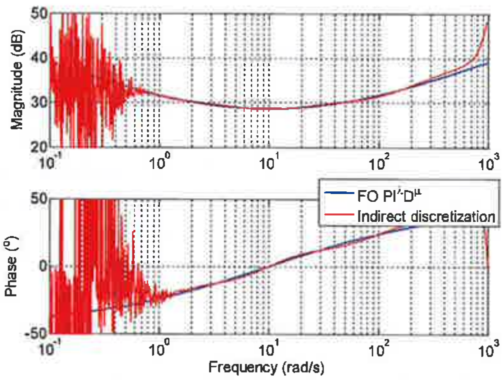


Figure 1. Problems when using Matlab *c2d* function in indirect discretization of a fractional order PI$^\lambda$D$^\mu$ controller transfer function

Consider the general rational continuous-time transfer function with *m* zeros and *n* poles, $m \le n$:

$$G(s) = K_c \frac{\prod_{j=1}^{m}\left(s - z_j^c\right)}{\prod_{i=1}^{n}\left(s - p_i^c\right)} \tag{7}$$

where K_c is the gain, z_j^c are the zeros, $j = \overline{1,m}$ and p_i^c are the poles, $i = \overline{1,n}$. Then, its discrete-time equivalent has the form:

$$G(z) = K_d \frac{\prod_{j=1}^{n}\left(z - z_j^d\right)}{\prod_{i=1}^{n}\left(z - p_i^d\right)} \tag{8}$$

where K_d is the corresponding discrete-time transfer function gain and z_j^d and p_i^d are the zeros and the poles, respectively. To determine the value of a discrete-time pole or zero, the relation in (6) is used as follows:

$$p_i^d = \frac{1+\alpha+\alpha p_i^c T_s}{1+\alpha - p_i^c T_s} \tag{9}$$

$$z_j^d = \frac{1+\alpha+\alpha z_j^c T_s}{1+\alpha - z_j^c T_s} \tag{10}$$

with a proper selection of the weighting factor α, as well as of the sampling time T_s. Once the discrete-time poles and zeros have been determined, the computation of the discrete-time gain K_d is based on:

$$K_d = K_c \frac{\prod_{j=1}^{m}\left(-z_j^c\right)\prod_{i=1}^{n}\left(1-p_i^d\right)}{\prod_{i=1}^{n}\left(-p_i^c\right)\prod_{j=1}^{n}\left(1-z_j^d\right)} \qquad (11)$$

The relation in (11) is obtained by making $G_c(s) = G_d(z)$ in steady-state regime, that is $G_c(0) = G_d(1)$.

Remarks:

1. To calculate the discrete-time gain K_d using (11), all pure integrators and differentiators need to be removed.
2. Continuous-time transfer function with dead time are converted in two steps. The rational part is converted using (9), (10) and (11), while the dead time is converted separately to the nearest integer number of samples: $e^{-s\tau_d} \Leftrightarrow z^{-d}$ with $\tau_d \cong dT_s$.
3. If $m<n$, then $(n-m)$ continuous-time zeros at $-\infty$ are converted into discrete-time zeros at $-\alpha$. In this case, $(n-m)$ zeros are added in discrete-time: $z_j^d = -\alpha$ with $j = \overline{m+1, n}$.

Figure 2 shows the ideal frequency response of the FO-$PI^\lambda D^\mu$ controller in (3), as well as the rational discrete-time approximation obtained using the proposed generating function.

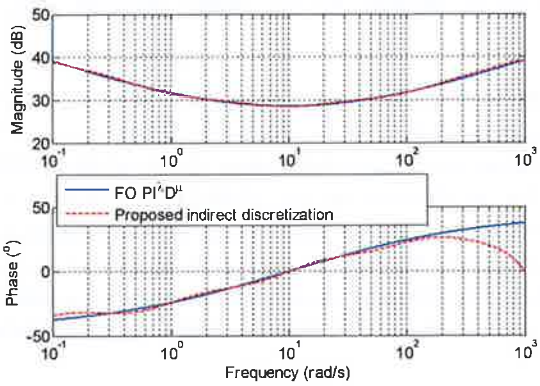


Figure 2. Frequency response of the $PI^\lambda D^\mu$ controller and the rational approximation obtained using the proposed indirect discretization method

To compute the discrete-time rational transfer function, the 10th order continuous-time rational transfer function obtained in Section 2 is discretized using the proposed method. For the discretization, the same sampling time is used as in Figure 1, T_s=0.00314 seconds, while α=1. With this choice of the weighting parameter α, the proposed generating function equals the Tustin operator, the same discretization method used in Section 2. Figure 2 shows that with the proposed generating function, an improved low frequency response is obtained in comparison to the result in Figure 1.

4. Case study: FO-PI$^{\lambda}$D$^{\mu}$ controller for DC motor speed

The experimental unit consists of the modular servo system built by Inteco [20] as indicated in Figure 3. The system consists of a DC motor equipped with a tachogenerator, which is used to measure the rotational speed of the DC motor, an inertia load, backlash, incremental encoder, and gearbox with output disk [21].

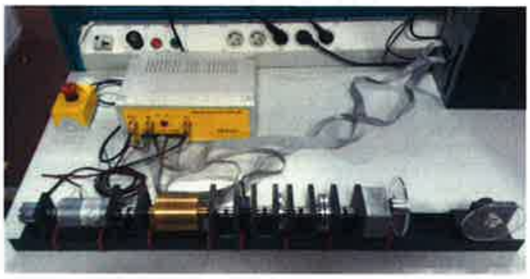


Figure 3. The modular servo system used as a case study

To control the rotational speed by varying the voltage supplied to the DC motor, the mathematical model of the modular servo system has been determined experimentally as:

$$P(s) = \frac{k}{Ts+1} \tag{12}$$

where k=158 and T=0.95 seconds are the motor nominal gain and time constant, respectively.

The next step consists in the tuning of a FO-PI$^{\mu}$D$^{\lambda}$ controller for the DC motor. The performance specifications are: a gain crossover frequency $\omega_{gc=}3$, a phase margin, φ_m=60^{o}, as well as robustness to gain variations. A bound B=-20dB is imposed for the sensitivity function, for all frequencies $\omega \leq 0.001$ rad/s, while for the complementary sensitivity function, a limit A=-20dB is considered, for all frequencies $\omega \geq 100$ rad/s. To tune the FO-PI$^{\mu}$D$^{\lambda}$ controller, the Matlab optimization toolbox is considered, using the *fmincon()* function [22]. The resulting controller parameters are determined as: k_p=0.007, k_i=5.5882, k_d=0.078, μ=0.78 and λ=0.9, with the final controller given as:

$$H_{FO-PID}(s) = 0.007\left(1 + \frac{5.5882}{s^{0.78}} + 0.078s^{0.9}\right) \tag{13}$$

The FO-PI$^{\mu}$D$^{\lambda}$ controller, the fractional order integrator and differentiator are first approximated using the Recursive Oustaloup Filter method [14] used with a low frequency bound of 0.0157 and a high frequency bound of 157 rad/sec, as well as an order of approximation N=2. The frequency response of the open loop system with the designed fractional order controller is given in Figure 4, considering the nominal DC motor model in (12). The phase margin and gain crossover frequencies are met. The Bode diagrams corresponding to the uncertain open loop system with +25% and -25% gain variations are also given in Figure 4. The phase margin in these two cases remains constant around 60°. Thus, the robustness to gain variations is also met. Finally, the Bode diagrams for the sensitivity function and complementary sensitivity function are analyzed in Figure 5a) and b), respectively. In this case also, the two performance specifications have been met.

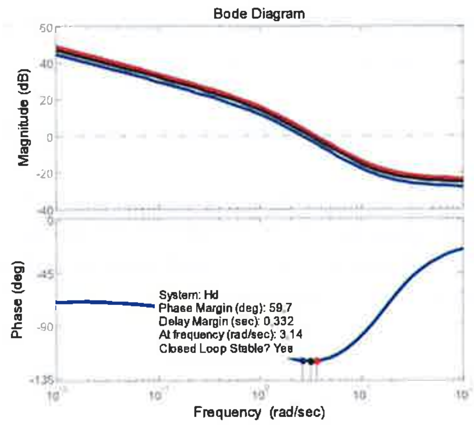
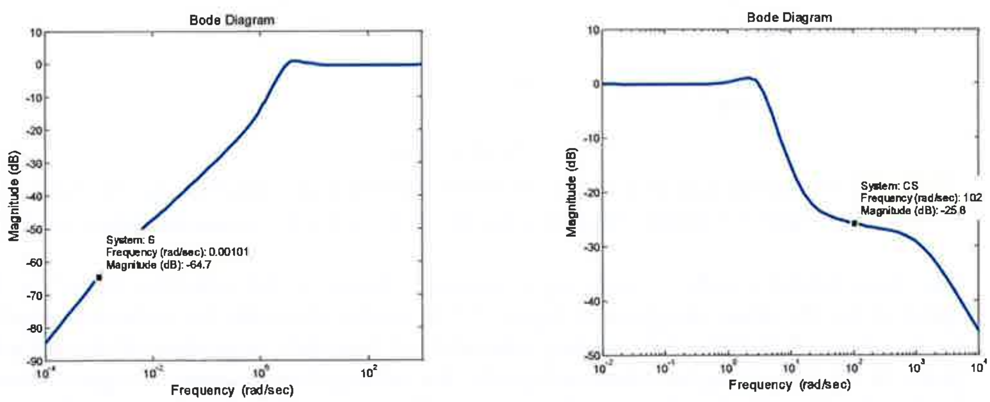


Figure 4. Bode diagrams of the nominal and uncertain open loop system with ±25% variation of the open loop gain (black for nominal system, blue for -25% gain variation, red for +25% gain variation)



a) b)

Figure 5. Bode diagrams of the a) sensitivity function b) complementary sensitivity function

The rational continuous-time approximation of the FO-PI$^{\mu}$D$^{\lambda}$ controller in (13) is computed as:

$$H_{FO-PID}(s)=\frac{\begin{array}{c}0.42\,(s+4.93)(s+3.76)(s+0.34)(s+0.28)(s+0.022)\\(s+0.018)(s^2+252.5s+1.6\cdot 10^4)(s^2+25.97s+266.8)\end{array}}{\begin{array}{c}(s+1368)(s+134.2)(s+86.3)(s+8.47)(s+5.45)(s+0.53)\\(s+0.34)(s+0.034)(s+0.0217)(s+0.00213)\end{array}} \quad (14)$$

To implement the designed FO-PI$^{\mu}$D$^{\lambda}$ controller, the proposed discretization method is applied to the rational continuous-time approximation in (14). The discrete-time poles, zeros and gain are computed according to (9), (10) and (11), yielding the final rational transfer function:

$$H_{FO-PID}(z)=\frac{\begin{array}{c}0.029\,(z-0.9089)(z-0.9293)(z-0.9932)(z-0.9944)(z-0.99956)\\(z-0.99963)(z^2-0.3729z+0.03481)(z^2-1.537z+0.6081)\end{array}}{\begin{array}{c}(z+0.15)(z-0.17)(z-0.29)(z-0.85)(z-0.9)(z-0.9894)\\(z-0.9932)(z-0.99932)(z-0.99956)(z-0.99995)\end{array}} \quad (15)$$

For the generating function, α=0.2 and the sampling period is T_s=0.02 seconds. Figure 6 shows the ideal frequency response of the FO-PI$^{\mu}$D$^{\lambda}$ controller in (13), along with the frequency response of its discrete-time rational approximation in (15). The Bode diagrams show a good agreement between the two.

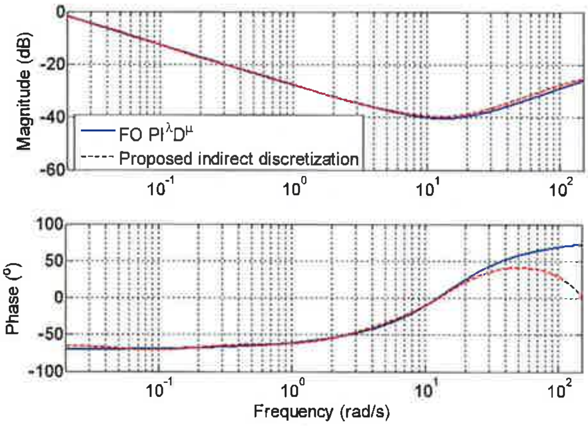


Figure 6. Frequency response of the FO-PI$^{\lambda}$D$^{\mu}$ controller designed for the DC motor and the rational approximation obtained using the proposed indirect discretization method

The experimental results considering a staircase change of the reference signal for the speed of the DC motor are given in Figure 7. The results show that the controller achieves an overshoot of 20-23% and a settling time of 2.1-2.3 seconds, regardless of the operating point. In the case of higher rotational speeds, the settling time increases to approximately 4 seconds. Overall, the digital realization in (15) ensures the closed loop performance specifications used for the design of the FO-PID controller in (13).

4. Conclusions

The major drawback when using a FO-PID controller consists in its rational approximation. For practical implementation, the digital realization of the controller is required. Several approaches have been proposed in this regard, each having their advantages and limitations. In this paper an indirect discretization method is proposed. The approach consists in obtaining first a continuous-time approximation of the ideal fractional order PID controller to which a new s to z transform is applied to obtain the discrete-time rational transfer function. The mapping of the poles and zeros of the analog transfer function to the corresponding poles and zeros of the digital transfer function is discussed. An experimental case study is considered to show that the discrete-time approximation based on the proposed s to z transform ensures a good fit of the ideal fractional order PID controller.

Future work includes an analysis of the effect of α upon the discretization results, along with a discrete to continuous-time transformation as a consequence of the proposed s to z transform presented in this paper.

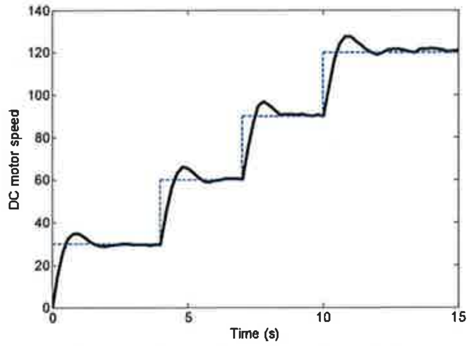


Figure 7. Closed loop experimental results using the digital approximation of the ideal FO-$PI^{\lambda}D^{\mu}$ controller

Acknowledgments

This work was supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS – UEFISCDI, project number PN-II-RU-TE-2014-4-0598, TE 86/2015.

References

[1] Åström, K.J, Hägglund, T., Revisiting the Ziegler–Nichols step response method for PID control, J. Process. Control., 14, 635–650, 2004

[2] Chen, Y.Q., Moore, K.L., *Relay feedback tuning of robust PID controllers with iso-damping property*, IEEE Trans. Syst. Man Cybern., 35, 23–31, 2005

[3] Johnson, M.A., Moradi, M.H., *PID Control: New Identification and Design Methods*, Springer-Verlag, London, 2005

[4] Wang, Q.G., Lee, T.H., Fung, H.W., Bi, Q., Zhang, Y., *PID tuning for improved performance*, IEEE T. Contr. Syst. T., 7, 457-65, 1999

[5] Su, Y.X., Sun, D., Duan, B.Y., *Design of an enhanced nonlinear PID controller*, Mechatronics, 15, 1005-1024, 2005

[6] Podlubny, I., *Fractional-order systems and $PI^{\mu}D^{\lambda}$-controllers*, IEEE Trans. Autom. Control., 44, 208-214, 1999

[7] De Keyser, R., Muresan, C.I., Ionescu, C., A *Novel Auto-tuning Method for Fractional Order PI/PD Controllers*, ISA Trans., 10.1016/j.isatra.2016.01.021, 2016

[8] Monje, C.A., Chen, Y.Q., Vinagre, B., Xue, D., Feliu, V., *Fractional order Systems and Controls: Fundamentals and Applications*, Springer Verlag, Berlin, 2010.

[9] Muresan, C.I., Folea, S., Mois, G., Dulf, E.H., *Development and Implementation of an FPGA Based Fractional Order Controller for a DC Motor*, Mechatronics, 23, 798-804, 2013

[10] Muresan, C.I., Ionescu, C., Folea, S., De Keyser, R., *Fractional Order Control of Unstable Processes: The Magnetic Levitation Study Case*, J. Nonlinear. Dyn., 80, 1761-1772, 2015

[11] Jesus, I.S., Barbosa, R.S., *Genetic optimization of fuzzy fractional PD+I controllers*, ISA Trans., 57, 220–230, 2015

[12] Kheirizad, I., Akbar Jalali, A., Khandani, K., *Stabilization of all-pole unstable delay systems by fractional-order [PI] and [PD] controllers*, T. I. Meas. Control., 35, 257-26, 2013

[13] Bettayeb, M., Boussalem, C., Mansouri, R., Al-Saggaf, U.M., *Stabilization of an inverted pendulum-cart system by fractional PI-state feedback*, ISA Trans., 53, 508–516, 2014

[14] Oustaloup, A., Levron, F., Mathieu, B., Nanot, F.M., *Frequency-band complex noninteger differentiator: characterization and synthesis*, IEEE Trans. Circuits Syst., I, Fundam. Theory Appl., 47, 25 -39, 2000

[15] Chen, Y.Q., *Oustaloup-Recursive-Approximation for Fractional Order Differentiators*, www.mathworks.com/matlabcentral/fileexchange/3802, 2003

[16] Ifeachor, E.C., Jervis, B.W., *Digital Signal Processing. A Practical Approach*, Pearson Education, Essex, 2004

[17] Al-Alaoui, M.A., *Filling the gap between the bilinear and the backward diference transforms: An interactive design approach*, Int. J. of Electrical Engineering Education, 34, 331-337, 1997

[18] Krishna, B.T., Reddy, K.V.V.S., *Design of fractional order digital differentiators and integrators using indirect discretization*, Fract. Calc. Appl. Anal., 11, 143-151, 2008

[19] Krishna, B.T., *Studies on fractional order differentiators and integrators: A survey*, Signal Process., 91, 386–426, 2011

[20] Inteco, *Modular Servo System-User's Manual*, available at www.inteco.com.pl, 2008

[21] Muresan, C.I., Dulf, E.H., Both, R., *Vector Based Tuning and Experimental Validation of Fractional Order PI/PD Controllers*, J. Nonlinear. Dyn., DOI: 10.1007/s11071-015-2328-2, 2015

[22] Dulf, E-H., Muresan, C.I., Ionescu, C., *Robust controller design using fractional calculus and Hinf norm*, submitted to Optimal Control Applications and Methods, under review, 2016