

# Convergence of the Mixed MFIE in the Energy Norm

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## Abstract

The convergence behavior of the solution of the mixed discretization of the magnetic field integral equation is investigated. It is proved that, when the scatterer is smooth and simply connected, the discretization achieves optimal convergence in the  $H_{\text{div}}^{-\frac{1}{2}}$  or energy norm. This norm is as sensitive to the charge as to the current. Hence, this convergence result explains why the mixed discretization leads to much more accurate solutions than the standard discretization, for which only convergence in the  $L_2$  norm has been proved.

*Key words:* MFIE, mixed discretization, convergence, energy norm.

## 1 Introduction

The magnetic field integral equation (MFIE) models the scattering of electromagnetic waves by a perfectly conducting scatterer. When the scatterer occupies the domain  $\Omega$  and has boundary  $\Gamma$ , the MFIE is given by

$$\lim_{\mathbf{r}' \rightarrow \mathbf{r}} \hat{\mathbf{n}}(\mathbf{r}) \times [\mathbf{h}[\mathbf{j}](\mathbf{r}') + \mathbf{h}^i(\mathbf{r}')] = 0, \forall \mathbf{r} \in \Gamma \quad (1)$$

where  $\hat{\mathbf{n}}(\mathbf{r})$  is the exterior surface normal to  $\Gamma$ ,  $\mathbf{1}$  is the 3-by-3 identity matrix and  $\mathbf{h}[\mathbf{j}](\mathbf{r})$  is the magnetic field in the point  $\mathbf{r}$ , generated by the surface current distribution  $\mathbf{j}(\mathbf{r})$ . For the modeling of the fields outside of  $\Omega$ , the limit should be taken such that  $\mathbf{r}$  approaches the boundary  $\Gamma$  from the *inside* of the scatterer. In this case, equation (1) is sometimes called the external MFIE. Essentially, it states that the tangential magnetic field just inside of the PEC should be zero.

To solve the external (or similarly the internal) MFIE, its solution is usually approximated as a linear combination of Rao-Wilton-Glisson (RWG) basis functions. Subsequently, the equation is tested with the same functions. This leads to what will henceforth be called the 'standard MFIE'. However, the literature is rife with evidence that the standard MFIE yields results that are, in general, much less accurate than those obtained using the electric field integral equation (EFIE), given the same mesh density. In addition, a low-frequency breakdown can be identified [1] that leads to nonphysical solutions when the frequency gets too low.

Recently, a different testing scheme for the MFIE was proposed [2], dubbed the mixed discretization, using rotated Buffa-Christiansen (BC) functions. As shown in [3], this scheme avoids the low-frequency breakdown of the standard MFIE for simply connected scatterers.

Therefore, it is well understood why the mixed MFIE works at low frequencies, while the standard MFIE does not. However, numerical results also show that the mixed MFIE leads to much improved accuracy (over the standard MFIE), rivaling the EFIE for comparable mesh density. The reasons for this behavior are less well understood and cannot be fully explained based on the mechanism of the low-frequency breakdown.

## 2 Optimal Convergence for the Mixed MFIE

In this contribution, an at least partial explanation for this behavior will be provided. In particular, it will be shown that the solution of the mixed MFIE converges in the  $H_{\text{div}}^{-\frac{1}{2}}$  norm, whereas the best known result for the standard MFIE is convergence in the  $L_2$  norm. The surface of the scatterer will be assumed to be smooth, such that the integral operator  $\mathbf{h}[\mathbf{j}](\mathbf{r}')$  becomes

$$\mathbf{h}[\mathbf{j}](\mathbf{r}) = \frac{1}{2}\mathbf{j}(\mathbf{r}) + \mathbf{C}[\mathbf{j}](\mathbf{r}), \quad (2)$$

where  $\mathbf{C}$  is a compact operator.

Using these assumptions, a discrete Inf-Sup condition for the mixed MFIE will be derived, based on the discrete Inf-Sup property of the RWG-BC dual finite element pair and the compactness of  $\mathbf{C}$ :

$$\inf_{\mathbf{u} \in X_h} \sup_{\mathbf{v} \in B_h} \frac{|\langle \mathbf{n} \times \mathbf{v}(\mathbf{r}), \mathbf{h}[\mathbf{u}](\mathbf{r}) \rangle|}{\|\mathbf{v}(\mathbf{r})\|_{H_{\text{div}}^{-\frac{1}{2}}} \|\mathbf{u}(\mathbf{r})\|_{H_{\text{div}}^{-\frac{1}{2}}}} \geq \beta > 0. \quad (3)$$

Here,  $X_h$  is the span of all RWGs defined on a given quasi-uniform mesh with maximal edge length  $h$ , while  $B_h$  is the span of all BC functions on the same mesh. The details of the derivation require that  $h$  is sufficiently small and that the frequency is not a resonance frequency of the scatterer's interior. Finally, the discrete Inf-Sup property (3) is used to prove optimal converge (up to constant factors) in the  $H_{\text{div}}^{-\frac{1}{2}}$  norm.

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