Rapidly Converging Electromagnetic Simulations in the Entire Frequency Spectrum without the Search for Global Loops

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***Abstract*—All known integral equation techniques for simulating scattering and radiation from arbitrarily shaped, perfect electrically conducting objects suffer from one or more of the following shortcomings: (i) they give rise to ill-conditioned systems when the frequency is low (ii) and/or when the discretization density is high, (iii) their applicability is limited to the quasi-static regime, (iv) they require a search for global topological loops, (v) they suffer from numerical cancelations in the solution when the frequency is very low. This work presents an equation that does not suffer from any of the above drawbacks when applied to smooth and closed objects. The new formulation is obtained starting from a Helmholtz decomposition of two discretizations of the electric field integral operator obtained by using RWGs and dual bases respectively. The new decomposition does not leverage loop and star/tree basis functions, but projectors that derive from them. Following the decomposition, the two discretizations are combined in a Calderon-like fashion resulting in a new overall equation that is shown to exhibit self-regularizing properties without suffering from the limitations of existing formulations. Numerical results show the usefulness of the proposed method both for closed and open structures.**

I. Introduction

Electric Field Integral Equations (EFIEs), while widely used, are known to be no panacea. The EFIE operator (EFIO) is composed of vector and scalar potential contributions that scale directly and inversely proportional to the frequency, respectively. These operators' divergent properties is the source of the ill-conditioning of the discretized EFIO for low-frequencies. This so-called low-frequency breakdown phenomenon in the past has been dealt with by using loop-star and loop-tree quasi-Helmholtz decompositions. When using these decompositions with the EFIO and after appropriate matrix scaling with suitably chosen powers of (kh) (where k is the wavenumber and h is the average mesh parameter), the low frequency breakdown is solved; that is, in the limit of k going to zero, the matrix condition number is constant. That said, these methods do not cure the undesirable scaling of the matrix condition number with h. Following their application, the matrix condition number scales as h^{-1}, h^{-2}, or h^{-3} (depending on the formulation). This the so called dense discretization breakdown. In addition to suffering from dense discretization breakdown, loop-star/tree decompositions also require the detection of global loops when the surface is a non-simply connected geometry, i.e. it contains holes and handles [1]. Existing general-purpose algorithms for finding global loops exhibit quadratic complexity. Their cost therefore scales worse than that of fast integral equation solvers, which exhibit quasi-linear complexity. Recently a new family of augmented equations that is immune to low frequency breakdown and that, remarkably, does not require the detection of global loops has been introduced [2]. Unfortunately, these formulations still suffer from dense discretization breakdown, since they inherit the spectral properties of the EFIO. Finally, several of the above schemes are susceptible to very low frequency cancelations in the solution vector. In fact, even if the equations are made well-conditioned, for plane wave scattering problems the physics dictates that the non-solenoidal and solenoidal components of the current scale as k and are frequency independent, respectively. If these two components are not separated during the solution process, numerical cancelations that deteriorate the accuracy of the far field computation ensue. This phenomenon has been first pointed out in [3].

In summary, to the best of our knowledge, there exists no integral equations that is simultaneously immune to low frequency and dense discretization breakdown, and free from very low frequency cancelation and the need to detect global loops. This paper presents a new equation that does not suffer from any of these drawbacks. It introduces a new basis-free loop-star decomposition that derives from projections and is used to rescale the standard EFIO and remove low frequency breakdown and very low frequency current cancelation phenomena. Moreover, the rescaled EFIO is self-regularizing, and when squared in a Calderon-like fashion, is immune from dense discretization breakdown. Different from the standard Calderón EFIO, however, our new operator does not have any static null-space. In conclusion, the resulting equation simultaneously is free from low frequency and dense discretization breakdown, very low frequency cancelations, and the need to detect global loops.

II. Background and Notation

Let Γ be the surface of an orientable PEC object residing in a background medium of permittivity ϵ and permeability μ and let $\hat{\boldsymbol{n}}_{\boldsymbol{r}}$ denote Γ's normal vector at $\boldsymbol{r}$. Surface Γ can be non-simply connected, i.e. it can potentially have holes and/or handles. The incident electric field $\boldsymbol{E}^i(\boldsymbol{r})$ impinges on Γ and induces the surface current density $\boldsymbol{J}(\boldsymbol{r})$, which satisfies the EFIE

$$\mathcal{T}(\boldsymbol{J}) = -\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^i \tag{1}$$

where $\mathcal{T}(\boldsymbol{J}) = k\,\mathcal{T}_s(\boldsymbol{J}) + \frac{1}{k}\,\mathcal{T}_h(\boldsymbol{J})$ with

$$\mathcal{T}_s(\boldsymbol{J}) = i\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \int_\Gamma \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}', \tag{2}$$

$$\mathcal{T}_h(\boldsymbol{J}) = i\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \nabla \int_\Gamma \frac{e^{ik|\boldsymbol{r}-\boldsymbol{r}'|}}{4\pi|\boldsymbol{r}-\boldsymbol{r}'|}\nabla_s \cdot \boldsymbol{J}(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}', \tag{3}$$

and the wavenumber $k = 2\pi/\lambda = \omega\sqrt{\epsilon\mu}$.

To solve the EFIE by the boundary element method, Γ is approximated by a mesh of planar triangles with average edge length h, and $\boldsymbol{J}(\boldsymbol{r})$ is approximated as $\boldsymbol{J}(\boldsymbol{r}) \approx \sum_{n=1}^{N} I_n \boldsymbol{f}_n(\boldsymbol{r})$ where $\boldsymbol{f}_n(\boldsymbol{r})$, $n = 1,\dots,N$ are Rao-Wilton-Glisson (RWG) div-conforming basis functions defined on the mesh's N internal edges. To determine the coefficients I_n, the above expression for $\boldsymbol{J}(\boldsymbol{r})$ is substituted into (1) and the resulting equation is Galerkin tested with the functions $\hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i$ yielding the $N \times N$ discretized EFIE system $\bar{\bar{\mathbf{Z}}}\bar{\mathbf{I}} = \left(k\bar{\bar{\mathbf{Z}}}_s + \frac{1}{k}\bar{\bar{\mathbf{Z}}}_h\right)\bar{\mathbf{I}} = \bar{\mathbf{V}}$ where $(\bar{\bar{\mathbf{Z}}}_s)_{i,j} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \mathcal{T}_s(\boldsymbol{f}_j)\rangle$, $(\bar{\bar{\mathbf{Z}}}_h)_{i,j} = \langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \mathcal{T}_h(\boldsymbol{f}_j)\rangle$, $(\bar{\mathbf{V}})_i = \langle \boldsymbol{f}_i, \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{E}^i\rangle$, and $(\bar{\mathbf{I}})_j = I_j$. The EFIO can be alternatively discretized by using the Buffa-Christiansen (BC) basis functions defined on the mesh's barycentric refinement. These functions, similar to RWGs, are div-conforming and defined on the mesh's N internal edges. In what follows they will be denoted by $\boldsymbol{f}_j^{BC}$ and their explicit definition can be found in [4]. The BC functions are also quasi curl-conforming in the sense that the mixed gram matrix $(\bar{\bar{\mathbf{G}}}_{mix})_{i,j} = \left\langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i, \boldsymbol{f}_j^{BC}\right\rangle$ between BC and curl-conforming rotated RWGs is well-conditioned. The BC discretized EFIO will be denoted $\bar{\bar{\mathbb{Z}}} = \left(k\bar{\bar{\mathbb{Z}}}_s + \frac{1}{k}\bar{\bar{\mathbb{Z}}}_h\right)$ where $(\bar{\bar{\mathbb{Z}}}_s)_{i,j} = \left\langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i^{BC}, \mathcal{T}_s(\boldsymbol{f}_j^{BC})\right\rangle$ and $(\bar{\bar{\mathbb{Z}}}_h)_{i,j} = \left\langle \hat{\boldsymbol{n}}_{\boldsymbol{r}} \times \boldsymbol{f}_i^{BC}\,\mathcal{T}_h(\boldsymbol{f}_j^{BC})\right\rangle$.

To construct the new EFIE, we need to define loop and star transformation matrices. For a general introduction to loop-star decompositions the reader should refer to [1]. In the following the loop functions will be denoted by $\{\boldsymbol{\Lambda}_i, i = 1,\dots,N_s\}$. The loop functions are associated with the inner nodes of the mesh. The loop to RWG transformation matrix (the matrix whose columns are the coefficients of the loop functions when expressed as linear combinations of the RWG functions) will be denoted by $\bar{\bar{\mathbf{\Lambda}}}$. The star functions [1], associated with the cells of the mesh, will be denoted by $\{\boldsymbol{\Sigma}_i, i = 1,\dots,N_{ns}\}$. The star to RWG transformation matrix (the matrix whose columns are the coefficients of the star functions when expressed as linear combinations of the RWG functions) will be denoted by $\bar{\bar{\mathbf{\Sigma}}}$. When Γ is not simply connected (i.e. it contains handles and/or holes), it is necessary to consider also the global loops. The global loops to RWG transformation matrix will be denoted by $\bar{\bar{\mathbf{H}}}$. The column dimension of $\bar{\bar{\mathbf{H}}}$ is $2N_{handles} + N_{holes}$, where $N_{handles}$ and N_{holes} are the number of handles and holes of Γ, respectively. The union of loop, star, and global loop functions (after eliminating one linearly dependent star and, on closed structures, one linearly dependent loop) is a basis equivalent to the RWG basis, i.e. $\bar{\mathbf{I}} = \bar{\bar{\mathbf{\Lambda}}}\bar{\mathrm{l}} + \bar{\bar{\mathbf{\Sigma}}}\bar{\mathrm{s}} + \bar{\bar{\mathbf{H}}}\bar{\mathrm{h}}$ for any vector $\bar{\mathbf{I}}$ of RWG coefficients. It should also be noted that both $\bar{\bar{\mathbf{\Lambda}}}\bar{\mathrm{l}}$ and $\bar{\bar{\mathbf{H}}}\bar{\mathrm{h}}$ are RWG coefficients of solenoidal functions. A dual decomposition exists for the BC functions: given an arbitrary BC coefficient vector $\bar{\mathbb{I}}$, the following decomposition holds $\bar{\mathbb{I}} = \bar{\bar{\mathbf{\Lambda}}}\bar{\mathbb{l}} + \bar{\bar{\mathbf{\Sigma}}}\bar{\mathbb{s}} + \bar{\bar{\mathbf{H}}}\bar{\mathbb{h}}$. It should be noted that both $\bar{\bar{\mathbf{\Sigma}}}\bar{\mathbb{s}}$ and $\bar{\bar{\mathbf{H}}}\bar{\mathbb{h}}$ are BC coefficients of solenoidal functions.

III. The new equation

We next describe a new EFIE that, contrary to currently available ones, is *concurrently* immune to the low frequency breakdown and the dense discretization breakdown, as well as very low frequency solution cancelation. Moreover, the new equation will not require an identification search of global loops, that is the matrix $\bar{\bar{\mathbf{H}}}$ defined above. The new formulation is obtained in two steps: (i) first, a quasi-Helmholtz decomposition is applied to the EFIO to cure it from low frequency breakdown. For this purpose, we introduce a new, basis-free decomposition, since a standard quasi-Helmholtz decomposition introduces a basis-related dense discretization breakdown (see [5]) and would require the explicit determination of $\bar{\bar{\mathbf{H}}}$ (the global loops). (ii) The new decomposed equation will be "squared" in a suitable, Calderón like fashion to cure it from dense discretization breakdown.

First of all, note that since it can be shown that $\bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Lambda}}} = \bar{\bar{\mathbf{0}}}$ and $\bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{H}}} = \bar{\bar{\mathbf{0}}}$ then for an arbitrary RWG vector $\bar{\mathbf{I}}$: $\bar{\bar{\mathbf{\Sigma}}}^T\bar{\mathbf{I}} = \bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Lambda}}}\bar{\mathrm{l}} + \bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Sigma}}}\bar{\mathrm{s}} + \bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{H}}}\bar{\mathrm{h}} = \bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Sigma}}}\bar{\mathrm{s}}$, so that we get $\bar{\mathrm{s}} = (\bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Sigma}}})^+\bar{\bar{\mathbf{\Sigma}}}^T\bar{\mathbf{I}}$ where $(\bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Sigma}}})^+$ denotes the pseudoinverse of $(\bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Sigma}}})$. Finally, the star component of $\bar{\mathbf{I}}$ is obtained by using the projection $\bar{\bar{\mathbf{P}}}^{\Sigma}\bar{\mathbf{I}}$, where the projector $\bar{\bar{\mathbf{P}}}^{\Sigma}$ is defined as $\bar{\bar{\mathbf{P}}}^{\Sigma} = \bar{\bar{\mathbf{\Sigma}}}(\bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Sigma}}})^+\bar{\bar{\mathbf{\Sigma}}}^T$ The loops and global loops components of $\bar{\mathbf{I}}$ can be obtained by the complementary projector $\bar{\bar{\mathbf{P}}}^{\Lambda H} = \bar{\bar{\mathbf{I}}} - \bar{\bar{\mathbf{\Sigma}}}(\bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Sigma}}})^+\bar{\bar{\mathbf{\Sigma}}}^T$. Finally, define the decomposition operator $\bar{\bar{\mathbf{M}}} = \bar{\bar{\mathbf{P}}}^{\Lambda H}\frac{1}{\sqrt{k}} + i\bar{\bar{\mathbf{P}}}^{\Sigma}\sqrt{k}$. Using the decomposition operator with the EFIO it is easily shown that $\bar{\bar{\mathbf{M}}}^T\bar{\bar{\mathbf{Z}}}\bar{\bar{\mathbf{M}}} = \left(\bar{\bar{\mathbf{P}}}^{\Lambda H}\bar{\bar{\mathbf{Z}}}_s\bar{\bar{\mathbf{P}}}^{\Lambda H} - \bar{\bar{\mathbf{Z}}}_h\right) + O(k)$, which is clearly immune from low frequency breakdown. The treatment for BC discretized matrix $\bar{\bar{\mathbb{Z}}}$ is the same provided that the role of the loop and star matrices is exchanged. Thus we will define the dual projectors $\bar{\bar{\mathbb{P}}}^{\Lambda} = \bar{\bar{\mathbf{\Lambda}}}(\bar{\bar{\mathbf{\Lambda}}}^T\bar{\bar{\mathbf{\Lambda}}})^+\bar{\bar{\mathbf{\Lambda}}}^T$ and $\bar{\bar{\mathbb{P}}}^{\Sigma H} = \bar{\bar{\mathbf{I}}} - \bar{\bar{\mathbf{\Lambda}}}(\bar{\bar{\mathbf{\Lambda}}}^T\bar{\bar{\mathbf{\Lambda}}})^+\bar{\bar{\mathbf{\Lambda}}}^T$ and the associated decomposition operator $\bar{\bar{\mathbb{M}}} = \bar{\bar{\mathbb{P}}}^{\Sigma H}\frac{1}{\sqrt{k}} + i\bar{\bar{\mathbb{P}}}^{\Lambda}\sqrt{k}$ so that also $\bar{\bar{\mathbb{M}}}^T\bar{\bar{\mathbb{Z}}}\bar{\bar{\mathbb{M}}}$ will be immune from low frequency breakdown similarly to the previous case. The procedure for the practical computation of $(\bar{\bar{\mathbf{\Sigma}}}^T\bar{\bar{\mathbf{\Sigma}}})^+$ and $(\bar{\bar{\mathbf{\Lambda}}}^T\bar{\bar{\mathbf{\Lambda}}})^+$ is omitted here for space limitations, but it can be found in [5].

For the standard EFIO both the low frequency and the dense discretization breakdown are solved by Calderón preconditioning: the EFIO discretized matrix $\bar{\bar{\mathbf{Z}}}$ is replaced by the Calderón preconditioned matrix $\bar{\bar{\mathbb{Z}}}\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{Z}}}$. This matrix is provably immune from both low frequency and dense discretization breakdown, however it has a null-space in statics and it can suffer from current cancelation at very low frequencies. The new equation proposed in this work is obtained by replacing the RWG- and BC-discretized EFIO operators $\bar{\bar{\mathbf{Z}}}$ and $\bar{\bar{\mathbb{Z}}}$ with the respective decomposed ones. After defining $\bar{\mathbf{I}} = \left(\bar{\bar{\mathbf{M}}}\right)\bar{\mathbf{Y}}$, our new equation reads

$$\left(\bar{\bar{\mathbb{M}}}^T\bar{\bar{\mathbb{Z}}}\bar{\bar{\mathbb{M}}}\right)\bar{\bar{\mathbf{G}}}_{mix}^{-1}\left(\bar{\bar{\mathbf{M}}}^T\bar{\bar{\mathbf{Z}}}\bar{\bar{\mathbf{M}}}\right)\bar{\mathbf{Y}} = \left(\bar{\bar{\mathbb{M}}}^T\bar{\bar{\mathbb{Z}}}\bar{\bar{\mathbb{M}}}\right)\bar{\bar{\mathbf{G}}}_{mix}^{-1}\bar{\bar{\mathbf{M}}}^T\bar{\mathbf{V}}. \quad (4)$$

IV. Properties of the New Equation

Let's first prove that the operator of the new equation has the same null-space of the EFIO and thus, in particular, it does not have the global loops spanned static null-space of the Calderón EFIE. It is sufficient to prove the statement in statics, in fact $\bar{\bar{\mathbf{G}}}_{mix}^{-1}$ is non-singular and, away from statics, also the matrices $\bar{\bar{\mathbf{M}}}$ and $\bar{\bar{\mathbb{M}}}$ are well-defined and non-singular. Let's study the static limit of $\bar{\bar{\mathbf{M}}}^T\bar{\bar{\mathbf{Z}}}\bar{\bar{\mathbf{M}}}$, i.e. let's prove that the operator $\left(\bar{\bar{\mathbf{P}}}^{\Lambda H}\bar{\bar{\mathbf{Z}}}_s\bar{\bar{\mathbf{P}}}^{\Lambda H} - \bar{\bar{\mathbf{P}}}^{\Sigma}\bar{\bar{\mathbf{Z}}}_h\bar{\bar{\mathbf{P}}}^{\Sigma}\right)$ does not have a null-space. Since $\bar{\bar{\mathbf{P}}}^{\Lambda H}\bar{\bar{\mathbf{P}}}^{\Sigma} = \bar{\bar{\mathbf{0}}}$ it is sufficient to prove that $\bar{\bar{\mathbf{P}}}^{\Lambda H}\bar{\bar{\mathbf{Z}}}_s\bar{\bar{\mathbf{P}}}^{\Lambda H}$ is non-singular on the range of $\left[\bar{\bar{\mathbf{\Lambda}}}, \bar{\bar{\mathbf{H}}}\right]$ and that $\bar{\bar{\mathbf{P}}}^{\Sigma}\bar{\bar{\mathbf{Z}}}_h\bar{\bar{\mathbf{P}}}^{\Sigma}$ is non singular on the range of $\bar{\bar{\mathbf{\Sigma}}}$. Since on these spaces both $\bar{\bar{\mathbf{Z}}}_s$ and $\bar{\bar{\mathbf{Z}}}_h$ in statics are symmetric positive definite matrices, then we get that $\forall\bar{\mathbf{v}}$ in the range of $\left[\bar{\bar{\mathbf{\Lambda}}}, \bar{\bar{\mathbf{H}}}\right]$

$$\bar{\mathbf{v}}^T\bar{\bar{\mathbf{P}}}^{\Lambda H}\bar{\bar{\mathbf{Z}}}_s\bar{\bar{\mathbf{P}}}^{\Lambda H}\bar{\mathbf{v}} = 0 \Rightarrow \bar{\bar{\mathbf{P}}}^{\Lambda H}\bar{\mathbf{v}} = \bar{\mathbf{0}}$$
$$\Rightarrow \bar{\mathbf{v}} \text{ is in the range of } \bar{\bar{\mathbf{\Sigma}}} \Rightarrow \bar{\mathbf{v}} = \bar{\mathbf{0}} \quad (5)$$

dually, $\forall\bar{\mathbf{q}}$ in the range of $\bar{\bar{\mathbf{\Sigma}}}$

$$\bar{\mathbf{q}}^T\bar{\bar{\mathbf{P}}}^{\Sigma}\bar{\bar{\mathbf{Z}}}_s\bar{\bar{\mathbf{P}}}^{\Sigma}\bar{\mathbf{q}} = 0 \Rightarrow \bar{\bar{\mathbf{P}}}^{\Sigma}\bar{\mathbf{v}} = \bar{\mathbf{0}}$$
$$\Rightarrow \bar{\mathbf{q}} \text{ is in the range of } \left[\bar{\bar{\mathbf{\Lambda}}}, \bar{\bar{\mathbf{H}}}\right] \Rightarrow \bar{\mathbf{q}} = \bar{\mathbf{0}} \quad (6)$$

from this we deduce that $\bar{\bar{\mathbf{M}}}^T\bar{\bar{\mathbf{Z}}}\bar{\bar{\mathbf{M}}}$ does not have a static null-space. The same statement is proved for $\bar{\bar{\mathbb{M}}}^T\bar{\bar{\mathbb{Z}}}\bar{\bar{\mathbb{M}}}$ by using the same approach. Finally we deduce that the newly proposed equation (4) has no static null-space since it is the product of three non-singular matrices.

Let's now study the conditioning behavior of (4) as a function of frequency and discretization. Since both $\bar{\bar{\mathbf{M}}}^T\bar{\bar{\mathbf{Z}}}\bar{\bar{\mathbf{M}}}$ and $\bar{\bar{\mathbb{M}}}^T\bar{\bar{\mathbb{Z}}}\bar{\bar{\mathbb{M}}}$ are immune from the low frequency breakdown, so will be (4), since $\bar{\bar{\mathbf{G}}}_{mix}^{-1}$ is a frequency independent matrix. The new equation is also immune from the dense discretization breakdown since, after some manipulations, it is easy to see that the singular value relevant blocks of the new equation are the same as those of the standard Calderón EFIO (which is immune from the dense discretization breakdown).

To complete the analysis of the properties of (4) we will study the frequency behavior of the solution for different types of excitation in order to show that the solution of (4) does not suffer from very low frequency cancelations.

If we assume that, as a function of k, the solenoidal part of the physical current $\bar{\mathbf{I}}$ scales as (R_s, I_s) (real part and imaginary part) and if we assume that the non-solenoidal part scales as (R_{ns}, I_{ns}), then the scattered far-field due to the solenoidal part will scale as $\left(R_s^F, I_s^F\right) = (kI_s, kR_s)$ while the scattered field due to the non-solenoidal part will scale as $\left(R_{ns}^F, I_{ns}^F\right) = (R_{ns}, I_{ns})$ [3]. Then from the relationship $\bar{\mathbf{Y}} = \bar{\bar{\mathbf{M}}}^{-1}\bar{\mathbf{I}}$ and the definition of $\bar{\bar{\mathbf{M}}}$ it follows that the solenoidal part of the solution $\bar{\mathbf{Y}}$ of (4) will scale as summarized in Table I. In the table, all possible kinds of excitation have been considered. For space limitation we omit the detailed definition of these excitations and the reason for their frequency scaling, but the reader can refer to [2]. From the table it is clear that all the necessary current components are retrieved by our scheme (real and imaginary for plane wave exitations, purely imaginary for capacitive and inductive excitations), so that the new equation always recover correctly both physical currents and far-fields.

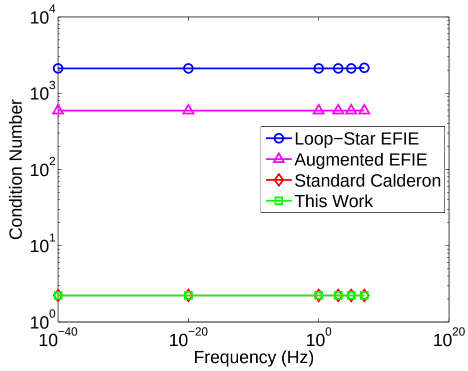

Fig. 1. Sphere: condition number of (4) as a function of the excitation frequency.

Source	$\left(\sqrt{k}R_s, \sqrt{k}I_s\right)$	$\left(\frac{I_{ns}}{\sqrt{k}}, \frac{R_{ns}}{\sqrt{k}}\right)$	Recovered terms
Plane Wave	$\left(\sqrt{k}, k\sqrt{k}\right)$	$\left(\sqrt{k}, k\sqrt{k}\right)$	R_s,I_{ns},R_{ns},I_s
Inductive	$\left(k^2\sqrt{k}, \frac{1}{\sqrt{k}}\right)$	$\left(\sqrt{k}, k^3\sqrt{k}\right)$	I_s,I_{ns}
Capacitive	$\left(k^4\sqrt{k}, k\sqrt{k}\right)$	$\left(\sqrt{k}, k^3\sqrt{k}\right)$	I_s,I_{ns}

TABLE I
Scalings of the Solution $\bar{\mathbf{Y}}$ of (4)

V. Numerical Results

The new equation was tested for a sphere and for a planar square ring. The first test involves a sphere of unit radius that is excited by a plane wave. Figs. 1 and 2 show the condition number of the system matrix of different formulations as a function of the excitation frequency and the discretization density, respectively. The proposed equation clearly is immune from low frequency and dense discretization breakdown. The behavior of the equation is also compared with that of loop-star-decomposed and Augmented EFIEs ([2], equation (9)), that are both suffering from the dense discretization breakdown (Fig. 2).

The fact that the new equation is immune from the very low frequency current cancelation is confirmed by Fig. 3 which show the far field calculated using (4) at 10^{-40}Hz. From Fig. 3 it is clear that although a standard Calderón equation can provide a stable solution till relatively low frequencies,

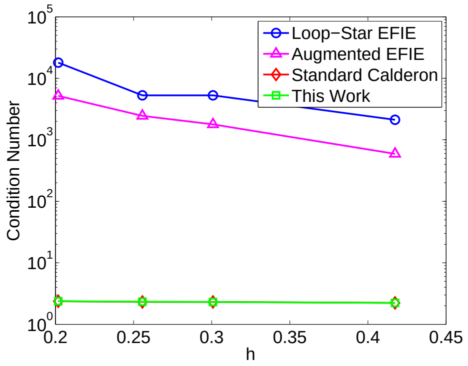

Fig. 2. Sphere: condition number of (4) as a function of the average mesh size h.

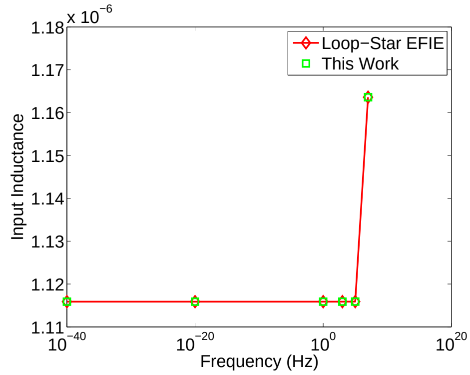

Fig. 4. Input inductance of the square inductor as a function of the frequency.

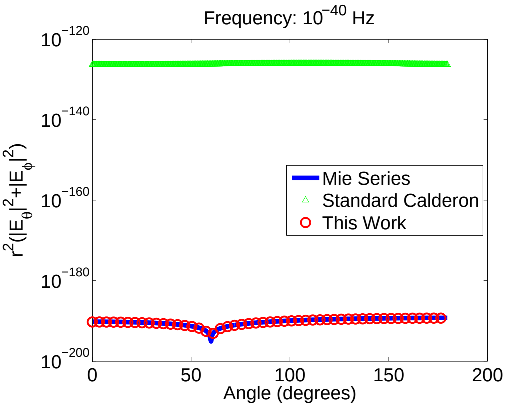

Fig. 3. Sphere: far field calculated when the frequency equals 10^{-40}Hz

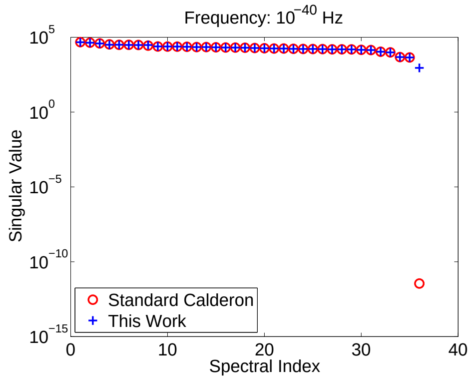

Fig. 5. Square inductor: singular values of the standard Calderón EFIO and of operator in (4).

the new equation (4) is immune from the very low frequency current cancelation and provides stable solutions even when the frequency is arbitrarily low.

To study the behavior of the new equation (4) when applied to inductive structures with voltage gap excitations, consider a square ring inductor with side length $1m$ and width $0.25m$; the voltage gap is located in the center of one of the ring's sides. Fig. 4 shows the absolute value of the input inductance as a function of frequency. The values obtained by solving (4) are in very good agreement with those obtained using the standard loop-star EFIE. The computed value of the static inductance is near 1.197μH, the approximate inductance value predicted using classical expressions. Fig. 5 compares the singular values of the system matrix produced by the standard Calderón approach and (4) for a frequency of 10^{-40}Hz. A static null-space of dimension one is expected for the Calderón EFIO, since the open structure has one hole. An almost zero singular value is evident in Fig. 5. In contrast, and as predicted by our theory, the new equation however does not have a static null-space and thus the global loop does not need to be detected.

VI. Conclusions

This paper presented an electric field integral equation that is immune from both low-frequency and dense discretization breakdown, does not require a search for global topological loops, and does not suffer from numerical cancelations in the solution when the frequency is very low. The computational cost of all calculations in the new formulation not required by the solvers it builds on scale linearly in the number of unknowns; hence the new formulation can be applied in tandem with fast methods without degradation in computational complexity. Numerical results demonstrated the beneficial properties of the new technique.

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