Nullspace of the Static MFIE Operator and its Effect on the Numerical Solution of BIE's

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Introduction

Boundary integral equations are often used to simulate scattering by closed perfect electrically conductors (PEC). Among the many available alternatives, the electric and magnetic field integral equation (EFIE and MFIE) are the most popular; both can be formulated in the frequency and time domains. The EFIE is more versatile than the MFIE: it applies to open structures and wires, and is easily modified to account for surface resistances and impedances. Moreover, the EFIE typically is more accurate than the MFIE. That said, there are situations were the MFIE is the more sensible choice. Indeed, the linear systems resulting upon discretization the MFIE generally are better conditioned than those resulting upon discretization the EFIE. This is because the MFIE is an equation of the second kind whereas the EFIE is an equation of the first kind. The spectrum of the former is bounded and has a finite non-zero accumulation point, while the latter has a spectrum accumulating both at zero and infinity. This does not mean that the MFIE is devoid of any spectrum related problems. In the moderate to high frequency regime, the MFIE is ill-posed at frequencies where the cavity formed by the scatterer supports resonant modes. The traces of these resonant fields reside in the nullspace of the MFIE operator. After discretization, the existence of these resonant fields results in ill-conditioned systems. In the low frequency regime, however, the MFIE generally is free of resonances when applied to simply connected geometries. However, applied to non-trivial topologies the picture becomes more complicated. Indeed, as will be shown in this contribution, the static MFIE operator has a nullspace when applied to torus-like geometries. And although it makes no sense to use the MFIE in the static regime, the presence of this nullspace definitely affects quasi-static simulations. In this paper, the construction of a basis for the nullspace of the static MFIE operator will be sketched and the effect of its existence on the MFIE-based simulation of non-static frequency domain and time-domain problems will be elucidated.

Equations and Discretization

Since this contribution focuses on the effects of a non-trivial nullspace of the MFIE operator on both frequency domain and time-domain computations, some notational conventions are introduced to distinguish both cases. Transient currents and fields are represented using bold upper case Roman symbols $(\boldsymbol{H}(\boldsymbol{r},t), \boldsymbol{J}(\boldsymbol{r},t))$. Their frequency domain counterparts are denoted by bold lower case symbols $(\boldsymbol{h}(\boldsymbol{r}), \boldsymbol{j}(\boldsymbol{r}))$; their frequency dependence is suppressed. Time-domain operators are denoted by calligraphic symbols $(\mathcal{K})$ and frequency domain operators by Roman capitals (K).

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In discretized form, frequency domain quantities are recognized by the absence of a temporal subscript. Frequency domain and time-domain MFIE's are disambiguated by including the prefixes FD and TD. All transient signals are assumed causal (i.e. they vanish for $t < 0$).

Consider a closed PEC scatterer with boundary Γ and exterior normal $\hat{\boldsymbol{n}}$, which is illuminated by an incident wave $\boldsymbol{H}^i(\boldsymbol{r}, t)$ or $\boldsymbol{h}^i(\boldsymbol{r})$. Enforcing the magnetic boundary condition on Γ yields the FD-MFIE

$$\hat{\boldsymbol{n}} \times \boldsymbol{h}^i(\boldsymbol{r}) = \left\{\frac{1}{2} + K\right\}[\boldsymbol{j}(\boldsymbol{r})] = \frac{\boldsymbol{j}(\boldsymbol{r})}{2} - \hat{\boldsymbol{n}} \times \int_\Gamma dS' \nabla \times \frac{e^{-jkR}}{4\pi R}\boldsymbol{j}(\boldsymbol{r}') \tag{1}$$

and the TD-MFIE

$$\hat{\boldsymbol{n}} \times \boldsymbol{H}^i(\boldsymbol{r}, t) = \left\{\frac{1}{2} + \mathcal{K}\right\}[\boldsymbol{J}(\boldsymbol{r}, t)] = \frac{\boldsymbol{J}(\boldsymbol{r}, t)}{2} - \hat{\boldsymbol{n}} \times \int_\Gamma dS' \nabla \times \frac{\boldsymbol{J}(\boldsymbol{r}', t - R/c)}{4\pi R}, \tag{2}$$

for all $\boldsymbol{r} \in \Gamma$ and $t > 0$. Schemes for discretizing these equations are described in [1, 2], and yield the FD-MFIE method of moments (MOM) system

$$\left(\frac{1}{2}\mathbf{I} + \mathbf{K}\right) \cdot \mathsf{J} = \mathsf{H}^i \tag{3}$$

and the TD-MFIE marching-on-in-time (MOT) system

$$\frac{1}{2}\mathbf{I} \cdot \mathsf{J}_j + \sum_{k=0}^{k_{\max}} \mathbf{K}_k \cdot \mathsf{J}_{j-k} = \mathsf{H}^i_j. \tag{4}$$

The latter system can be solved for all J_j, starting with J_0. Both systems typically are solved iteratively. The condition number of the matrices that need to be inverted (i.e. $\mathbf{K} + \mathbf{I}/2$ and $\mathbf{K}_0 + \mathbf{I}/2$) is therefore of utmost importance. Moreover, the MOT system can be unstable; this means that the excitation will couple to spurious non-decaying regime solutions that pollute the physical solution. This effect is most pronounced after the forcing term has decayed.

Nullspace of the static MFIE operator

In the zero frequency limit, the FD-MFIE becomes

$$\hat{\boldsymbol{n}} \times \boldsymbol{h}^i(\boldsymbol{r}) = \left\{\frac{1}{2} + K^s\right\}[\boldsymbol{j}(\boldsymbol{r})] = \frac{\boldsymbol{j}(\boldsymbol{r})}{2} - \hat{\boldsymbol{n}} \times \int_\Gamma dS' \nabla \times \frac{1}{4\pi R}\boldsymbol{j}(\boldsymbol{r}'). \tag{5}$$

In the case where Γ is an N-torus, i.e. a generalized torus with N holes, the static MFIE operator $1/2+K^s$ has a non-trivial nullspace comprising N linear independent current configurations.

The construction of these nullspace elements will now be sketched. This construction follows the theory of [3], applied to the MFIE operator. Denote the interior of the PEC by Ω^- and its exterior by Ω^+. Now let L_i be a closed loop in Ω^-, circling only hole i (once) and denote by $\hat{\boldsymbol{l}}$ the tangential unit vector along this loop. The magnetic field caused in Ω^+ by a current of unit amplitude running along L_i, in the absence of the PEC is

$$\boldsymbol{h}_{i,0}(\boldsymbol{r}) = \int_{L_i} dl' \nabla \frac{1}{4\pi R} \times \hat{\boldsymbol{l}}' dl'. \tag{6}$$

A second contribution is defined by $\boldsymbol{h}_{i,1}(\boldsymbol{r}) = \nabla\psi(\boldsymbol{r})$ where $\psi(\boldsymbol{r})$ is the unique bounded solution in Ω^+ of

$$\nabla^2\psi = 0, \quad \frac{\partial\psi}{\partial n} = -\hat{\boldsymbol{n}} \cdot \boldsymbol{h}_{i,0}. \tag{7}$$

Since $\int_\Gamma dS' \hat{\boldsymbol{n}} \cdot \boldsymbol{h}_{i,0}(\boldsymbol{r}) = \int_{\Omega^+} dV' \nabla \cdot \boldsymbol{h}_{i,0}(\boldsymbol{r}) = 0$, the Neumann-condition is viable. The field $\boldsymbol{h}_i = \boldsymbol{h}_{i,0} + \boldsymbol{h}_{i,1}$ in Ω^+ now obeys

$$\nabla \cdot \boldsymbol{h}_i = 0, \quad \nabla \times \boldsymbol{h}_i = 0, \quad \hat{\boldsymbol{n}} \cdot \boldsymbol{h}_i = 0. \tag{8}$$

If the surface current $\boldsymbol{j}_i(\boldsymbol{r}) = \hat{\boldsymbol{n}} \times \boldsymbol{h}_i(\boldsymbol{r})$ is introduced on Γ and the magnetic field is extended by zero in Ω^-, this configuration of bounded fields and sources fulfills the Maxwell equations and jump conditions in all of space *and* obeys the PEC boundary conditions for the magnetic field. Therefore, the current $\boldsymbol{j}_i(\boldsymbol{r})$ resides in the nullspace of the static MFIE operator. Furthermore, it can be proven that the N currents $\boldsymbol{j}_i(\boldsymbol{r})$ thus constructed form a basis for the nullspace of the static MFIE.

Numerical Results

Both the FD-MFIE-MOM and TD-MFIE-MOT system are affected by the presence of a non-trivial nullspace of the static MFIE operator. In the case of the FD-MFIE equation, the nullspace of the static operator resides approximately in the nullspace of the low-frequency operator. The FD-MFIE-MOM system thus becomes ill-conditioned. In the case of the TD-MFIE equation, the presence of the nullspace results in a constant amplitude non-physical tail superposed on the true solution.

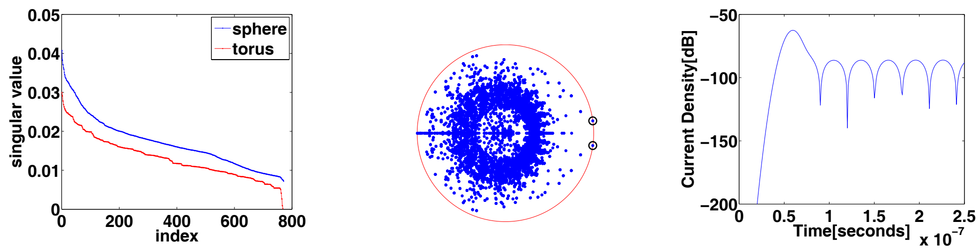
Consider a torus with large radius of 1 meter, and small radius of 0.25 meter which is discretized using 768 unknowns. In the FD simulations, the torus is illuminated by a plane wave

$$\boldsymbol{h}^i(\boldsymbol{r}) = \hat{\boldsymbol{z}} e^{-j\omega/c\hat{\boldsymbol{x}}\cdot\boldsymbol{r}} \tag{9}$$

with $\omega = 29.979$ MHz. To provide a comparison, a sphere of radius 1 meter discretized using 771 unknowns was illuminated by the same field. A singular value decomposition of the system matrices was performed (Fig. 1(a)). The singular value spectrum of the torus' system matrix shows a singular vector that is approximately in the nullspace. The corresponding singular vector was computed and is plotted in Fig. 2(a). The TD simulations were done using an incident field of the form

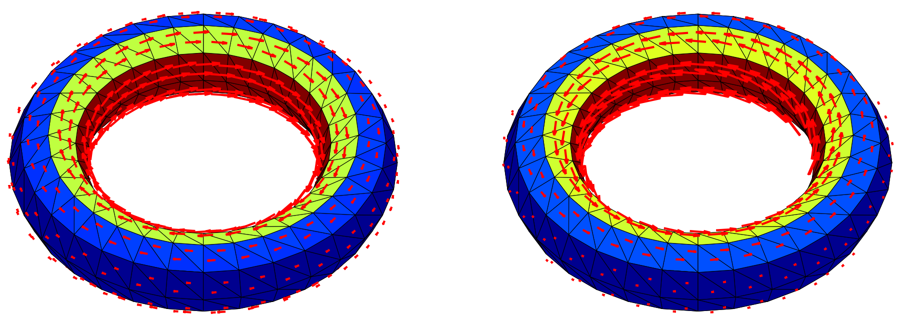
$$\boldsymbol{H}^i(\boldsymbol{r}, t) = \frac{4}{T\sqrt{\pi}} \hat{\boldsymbol{z}} e^{-\gamma^2} \tag{10}$$

with $\gamma = \frac{4}{T}(ct - ct_0 - \hat{\boldsymbol{x}} \cdot \boldsymbol{r})$, $T = 12.0$ meter, $t_0 = 60.042$ ns, and using a time step $\Delta t = 1.0007$ ns. A polynomial eigenvalue analysis [4] was performed on the resulting system, revealing the presence of a slowly oscillating non-decaying mode



(a) (b) (c)

Figure 1: Singular value spectrum of MFIE-MOM system for the torus and the sphere (a). Polynomial eigenvalues (b) and solution (c) of MFIE-MOT system for the torus.



(a) (b)

Figure 2: Singular vector of the MFIE-MOM system belonging to the smallest singular value (a) and polynomial eigenvector belonging to the pole near 1 of the MFIE-MOT system (b).

(Fig. 1(b)). This mode can be found in the solution of the TD-MFIE-MOT system (Fig. 1(c)). The eigenvector corresponding to this pole was computed and plotted. Fig. 2(b) shows this is the same current distribution causing the near zero singular value of the frequency domain system matrix.

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