BIDIRECTIONALITY IN THE WAVEGUIDE STURM-LIOUVILLE PROBLEM

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Abstract: In this paper we present an operator based proof of the bidirectionality of general reciprocal waveguides. The proof starts from the vectorial, coupled, singular and non-Hermitian Sturm-Liouville problem that governs the modal field propagation. This paper is in honor of Don Dudley, one of the most eminent electromagneticists.

INTRODUCTION

In his book [1] Don Dudley introduces the electromagnetic community to the power of Sturm-Liouville theory for the solution of boundary value problems that arise in electromagnetics. One such Sturm-Liouville problem is that of finding eigenmodes in waveguiding structures.

The eigenmode problem is a very difficult Sturm-Liouville problem. It is often singular because waveguides such as optical fibers or microstrip lines are open structures. It is a vectorial Sturm-Liouville problem and a coupled Sturm-Liouville problem. In general it will also not be Hermitian unless the waveguide consists of lossless materials. Available mathematical literature does not cover such coupled, singular, vectorial non-Hermitian Sturm-Liouville problems. This often impedes the derivation of general conclusions regarding the eigenmodes of general waveguides.

In this contribution we nevertheless want to try to derive one such general property of waveguides. This property is bidirectionality. Bidirectionality means that for each eigenmode propagating in one direction in a waveguide there exists another eigenmode propagating in the other direction with the same propagation coefficient. For waveguides build from isotropic materials this bidirectionality property is evident. However, for waveguides build from general linear materials such as anisotropic or even bianisotropic materials this becomes much less obvious. Actually in general such waveguides will not be bidirectional at all.

The occurrence of bidirectionality can have different origins. One reason could be some kind of symmetry in the cross-section of the waveguide. We will not focus on this kind of origin of bidirectionality but rather refer to [2] and [3]. Bidirectionality due to symmetry is rather easily derived from symmetry properties of the Sturm-Liouville problem. Propagating eigenmodes in a lossless waveguide are also bidirectional [3].

In this paper we will look at reciprocity as the origin for bidirectionality. A waveguide build from reciprocal materials is bidirectional. This problem has received the attention of many scientists during the past 50 years. In 1958-1959 Harrington and Villeneuve [4] used a proof based on antenna theory. In 1991 McIsaac [2] said that the property was simply not true. In 1996 the first author [5] gave a proof based on the Lorentz reciprocity theorem and a continuity argument. In 2006 Yaghjian [6] came with a proof again based on antenna theory but without interchanging the antennas contrary to what was done in [4]. For a discussion of these papers we refer to [3].

All these proofs are indirect proofs, if they are correct then a direct proof from Maxwell's equations should be possible also. Already in 1999 the first author mentions the directions of such a proof in his book [3]. In this contribution we want to highlight this operator based proof and present some new viewpoints.

Although our discussion is valid for waveguides build from bianisotropic materials and also for periodic waveguides [7] we will limit ourselves for the sake of the simplicity of the equations to uniform waveguides build from reciprocal anisotropic waveguides. The bidirectionality concept can also be extended to non-reciprocal materials. In that case there is mutual bidirectionality between a waveguide and its adjoint waveguide build from the adjoint reciprocal materials.

AN OPERATOR BASED PROOF OF BIDIRECTIONALITY

Assume a waveguide oriented along the z-axis. The material parameters of the waveguide then only depend on the x- and y-coordinates. The xy-plane is called the transversal plane and the position vector in this plane is $\mathbf{r}_t$. We assume an $e^{j\omega t}$ time dependence. For eigenmodes the electric and magnetic field are written as $\mathbf{e}(\mathbf{r}) = \mathbf{E}(\mathbf{r}_t)e^{-\gamma z}$ and $\mathbf{h}(\mathbf{r}) = \mathbf{H}(\mathbf{r}_t)e^{-\gamma z}$, with γ the modal propagation coefficient and $\mathbf{E}(\mathbf{r}_t)$ and $\mathbf{H}(\mathbf{r}_t)$ the modal fields. Inserting this modal representation in the sourceless curl Maxwell equations leads to the following eigenvalue problem

$$(\nabla_t \times \mathbf{E}_(\mathbf{r}_t)) \cdot \mathbf{u}_z = -j\omega\mu_{zz}(\mathbf{r}_t)H_z(\mathbf{r}_t) - j\omega\boldsymbol{\mu}_{zt}(\mathbf{r}_t)\cdot\mathbf{H}_t(\mathbf{r}_t), \tag{1}$$

$$-\gamma\mathbf{u}_z \times \mathbf{E}_t(\mathbf{r}_t) + \nabla_t \times E_z(\mathbf{r}_t)\mathbf{u}_z = -j\omega\overline{\overline{\mu}}_{tt}(\mathbf{r}_t)\cdot\mathbf{H}_t(\mathbf{r}_t) - j\omega\boldsymbol{\mu}_{tz}(\mathbf{r}_t)H_z(\mathbf{r}_t), \tag{2}$$

$$(\nabla_t \times \mathbf{H}_(\mathbf{r}_t)) \cdot \mathbf{u}_z = j\omega\epsilon_{zz}(\mathbf{r}_t)E_z(\mathbf{r}_t) + j\omega\boldsymbol{\epsilon}_{zt}(\mathbf{r}_t)\cdot\mathbf{E}_t(\mathbf{r}_t), \tag{3}$$

$$-\gamma\mathbf{u}_z \times \mathbf{H}_t(\mathbf{r}_t) + \nabla_t \times H_z(\mathbf{r}_t)\mathbf{u}_z = j\omega\overline{\overline{\epsilon}}_{tt}(\mathbf{r}_t)\cdot\mathbf{E}_t(\mathbf{r}_t) + j\omega\boldsymbol{\epsilon}_{tz}(\mathbf{r}_t)E_z(\mathbf{r}_t), \tag{4}$$

where we have split vectors $\mathbf{a} = \mathbf{a}_t + a_z\mathbf{u}_z$ and dyadics $\overline{\overline{a}} = \overline{\overline{a}}_{tt} + \mathbf{a}_{tz}\mathbf{u}_z + \mathbf{u}_z\mathbf{a}_{zt} + a_{zz}\mathbf{u}_z\mathbf{u}_z$ in transversal and longitudinal parts. From this coupled system of equations we can eliminate the longitudinal fields $E_z(\mathbf{r}_t)$ and $H_z(\mathbf{r}_t)$ leading finally to

$$-\gamma\begin{pmatrix} \mathbf{E}_t(\mathbf{r}_t) \\ \mathbf{u}_z \times \mathbf{H}_t(\mathbf{r}_t) \end{pmatrix} = -\begin{pmatrix} \overline{\overline{P}}_{tt}(\mathbf{r}_t) & \overline{\overline{Z}}_{tt}(\mathbf{r}_t) \\ \overline{\overline{Y}}_{tt}(\mathbf{r}_t) & \overline{\overline{Q}}_{tt}(\mathbf{r}_t) \end{pmatrix}\cdot\begin{pmatrix} \mathbf{E}_t(\mathbf{r}_t) \\ \mathbf{u}_z \times \mathbf{H}_t(\mathbf{r}_t) \end{pmatrix}, \tag{5}$$

with the operator dyadic $\overline{\overline{P}}_{tt}(\mathbf{r}_t)$ defined as

$$\overline{\overline{P}}_{tt}(\mathbf{r}_t) = \nabla_t\frac{\boldsymbol{\epsilon}_{zt}(\mathbf{r}_t)}{\epsilon_{zz}(\mathbf{r}_t)} + \mathbf{u}_z\mathbf{u}_z\,{}^{\times}_{\times}\,\frac{\boldsymbol{\mu}_{tz}(\mathbf{r}_t)}{\mu_{zz}(\mathbf{r}_t)}\nabla_t, \tag{6}$$

$$\overline{\overline{Z}}_{tt}(\mathbf{r}_t) = \nabla_t\frac{1}{j\omega\epsilon_{zz}(\mathbf{r}_t)}\nabla_t - j\omega\mathbf{u}_z\mathbf{u}_z\,{}^{\times}_{\times}\,\overline{\overline{\mu}}_{tt}(\mathbf{r}_t), \tag{7}$$

$$\overline{\overline{Y}}_{tt}(\mathbf{r}_t) = \mathbf{u}_z\mathbf{u}_z\,{}^{\times}_{\times}\,\nabla_t\frac{1}{j\omega\mu_{zz}(\mathbf{r}_t)}\nabla_t - j\omega\mathbf{u}_z\mathbf{u}_z\,{}^{\times}_{\times}\,\overline{\overline{\epsilon}}_{tt}(\mathbf{r}_t), \tag{8}$$

$$\overline{\overline{Q}}_{tt}(\mathbf{r}_t) = \mathbf{u}_z\mathbf{u}_z\,{}^{\times}_{\times}\,\nabla_t\frac{\boldsymbol{\mu}_{zt}(\mathbf{r}_t)}{\mu_{zz}(\mathbf{r}_t)} + \frac{\boldsymbol{\epsilon}_{tz}(\mathbf{r}_t)}{\epsilon_{zz}(\mathbf{r}_t)}\nabla_t, \tag{9}$$

where Gibbs dyadic products are used as defined in [8]. For bianisotropic media expressions for these operators can be found in [3]. Using a suitable definition of the transposed of a dyadic operator we can, after tedious manipulations, show that

$$\overline{\overline{Z}}_{tt} = \overline{\overline{Z}}_{tt}^T \qquad \overline{\overline{Y}}_{tt} = \overline{\overline{Y}}_{tt}^T \qquad \overline{\overline{P}}_{tt} = -\overline{\overline{Q}}_{tt}^T. \tag{10}$$

The formal eigenvalue problem now becomes

$$\det\begin{pmatrix} \overline{\overline{P}}_{tt}(\mathbf{r}_t) - \gamma\overline{\overline{I}}_{tt} & \overline{\overline{Z}}_{tt}(\mathbf{r}_t) \\ \overline{\overline{Y}}_{tt}(\mathbf{r}_t) & \overline{\overline{Q}}_{tt}(\mathbf{r}_t) - \gamma\overline{\overline{I}}_{tt} \end{pmatrix} = 0, \tag{11}$$

which is equivalent to

$$\det\begin{pmatrix} -\overline{\overline{Q}}_{tt}^T(\mathbf{r}_t) + \gamma\overline{\overline{I}}_{tt} & \overline{\overline{Z}}_{tt}^T(\mathbf{r}_t) \\ \overline{\overline{Y}}_{tt}^T(\mathbf{r}_t) & -\overline{\overline{P}}_{tt}^T(\mathbf{r}_t) + \gamma\overline{\overline{I}}_{tt} \end{pmatrix} = 0. \tag{12}$$

The legitimity of these operations can be understood when we represent the dyadic operator by some matrices resulting from some kind of discretization of the operators such as finite differences or decompositions in some basis. Taking into account (10) completes the proof.

Note that the modal fields of the two modes propagating in opposite direction do not have a simple relation.

CONCLUSIONS

In this paper we gave an operator based proof for the bidirectionality of reciprocal waveguides starting from the singular, vectorial, non-reciprocal, coupled Sturm-Liouville problem describing the modal propagation.

REFERENCES

[1] D. Dudley, *Mathematical Foundations for Electromagnetic Theory*, IEEE Press Series on Electromagnetic Waves, D. Dudley, Series Editor, New York, 1994.

[2] P.R. McIsaac, "Mode orthogonality in reciprocal and nonreciprocal waveguides," *IEEE Trans. on Microwave Theory and Techn.*, Vol. 39, pp. 1808–1816, 1991.

[3] F. Olyslager, *Electromagnetic waveguides and transmission lines*, Oxford University Press, Oxford, 1999.

[4] A.T. Villneuve, "Orthogonality relationships for waveguides and cavities with inhomogeneous anisotropic media," *IRE Trans. on Microw. Theory and Techn.* Vol. 7, pp. 441–446, 1959.

[5] F. Olyslager, "Properties of and generalized full-wave transmission line models for (bi)(an)isotropoc waveguides," *IEEE Trans. on Microwave Theory and Techn.*, Vol. 44, pp. 2064–2075, 1996.

[6] A.D. Yaghjian, Proc. Natl. Radio Science Meeting, p. 33, Boulder, Colorado, 2006.

[7] D. Pissoort and F. Olyslager, "Study of eigenmodes in periodic waveguides using the Lorentz reciprocity theorem," *IEEE Trans. on Microwave Theory and Techn.*, Vol. 52, pp. 542–553, 2004.

[8] I.V. Lindell, *Methods for Electromagnetic Field Analysis*, 2nd ed., Oxford: University Press, 1995.

O21-56: Memorial Session for the late Donald Dudley

Friday, July 27, 2007 • Drawing Room

Organizers/Co-Chairs: *K. Langenberg, Kassel University, Germany*
E. Heyman, Tel Aviv University, Israel

8:30 AM EMTS269: "Donald G. Dudley: A Life Remembered" (Invited)
W.R. Stone, Stoneware Limited, USA
Presenter: Ross Stone, Stoneware Limited, USA

8:50 AM EMTS273: "Observations on Interior Resonances and the Equivalence Principle in a Simple Setting" (Invited)
C.M. Butler, Clemson University, USA
Presenter: Chalmers Butler, Clemson University, USA

9:10 AM EMTS92: "Large-scale Nonlinear Inversion Approaches for Geophysical Applications" (Invited)
T.M. Habashy, A. Abubakar, Schlumberger-Doll Research, USA
Presenter: Tarek Habashy, Schlumberger-Doll Research, USA

9:30 AM EMTS274: "Closed-Form Expressions for Layered Media Green's Functions in Terms of Spherical and Cylindrical Waves" (Invited)
A.C. Cangellaris, V. Kourkoulos, University of Illinois at Urbana-Champaign, USA
Presenter: Andreas Cangellaris, University of Illinois at Urbana-Champaign, USA

9:50 AM EMTS247: "A Phase-Space Beam Summation Representation for 3D Radiation from a Line Source Distribution" (Invited)
M. Katsav, E. Heyman, Tel Aviv University, Israel
Presenter: Ehud Heyman, Tel Aviv University, Israel

10:10 AM EMTS143: "Recent Developments in the Thin-Wire Problem" (Invited)
A.G. Tijhuis, M.C. van Beurden, Eindhoven University of Technology, The Netherlands
Presenter: Anton Tijhuis, Eindhoven University of Technology, The Netherlands

10:50 AM EMTS178: "Discrete Green's Theorem, Green's functions and stable radiative FDTD boundary conditions" (Invited)
J.M. Arnold, University of Glasgow, UK;
B.P. de Hon, Technical University of Eindhoven, The Netherlands
Presenter: John Arnold, University of Glasgow, UK

11:10 AM EMTS250: "Bidirectionality in the waveguide Sturm-Liouville problem" (Invited)
F. Olyslager, H. Rogier, Ghent University, Belgium
Presenter: Hendrik Rogier, Ghent University, Belgium

11:30 AM EMTS119: "Electromagnetic information theory" (Invited)
E.A. Marengo, F.K. Gruber, Northeastern University, USA
Presenter: Edwin Marengo, Northeastern University, USA

11:50 AM EMTS158: "PHYSICAL LIMITATIONS ON D/Q FOR ANTENNAS" (Invited)
M. Gustafsson, C. Sohl, G. Kristensson, Lund University, Sweden
Presenter: Mats Gustafsson, Lund University, Sweden

12:10 PM EMTS224: "Wave Field Imaging for Homeland Security" (Invited)
K.J. Langenberg, K. Mayer, University of Kassel, Germany;
C. Sklarczyk, Fraunhofer Institut für zerstörungsfreie Prüfverfahren, Germnay
Presenter: Karl Langenberg, University of Kassel, Germany

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