1-systems of $Q(6,q)$ and $Q^-(7,q)$: recent results

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In [2], m-systems of polar spaces were introduced by Shult and Thas and a description of several classes of examples is given. However, few other examples are known until now. First, we briefly discuss the construction of a new class of 1-systems of $Q(6,q)$, q odd, starting from a generalization of a flock of a quadratic cone in $PG(3,q)$. Secondly, we mention a uniqueness result about 1-systems of the quadric $Q^-(7,q)$.

An *i-flock* of a quadratic cone $LQ(2,q)$ with line vertex in $PG(4,q)$, q odd, is defined as a partition of $LQ(2,q)\setminus L$ in q^2 mutually disjoint conics such that the planes of the elements of the i-flock pairwise intersect in internal points of $LQ(2,q)$. It can be shown that to every i-flock of $LQ(2,q)$ a locally hermitian 1-system of $Q(6,q)$ is associated and conversely.

Applying the theory of the i-flocks to the semi-classical non-hermitian spread $\mathcal{S}_{[9]}$ of the hexagon $H(q)$, q odd and $q \equiv 1 \bmod 3$ (see [1]), which is locally hermitian at some line L, we find an interesting geometric construction of $\mathcal{S}_{[9]}$ starting from a rational normal cubic scroll $\mathcal{R}^3$ having L as directrix line. Apparently the conics on $\mathcal{R}^3$ determine the q^2 conic planes of the i-flock associated with $\mathcal{S}_{[9]}$ and hence the i-flock can be reconstructed from the cubic scroll. Surprisingly this geometric construction not only yields the 1-system $\mathcal{S}_{[9]}$; it turns out that different cubic scrolls may give rise to non-isomorphic locally hermitian 1-systems of $Q(6,q)$. In particular, there are $(q-3)/2$ orbits in the set of all non-hermitian locally hermitian 1-systems of $Q(6,q)$ constructed from a cubic scroll, under the subgroup of $PGL(7,q)$ fixing $Q(6,q)$.

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It can be shown that a locally hermitian non-hermitian 1-system of $Q(6,q)$, q odd, is semi-classical if and only if it arises from a rational normal cubic scroll $\mathcal{R}^3$ with directrix line $L \subseteq Q(6,q)$ and with the property that all points of $\mathcal{R}^3 \setminus L$ are internal points of $Q(6,q)$. As it is possible to determine all such cubic scrolls, this yields a complete characterization and determination of the locally hermitian semi-classical 1-systems of $Q(6,q)$ for q odd.

Now, let $\mathcal{M}$ be a 1-system of $Q^-(7,q)$. Then it can be shown that every line of $Q^-(7,q)$ contains 0, 1, 2 or $q+1$ points on lines of $\mathcal{M}$ and that the latter holds if and only if the line itself belongs to $\mathcal{M}$. For q odd, this implies that every plane of $Q^-(7,q)$ either contains a line of $\mathcal{M}$ or an irreducible conic of points on lines of $\mathcal{M}$. This observation enabled us to prove that $Q^-(7,q)$ has a unique 1-system for q odd, up to a projectivity. This is an important result because 1-systems of $Q^-(7,q)$ were candidates to yield new generalized quadrangles, as described in [3].

References

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