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Ana Rita Garcia Alves Semigroup Operators and Applications

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Doutora Maria Paula Marques Smith
e da
Doutora Maria Paula Freitas de Sousa Mendes Martins

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ABSTRACT

In this thesis, some algebraic operators are studied and some examples of their application in semigroup theory are presented. This study contains properties of the following algebraic operators: direct product, semidirect product, wreath product and λ-semidirect product. Characterisations of certain semigroups are provided using the operators studied.

RESUMO

Nesta tese estudamos alguns operadores algébricos e apresentamos exemplos de suas aplicações. No estudo efetuado estabelecemos propriedades dos seguintes operadores algébricos: produto direto, produto semidireto, produto de wreath e produto λ-semidireto. São também estabelecidas caracterizações de certos semigrupos usando os operadores estudados.

Contents

Introduction

The main objectives of this dissertation are the study of some algebraic operators and of their importance for the development of semigroup theory, and the presentation of some examples of their application in this theory. Some of this operators are universal in the sense they are used in classes of any kind of algebras. An example of this is the direct product. Other operators were introduced only for classes of semigroups. That is the case, for example, of the λ-semidirect product. The studies about this last kind of operators can be found in several articles and in certain cases with very different terminology and notation. Thus, in the present study, we present a brief review of this knowledge.

In the preliminary phase, we study basic concepts and results concerning arbitrary semigroups as well as regular semigroups, orthodox and inverse semigroups, which are necessary to understand the subsequent chapters. For all the notations, terminologies and notions not defined in this thesis, and for the proofs of the results presented in Chapter 1, the reader is referred to [5], [6], [7] and [8]. The following chapters contain a study of the direct product, semidirect product, wreath product and λ-semidirect product: some properties and some applications.

1 | Preliminaries

1.1 Basic definitions

A *semigroup* is a pair $(S, \cdot)$ composed of a non-empty set S and an associative binary operation $\cdot$, that is, a binary operation $\cdot$ that satisfies

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z,$$

for all $x, y, z \in S$. This algebraic structure can be found, in a natural way, in mathematics and some examples are $(\mathbb{N}, +)$ and $(\mathbb{N}, \times)$, since the sum and the multiplication of natural numbers satisfy the associative law. Usually, the product of two elements x and y is simply denoted by xy and we write S to denote a semigroup $(S, \cdot)$ when it is not necessary to clarify the nature of the operation.

If a semigroup S satisfies the commutative law, we say that S is a *commutative semigroup*. For example, the multiplication of integer numbers is commutative and so $(\mathbb{Z}, \cdot)$ is a commutative semigroup.

An element $e \in S$ is said to be an *identity* of S if, for all $x \in S$,

$$xe = ex = x.$$

Note that a semigroup S can have at most one identity. When it exists, this element is denoted by 1_S and S is said to be a *monoid*. An element $f \in S$ is said to be a *zero* of S if, for every $x \in S$,

$$xf = fx = f.$$

Moreover, a semigroup S has at most one zero. When it exists, this element is denoted by 0_S. If a semigroup S has no identity then it is possible to extend the multiplication on S to $S \cup \{1\}$ by setting

$$\forall x \in S, \quad x1 = 1x = x \quad \text{and} \quad 11 = 1.$$

Then $(S \cup \{1\}, \cdot)$ is a semigroup with identity 1. This monoid is denoted by S^1. Analogous to the above construction, if a semigroup S has no zero, we can extend the multiplication on S to $S \cup \{0\}$ by

$$\forall x \in S, \quad x0 = 0x = 0 \quad \text{and} \quad 00 = 0.$$

Then $(S \cup \{0\}, \cdot)$ is a semigroup with zero 0. It is denoted by S^0.

Examples of classes of semigroups are the class of left zero semigroups and the class of right zero semigroups. An element x of a semigroup S is called a *left zero* (respectively, *right zero*) if $xy = x$, for all $y \in S$ (respectively, $yx = x$, for all $y \in S$). A semigroup consisting of only left zero elements (respectively, right zero elements) is called a *left zero semigroup* (respectively, *right zero semigroup*).

An element $e \in S$ is called an *idempotent* of S if $e^2 = e$. A semigroup may contain no idempotents. When a semigroup S contains idempotents, the set of idempotents is denoted by $E(S)$. An important class of semigroups is the class of bands. A semigroup S is said to be a *band* if all its elements are idempotents. A band is called a *semilattice* if it is commutative. A semigroup S is said to be a *rectangular band* if it is a band and satisfies $aba = a$, for every $a, b \in S$. It is easy to check that an alternative definition of rectangular band is the following: S is a rectangular band if it is a band and satisfies $abc = ac$, for all $a, b, c \in S$. In fact, the conditions $aba = a$ and $abc = ac$ are equivalent on a band S. Clearly, the second condition implies the first one. Also, if the first condition is satisfied then, for any $a, b, c \in S$,

$$abc = ab(cac) = (a(bc)a)c = ac.$$

It is known that every band is determined by a semilattice Y, a family of rectangular bands indexed by Y and a family of homomorphisms satisfying certain conditions. A band S is called a *left normal band* if $axy = ayx$, for every $a, x, y \in S$.

1.1.1 Subsemigroups

Let S be a semigroup. A non-empty subset T of S is said to be a *subsemigroup* of S if, for every $x, y \in S$,

$$x, y \in T \Rightarrow xy \in T.$$

A subsemigroup of S is a *subgroup* of S if it is a group under the semigroup operation.

Proposition 1.1. *A non-empty subset T of a semigroup S is a subgroup if and only if $Tx = xT = T$, for all $x \in T$, where $Tx = \{yx : y \in T\}$ and $xT = \{xy : y \in T\}$.*

We now consider an important class of subsemigroups of S. A non-empty subset I of S is said to be:

- a *left ideal* of S if, for all $i \in I$ and all $s \in S$, $si \in I$, that is, if $SI \subseteq I$;
- a *right ideal* if $IS \subseteq I$;
- an *ideal* if it is both a left and a right ideal.

For each $a \in S$, it is easy to prove that the smallest left ideal of S containing a is $Sa \cup \{a\}$. This left ideal is called the *principal left ideal generated by* a. We denote it by S^1a.

Similarly, the *principal right ideal generated by* a, aS^1, and the *principal ideal generated by* a, S^1aS^1, are defined.

1.1.2 Homomorphisms

Let S and T be semigroups. A map $\varphi : S \to T$ is called a *homomorphism* (or a *morphism*) if, for every $x, y \in S$,

$$(xy)\varphi = (x\varphi)(y\varphi). \tag{1.1}$$

If S and T are monoids with identities 1_S and 1_T, respectively, φ is said to be a *monoid-morphism* if (1.1) is satisfied and $1_S\varphi = 1_T$.

If φ is injective, φ is said to be a *monomorphism* (or an *embedding*). If there exists an embedding from S into T, we say that S is *embeddable* into T. If φ is surjective, φ is said to be an *epimorphism*. A morphism φ is said to be an *isomorphism* if it is bijective. If there exists an isomorphism from S into T, we say that S and T are *isomorphic* and we write $S \simeq T$.

A morphism φ from S into itself is called an *endomorphism* and an endomorphism φ is called an *automorphism* if it is bijective. We denote by $\mathrm{End}(S)$ the set of all endomorphisms on S and by $\mathrm{Aut}(S)$ the set of all automorphisms on S.

1.1.3 Compatible equivalence relations

A binary relation ρ on a semigroup S is a subset of the cartesian product $S \times S$. If $x, y \in S$ are ρ-related, we simply write $(x, y) \in \rho$ or $x \,\rho\, y$.

A binary relation ρ on a semigroup S is said to be an *equivalence relation* if it is reflexive, symmetric and transitive. For an equivalence relation ρ on S, the sets $[x]_\rho = \{a \in S : \ (x, a) \in \rho\}$ are called *equivalence ρ-classes* or, simply, *ρ-classes*.

A family $\pi = \{A_i : \ i \in I\}$ of subsets of S is called a partition of S if

(1) For each $i \in I$, $A_i \neq \emptyset$;

(2) For all $i, j \in I$, if $i \neq j$ then $A_i \cap A_j = \emptyset$;

(3) $\displaystyle\bigcup_{i \in I} A_i = S$.

Observe that the set $\{[x]_\rho : x \in S\}$ is a partition of the semigroup S. This set is called the *quotient set* and is denoted by S/ρ.

We define some well-known equivalence relations on a semigroup. Principal ideals of a semigroup S allow us to define on S five equivalence relations which are called *Green's relations*. We present the definition of four of them, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$ and $\mathcal{D}$, as well as various results that proved to be relevant for our study. The relations $\mathcal{L}$ and $\mathcal{R}$ are defined by

- For all $a, b \in S$, $a \,\mathcal{L}\, b \Leftrightarrow S^1 a = S^1 b$;

- For all $a, b \in S$, $a \mathcal{R} b \Leftrightarrow aS^1 = bS^1$.

The following results highlight some fundamental properties of Green's relations $\mathcal{L}$ and $\mathcal{R}$.

Proposition 1.2. [7, Cf. Proposition 2.1.1] *Let a, b be elements of a semigroup S. Then*

(1) *$a \mathcal{L} b$ if and only if there exist $x, y \in S^1$ such that $xa = b$ and $yb = a$;*

(2) *$a \mathcal{R} b$ if and only if there exist $u, v \in S^1$ such that $au = b$ and $bv = a$.*

A binary relation ρ on a semigroup S is said to be *left compatible* (with the multiplication) if, for all $a, b, c \in S$,

$$(a, b) \in \rho \Rightarrow (ca, cb) \in \rho,$$

and *right compatible* (with the multiplication) if, for all $a, b, c \in S$,

$$(a, b) \in \rho \Rightarrow (ac, bc) \in \rho.$$

If ρ is left and right compatible, ρ is called *compatible* (with the multiplication). A compatible equivalence is called a *congruence*.

Proposition 1.3. *A relation ρ on a semigroup S is a congruence if and only if*

$$(\forall a, b, c, d \in S) \;\; [(a, b) \in \rho \wedge (c, d) \in \rho \Rightarrow (ac, bd) \in \rho]. \tag{1.2}$$

Proof: Suppose that ρ is a congruence on S. Let $(a, b), (c, d) \in \rho$. By right compatibility, $(ac, bc) \in \rho$ and, by left compatibility, $(bc, bd) \in \rho$. By transitivity, $(ac, bd) \in \rho$. Thus (1.2) is satisfied.

Conversely, suppose that (1.2) holds. If $(a, b) \in \rho$ and $c \in S$ then, by reflexivity, $(c, c) \in \rho$ and so $(ac, bc) \in \rho$ and $(ca, cb) \in \rho$. Hence ρ is left and right compatible and therefore ρ is a congruence. □

If ρ is a congruence on a semigroup S, we can algebrize the quotient set S/ρ in order to obtain a semigroup. On S/ρ, define

$$[a]_\rho [b]_\rho = [ab]_\rho. \tag{1.3}$$

First, note that this definition does not depend on the choice of the representatives of the ρ-classes $[a]_\rho$ and $[b]_\rho$. In fact, if $a' \in [a]_\rho$ and $b' \in [b]_\rho$ then $(a', a) \in \rho$ and $(b', b) \in \rho$. Since ρ is a congruence, $(a'b', ab) \in \rho$. Hence, $[ab]_\rho = [a'b']_\rho$ and so the equality (1.3) defines an operation on S/ρ. Moreover, this operation is associative and therefore $(S/\rho, \cdot)$ is a semigroup.

The relations $\mathcal{L}$ and $\mathcal{R}$ are not congruences. However, they have the following property.

Proposition 1.4. [7, Cf. Proposition 2.1.2] *$\mathcal{L}$ is right compatible with the multiplication and $\mathcal{R}$ is left compatible with the multiplication.*

We know that the intersection of two equivalence relations is an equivalence. The same does not apply to the union of equivalence relations ρ and σ, say. However the intersection of all equivalence relations on an arbitrary semigroup S that contain ρ and σ is the least equivalence relation on S that contains ρ and σ. So, the set $\mathcal{E}(S)$ of all equivalence relations on S, together with inclusion $\subseteq$, is a lattice where $\rho \wedge \sigma = \rho \cap \sigma$ and $\rho \vee \sigma = \bigcap_{\substack{\tau \in \mathcal{E}(S) \\ \tau \supseteq \rho, \sigma}} \tau$. The following proposition is a well-known result:

Proposition 1.5. [6, Cf. Corollary I.5.15] *If ρ and σ are equivalences on a semigroup S such that $\rho \circ \sigma = \sigma \circ \rho$ then $\rho \vee \sigma = \rho \circ \sigma$.*

We are now ready to introduce the definition of two more Green's relations on a semigroup S: $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$. Since $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ [7, Proposition 2.1.3] it follows from Proposition 1.5 that $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$, that is

$$(\forall a, b \in S) \ \ [a \, \mathcal{D} \, b \Leftrightarrow \exists z \in S : \ a \, \mathcal{L} \, z \ \wedge \ z \, \mathcal{R} \, b]$$

[5, Proposition II.1.2].

The following theorem highlights the multiplicative properties of $\mathcal{H}$-classes.

Theorem 1.6. [7, Theorem 2.2.5] **(Green's Theorem)** *If H is an $\mathcal{H}$-class in a semigroup S then either $H^2 \cap H = \emptyset$ or $H^2 = H$ and H is a subgroup of S.*

We denote by L_a (respectively, R_a, H_a, D_a) the $\mathcal{L}$-class (respectively, $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class) that contains the element a.

Corollary 1.7. [7, Corollary 2.2.6] *If e is an idempotent of a semigroup S then H_e is a subgroup of S. No $\mathcal{H}$-class in S can contain more than one idempotent.*

1.2 Regular semigroups

An element $a \in S$ is said to be *regular* if there exists $x \in S$ such that $axa = a$. An element x satisfying $axa = a$ is called an *associate* of a. The set of all associate elements of a is denoted by $A(a)$. If $x \in A(a)$ then the element $x' := xax$ is such that $x' = x'ax'$ and $a = ax'a$. Such an element is called an *inverse* of a. The set of all inverses of a is denoted by $V(a)$.

As a consequence of the definitions of idempotent element, regular element and inverse of an element, we have

Proposition 1.8. *Let S be a semigroup and $a \in S$.*

(1) *If $x \in A(a)$ then $ax \in E(S)$ and $xa \in E(S)$.*

(2) *If $e \in E(S)$ then $e \in V(e)$.*

A semigroup S is said to be a *regular semigroup* if all its elements are regular, that is, if $A(x) \neq \emptyset$, for every $x \in S$. From this definition and (1) of Proposition 1.8, it follows that

Proposition 1.9. *If S is a regular semigroup then $E(S) \neq \emptyset$.*

A consequence of the definition of regular element is presented by the following result and is a useful tool for further results.

Proposition 1.10. *Let S be a regular semigroup. Then, for all $s \in S$, there exist $e, f \in E(S)$ and $x \in S$ such that $xs = e$ and $sx = f$.*

An important result on the study of regular semigroups is the Lallement's Lemma:

Lemma 1.11. [7, Cf. Theorem 2.4.3]**(Lallement's Lemma)** *Let ρ be a congruence on a regular semigroup S, and let $[a]_\rho$ be an idempotent in S/ρ. Then there exists an idempotent e in S such that $[e]_\rho = [a]_\rho$.*

In a regular semigroup S, for any $a \in S$, $a = (aa')a \in Sa$ ($a' \in A(a)$) and, similarly, $a \in aS$. Thus, whenever considering Green's relations $\mathcal{L}$ and $\mathcal{R}$ on a regular semigroup S we can define, more simply,

- For all $a, b \in S$, $a\,\mathcal{L}\,b \Leftrightarrow Sa = Sb$;
- For all $a, b \in S$, $a\,\mathcal{R}\,b \Leftrightarrow aS = bS$.

Also, we can establish Proposition 1.2 for regular semigroups.

Proposition 1.12. *Let a, b be elements of a regular semigroup S. Then*

(1) *$a\,\mathcal{L}\,b$ if and only if there exist $x, y \in S$ such that $xa = b$ and $yb = a$;*

(2) *$a\,\mathcal{R}\,b$ if and only if there exist $u, v \in S$ such that $au = b$ and $bv = a$.*

Using this proposition, the next two corollaries can be easily proved. Corollary 1.14 characterises $\mathcal{L}$ and $\mathcal{R}$ on the set of idempotents of S.

Corollary 1.13. *Let S be a regular semigroup, $a \in S$ and $x \in A(a)$. Then $xa\,\mathcal{L}\,a$ and $a\,\mathcal{R}\,ax$.*

Corollary 1.14. *Let S be a regular semigroup and $e, f \in E(S)$. Then*

(1) *$e\,\mathcal{L}\,f$ if and only if $ef = e$ and $fe = f$;*

(2) *$e\,\mathcal{R}\,f$ if and only if $ef = f$ and $fe = e$.*

For each congruence ρ on a regular semigroup S, there are two sets that play an important role in the definition of congruence: the kernel and the trace of ρ. For a congruence ρ on S,

- the *kernel* of ρ is denoted by $\ker\rho$ and is given by

$$\ker\rho := \bigcup_{e \in E(S)} [e]_\rho;$$

- the *trace* of ρ is denoted by $\operatorname{tr}\rho$ and is the restriction of ρ to $E(S)$: $\operatorname{tr}\rho = \rho|_{E(S)}$.

Each congruence ρ on a regular semigroup can therefore be associated to the ordered pair $(\ker\rho, \operatorname{tr}\rho)$. In [11] the authors provide a characterisation of such pair and proved that the pair $(\ker\rho, \operatorname{tr}\rho)$ uniquely determines ρ:

Proposition 1.15. [11, Corollary 2.11] *A congruence on a regular semigroup S is uniquely determined by its kernel and its trace.*

We observe that in view of Lallement's Lemma we have

$$ker\rho = \{s \in S : \ (s, s^2) \in \rho\}.$$

A congruence ρ on a semigroup S is said to be a *group congruence* if S/ρ is a group. If ρ is a group congruence on a regular semigroup S, the trace of ρ is the universal congruence in $E(S)$ and $\ker\rho = 1_{S/\rho}$.

We end this section addressing a special class of semigroups. Let S be a semigroup and let $G(S)$ be the group generated by the elements of S, as generators, and all identities $ab = c$ which hold in S, as relations. The mapping $\alpha : S \to G(S)$ defined by $s\alpha = s$, for all $s \in S$, is a homomorphism and is such that, for any group H and any semigroup homomorphism $h : S \to H$, there exists a unique group homomorphism $g : G(S) \to H$ satisfying $h = \alpha g$. The group $G(S)$, together with the homomorphism α, is called the *universal group of S*.

We have the following result for regular semigroups:

Proposition 1.16. [5, Cf. Proposition IX.4.1] *For a regular semigroup S, the following statements are equivalent:*

(1) $e \in E(S)$ *and* $ea \in E(S) \Rightarrow a \in E(S)$*;*

(2) $e \in E(S)$ *and* $ae \in E(S) \Rightarrow a \in E(S)$*;*

(3) $E(S) = (1_{G(S)})\alpha^{-1}$*, where* $(G(S), \alpha)$ *is the universal group of* S*.*

A regular semigroup that satisfies the equivalent conditions of Proposition 1.16 is called an *E-unitary semigroup*.

1.2.1 Inverse semigroups

A semigroup $(S, \cdot)$ is said to be a *U-semigroup* if a unary operation $x \mapsto x'$ is defined on S such that, for all $x \in S$,

$$(x')' = x.$$

Clearly, every semigroup is a U-semigroup for the unary operation $a \mapsto a' = a$. We now see a special case where the unary and the binary operations interact with each other. If S is a U-semigroup and the unary operation $x \mapsto x'$ satisfies, for all $x \in S$, the axiom $xx'x = x$, we say that S is an *I-semigroup*. In an I-semigroup S, given $x \in S$, since $x' \in S$, we have

$$x'xx' = x'(x')'x' = x'.$$

Thus $x' \in V(x)$. Because of this, x' is usually denoted by x^{-1}. An important class of I-semigroups is the class of inverse semigroups. A semigroup S is said to be an *inverse semigroup* if it is an I-semigroup and its idempotents commute. So the set of idempotents of an inverse semigroup S is a commutative inverse subsemigroup of S. Since the inverse of an element $x \in S$ is, in particular, an associate of x, we have:

Proposition 1.17. *Every inverse semigroup is regular.*

The converse of Proposition 1.17 does not hold. For example, a left zero semigroup with two elements a and b, say, is a regular semigroup but it is not an inverse semigroup since its idempotents do not commute ($ab = a$ and $ba = b$) . However, if all idempotents of a regular semigroup S commute then S is an inverse semigroup. Some characterisations of an inverse semigroup are listed in the next result:

Theorem 1.18. [7, Cf. Theorem 5.1.1] *Let S be a semigroup. The following statements are equivalent:*

(1) *S is an inverse semigroup;*

(2) *Every $\mathcal{L}$-class and every $\mathcal{R}$-class contains exactly one idempotent;*

(3) *Every element of S has a unique inverse.*

We now present some properties of inverse semigroups.

Proposition 1.19. [7, Cf. Proposition 5.1.2] *Let S be an inverse semigroup. Then*

(1) *For all $a, b \in S$, $(ab)^{-1} = b^{-1}a^{-1}$;*

(2) *For every $a \in S$ and every $e \in E(S)$, $aea^{-1} \in E(S)$ and $a^{-1}ea \in E(S)$;*

(3) *For all $a, b \in S$, $(a, b) \in \mathcal{L} \Leftrightarrow a^{-1}a = b^{-1}b$ and $(a, b) \in \mathcal{R} \Leftrightarrow aa^{-1} = bb^{-1}$;*

(4) *If $e, f \in E(S)$, then $(e, f) \in \mathcal{D}$ if and only if there is $a \in S$ such that $aa^{-1} = e$ and $a^{-1}a = f$.*

A group is an inverse semigroup. Proposition 1.1 characterises a group and the next result gives a characterisation of a group in terms of an inverse semigroup.

Proposition 1.20. *A semigroup S is a group if and only if S is an inverse semigroup with a unique idempotent.*

The next result provides a representation for left cosets of a group. It will be useful for proving some auxiliar results on Chapter 3.

Lemma 1.21. *Every non-empty subset X of a group G is a left coset of G if and only if $X = XX^{-1}X$, where $X^{-1} = \{x^{-1} : x \in X\}$.*

Proof: Let X be a non-empty subset of a group G. Suppose that X is a left coset of G. Then there is a subgroup H of G such that $X = aH$, for some $a \in G$. Then

$$\begin{aligned} x \in XX^{-1}X &\Rightarrow x = ah_1(ah_2)^{-1}ah_3, \text{ with } h_1, h_2, h_3 \in H \\ &\Rightarrow x = ah_1h_2^{-1}a^{-1}ah_3 = a(h_1h_2^{-1}h_3), \text{ with } h_1h_2^{-1}h_3 \in H \\ &\Rightarrow x \in aH = X. \end{aligned}$$

Also, for any $x \in X$, $x = xx^{-1}x$ and so $x \in XX^{-1}X$. Hence, $X = XX^{-1}X$.

Conversely, suppose that $X = XX^{-1}X$. Let $H = \{y^{-1}z : y, z \in X\}$. We show that H is a subgroup of G:

- The set H is non-empty since $X \neq \emptyset$;
- For $a, b \in H$, there exist $y_1, y_2, z_1, z_2 \in X$ such that $a = y_1^{-1}z_1$ and $b = y_2^{-1}z_2$. Then $ab = y_1^{-1}(z_1y_2^{-1}z_2)$, with $y_1 \in X$ and $z_1y_2^{-1}z_2 \in XX^{-1}X = X$. Hence $ab \in H$.
- For $a \in H$, there exist $y, z \in X$ such that $a = y^{-1}z$. Then $a^{-1} = (y^{-1}z)^{-1} = z^{-1}y$ and therefore $a^{-1} \in H$.

Moreover, for every $x \in X$, $X = xH$, since

- for $a \in X$, we have $a = 1_G a = xx^{-1}a$ and $x^{-1}a \in H$;

and

- for every $y, z \in X$, $x(y^{-1}z) = xy^{-1}z \in XX^{-1}X = X$.

Hence X is a left coset of the subgroup H of G. □

The natural partial order

A binary relation ρ on a set S is said to be a *partial order* if it is reflexive, antisymmetric and transitive. A partial order on S is denoted by $\leq_S$ or, simply, by $\leq$.

Let S be an inverse semigroup. Note that $E(S)$ is a non-empty set. The binary relation $\leq$ defined on $E(S)$ by

$$(\forall e, f \in E(S)) \;\; [e \leq f \Leftrightarrow ef = fe = e] \tag{1.4}$$

is a partial order. This partial order can be extended to the whole semigroup in the following way:

$$(\forall a, b \in S) \;\; [a \leq b \Leftrightarrow \exists e \in E(S): \;\; a = eb]. \tag{1.5}$$

In fact, if $a, b \in E(S)$ and $a \leq b$ then there exits $g \in E(S)$ such that $a = gb$. Since a, b and g are idempotents and the idempotents commute, we have

$$a = gb = (gb)b = ab = ba.$$

So the partial order defined by (1.5), when restricted to the set of idempotents of S, coincides with the one defined by (1.4). We call *natural partial order* to the binary relation $\leq$ defined on S by (1.5). This relation is compatible with the multiplication. In fact, for $a, b, c, d \in S$ such that $a \leq b$ and $c \leq d$, we have

$$a \leq b \Leftrightarrow \exists e \in E(S) : \quad a = eb$$

and

$$c \leq d \Leftrightarrow \exists f \in E(S) : \quad c = fd.$$

Then

$$\begin{aligned} ac &= ebfd \\ &= eb(b^{-1}b)fd \\ &= (ebfb^{-1})bd. \end{aligned}$$

Since $e, bfb^{-1} \in E(S)$, $ebfb^{-1} \in E(S)$ and so $ac \leq bd$. Also, the relation $\leq$ is compatible with the inversion, that is, if $a \leq b$ then $a^{-1} \leq b^{-1}$. Let $a, b \in S$ be such that $a \leq b$. Then $a = eb$, for some $e \in E(S)$, and so

$$a^{-1} = (eb)^{-1} = b^{-1}e^{-1} = b^{-1}(bb^{-1})e = (b^{-1}eb)b^{-1}.$$

Since $b^{-1}eb \in E(S)$, we obtain $a^{-1} \leq b^{-1}$.

In the following result, some alternative characterisations of the natural partial order on inverse semigroups are presented.

Proposition 1.22. [7, Cf. Proposition 5.2.1] *On an inverse semigroup S, the following statements are equivalent, for all $a, b \in S$,*

(1) $a \leq b$;

(2) $\exists e \in E(S) : \ a = be$;

(3) $aa^{-1} = ba^{-1}$;

(4) $aa^{-1} = ab^{-1}$;

(5) $a^{-1}a = b^{-1}a$;

(6) $a^{-1}a = a^{-1}b$;

(7) $a = ab^{-1}a$;

(8) $a = aa^{-1}b$.

Congruences

A congruence ρ on an inverse semigroup S has the following useful properties.

Proposition 1.23. [8, Proposition 2.3.4] *Let ρ be a congruence on an inverse semigroup S.*

(1) *If $(s,t) \in \rho$ then $(s^{-1}, t^{-1}) \in \rho$, $(s^{-1}s, t^{-1}t) \in \rho$ and $(ss^{-1}, tt^{-1}) \in \rho$.*

(2) *If $e \in E(S)$ and $(s,e) \in \rho$ then $(s, s^{-1}) \in \rho$, $(s, s^{-1}s) \in \rho$ and $(s, ss^{-1}) \in \rho$.*

Proposition 1.24. *Let ρ be a congruence on an inverse semigroup S. Then*

(1) *S/ρ is an inverse semigroup;*

(2) *$\ker\rho$ is an inverse subsemigroup of S.*

Proof:

(1) S/ρ is clearly a semigroup, since S is a semigroup. Let $[x]_\rho \in S/\rho$. Then $x \in S$. Since S is an inverse semigroup, there exists a unique inverse of x, $x^{-1} \in S$. So $[x^{-1}]_\rho \in S/\rho$ and $[x^{-1}]_\rho \in V([x]_\rho)$. Let $[a]_\rho, [b]_\rho \in E(S/\rho)$. By Lemma 1.11, $[a]_\rho = [e]_\rho$ and $[b]_\rho = [f]_\rho$, for some $e, f \in E(S)$. Since S is inverse, $ef = fe$ Thus, $[e]_\rho[f]_\rho = [f]_\rho[e]_\rho$, that is, $[a]_\rho[b]_\rho = [b]_\rho[a]_\rho$. Hence S/ρ is an inverse semigroup.

(2) Since the idempotents of S commute and $ef \in E(S)$, for all $e, f \in E(S)$, $\ker\rho$ is a subsemigroup of S. Let $x \in \ker\rho$. Then $x \in [e]_\rho$, for some $e \in E(S)$. Since S is an inverse semigroup, there exists $x^{-1} \in S$ the unique inverse of x. We show that $x^{-1} \in \ker\rho$. We have

$$\begin{aligned}(x,e) \in \rho \ &\Rightarrow (x^{-1}, e^{-1}) \in \rho \quad \text{(Proposition 1.23)}\\ &\Rightarrow (x^{-1}, e) \in \rho\end{aligned}$$

and so $x^{-1} \in [e]_\rho$, $e \in E(S)$. Then $x^{-1} \in \ker\rho$. Since $\ker\rho \subseteq S$, it is clear that the idempotents of $\ker\rho$ commute. Thus $\ker\rho$ is an inverse subsemigroup of S. □

Let S be a semigroup. A congruence ρ on S is said to be an *inverse semigroup congruence* if S/ρ is an inverse semigroup. Other kind of congruences is the idempotent-separating congruences. We say that an equivalence relation ρ on S is *idempotent-separating* or that *separates idempotents* if $\operatorname{tr}\rho = \operatorname{id}_{E(S)}$, that is, no ρ-class has more than one idempotent. We now present some results about idempotent-separating congruences on inverse semigroups. By Corollary 1.7, in any semigroup S, H_e is a subgroup of S, for all $e \in E(S)$. Then the equivalence $\mathcal{H}$ separates idempotents and therefore every congruence contained in $\mathcal{H}$ separates idempotents. The condition of a congruence ρ being contained in $\mathcal{H}$ is also necessary for ρ to be idempotent-separating.

Proposition 1.25. [8, Cf. Proposition 3.2.12] *If S is an inverse semigroup then a congruence ρ on S is idempotent-separating if and only if $\rho \subseteq \mathcal{H}$.*

The maximum idempotent-separating congruence on an inverse semigroup S is given by

$$\mu_S = \{(a, b) \in S \times S : \ (\forall e \in E(S)) \ a^{-1}ea = b^{-1}eb\}$$

[6, Theorem V.3.2].

Before showing that the maximum idempotent-separating congruence, $\mu_{S/\mu}$, on S/μ_S, with S an inverse semigroup, is the identity congruence, we have to recall the definition of a congruence on the quotient semigroup. On an inverse semigroup S, if ρ and τ are both congruences on S and $\rho \supseteq \tau$ then the relation

$$\rho/\tau := \{([a]_\tau, [b]_\tau) \in S/\tau \times S/\tau : \ (a, b) \in \rho\}$$

is a congruence on S/τ [6, Theorem V.5.6].

To prove that μ_{S/μ_S} is the identity congruence we must show that, for all $([a]_{\mu_S}, [b]_{\mu_S}) \in \mu_{S/\mu_S}$, $[a]_{\mu_S} = [b]_{\mu_S}$. Suppose that $([a]_{\mu_S}, [b]_{\mu_S}) \in \mu_{S/\mu_S}$. Then every idempotent in S/μ_S has the form $[e]_{\mu_S}$, with $e \in E(S)$, we obtain

$$[a]_{\mu_S}^{-1}[e]_{\mu_S}[a]_{\mu_S} = [b]_{\mu_S}^{-1}[e]_{\mu_S}[b]_{\mu_S}$$

and so

$$[a^{-1}ea]_{\mu_S} = [b^{-1}eb]_{\mu_S}.$$

Since $a^{-1}ea, b^{-1}eb \in E(S)$ and μ_S is idempotent-separating, it follows that $a^{-1}ea = b^{-1}eb$. Hence $(a, b) \in \mu_S$, that is, $[a]_{\mu_S} = [b]_{\mu_S}$.

An inverse semigroup S is said to be *fundamental* if μ_S is the identity congruence.

The considerations made above prove the next theorem.

Theorem 1.26. [6, Cf. Theorem V.3.4] *Let S be an inverse semigroup and μ_S be the maximum idempotent-separating congruence on S. Then S/μ_S is fundamental.*

Observe that not all elements of an inverse semigroup S commute with all idempotents of S and so we define the *centraliser* of $E(S)$ in S:

$$Z(E(S)) = \{x \in S : \ xe = ex, \ \text{for all } e \in E(S)\}.$$

Let S be an inverse semigroup. An inverse subsemigroup of S is said to be *full* if it contains all the idempotents of S.

Proposition 1.27. *Let S be an inverse semigroup. Then $Z(E(S))$ is a full inverse subsemigroup of S.*

Proof: Let $x, y \in Z(E(S))$. Then, for any $e \in E(S)$,

$$\begin{aligned} (xy)e &= x(ey) \quad (y \in Z(E(S))) \\ &= e(xy) \quad (x \in Z(E(S))), \end{aligned}$$

and so $xy \in Z(E(S))$. Therefore $Z(E(S))$ is a subsemigroup of S.

Let $x \in Z(E(S))$ and $x^{-1} \in S$ be the inverse of x. Then

$$\begin{aligned} x^{-1}e &= x^{-1}xx^{-1}e \\ &= x^{-1}exx^{-1} \quad \text{(idpts commute)} \\ &= x^{-1}xex^{-1} \quad (x \in Z(E(S))) \\ &= ex^{-1}. \quad \text{(idpts commute)} \end{aligned}$$

Hence $x^{-1} \in Z(E(S))$. Thus $Z(E(S))$ is inverse.

Let $f \in E(S)$. Since S is inverse, all its idempotents commute and so, for any $e \in E(S)$, $ef = fe$. Then $f \in Z(E(S))$. Hence $Z(E(S))$ is full. $\square$

A semigroup S is said to be a *Clifford semigroup* if it is regular and $Z(E(S)) = S$. A Clifford semigroup is an inverse semigroup. Using this definition, it is easy to prove the following result:

Proposition 1.28. *Let S be an inverse semigroup. Then $Z(E(S))$ is a Clifford semigroup.*

Proof: By Proposition 1.27, $Z(E(S))$ is a regular semigroup. Since $Z(E(S)) \subseteq S$ and $E(Z(E(S))) \subseteq E(S)$, it is obvious, from the definition of $Z(E(S))$, that

$$\forall e \in E(Z(E(S)))\ \ \forall a \in Z(E(S))\ ea = ae.$$

Hence $Z(E(S))$ is Clifford semigroup. $\square$

Proposition 1.29. *Let S be an inverse semigroup and μ_S be the maximum idempotent-separating congruence on S. Then μ_S is the unique idempotent-separating congruence such that $\ker \mu_S = Z(E(S))$.*

Proof: We show that $\ker \mu_S = Z(E(S))$. Let $a \in \ker \mu_S$ and $f \in E(S)$. Then $(a, e) \in \mu_S$, for some $e \in E(S)$, and so,

$$a^{-1}fa = e^{-1}fe,$$

that is,

$$a^{-1}fa = efe = e^2f = ef.$$

We have

$$\begin{aligned} (a^{-1}fa)^{-1}f(a^{-1}fa) &= (ef)^{-1}f(ef) \\ &= effef \\ &= ef \end{aligned}$$

and

$$\begin{aligned} (fe)^{-1}f(fe) &= feffe \\ &= fe. \end{aligned}$$

Since the idempotents of S commute, $ef = fe$ and so

$$(a^{-1}fa)^{-1}f(a^{-1}fa) = (fe)^{-1}f(fe),$$

which gives $(a^{-1}fa, fe) \in \mu_S$. From μ_S being an idempotent-separating congruence and $a^{-1}fa, fe \in E(S)$, we obtain $a^{-1}fa = fe$. By Proposition 1.25, $\mu_S \subseteq \mathcal{H}$ and so, from $(a, e) \in \mu_S$, we obtain $ae = a$. Thus

$$\begin{aligned} fa &= faa^{-1}a \\ &= aa^{-1}fa \\ &= aef \\ &= af. \end{aligned}$$

Hence $a \in Z(E(S))$. Conversely, let $a \in Z(E(S))$. We have

$$a = a(a^{-1}a) = (a^{-1}a)a$$

and so

$$aa^{-1} = (a^{-1}a)(aa^{-1})$$

giving

$$aa^{-1} \leq a^{-1}a.$$

Since $Z(E(S))$ is an inverse subsemigroup of S, $a^{-1} \in Z(E(S))$ and so

$$a^{-1} = a^{-1}(aa^{-1}) = (aa^{-1})a^{-1}$$

whence

$$a^{-1}a = (aa^{-1})(a^{-1}a)$$

and therefore

$$a^{-1}a \leq aa^{-1}.$$

Thus $aa^{-1} = a^{-1}a$. For any $e \in E(S)$,

$$\begin{aligned} a^{-1}ea &= a^{-1}ae && (a \in Z(E(S))) \\ &= aa^{-1}e && (a^{-1}a = aa^{-1}) \\ &= eaa^{-1} && \text{(idpts commute)} \\ &= e(aa^{-1})(aa^{-1}) \\ &= (aa^{-1})^{-1}e(aa^{-1}) && \text{(idpts commute)} \end{aligned}$$

and therefore $(a, aa^{-1}) \in \mu_S$. Since $aa^{-1} \in E(S)$, it follows that $a \in \ker \mu_S$.

The uniqueness of μ_S follows directly from Proposition 1.15. □

As a consequence of Proposition 1.28 and Proposition 1.29, we have the following result:

Corollary 1.30. *Let S be an inverse semigroup. Then the inverse semigroup $\ker \mu_S$ is a Clifford semigroup.*

1.2.2 Orthodox semigroups

By Proposition 1.9, in regular semigroups there always exist elements which are idempotents and so we can consider the non-empty set of idempotents $E(S)$. Although $E(S)$ is not necessarily a subsemigroup of S, there are regular semigroups in which the idempotents form a subsemigroup – it is the case, for example, of bands. Thus it makes sense to define the following concept. A semigroup S is called *orthodox* if it is regular and if its idempotents constitute a subsemigroup of S. Next, we present some characterisations of this class of semigroups.

Theorem 1.31. [7, Cf. Theorem 6.2.1] *Let S be a regular semigroup. The following statements are equivalent:*

(1) S is orthodox;

(2) For every $a, b \in S$, $V(b)V(a) \subseteq V(ab)$;

(3) For all $e \in E(S)$, $V(e) \subseteq E(S)$.

A further characterisation of orthodox semigroups is the following:

Theorem 1.32. [7, Theorem 6.2.4] *A regular semigroup S is orthodox if and only if*

$$(\forall a, b \in S) \ \ [V(a) \cap V(b) \neq \emptyset \Rightarrow V(a) = V(b)].$$

Orthodox semigroups are not necessarily inverse. They can, however, be factorised into inverse semigroups as the next proposition shows.

Proposition 1.33. [7, Theorem 6.2.5] *Let S be an orthodox semigroup. The relation*

$$\gamma = \{(x, y) \in S \times S : \ V(x) = V(y)\}$$

is the smallest inverse semigroup congruence on S.

In an orthodox semigroup S, the congruence γ satisfies an important property:

Proposition 1.34. *Let S be an orthodox semigroup. Then, for all $x \in S$,*

$$(x, e) \in \gamma \wedge e \in E(S) \ \Rightarrow \ x \in E(S).$$

Proof: Let $x \in S$ and $e \in E(S)$ be such that $(x, e) \in \gamma$. Then, by Lemma 1.33, $V(x) = V(e)$ and so, since $e \in V(e)$, $e \in V(x)$, that is, $x \in V(e)$. It now follows from (3) of Theorem 1.31 that $x \in E(S)$. □

A congruence on a semigroup S with idempotents is called *idempotent-pure* if $[e]_\rho \subseteq E(S)$, for all $e \in E(S)$.

Corollary 1.35. *The smallest inverse congruence on an orthodox semigroup is idempotent-pure.*

It follows immediately from (3) in Proposition 1.16 that E-unitary regular semigroups are orthodox. E-unitary semigroups are exactly the semigroups for which the band of idempotents is a σ-class, where σ is the least group congruence on the semigroup:

Proposition 1.36. *Let S be an E-unitary semigroup and σ_S be the least group congruence on S. Then $E(S)$ is a σ_S-class and hence $E(S)$ is the kernel of σ_S.*

Proof: We show that $E(S)$ is the identity class of the group S/σ_S. We have

$$\begin{aligned} a \in 1_{S/\sigma_S} \quad & \Rightarrow [a]_{\sigma_S} = 1_{S/\sigma_S} \in E(S/\sigma_S) \\ & \Rightarrow [a]_{\sigma_S} = [e]_{\sigma_S}, \text{ for some } e \in E(S) \quad \text{(Lemma 1.11)} \\ & \Rightarrow (a, e) \in \sigma_S \\ & \Rightarrow afe' \in E(S), \text{ for some } f \in E(S) \quad \text{[15, Lemma 1.3]} \\ & \Rightarrow a(fe) \in E(S). \end{aligned}$$

Since fe is an idempotent and S is E-unitary, we obtain that a is also an idempotent of S. Then $1_{S/\sigma_S} \subseteq E(S)$. The converse is clear since, for any $e \in E(S)$, $[e]_{\sigma_S}$ is an idempotent of the group S/σ_S, and so $[e]_{\sigma_S} = 1_{S/\sigma_S}$, giving $e \in 1_{S/\sigma_S}$. Then $E(S) \subseteq 1_{S/\sigma_S}$. Hence $E(S)$ is the identity of the group S/σ_S. □

The next result is very useful to prove a result on Chapter 4.

Proposition 1.37. *Let S be an E-unitary regular semigroup such that $E(S)$ is a left normal band. Let σ_S be the least group congruence on S. Then $(s,t) \in \sigma_S$ if and only if $st' \in E(S)$, for $t' \in V(t)$.*

Proof: Let S be an E-unitary regular semigroup, such that $E(S)$ is a left normal band, and σ_S be the least group congruence on S. From [15, Lemma 1.3], it follows that the following statements are equivalent:

(i) $(s,t) \in \sigma_S$;

(ii) $set' \in E(S)$, for some $e \in E(S)$ and some $t' \in V(t)$.

By [14, Lemma 2.6], (ii) is equivalent to

(iii) $st' \in E(S)$, for $t' \in V(t)$

and so (i) and (iii) are equivalent. □

2 | Direct Product

In semigroup theory, given a non-empty family of semigroups it is possible to construct a new semigroup. One of the methods for such a construction and the simplest one is the direct product of semigroups. It is given by the cartesian product of the underlying sets and an operation defined componentwise.

Let $\mathcal{S} = \{S_i : \ i \in I\}$ be a non-empty family of semigroups. On the cartesian product $\prod_{i \in I} S_i$, the operation defined by

$$(x_i)_{i\in I}(y_i)_{i\in I} = (x_i y_i)_{i\in I},$$

for all $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$, is easily seen to be associative. The resulting semigroup $\left(\prod_{i\in I} S_i, \cdot\right)$ is called the *direct product of $\mathcal{S}$*.

Example 2.1. Consider the semigroups $S_1 = (\mathbb{N}, +)$ and $S_2 = (\mathbb{Z}, \times)$. The operation of the direct product $S_1 \times S_2$ is given by

$$(n, x)(m, y) = (n + m, x \times y).$$

In the next result, we show how the notion of direct product can be used to provide a characterisation of rectangular bands.

Theorem 2.2. *The direct product of a left zero semigroup by a right zero semigroup is a rectangular band. Conversely, every rectangular band is isomorphic to the direct product of a left zero semigroup by a right zero semigroup.*

Proof: Let I be a left zero semigroup and Λ be a right zero semigroup. Consider the direct product $I \times \Lambda$. For $(i, \lambda), (j, \mu) \in I \times \Lambda$, we have

$$(i, \lambda)(i, \lambda) = (i, \lambda)$$

and

$$(i, \lambda)(j, \mu)(i, \lambda) = (iji, \lambda\mu\lambda) = (i, \lambda).$$

Then the direct product $I \times \Lambda$ is a rectangular band.

Now, let B be a rectangular band. Since every element of B is idempotent and $x = xyx$, for every $x, y \in B$,

$$x = x^2 = xyx \qquad \text{and} \qquad y = y^2 = yxy.$$

It follows that $x \,\mathcal{R}\, xy$ and $xy \,\mathcal{L}\, y$. Hence $x \,\mathcal{D}\, y$, for all $x, y \in B$. By the definition of $\mathcal{D}$-class, we can conclude that the intersection of any $\mathcal{R}$-class and any $\mathcal{L}$-class is non-empty. Since B is a band, it follows by Corollary 1.7 that any $\mathcal{H}$-class has a unique idempotent and hence a single element.

Let $\theta : B \to B/\mathcal{R} \times B/\mathcal{L}$ be defined by $a\theta = (R_a, L_a)$. We show that θ is a bijection. Let $a, b \in B$ be such that $a\theta = b\theta$. Then

$$\begin{aligned}
(R_a, L_a) = (R_b, L_b) \;&\Rightarrow\; a \,\mathcal{R}\, b \quad \wedge \quad a \,\mathcal{L}\, b \\
&\Rightarrow\; a \,\mathcal{H}\, b \qquad\qquad\qquad \text{(definition of } \mathcal{H}\text{-class)} \\
&\Rightarrow\; a = b. \qquad\qquad\qquad \text{(each } \mathcal{H}\text{-class has a unique idempotent)}
\end{aligned}$$

So θ is injective.

Let $(R_b, L_a) \in B/\mathcal{R} \times B/\mathcal{L}$. By the above, $b \,\mathcal{R}\, ba$ and $ba \,\mathcal{L}\, a$ and so

$$(R_b, L_a) = (R_{ba}, L_{ba}) = (ba)\theta.$$

Thus θ is surjective.

Since, for all $a, a', b, b' \in B$,

$$a \,\mathcal{R}\, a' \Leftrightarrow R_a = R_{a'}$$

and

$$b \,\mathcal{L}\, b' \Leftrightarrow L_b = L_{b'},$$

the equalities

$$R_a R_b = R_a \qquad \text{and} \qquad L_a L_b = L_b$$

define operations on $B/\mathcal{R}$ and on $B/\mathcal{L}$, respectively. These operations make $B/\mathcal{R}$ a left zero semigroup and $B/\mathcal{L}$ a right zero semigroup, respectively.

We now consider the direct product $B/\mathcal{R} \times B/\mathcal{L}$ and show that θ is a homomorphism. In fact, for all $a, b \in B$,

$$(ab)\theta = (R_{ab}, L_{ab}) = (R_a, L_b)$$

and

$$(a\theta)(b\theta) = (R_a, L_a)(R_b, L_b) = (R_a R_b, L_a L_b) = (R_a, L_b).$$

Thus θ is an isomorphism. $\square$

3 | Semidirect Product

The construction of the direct product of semigroups is generalized by the so called semidirect product of semigroups. This notion of semidirect product was used for semigroups by Neumann in [9] as a tool to define another operator – the wreath product of semigroups – which we will study in the next chapter.

3.1 Definitions and basic results

Let S and T be semigroups. The semigroup S is said to *act on T by endomorphisms on the left* if, for every $s \in S$, there is a mapping $a \mapsto {}^{s}a$ from T to itself such that, for all $s, r \in S$ and for all $a, b \in T$,

(SP1) ${}^{s}(ab) = {}^{s}a\,{}^{s}b$;

(SP2) ${}^{sr}a = {}^{s}({}^{r}a)$.

If S is a monoid, we say that the *monoid S acts on T by endomorphisms on the left* if the semigroup S acts on T by endomorphisms on the left and, for all $a \in T$,

(SP3) ${}^{1_S}a = a$.

If S acts on T by endomorphisms on the left and the mapping $a \mapsto {}^{s}a$ is a bijection, we say that S *acts on T by automorphisms on the left*.

Let S and T be semigroups such that S acts on T by endomorphisms on the left. On the cartesian product $T \times S$, consider the operation defined by

$$(a, s)(b, r) = (a\,{}^{s}b, sr), \tag{3.1}$$

for all $a, b \in T$ and for all $s, r \in S$. We show that this operation is associative. Let $(a,s),(b,r),(c,u) \in T \times S$. Then

$$\begin{aligned}
(a,s)\left((b,r)(c,u)\right) &= (a,s)(b\,{}^{r}c, ru) \\
&= (a\,{}^{s}(b\,{}^{r}c), s(ru)) \\
&= (a\,{}^{s}b\,{}^{s}({}^{r}c), sru) && \text{(SP1)} \\
&= (a\,{}^{s}b\,{}^{sr}c, (sr)u) && \text{(SP2)} \\
&= (a\,{}^{s}b, sr)(c,u) \\
&= \left((a,s)(b,r)\right)(c,u).
\end{aligned}$$

Hence $T \times S$ equipped with the multiplication given by (3.1) is a semigroup. This semigroup is called the *semidirect product of T by S* and is denoted by $T * S$.

Proposition 3.1. *Let S be a monoid acting on a monoid T by endomorphisms on the left. Then the semidirect product of T by S is a monoid with identity $(1_T, 1_S)$.*

Proof: Suppose that S and T are monoids. For any $(a,b) \in T * S$,

$$(a,b)(1_T, 1_S) = (a\,{}^{b}1_T, b1_S) = (a1_T, b) = (a,b)$$

and

$$(1_T, 1_S)(a,b) = (1_T\,{}^{1_S}a, 1_S b) = (1_T a, b) = (a,b).$$

So $(1_T, 1_S)$ is the identity of $T * S$. □

We observe that for arbitrary semigroups S and T, S acts on T by endomorphisms on the left. In fact, for every $s \in S$, the mapping $a \mapsto {}^{s}a = a$ from T to itself satisfies conditions (SP1) and (SP2). Hence we can consider the semidirect product $T * S$. Moreover, since, for all $a, b \in T$ and $s, r \in S$,

$$(a,s)(b,r) = (a\,{}^{s}b, sr) = (ab, sr),$$

we have that the direct product of any two semigroups T and S is a semidirect product of T by S.

We define now an operator called reverse semidirect product and we will prove that under certain conditions the semidirect product and the reverse semidirect product coincide up to isomorphisms.

Let S and T be semigroups. The semigroup S is said to *act reversely on* T *by endomorphisms on the left* if, for each $s \in S$, there is a mapping $a \mapsto {}_s a$ of T, such that, for all $a, b \in T$ and $s, r \in T$

(RP1) ${}_s(ab) = {}_s a \, {}_s b$;

(RP2) ${}_s({}_r a) = {}_{rs} a$.

If S is a monoid, we say that the *monoid* S *acts reversely on* T *by endomorphisms on the left* if the semigroup S acts reversely on T by endomorphisms on the left and, for all $a \in T$,

(RP3) ${}_{1_S} a = a$.

If S acts reversely on T by endomorphism on the left and, for every $s \in S$, the mapping $a \mapsto {}_s a$, for all $a \in T$, is a bijection, we say that S *acts reversely on* T *by automorphisms on the left.*

Let S be a semigroup acting reversely on a semigroup T by endomorphisms on the left. On the cartesian product $T \times S$, consider the operation defined by

$$(a, s)(b, r) = ({}_r ab, sr) \tag{3.2}$$

for all $a, b \in T$ and all $s, r \in S$. We show that this operation is associative. Let $(a, s), (b, r), (c, u) \in T \times S$. Then

$$\begin{aligned}
(a, s)((b, r)(c, u)) &= (a, s)({}_u bc, ru) \\
&= ({}_{ru} a \, {}_u bc, s(ru)) \\
&= ({}_u({}_r a) \, {}_u bc, (sr)u) && \text{(RP2)} \\
&= ({}_u({}_r ab)c, (sr)u) && \text{(RP1)} \\
&= ({}_r ab, sr)(c, u) \\
&= ((a, s)(b, r))(c, u).
\end{aligned}$$

Hence $T \times S$ together with the multiplication given by (3.2) is a semigroup. This semigroup is called the *reverse semidirect product of T by S* and is denoted by $T *_r S$.

The next example shows that, in general, these two operators do not coincide.

Example 3.2. Let $T = \{a, b\}$ be a left zero semigroup and $S = \{x, y\}$ be a right zero semigroup. On the one hand, the semigroup S acts reversely on T by endomorphisms on the left since

$$_xa = {}_xb = a, \qquad {}_ya = {}_yb = b$$

define mappings of T that satisfy (RP1) and (RP2). Therefore we can construct the reverse semidirect product $T *_r S$. For $(c, z), (d, w) \in T *_r S$, we have

$$(c, z)(d, w) = ({}_wcd, zw) = ({}_wc, w).$$

Hence $E(T *_r S) = \{(a, x), (b, y)\}$.

On the other hand, if, for every $s \in S$, the mapping $a \mapsto {}^sa$ satisfies (SP1) and (SP2), then, for all $(c, z), (d, w) \in T * S$,

$$(c, z)(d, w) = (c\,{}^zd, zw) = (c, w)$$

and so $T * S$ is a band. Hence no semidirect product of T by S is isomorphic to the reverse semidirect product constructed above.

If S is a semigroup that acts reverserly on a semigroup T by automorphisms on the left, then the mapping $a \mapsto {}^sa := b$, where b is the unique element in T such that ${}_sb = a$, is a bijection that satisfies (SP1) and (SP2), for all $a \in T$ and $s, t \in S$. To prove this, observe first that, for all $s \in S$ and $a \in T$,

$${}^s({}_sa) = a \qquad \text{and} \qquad {}_s({}^sa) = a. \tag{3.3}$$

We have:

(SP1) For all $a, b \in T$ and all $s \in S$,

$$\begin{aligned}
{}^{s}a\,{}^{s}b &= {}^{s}(\,{}_{s}(\,{}^{s}a\,{}^{s}b)) && \text{(3.3)}\\
&= {}^{s}(\,{}_{s}(\,{}^{s}a)\,{}_{s}(\,{}^{s}b)) && \text{(RP1)}\\
&= {}^{s}(ab). && \text{(3.3)}
\end{aligned}$$

(SP2) For all $a, b \in T$ and $s, r \in S$,

$$\begin{aligned}
{}^{sr}a = b &\Leftrightarrow {}_{sr}b = a\\
&\Leftrightarrow {}_{r}(\,{}_{s}b) = a && \text{(RP2)}\\
&\Leftrightarrow {}_{s}b = {}^{r}a\\
&\Leftrightarrow b = {}^{s}(\,{}^{r}a),
\end{aligned}$$

and so

$${}^{sr}a = {}^{s}(\,{}^{r}a).$$

Hence we can consider the semidirect product $T * S$ with relation to $a \mapsto {}^{s}a := b$ where $b \in T$ is such that ${}_{s}b = a$. The next result shows that the semigroups $T *_r S$ and $T * S$ are isomorphic.

Theorem 3.3. *Let S and T be semigroups such that S acts reversely on T by automorphisms on the left and $T * S$ be the semidirect product associated to the action defined by $a \mapsto {}^{s}a = b$, where $s \in S$ and, for all $a \in T$, b is the unique element such that ${}_{s}b = a$. Then the mapping $\varphi : T *_r S \to T * S$ defined by $(a, s)\varphi = ({}^{s}a, s)$ is an isomorphism.*

Proof: Let $(a, s) \in T *_r S$. Then $a \in T$ and $s \in S$ and so $({}^{s}a, s) \in T * S$. Also, if $(a, s) = (b, r)$ then $s = r$, $a = b$ and, since $a \mapsto {}^{s}a$ is a mapping, ${}^{s}a = {}^{r}b$. So $({}^{s}a, s) = ({}^{r}b, r)$. The equality $(a, s)\varphi = ({}^{s}a, s)$ defines a mapping from $T *_r S$ into $T * S$. We show that φ is an isomorphism from $T *_r S$ to $T * S$. Let $(a, s), (b, r) \in T *_r S$. We have

$$\begin{aligned}
(a, s)\varphi = (b, r)\varphi &\Rightarrow ({}^{s}a, s) = ({}^{r}b, r)\\
&\Rightarrow {}^{s}a = {}^{r}b \text{ and } s = r\\
&\Rightarrow {}_{s}(\,{}^{s}a) = {}_{r}(\,{}^{r}b) \text{ and } s = r\\
&\Rightarrow a = b \text{ and } s = r\\
&\Rightarrow (a, s) = (b, r).
\end{aligned}$$

Thus φ is injective. Let $(b, r) \in T * S$. Since $b = {}^{r}({}_{r}b)$, it follows that

$$(b, r) = ({}^{r}({}_{r}b), r) = ({}_{r}b, r)\varphi,$$

with $({}_{r}b, r) \in T *_{r} S$ and so φ is surjective. Let $(a, s), (b, r) \in T *_{r} S$. We have

$$\begin{aligned}
((a, s)(b, r))\varphi &= ({}_{r}ab, sr)\varphi \\
&= ({}^{sr}({}_{r}ab), sr) \\
&= ({}^{sr}({}_{r}a)\,{}^{sr}b, sr) && \text{(SP1)} \\
&= ({}^{s}({}^{r}({}_{r}a))\,{}^{s}({}^{r}b), sr) && \text{(SP2)} \\
&= ({}^{s}a\,{}^{s}({}^{r}b), sr) \\
&= ({}^{s}a, s)({}^{r}b, r) \\
&= (a, s)\varphi(b, r)\,\varphi.
\end{aligned}$$

So φ is a morphism. Hence φ is an isomorphism. $\square$

3.2 Regularity on semidirect product

As the example below shows, the semidirect product of two regular semigroups is not necessarily a regular semigroup.

Example 3.4. Let $S = \{x, y\}$ be a two-element left zero semigroup and $T = \{a, b\}$ be a two-element right zero semigroup. Being bands, these semigroups are clearly regular semigroups. Moreover, S acts on T by endomorphisms on the left since

$${}^{x}a = {}^{x}b = a, \qquad {}^{y}a = {}^{y}b = b$$

define mappings from T to itself that satisfy (SP1) and (SP2). We prove that the semidirect product $T * S$ is not regular. For every $(\alpha, \beta) \in T * S$, we have

$$\begin{aligned}(a, y)(\alpha, \beta)(a, y) &= (a\,{}^{y}\alpha, y\beta)(a, y) \\ &= (ab, y)(a, y) \\ &= (b, y)(a, y) \\ &= (b\,{}^{y}a, yy) \\ &= (bb, y) \\ &= (b, y) \\ &\neq (a, y).\end{aligned}$$

Hence the element (a, y) has no associate in $T * S$ and therefore the semidirect product is not regular.

In this section, we study the question of the regularity of the semidirect product of monoids.

In the case where the second component of the semidirect product is a group, the structure of the semidirect product is, in some cases, determined by the structure of the first component, as it is established in the next theorem.

Theorem 3.5. *Let S be a group acting on a semigroup T by endomorphisms on the left. Then*

(1) *if T is a regular semigroup then $T * S$ is a regular semigroup;*

(2) *if T is an inverse semigroup then $T * S$ is an inverse semigroup;*

(3) *if T is a group then $T * S$ is a group.*

Proof:

(1) Let $(t, s) \in T * S$. Then, for $t' \in A(t)$, we have

$$(t, s)({}^{s^{-1}}t', s^{-1})(t, s) = (t\,{}^{ss^{-1}}t', ss^{-1})(t, s) = (tt', 1_S)(t, s) = (tt't, s) = (t, s).$$

Hence $({}^{s^{-1}}t', s^{-1}) \in A((t, s))$.

(2) For $(t,s) \in T * S$, by the proof of (1), we have that $({}^{s^{-1}}(t^{-1}), s^{-1}) \in A((t,s))$. Then

$$({}^{s^{-1}}(t^{-1}), s^{-1})(t,s)({}^{s^{-1}}(t^{-1}), s^{-1}) \in V((t,s)),$$

that is,

$$({}^{s^{-1}}(t^{-1}), s^{-1}) \in V((t,s)).$$

Suppose now that $(t', s') \in V((t,s))$. Then

$$(t,s) = (t\,{}^{s}t'\,{}^{ss'}t, ss's) \qquad \text{and} \qquad (t',s') = (t'\,{}^{s'}t\,{}^{s's}t', s'ss')$$

and so

$$s' = s^{-1}, \qquad t = t\,{}^{s}(t')t, \qquad t' = t'\,{}^{s^{-1}}tt'.$$

Hence

$${}^{s}(t')t\,{}^{s}(t') \in V(t) = \{t^{-1}\}$$

and

$$t' = {}^{s^{-1}s}(t') = {}^{s^{-1}}({}^{s}(t'\,{}^{s^{-1}}tt')) = {}^{s^{-1}}({}^{s}(t')t\,{}^{s}(t')) = {}^{s^{-1}}(t^{-1}).$$

We have shown that $(t', s') = ({}^{s^{-1}}(t^{-1}), s^{-1})$ and so $(t,s)^{-1} = ({}^{s^{-1}}(t^{-1}), s^{-1})$.

(3) Let $(t,s) \in T * S$. By the proof of (2), we have that $({}^{s^{-1}}(t^{-1}), s^{-1}) = (t,s)^{-1}$. Then

$$\begin{aligned} (t,s)(t,s)^{-1} &= (t,s)({}^{s^{-1}}(t^{-1}), s^{-1}) \\ &= (t\,{}^{ss^{-1}}(t^{-1}), ss^{-1}) \\ &= (tt^{-1}, 1_S) \\ &= (1_T, 1_S) \end{aligned}$$

and

$$\begin{aligned} (t,s)^{-1}(t,s) &= ({}^{s^{-1}}(t^{-1}), s^{-1})(t,s) \\ &= ({}^{s^{-1}}(t^{-1})\,{}^{s^{-1}}t, s^{-1}s) \\ &= ({}^{s^{-1}}(t^{-1}t), 1_S) \\ &= ({}^{s^{-1}}1_T, 1_S) \\ &= (1_T, 1_S). \end{aligned}$$

□

Lemma 3.6. *Let S and T be monoids such that S acts on T by endomorphisms on the left. The following statements are equivalent:*

(1) $\forall a \in T \ \ \forall s \in S \ \ \exists e \in E(S): \ \ sS = eS$ *and* $a \in T\ {}^{e}a$*;*

(2) $\forall a \in T \ \ \forall s \in S \ \ \exists s' \in V(s): \ \ a \in T\ {}^{ss'}a$.

Proof:

$[1 \Rightarrow 2]$ Let $a \in T$ and $s \in S$. Let $e \in E(S)$ be such that $sS = eS$ and $a \in T\ {}^{e}a$. Then there exist $s', e' \in S$ such that

$$e = ss' \qquad \text{and} \qquad s = ee',$$

whence

$$e = ee = ss'e \qquad \text{and} \qquad s = ee' = eee' = es.$$

Consider $s'' = s'e$. We have

- $ss'' = ss'e = e$;
- $s''ss'' = s'ee = s'e = s''$;
- $ss''s = es = s$.

Thus $s'' \in V(s)$. Since $ss'' = e$, by (1), we obtain $a \in T\ {}^{ss''}a$.

$[2 \Rightarrow 1]$ This is clear since, for each $a \in T$, $s \in S$ and $s' \in V(s)$, $ss' \in E(S)$ and $sS = (ss')S$. $\square$

Using this result, we can obtain a characterisation of regular semidirect products of monoids.

Theorem 3.7. *Let S and T be monoids such that S acts on T by endomorphisms on the left. The semidirect product $T * S$ is regular if and only if*

(1) *the monoids T and S are regular; and*

(2) $\forall a \in T \ \ \forall s \in S \ \ \exists e \in E(S): \quad sS = eS$ *and* $a \in T\ {}^{e}a$.

Proof: Suppose that $T * S$ is regular. Let $a \in T$ and $s \in S$. Then there exist $a' \in T$ and $s' \in S$ such that $(a', s') \in V((a, s))$. It follows that

$$\begin{aligned}(a,s) = (a,s)(a',s')(a,s) \ &\Leftrightarrow (a,s) = (a\ {}^{s}a', ss')(a,s)\\ &\Leftrightarrow (a,s) = (a\ {}^{s}a'\ {}^{ss'}a, ss's)\end{aligned}$$

and so

$$a = a\ {}^{s}a'\ {}^{ss'}a \qquad \text{and} \qquad s = ss's. \tag{3.4}$$

It also follows that

$$\begin{aligned}(a',s') = (a',s')(a,s)(a',s') \ &\Leftrightarrow (a',s') = (a'\ {}^{s'}a, s's)(a',s')\\ &\Leftrightarrow (a',s') = (a'\ {}^{s'}a\ {}^{ss'}a', s'ss')\end{aligned}$$

and so

$$a' = a'\ {}^{s'}a\ {}^{ss'}a' \qquad \text{and} \qquad s' = s'ss'. \tag{3.5}$$

From $a = a\ {}^{s}a'\ {}^{ss'}a$, we obtain that $a \in T\ {}^{ss'}a$ and so, by Lemma 3.6, we conclude that (2) holds.

The identities $s = ss's$ and $s' = s'ss'$ guarantee that S is regular. Now, take $s = 1_S$. Then $s' = 1_S$ and the identities (3.4) and (3.5) are, respectively,

$$a = aa'a \qquad \text{and} \qquad a' = a'aa'.$$

So $a' \in V(a)$. Thus the monoid T is regular.

Conversely, suppose that (1) and (2) hold. Let $(a, s) \in T * S$. From (2) and Lemma 3.6, we can choose $s' \in V(s)$ such that $a \in T\ {}^{ss'}a$. Then $a = u\ {}^{ss'}a$, for some $u \in T$. Let $v \in V(a)$. Consider $a' = {}^{s'}v$. We have

$$\begin{aligned}(a,s)(a',s')(a,s) \ &= (a\ {}^{s}a', ss')(a,s)\\ &= (a\ {}^{s}a'\ {}^{ss'}a, ss's)\\ &= (u\ {}^{ss'}a\ {}^{ss'}v\ {}^{ss'}a, s)\\ &= (u\ {}^{ss'}(ava), s)\\ &= (u\ {}^{ss'}a, s)\\ &= (a,s)\end{aligned}$$

and

$$\begin{aligned}(a',s')(a,s)(a',s') &= (a'\,{}^{s'}a, s's)(a',s') \\ &= (a'\,{}^{s'}a\,{}^{s's}a', s'ss') \\ &= ({}^{s'}v\,{}^{s'}a\,{}^{s'ss'}v, s') \\ &= ({}^{s'}(vav), s') \\ &= ({}^{s'}v, s') \\ &= (a', s').\end{aligned}$$

Hence $(a',s') \in V((a,s))$ and so $T * S$ is regular. □

Corollary 3.8. *For monoids S and T, a sufficient condition for the semidirect product $T * S$ to be regular is that S and T are regular and that $a \in T\,{}^{e}a$, for every $a \in T$ and every $e \in E(S)$.*

Proof: Since S is regular, it is obvious that for each $s \in S$, there exists $e \in E(S)$ such that $sS = eS$ (take $e = ss'$). Thus (1) and (2) of Theorem 3.7 hold. Hence $T * S$ is regular. □

The following example shows that the sufficient condition of Corollary 3.8 is not a necessary condition.

Example 3.9. Let $S = \{1_S, a, b\}$ be a monoid with identity 1_S and such that $xa = a$ and $xb = b$, for all $x \in S$:

	1_S	a	b
1_S	1_S	a	b
a	a	a	b
b	b	a	b

Let $T = \{1_T, e, f, 0_T\}$ be a semilattice with identity 1_T and zero 0_T and such that $ef = f$:

	1_T	e	f	0_T
1_T	1_T	e	f	0_T
e	e	e	f	0_T
f	f	f	f	0_T
0_T	0_T	0_T	0_T	0_T

Since S and T are bands, the semigroups S and T are regular.

We show that each $s \in S$ determines a mapping $x \mapsto {}^s x$ from T to itself that satisfies conditions (SP1), (SP2) and (SP3). Define:

$$\begin{array}{llll} {}^{1_S}1_T = 1_T, & {}^{1_S}f = f, & {}^{1_S}e = e, & {}^{1_S}0_T = 0_T, \\ {}^{a}1_T = 1_T, & {}^{a}f = 0_T, & {}^{a}e = e, & {}^{a}0_T = 0_T, \\ {}^{b}1_T = 1_T, & {}^{b}f = e, & {}^{b}e = e, & {}^{b}0_T = 0_T. \end{array}$$

We have

- ${}^s(1_T 1_T) = 1_T = {}^s1_T\, {}^s1_T$, for all $s \in S$;
- ${}^s(e1_T) = {}^s(1_T e) = {}^s(ee) = e = {}^se\, {}^se = {}^s1_T\, {}^se = {}^se\, {}^s1_T$, for all $s \in S$;
- ${}^b(fx) = {}^b(xf) = {}^bf = e = {}^bx\, {}^bf = {}^bf\, {}^bx$, for all $x \in T \setminus \{0_T\}$;
- ${}^a(fx) = {}^a(xf) = {}^af = 0_T = {}^ax\, {}^af = {}^af\, {}^ax$, for all $x \in T \setminus \{0_T\}$;
- ${}^s(0_T x) = {}^s(x0_T) = {}^s0_T = 0_T = {}^sx\, {}^s0_T = {}^s0_T\, {}^sx$, for every $s \in S$ and every $x \in T$;
- ${}^{1_S}(xy) = xy = {}^{1_S}x\, {}^{1_S}y$, for every $x, y \in T$;
- ${}^{sr}1_T = 1_T = {}^s({}^r1_T)$, for all $s, r \in S$;
- ${}^{sr}e = e = {}^s({}^re)$, for all $s, r \in S$;
- ${}^{sr}0_T = 0_T = {}^s({}^r0_T)$, for all $s, r \in S$;
- ${}^{a1_S}f = {}^{1_Sa}f = {}^{ba}f = {}^af = 0_T = {}^b({}^af) = {}^{1_S}({}^af) = {}^a({}^{1_S}f)$;
- ${}^{b1_S}f = {}^{1_Sb}f = {}^{ab}f = {}^bf = e = {}^a({}^bf) = {}^{1_S}({}^bf) = {}^b({}^{1_S}f)$;
- ${}^{1_S1_S}x = x = {}^{1_S}({}^{1_S}x)$, for all $x \in T$.

Then S acts on T by endomorphisms on the left. Since $aS = \{a, b\} = bS$, it follows that $a\,\mathcal{R}\,b$. Observe that

- if $y = 1_T$ then $1_T \in T\, {}^b1_T = T$;
- if $y \in \{e, f\}$ then $y \in T\, {}^by = Te = \{e, f, 0_T\}$;

- if $y = 0_T$ then $0_T \in T\,^b 0_T = \{0_T\}$.

So $y \in T\,^b y$, for all $y \in T$. Consequently, (1) and (2) of Theorem 3.7 are satisfied. Therefore the semidirect product $T * S$ is regular. However, $a \in E(S)$ and $T\,^a f = \{0_T\}$ and so $f \notin T\,^a f$. Then the hypotesis of Corollary 3.8 is not satisfied and so it is not a necessary condition.

Corollary 3.10. *Under the conditions of Theorem 3.7, if S is an inverse monoid then $T * S$ is regular if and only if*

(1) *T is regular; and*

(2) *$a \in T\,^e a$, for every $a \in T$ and every $e \in E(S)$.*

Proof: Suppose that $T * S$ is regular and let $a \in T$ and $e \in E(S)$. Then (2) of Theorem 3.7 holds and so, by Lemma 3.6, taking $s = e$, we obtain that $a \in T\,^{ee'} a$, for some $e' \in V(e)$. Since S is inverse this means that $a \in T\,^{ee^{-1}} a$, that is, $a \in T\,^e a$. The regularity of T follows from (1) of Theorem 3.7.

Conversely, suppose that (1) and (2) hold. Since every right ideal sS of an inverse semigroup S has a unique idempotent generator ss^{-1} (that is, for all $s \in S$, $sS = ss^{-1}S$), (1) and (2) of Theorem 3.7 are satisfied and so $T * S$ is regular. □

Note that the semidirect product of inverse semigroups is not, in general, an inverse semigroup. This is clear from the following example.

Example 3.11. Let $S = \{1_S, a\}$ be a commutative monoid with one non-identity idempotent a:

$$\begin{array}{c|cc} & 1_S & a \\ \hline 1_S & 1_S & a \\ a & a & a \end{array}.$$

Let $T = \{1_T, e, 0_T\}$ be a commutative monoid with zero and a non-identity idempotent e:

$$\begin{array}{c|ccc} & 1_T & e & 0_T \\ \hline 1_T & 1_T & e & 0_T \\ e & e & e & 0_T \\ 0_T & 0_T & 0_T & 0_T \end{array}.$$

Since S and T are commutative bands, both S and T are inverse monoids.

We show that each $s \in S$ determines a mapping $x \mapsto {}^{s}x$ from T to itself that satisfies (SP1), (SP2) and (SP3). Define:

$$\begin{array}{lll} {}^{1_S}1_T = 1_T, & {}^{1_S}e = e, & {}^{1_S}0_T = 0_T, \\ {}^{a}1_T = 1_T, & {}^{a}e = e, & {}^{a}0_T = e. \end{array}$$

We have

- ${}^{a}(1_T e) = {}^{a}(e 1_T) = {}^{a}(ee) = {}^{a}e = e = {}^{a}e\, {}^{a}e = {}^{a}e\, {}^{a}1_T = {}^{a}1_T\, {}^{a}e$;
- ${}^{a}(0_T e) = {}^{a}(e 0_T) = {}^{a}(0_T 0_T) = {}^{a}0_T = e = {}^{a}0_T\, {}^{a}0_T = {}^{a}e\, {}^{a}0_T = {}^{a}0_T\, {}^{a}e$;
- ${}^{a}(1_T 0_T) = {}^{a}(0_T 1_T) = e = {}^{a}0_T\, {}^{a}1_T = {}^{a}1_T\, {}^{a}0_T$;
- ${}^{a}(1_T 1_T) = {}^{a}1_T = 1_T = {}^{a}1_T\, {}^{a}1_T$;
- ${}^{1_S}(xy) = xy = {}^{1_S}x\, {}^{1_S}y$, for every $x, y \in T$;
- ${}^{aa}1_T = {}^{1_S a}1_T = {}^{a 1_S}1_T = {}^{a}1_T = 1_T = {}^{a}({}^{1_S}1_T) = {}^{1_S}({}^{a}1_T) = {}^{a}({}^{a}1_T)$;
- ${}^{aa}e = {}^{1_S a}e = {}^{a 1_S}e = {}^{a}e = e = {}^{a}({}^{1_S}e) = {}^{1_S}({}^{a}e) = {}^{a}({}^{a}e)$;
- ${}^{aa}0_T = {}^{1_S a}0_T = {}^{a 1_S}0_T = {}^{a}0_T = e = {}^{a}({}^{1_S}0_T) = {}^{1_S}({}^{a}0_T) = {}^{a}({}^{a}0_T)$;
- ${}^{1_S}x = x$, for all $x \in T$.

Thus S acts on T by endomorphisms on the left. Observe that

- $1_T \in T\, {}^{s}1_T = T1_T = T$, for every $s \in E(S)$;
- $e \in T\, {}^{s}e = Te = \{e, 0_T\}$, for every $s \in E(S)$;
- $0_T \in T\, {}^{1_S}0_T = T0_T = \{0_T\}$ and $0_T \in T\, {}^{a}0_T = Te = \{e, 0_T\}$.

Then $t \in T\, {}^{s}t$, for every $t \in T$ and every $s \in E(S)$. By Corollary 3.10, the semidirect product $T * S$ is regular. Since

$$(e, a)(e, a) = (e\, {}^{a}e, a^2) = (ee, a^2) = (e, a),$$

it follows that $(e,a) \in V((e,a))$.

We have

$$\begin{aligned}(0_T,a)(e,a)(0_T,a) &= (0_T\ {}^a e, a^2)(0_T,a)\\ &= (0_T,a)(0_T,a)\\ &= (0_T\ {}^a 0_T, a^2)\\ &= (0_T,a)\end{aligned}$$

and

$$\begin{aligned}(e,a)(0_T,a)(e,a) &= (e\ {}^a 0_T, a^2)(e,a)\\ &= (ee,a)(e,a)\\ &= (e,a)(e,a)\\ &= (e,a).\end{aligned}$$

Then $(0_T,a) \in V((e,a))$. Therefore (e,a) and $(0_T,a)$ are both inverses of $(e,a) \in T * S$ and so the regular monoid $T * S$ is not inverse.

The next result establishes a characterisation of semidirect products of monoids which are inverse monoids.

Theorem 3.12. *A semidirect product $T * S$ of two monoids T and S is an inverse monoid if and only if*

(1) *the monoids S and T are inverse, and*

(2) $\forall e \in E(S)\ \forall a \in T,\ {}^e a = a.$

Proof:

(i) First of all, we show that condition (2) is equivalent to

(2') the map $a \mapsto {}^s a$ is an automorphism of T.

Suppose that (2) holds. Let $s \in S$. If S is regular then, by Proposition 1.10, there exist $e, f \in E(S)$ and $x \in S$ such that $xs = e$ and $sx = f$. Therefore, for any $a \in T$,

$${}^x({}^s a) = {}^{xs} a = {}^e a = a = a\,\mathrm{id}_T$$

and

$$ {}^{s}({}^{x}a) = {}^{sx}a = {}^{f}a = a = a\,\mathrm{id}_{T}\,. $$

Thus $a \mapsto {}^{s}a$ is an automorphism. Suppose that (2') holds and let $e \in E(S)$. Then ${}^{e}({}^{e}a) = {}^{e^2}a = {}^{e}a$ and since $a \mapsto {}^{e}a$ is injective, we obtain ${}^{e}a = a$.

(ii) Suppose that $T * S$ is an inverse monoid. By Theorem 3.7, S and T are regular monoids.

Let $a \in T$ and $s \in S$. Since $T * S$ is inverse, the elements $(a, 1_{S})$ and $(1_{T}, s)$ of $T * S$ have a unique inverse $(a', 1_{S})$ and $(1_{T}, s')$, respectively. By (3.4) and (3.5), a' is the unique inverse of a and s' is the unique inverse of s. Hence both S and T are inverse semigroups and so (1) holds.

Now, let $e \in E(S)$ and $a \in T$. We show that ${}^{e}a = a$. Since $T * S$ is an inverse monoid, the element (a, e) of $T * S$ has a unique inverse (b, s). According to the proof of Theorem 3.7, we know that $s \in V(e)$ and since S is inverse, we have $s = e$. Then $(b, e) \in V((a, e))$, that is, $(a, e) \in V((b, e))$. From (3.4) and (3.5), we can deduce that

$$ a = a\,{}^{e}b\,{}^{e}a \qquad \text{and} \qquad b\,{}^{e}a\,{}^{e}b = b. $$

Hence

$$ {}^{e}a = {}^{e}(a\,{}^{e}b\,{}^{e}a) = {}^{e}a\,{}^{e}b\,{}^{e}a. $$

It follows that

$$ ({}^{e}a, e) = ({}^{e}a, e)(b, e)({}^{e}a, e) $$

and

$$ (b, e) = (b, e)({}^{e}a, e)(b, e). $$

Thus $({}^{e}a, e) \in V((b, e))$. Since $T * S$ is inverse, we can conclude that $({}^{e}a, e) = (a, e)$, that is, ${}^{e}a = a$. Consequently, (2) holds.

Conversely, suppose that the monoids S and T satisfy (1) and (2). By Corollary 3.10, $T * S$ is regular since $a = {}^{e}a = 1_{T}\,{}^{e}a \in T\,{}^{e}a$. To show that $T * S$ is inverse, it suffices to prove that the idempotents of $T * S$ commute. Let $(e,s) \in E(T * S)$. Then

$$\begin{aligned}(e,s)(e,s) = (e,s) \ &\Rightarrow (e\ {}^{s}e, s^2) = (e,s)\\ &\Rightarrow e\ {}^{s}e = e \quad \text{and} \quad s^2 = s\\ &\Rightarrow e^2 = e \quad \text{and} \quad s^2 = s.\end{aligned}$$

Consequently, if $(e,s),(f,u) \in E(T * S)$ then

$$ef = fe \in T$$

and

$$su = us \in S.$$

We have

$$\begin{aligned}(e,s)(f,u) \ &= (e\ {}^{s}f, su)\\ &= (ef, su)\\ &= (fe, us)\\ &= (f\ {}^{u}e, us)\\ &= (f,u)(e,s).\end{aligned}$$

Thus $T * S$ is an inverse monoid. □

3.3 An application of semidirect product

As an application of a semidirect product we prove a structure theorem for a class of regular semigroups: the class of uniquely unit orthodox semigroups.

Let S be a regular semigroup with identity element 1_S. We denote the group of units of S by H_{1_S}. An element $u \in S$ is said to be a *unit associate of* x if $u \in A(x) \cap H_{1_S}$. For each $x \in S$, the set of all unit associates of x is denoted by $U(x)$. The monoid S is said to be a *unit regular monoid* if $U(x) \neq \emptyset$, for all $x \in S$, and S is called *unit orthodox* if it is unit regular and orthodox. Moreover, S is said to be *uniquely unit orthodox* whenever S is orthodox and $U(x)$ is singleton, for all $x \in S$.

We need to state some auxiliar results:

Lemma 3.13. *Let S be a unit orthodox semigroup and $x \in S$. If $u, v, w \in U(x)$ then $uv^{-1}w \in U(x)$.*

Proof: Let $x \in S$ and $u, v, w \in U(x)$. Observe that $vxw, wxu \in V(x)$, since

$$\begin{aligned}(vxw)x(vxw) &= v(xwx)vxw \\ &= v(xvx)w \\ &= vxw,\end{aligned}$$

$$\begin{aligned}x(vxw)x &= (xvx)wx \\ &= xwx \\ &= x,\end{aligned}$$

$$\begin{aligned}(wxu)x(wxu) &= w(xux)wxu \\ &= w(xwx)u \\ &= wxu\end{aligned}$$

and

$$\begin{aligned}x(wxu)x &= (xwx)ux \\ &= xux \\ &= x.\end{aligned}$$

Then $vxw, wxu \in V(x)$. Since S is orthodox and $x \in V(vxw) \cap V(wxu)$, by Theorem 1.32, $V(vxw) = V(wxu)$ and so, by Lemma 1.33, $(vxw, wxu) \in \gamma$, γ being the smallest inverse semigroup congruence on S. Therefore $(xw, v^{-1}wxu) \in \gamma$. By Corollary 1.35, γ is idempotent-pure and since $xw \in E(S)$, it follows that $v^{-1}wxu \in E(S)$. Then

$$\begin{aligned}(v^{-1}wxu)(v^{-1}wxu) = v^{-1}wxu &\Rightarrow wxuv^{-1}wxu = wxu \\ &\Rightarrow xuv^{-1}wxu = xu \\ &\Rightarrow x(uv^{-1}w)x = x.\end{aligned}$$

Thus $uv^{-1}w \in A(x)$. Since $uv^{-1}w \in H_{1_S}$, it follows that $uv^{-1}w \in U(x)$. $\square$

As a consequence of this result, we have:

Corollary 3.14. *Let S be a unit orthodox semigroup and $x \in S$. Then $U(x)$ is a coset of some subgroup of H_{1_S}.*

Proof: Since S is unit orthodox, $U(x)$ is a non-empty subset of H_{1_S}. By Lemma 3.13, $U(x)U(x)^{-1}U(x) \subseteq U(x)$ and, since $a = aa^{-1}a$, for all $a \in U(x)$, $U(x) \subseteq U(x)U(x)^{-1}U(x)$. Then, $U(x) = U(x)U(x)^{-1}U(x)$ and hence, by Lemma 1.21, $U(x)$ is a (left) coset of some subgroup of H_{1_S}. $\square$

Corollary 3.15. *Let S be a unit orthodox semigroup and $e \in E(S)$. Then $U(e)$ is a subgroup of H_{1_S}.*

Proof: Since $e1_Se = e$ and $1_S \in H_{1_S}$, $1_S \in U(e)$. Let $u, w \in U(e)$. By Lemma 3.13, $uw = u1_S^{-1}w \in U(e)$ and so $U(e)$ is a subsemigroup of H_{1_S}. Since, by Corollary 3.14, $U(e)$ is a coset, we can conclude that $U(e)$ is a subgroup of H_{1_S}. $\square$

The following lemma shows that the set $U(x)$ is a coset of $U(xu)$.

Lemma 3.16. *Let S be a unit orthodox semigroup. Let $x \in S$. Then, for all $u \in U(x)$,*

$$U(x) = uU(xu).$$

Proof: Let $x \in S$ and $u, v \in U(x)$. We have

$$xu(u^{-1}v)xu = (xvx)u = xu$$

then $u^{-1}v \in U(xu)$. It follows that $v \in uU(xu)$. Thus $U(x) \subseteq uU(xu)$.

Now, let $w \in U(xu)$. Then $(xu)w(xu) = xu$ and so $x(uw)x = x$. Hence $uw \in U(x)$ and consequently $uU(xu) \subseteq U(x)$. $\square$

Finally, we can establish a characterisation for uniquely unit orthodox semigroups.

Theorem 3.17. *Let S be a unit orthodox monoid. The monoid S is uniquely unit orthodox if and if only, for every $e \in E(S)$, the subgroup $U(e)$ is trivial.*

Proof: Let S be a uniquely unit orthodox. Then $U(x)$ is singleton, for every $x \in S$. By Corollary 3.15, we have $U(e) = \{1_S\}$, for all $e \in E(S)$.

Conversely, suppose that $U(e) = \{1_S\}$, for every $e \in E(S)$. Let $x \in S$ and $u, v \in U(x)$. By Lemma 3.16, we deduce that

$$\{u\} = uU(xu) = U(x) = vU(xv) = \{v\}.$$

Since $xu, xv \in E(S)$, we have $u = v$ and so $U(x)$ is singleton. $\square$

In the next result, we construct a uniquely unit orthodox semigroup using the notion of semidirect product.

Theorem 3.18. *Let B be a band with an identity and G be a group. Let G act on B by automorphisms on the left. Then the semidirect product $B * G$ is a uniquely unit orthodox semigroup such that*

(i) $E(B * G) \simeq B$;

(ii) $H_{1_{B*G}} \simeq G$.

Proof: First, we determine the set of idempotents and the set of units of $B * G$.

If (e, x) is an idempotent of $B * G$ then $x = 1_G$. Conversely,

$$(e, 1_G)(e, 1_G) = (e\ {}^{1_G}e, 1_G 1_G) = (e^2, 1_G) = (e, 1_G).$$

Hence $E(B * G) = \{(e, 1_G) : e \in B\}$.

Let $x \in G$ and $e \in B$. Note that ${}^{1_G}e = e$. We have

$$\begin{aligned} e\ {}^{x}1_B &= {}^{1_G}e\ {}^{x}1_B \\ &= ({}^{xx^{-1}}e)({}^{x}1_B) \\ &= {}^{x}(\ {}^{x^{-1}}e 1_B) \\ &= {}^{x}(\ {}^{x^{-1}}e) \\ &= {}^{xx^{-1}}e \\ &= {}^{1_G}e \\ &= e \end{aligned}$$

and similarly ${}^{x}1_{B}e = e$. Thus ${}^{x}1_{B} = 1_{B}$. It follows that the identity of $B * G$ is $(1_{B}, 1_{G})$, since

$$(e, x)(1_{B}, 1_{G}) = (e\ {}^{x}1_{B}, x1_{G}) = (e, x)$$

and

$$(1_{B}, 1_{G})(e, x) = (1_{B}\ {}^{1_{G}}e, 1_{G}x) = (e, x).$$

So, if (e, x) is a unit of $B * G$ then e is a unit of B and therefore, for $f \in U(e)$,

$$\begin{aligned} ef = 1_{B} &\Rightarrow e^2 f = e \\ &\Rightarrow 1_{B} = ef = e \end{aligned}$$

giving $(e, x) = (1_{B}, x)$.

Conversely, since $(1_{B}, x)^{-1} = (1_{B}, x^{-1})$, every element of the form $(1_{B}, x)$, $x \in G$, is a unit of $B * G$. Consequently, the set of units of $B * G$, $H_{1_{B*G}}$, is

$$H_{1_{B*G}} = \{(1_{B}, x) :\ x \in G\}.$$

Now, we show that the semigroup $B * G$ is uniquely unit orthodox. The semigroup $B * G$ is clearly orthodox:

$$(e, 1_{G})(f, 1_{G}) = (e\,{}^{1_{G}}f, 1_{G}1_{G}) = (ef, 1_{G}) \in E(B * G),$$

for all $e, f \in B$. Also, given $(e, x) \in B * G$, we have

$$\begin{aligned} (e, x)(1_{B}, x^{-1})(e, x) &= (e\ {}^{x}1_{B}, xx^{-1})(e, x) \\ &= (e1_{B}, 1_{G})(e, x) \\ &= (e, 1_{G})(e, x) \\ &= (e\ {}^{1_{G}}e, 1_{G}x) \\ &= (e^2, x) \\ &= (e, x). \end{aligned}$$

So $(1_{B}, x^{-1}) \in U((e, x)) = A((e, x)) \cap H_{1_{B*G}}$, giving $U((e, x)) \neq \emptyset$. Hence $B * G$ is a unit orthodox semigroup. Moreover, given $(e, 1_{G}) \in E(B * G)$, if $(1_{B}, y) \in U((e, 1_{G})) = A((e, 1_{G})) \cap H_{1_{B*G}}$ then

$$(e, 1_{G})(1_{B}, y)(e, 1_{G}) = (e, 1_{G})$$

and so $y = 1_G$. Thus $U((e, 1_G)) \subseteq \{(1_B, 1_G)\}$. Since the other inclusion is trivial, we obtain, by Theorem 3.17, that $B * G$ is uniquely unit orthodox.

(i) The mapping $\alpha : B \to E(B * G)$ defined by $e\alpha = (e, 1_G)$ is a bijection. Since

$$e\alpha\, f\alpha = (e, 1_G)(f, 1_G) = (ef, 1_G) = (ef)\alpha,$$

for all $e, f \in B$, it follows that α is an isomorphism. So $B \simeq E(B * G)$.

(ii) Consider $\theta : H_{1_{B*G}} \to G$ defined by $(1_B, x)\theta = x$. Clearly, θ is bijective. Let $x, y \in G$. We have

$$\begin{aligned} ((1_B, x)(1_B, y))\theta &= (1_B\ {}^{x}1_B, xy)\theta \\ &= (1_B 1_B, xy)\theta \\ &= (1_B, xy)\theta \\ &= xy \\ &= (1_B, x)\theta\ (1_B, y)\theta. \end{aligned}$$

Thus θ is a homomorphism and so $H_{1_{B*G}} \simeq G$. □

Now, we show that every uniquely unit orthodox semigroup can be so constructed.

Theorem 3.19. *Let S be a uniquely unit orthodox with band of idempotents E. Let $U(a) = \{u_a\}$, for every $a \in S$. Let ${}^{u}e = ueu^{-1}$, for every $u \in H_{1_S}$ and every $e \in E$. Then*

(i) *H_{1_S} acts on E by automorphisms on the left; and*

(ii) *$S \simeq E * H_{1_S}$, under the mapping $a \mapsto (au_a, u_a^{-1})$.*

Proof:

(i) Let $u \in H_{1_S}$. We have, for every $e, f \in E$,

- ${}^{u}(ef) = u(ef)u^{-1} = (ueu^{-1})(ufu^{-1}) = {}^{u}e\ {}^{u}f$;
- ${}^{uv}e = uve(uv)^{-1} = uvev^{-1}u^{-1} = u({}^{v}e)u^{-1} = {}^{u}({}^{v}e)$;

- ${}^{u}e = {}^{u}f \Leftrightarrow ueu^{-1} = ufu^{-1} \Rightarrow u^{-1}ueu^{-1}u = u^{-1}ufu^{-1}u \Rightarrow e = f$;
- $u^{-1}fu \in E$ and ${}^{u}(u^{-1}fu) = uu^{-1}fuu^{-1} = f$.

Thus H_{1_S} acts on E by automorphisms on the left.

(ii) By (i), we can define the semidirect product $E * H_{1_S}$. Consider $\theta : S \to E * H_{1_S}$ defined by $a\theta = (au_a, u_a^{-1})$. Since $au_a \in E$, θ is well-defined. Let $a, b \in S$. Then

$$\begin{aligned} a\theta = b\theta \ &\Rightarrow (au_a, u_a^{-1}) = (bu_b, u_b^{-1}) \\ &\Rightarrow au_a u_a^{-1} = bu_b u_b^{-1} \\ &\Rightarrow a = b \end{aligned}$$

and so θ is injective. Let $(e, x) \in E * H_{1_S}$. Since $(ex)x^{-1}(ex) = e^2x = ex$, it follows that $u_{ex} = x^{-1}$. Then

$$(ex)\theta = (exu_{ex}, u_{ex}^{-1}) = (exx^{-1}, x) = (e, x).$$

Hence θ is surjective. We proceed to show that θ is a homomorphism. In order to do that we show first that, for every $a, b \in S$, $u_b u_a = u_{ab}$. Since $u_b b u_b \in V(b)$, $u_a a u_a \in V(a)$ and S is orthodox, it follows from Theorem 1.31 that

$$u_b b u_b u_a a u_a \in V(ab).$$

Thus

$$\begin{aligned} ab \ &= (ab)(u_b b u_b u_a a u_a)(ab) \\ &= a(bu_b b)u_b u_a(au_a a)b \\ &= (ab)u_b u_a(ab) \end{aligned}$$

and so $u_b u_a \in U(ab) = \{u_{ab}\}$. Consequently, $u_b u_a = u_{ab}$. Now, let $a, b \in S$. We have

$$\begin{aligned} (a\theta)(b\theta) \ &= (au_a, u_a^{-1})(bu_b, u_b^{-1}) \\ &= (au_a \ {}^{u_a^{-1}}(bu_b), u_a^{-1}u_b^{-1}) \\ &= (au_a u_a^{-1} bu_b u_a, u_a^{-1}u_b^{-1}) \\ &= (abu_b u_a, (u_b u_a)^{-1}) \\ &= (abu_{ab}, u_{ab}^{-1}) \\ &= (ab)\theta. \end{aligned}$$

Thus θ is an isomorphism from S to $E * H_{1_S}$. $\square$

4 | Wreath Product

In this chapter, we present a construction that was defined for semigroups by Neumann in [9]. According to Charles Wells in [16], this construction has been used in group theory for many years and its use in semigroup theory only begun fifty years ago.

4.1 A special semidirect product

Let S and T be semigroups and T^S be the set of all mappings from S into T. Together with the multiplication defined by

$$\forall f, g \in T^S \ \ \forall s \in S, \ s(fg) = (sf)(sg),$$

T^S is a semigroup – this is a consequence of T being a semigroup. Observe that T^S is a monoid if T is a monoid; the identity of T^S being the constant map $s \mapsto 1_T$, for all $s \in S$.

For all $s \in S$ and all $f \in T^S$, consider ${}^s f : S \to T$ defined by

$$x \ {}^s f = (xs)f,$$

for all $x \in S$. We show that S acts on T^S by endomorphisms on the left via the mapping $s \mapsto {}^s f$, for all $s \in S$ and all $f \in T^S$. Let $f, g \in T^S$ and $s \in S$. Then, for any $x \in S$,

$$x \ {}^s(fg) = (xs)(fg) = (xs)f \, (xs)g = (x \ {}^s f)(x \ {}^s g)$$

giving ${}^s(fg) = {}^s f \ {}^s g$. Also, if $s, r \in S$ and $f \in T^S$ then, for any $x \in S$,

$$x \ {}^{sr} f = (x(sr))f = ((xs)r)f = (xs) \ {}^r f = x \ {}^s({}^r f),$$

that is, ${}^{sr}f = {}^{s}({}^{r}f)$. If S is a monoid then, for any $f \in T^S$, ${}^{1_S}f = f$. We can, therefore, consider the semidirect product $T^S * S$ with respect to $s \mapsto {}^{s}f$. This semidirect product is called the *wreath product of T by S* and is denoted by $T \operatorname{Wr} S$.

Proposition 4.1. *Let S be a monoid acting on a monoid T by endomorphisms on the left. Then $T \operatorname{Wr} S$ is a monoid with identity $(f, 1_S)$, where $f : S \to T$ is the constant mapping $xf = 1_T$, for every $x \in S$.*

Proof: Suppose that S and T are monoids. Let $f \in T^S$ be the constant map $xf = 1_T$, for all $x \in S$. Then, for any $(g, s) \in T \operatorname{Wr} S$,

$$(f, 1_S)(g, s) = (f\ {}^{1_S}g, 1_S s) = (fg, s) \tag{4.1}$$

and

$$(g, s)(f, 1_S) = (g\ {}^{s}f, s1_S) = (g\ {}^{s}f, s). \tag{4.2}$$

Let $x \in S$. We have

$$x(fg) = (xf)(xg) = 1_T(xg) = xg$$

and

$$x(g\ {}^{s}f) = (xg)((xs)f) = (xg)1_T = xg,$$

and so it follows from (4.1) that

$$(f, 1_S)(g, s) = (g, s)$$

and from (4.2) that

$$(g, s)(f, 1_S) = (g, s).$$

Thus $(f, 1_S)$ is the identity of $T \operatorname{Wr} S$. $\square$

4.2 Regularity on wreath product

In this section, we establish some results about the regularity of the wreath product of monoids. We start with the following lemma.

Lemma 4.2. *Let T be a semigroup and X be a non-empty set. Then*

(1) *T is regular if and only if T^X is a regular semigroup;*

(2) *T is an inverse semigroup if and only if T^X is an inverse semigroup.*

Proof:

(1) Suppose that T is regular. Let $f \in T^X$. We define $g \in T^X$ as follows: let $x \in X$, $t = xf$ and t' be an arbitrarily fixed associate of t. Define $xg = t'$. Then

$$\begin{aligned} x(fgf) &= (xf)(xg)(xf) \\ &= tt't \\ &= t \\ &= xf. \end{aligned}$$

Since x is an arbitrary element of X, we obtain $fgf = f$, that is, $g \in A(f)$. Thus the semigroup T^X is regular.

Conversely, suppose that T^X is regular and let $t \in T$. Let f be the constant map defined by $xf = t$, for all $x \in X$. By hypothesis, there exists $f' \in T^X$ such that $f = ff'f$. Since, for any $x \in X$,

$$\begin{aligned} f = ff'f &\Rightarrow xf = (xf)(xf')(xf) \\ &\Leftrightarrow t = t(xf')t, \end{aligned}$$

it follows that $xf' \in A(t)$, for any $x \in X$. Thus the semigroup T is regular.

(2) Let T be an inverse semigroup. Then T is regular and so, by (1), T^X is regular. Let $f, g \in T^X$ be idempotents. We show that $fg = gf$. Let $x \in X$. Since f and g are idempotent mappings, both xf and xg are idempotents of T and since the semigroup T is inverse, xf and xg commute. We then have

$$\begin{aligned} x(fg) &= (xf)(xg) \\ &= (xg)(xf) \\ &= x(gf). \end{aligned}$$

Thus the idempotents of T^X commute. So the semigroup T^X is inverse.

Conversely, suppose that T^X is an inverse semigroup. By (1), T is regular. Let $t, u \in E(T)$. Consider the constant maps $f, g \in T^X$ defined by $xf = t$ and $xg = u$, for all $x \in X$. Clearly, $f, g \in E(T^X)$ and since T^X is inverse, $fg = gf$. We have

$$tu = (xf)(xg) = x(fg) = x(gf) = (xg)(xf) = ut.$$

Thus the idempotents of T commute and so T is inverse. $\square$

Since the wreath product $T \operatorname{Wr} S$ of two monoids is a semidirect product $T^S * S$ of the monoids T^S and S, we can apply Theorem 3.7 and obtain that the wreath product $T \operatorname{Wr} S$ is regular if and only if

(1) S and T^S are regular monoids; and

(2) $\forall f \in T^S \;\; \forall s \in S \;\; \exists e \in E(S): \;\; sS = eS$ and $f \in T^S\, {}^e f$.

By Lemma 4.2, (1) is equivalent to S and T being regular monoids. Now, let $f \in T^S$ and $s \in S$. By (2),

$$\exists e \in E(S): \;\; sS = eS \qquad \text{and} \qquad f \in T^S\, {}^e f.$$

We have

$$\begin{aligned} f \in T^S\, {}^e f \;\; &\Leftrightarrow \exists g \in T^S: \;\; f = g\, {}^e f \\ &\Leftrightarrow \exists g \in T^S \; \forall x \in S, \; xf = x(g\, {}^e f) \\ &\Leftrightarrow \exists g \in T^S \; \forall x \in S, \; xf = (xg)(x\, {}^e f) \\ &\Leftrightarrow \exists g \in T^S \; \forall x \in S, \; xf = xg\,(xe)f \\ &\Leftrightarrow \forall x \in S \; xf \in T(xe)f. \end{aligned}$$

Hence we have the following theorem.

Theorem 4.3. *Let S and T be monoids such that S acts on T by endomorphisms on the left. Then the wreath product $T \operatorname{Wr} S$ is regular if and only if*

(1) *S and T are regular monoids; and*

(2) $\forall f \in T^S \;\; \forall s \in S \;\; \exists e \in E(S): \quad sS = eS$ *and* $xf \in T(xe)f$, *for all* $x \in S$.

Proposition 4.4. *Let S and T be regular monoids such that S acts on T by endomorphisms on the left. If the wreath product $T \,\mathrm{Wr}\, S$ is regular then either*

(1) *T is a group; or*

(2) $\forall s, r \in S \;\; \exists e \in E(S), \quad sS = eS$ *and* $re = r$.

Proof: Suppose that $T \,\mathrm{Wr}\, S$ is regular and that T is not a group. Then there exists $t \in T$ such that $Tt \neq T$. Let $r \in S$ and define $f_r : S \to T$ by

$$xf_r = \begin{cases} 1_T & \text{if } x = r \\ t & \text{otherwise} \end{cases}.$$

Let $s \in S$. Since $T \,\mathrm{Wr}\, S$ is regular, (f_r, s) has an inverse, (g, s'), say. We have

$$\begin{aligned}(f_r, s)(g, s')(f_r, s) = (f_r, s) \;\; &\Leftrightarrow (f_r\, {}^s g, ss')(f_r, s) = (f_r, s) \\ &\Leftrightarrow (f_r\, {}^s g\, {}^{ss'} f_r, ss's) = (f_r, s)\end{aligned}$$

and so, for any $u \in S$,

$$u(f_r\, {}^s g\, {}^{ss'} f_r) = uf_r \qquad \text{and} \qquad ss's = s,$$

that is,

$$uf_r\,(us)g\,(uss')f_r = uf_r \qquad \text{and} \qquad ss's = s.$$

Taking $u = r$, we obtain

$$(rs)g\,(rss')f_r = 1_T.$$

Since $(rs)g \in T$, the supposition that $(rss')f_r = t$ leads to $Tt = T$ which contradicts the hypothesis of having $Tt \neq T$. Thus $(rss')f_r = 1_T$ (by the definition of f_r) and so we must have $rss' = r$, that is, $re = r$, where $e = ss' \in E(S)$. Clearly, $sS = eS$. □

In the next result, we show that if (2) of the previous proposition is replaced by the condition of S being a group we have a stronger result.

Theorem 4.5. *Let S be a regular monoid acting on a regular monoid T by endomorphisms on the left. Then the wreath product $T \operatorname{Wr} S$ is regular if and only if S or T is a group.*

Proof: Suppose that $T \operatorname{Wr} S$ is regular and T is not a group. Then (2) of Proposition 4.4 is satisfied. Consider $r = 1_S$. Thus, for any $s \in S$, there exists $e \in E(S)$ such that $eS = sS$ and $1_S e = 1_S$. Therefore

$$S = 1_S S = 1_S eS = eS = sS,$$

for all $s \in S$. Hence $1_S \in sS$, for every $s \in S$, that is, every element of S has an inverse. Consequently, S is a group.

Conversely, suppose that S is a group. Let $s \in S$. Since S is a group, we have $sS = S = 1_S S$ and $1_S \in E(S)$. Let $f \in T^S$. For any $x \in S$,

$$xf = 1_T(xf) = 1_T(x1_S)f \in T(x1_S)f.$$

Hence (2) of Theorem 4.3 is satisfied and so, since S and T are regular monoids, $T \operatorname{Wr} S$ is regular.

Now, suppose that T is a group. Let $s \in S$. Since S is a regular semigroup, we can consider $s' \in A(s)$. Then $ss' \in E(S)$ and

$$sS = ss'sS \subseteq ss'S \subseteq sS,$$

giving $sS = ss'S$. Since T is a group, $aT = Ta = T$, for every $a \in T$. For any $x \in S$ and any $f \in T^S$, we have $xf, (xss')f \in T$ and so $xf \in T(xss')f$. Hence (2) of Theorem 4.3 is satisfied and so, since S and T are regular monoids, $T \operatorname{Wr} S$ is a regular monoid. $\square$

Theorem 4.6. *Let S be an inverse monoid and T be a monoid such that S acts on T by endomorphisms on the left. Then the wreath product $T \operatorname{Wr} S$ is regular if and only if*

(1) *T is a group; or*

(2) *T is regular and $se = s$, for all $s \in S$ and all $e \in E(S)$.*

Proof: Suppose that the wreath product $T \operatorname{Wr} S$ is regular and T is not a group. Then, by Theorem 4.3, T is regular. Now let $s \in S$ and $e \in E(S)$. Using (2) of Proposition 4.4, we have that

$$\exists f \in E(S): \quad eS = fS \quad \text{and} \quad sf = s. \tag{4.3}$$

Since S is inverse, each $\mathcal{R}$-class of S contains a unique idempotent and so $e = f$. From (4.3), it follows that $se = s$.

Conversely, suppose that T is regular and $se = s$, for every $s \in S$ and every $e \in E(S)$. By hypothesis, S and T are regular monoids. Let $f \in T^S$ and $s \in S$. Since S is inverse, we can consider $s^{-1} \in V(s)$ and the idempotent ss^{-1}. We have, for all $x \in S$,

$$1_T(xss^{-1})f = 1_T(xf) = xf,$$

which gives $xf \in T(xss^{-1})f$. Thus (2) of Theorem 4.3 is satisfied. Since, by hypothesis, S and T are regular monoids, we obtain, by Theorem 4.3, that $T \operatorname{Wr} S$ is regular.

Now, suppose that T is a group. From Theorem 4.5, it follows immediately that the monoid $T \operatorname{Wr} S$ is regular. $\square$

Proposition 4.7. *Let S and T be monoids such that S acts on T by endomorphisms on the left. Then the wreath product $T \operatorname{Wr} S$ is an inverse monoid if and only if*

(1) *S and T are inverse monoids; and*

(2) *either $|T| = 1$ or $se = s$, for all $s \in S$ and all $e \in E(S)$.*

Proof: Let S and T be monoids such that S acts on T by endomorphisms on the left. By Theorem 3.12 and its proof, $T \operatorname{Wr} S$ is an inverse monoid if and only if the monoids S and T^S are inverse and S acts on T^S by automorphisms on the left. By Lemma 4.2, T^S being an inverse monoid is equivalent to T being an inverse monoid. Suppose that $T \operatorname{Wr} S$ is an inverse monoid and $|T| \neq 1$. Then there exists $t \in T$ such that $t \neq 1_T$. We show that S acts on T^S by automorphisms on the left if and only if $se = s$, for all $s \in S$ and all $e \in E(S)$.

First, suppose that S acts on T^S by automorphisms on the left. Let $e \in E(S)$. Define $f : S \to T$ by

$$(\forall x \in S) \qquad xf = \begin{cases} 1_T & \text{if } x \in Se \\ t & \text{if } x \notin Se \end{cases} .$$

Let $x \in S$. From Theorem 3.12, ${}^e f = f$, for all $f \in T^S$ and all $e \in E(S)$. It follows that

$$xf = x\,{}^e f = (xe)f = 1_T.$$

Then $x \in Se$, that is, $x = se$, for some $s \in S$. Therefore $xe = (se)e = se = x$.

Now, suppose that $se = s$, for all $s \in S$ and all $e \in E(S)$. Let $f \in T^S$. Since S is regular, for all $u \in S$, there exist $a, b \in E(S)$ and $r \in S$ such that $ru = a$ and $ur = b$. Since $se = s$, for all $s \in S$ and all $e \in E(S)$,

$$x\,{}^r({}^u f) = x\,{}^{ru} f = x\,{}^a f = (xa)f = xf = (xf)\,\mathrm{id}_{T^S}$$

and

$$x\,{}^u({}^r f) = x\,{}^{ur} f = x\,{}^b f = (xb)f = xf = (xf)\,\mathrm{id}_{T^S},$$

for any $x \in S$. Thus $f \mapsto {}^s f$ is an automorphism and so S acts on T^S by automorphisms on the left. $\square$

Corollary 4.8. *The wreath product of two monoids S and T is an inverse monoid if and only if either*

(1) *S is an inverse monoid and $|T| = 1$; or*

(2) *S is a group and T is an inverse monoid.*

Proof: Let $T \operatorname{Wr} S$ be an inverse monoid. By Proposition 4.7, S and T are inverse monoids. If $|T| \neq 1$ then $se = s$, for all $s \in S$ and all $e \in E(S)$. Taking $s = 1_S$, we have

$$1_S = 1_S e = e,$$

for any $e \in E(S)$. Since S is a regular monoid and has a unique idempotent, S is a group.

Conversely, suppose that S is an inverse monoid and $|T| = 1$. In particular, T is an inverse monoid. Thus, by Propostion 4.7, the wreath product $T \operatorname{Wr} S$ is an inverse monoid.

Now, suppose that S is a group and T is an inverse monoid. Then S and T are both inverse monoids, 1_S is the unique idempotent of S and $s1_S = s$, for every $s \in S$. Hence, by Proposition 4.7, $T \operatorname{Wr} S$ is an inverse monoid. □

4.3 An application of the wreath product

In [2], for a regular monoid S, the author establishes a wreath product embedding which depends on a certain group congruence on S. As an application of this result, a wreath product embedding for E-unitary regular semigroups with left normal band of idempotents is constructed. These results are presented in this section.

Theorem 4.9. *Let S be a regular monoid and ρ be a group congruence on S such that, for each ρ-class $[s] \in S/\rho$, there exist elements $r \in [s]$, $r' \in V(r)$ such that $tr'r = t$, for all $t \in [s]$. Then S is embeddable in $[1_S] \operatorname{Wr} S/\rho$.*

Proof: Observe that for each idempotent e of S, $[e] \in E(S/\rho)$ and so, since S/ρ is a group $E(S) \subseteq [1_S]$. Let $x, y \in [1_S]$. Then $(x, 1_S), (y, 1_S) \in \rho$ and therefore $(xy, 1_S) \in \rho$. So $[1_S]$ is a semigroup and we can therefore define the wreath product $[1_S] \operatorname{Wr} S/\rho$. Moreover, for any $x \in [1_S]$,

$$x = xx'x \,\rho\, 1_S x' 1_S = x',$$

and so $x' \in [1_S]$. Thus the semigroup $[1_S]$ is regular.

For each $[s] \in S/\rho$, fix $s_0 \in [s]$, $s'_0 \in V(s_0)$ such that $ts'_0s_0 = t$, for all $t \in [s]$, and let $(1_S)_0 = 1_S$ which gives $(1_S)'_0 = 1_S$, since $(1_S)'_0 \in V((1_S)_0) = V(1_S) = \{1_S\}$. For each $s \in S$, consider the correspondence $[u] \rightsquigarrow u_0s(us)'_0$ with domain S/ρ. Clearly, if $[u], [v] \in S/\rho$ are such that $[u]=[v]$ then $u_0s(us)'_0 = v_0s(vs)'_0$. Also,

$$\begin{aligned} s\in S \;&\Rightarrow \forall u\in S\ (u_0,u)\in\rho \quad \text{and} \quad (s,s)\in\rho \\ &\Rightarrow \forall u\in S\ (u_0s,us)\in\rho \\ &\Rightarrow \forall u\in S\ (u_0s,(us)_0)\in\rho \\ &\Rightarrow \forall u\in S\ (u_0s(us)'_0,(us)_0(us)'_0)\in\rho \\ &\Rightarrow \forall u\in S\ (u_0s(us)'_0,1_S)\in\rho \\ &\Rightarrow \forall u\in S\ u_0s(us)'_0\in[1_S]. \end{aligned}$$

Thus, for each $s\in S$, $f_s : S/\rho \to [1_S]$ defined by $[u]f_s = u_0s(us)'_0$, for all $[u]\in S/\rho$, is a mapping, that is, for all $s\in S$, $f_s\in [1_S]^{S/\rho}$. We now show that the equality $s\varphi=(f_s,[s])$, for all $s\in S$, is a monomorphism from S to $[1_S]\operatorname{Wr} S/\rho$. Clearly, φ is well-defined. Let $s,t\in S$. Then

$$\begin{aligned} s\varphi=t\varphi \;&\Leftrightarrow (f_s,[s])=(f_t,[t]) \\ &\Rightarrow [1_S]f_s=[1_S]f_t \quad \text{and} \quad [s]=[t] \\ &\Rightarrow (1_S)_0s(1_Ss)'_0=(1_S)_0t(1_St)'_0 \quad \text{and} \quad s_0=t_0 \\ &\Rightarrow 1_Sss'_0=1_Stt'_0 \quad \text{and} \quad s_0=t_0 \\ &\Rightarrow ss'_0=tt'_0 \quad \text{and} \quad s'_0=t'_0 \\ &\Rightarrow ss'_0=ts'_0 \\ &\Rightarrow s=ss'_0s_0=ts'_0s_0=t \end{aligned}$$

and so φ is injective. Let $s,t\in S$. We have

$$\begin{aligned} (s\varphi)(t\varphi) \;&= (f_s,[s])(f_t,[t]) \\ &= (f_s\,{}^{[s]}f_t,[s][t]). \end{aligned}$$

Since, for all $[u]\in S/\rho$,

$$\begin{aligned} [u](f_s\,{}^{[s]}f_t) \;&= [u]f_s\,[u]\,{}^{[s]}f_t \\ &= [u]f_s\,[us]f_t \\ &= u_0s(us)'_0(us)_0t((us)t)'_0 \\ &= u_0st(u(st))'_0 \\ &= [u]f_{st}, \end{aligned}$$

we have

$$(s\varphi)(t\varphi) = (f_{st}, [st]) = (st)\varphi.$$

$\square$

We now look at the case where S is an E-unitary regular semigroup in which the band $E(S)$ is a left normal band.

Let σ_S be the least group congruence on S. Consider

$$C(S) = \{H \in \mathcal{P}(S)\backslash\{\emptyset\} : \quad HE(S) \subseteq H \subseteq [s]_{\sigma_S}, \quad \text{for some} \quad s \in S\}.$$

Being a set of subsets of S, it is natural to consider in $C(S)$ the multiplication defined by

$$HK = \{hk : \ h \in H, \ k \in K\},$$

for all $H, K \in C(S)$.

Proposition 4.10. *Let $C(S)$ be defined as above. Then*

(1) *$C(S)$ is an E-unitary regular semigroup;*

(2) *The mapping $\varphi : S \to C(S)$ defined by $s\varphi = sE(S)$ is an embedding.*

Proof:

(1) (i) Let $H, K \in C(S)$. Then $H \subseteq [s]_{\sigma_S}$, for some $s \in S$, and $KE(S) \subseteq K \subseteq [t]_{\sigma_S}$, for some $t \in S$. So

$$HKE(S) \subseteq HK \subseteq [s]_{\sigma_S}[t]_{\sigma_S} = [st]_{\sigma_S},$$

that is, $HK \in C(S)$. Thus $C(S)$ is a semigroup.

(ii) We show that

$$H' = \{h' \in S : \quad h' \text{ is an inverse of some } h \in H\}$$

is an inverse of H in $C(S)$. First, we show that $H' \in C(S)$. Let $h \in H$, $h' \in V(h)$ and $e \in E(S)$. Since $E(S)$ is a left normal band and $hh' \in E(S)$,

$$\begin{aligned}(h'eh)(h'eh) &= h'e(hh')eh \\ &= h'e^2(hh')h \\ &= h'eh\end{aligned}$$

and so $h'eh \in E(S)$. Since $H \in C(S)$, $HE(S) \subseteq H$ and $hh'eh \in H$. We have

$$\begin{aligned}(h'e)(hh'eh)(h'e) &= (h'eh)(h'eh)h'e \\ &= h'ehh'e \\ &= h'(hh')e(hh')e \\ &= h'(hh')^2e^2 \\ &= h'(hh')e \\ &= h'e\end{aligned}$$

and

$$\begin{aligned}(hh'eh)(h'e)(hh'eh) &= h(h'eh)(h'eh)(h'eh) \\ &= hh'eh.\end{aligned}$$

Therefore $h'e \in V(hh'eh)$. Note that

$$\begin{aligned}h'ehh' &= h'hh'ehh' \\ &= h'(hh')^2e \\ &= h'hh'e \\ &= h'e.\end{aligned}$$

Hence $h'e = h'ehh' \in V(hh'eh)$. Since $hh'eh \in H$, $h'e \in V(hh'eh)$ and from the definition of H', it follows that $h'e \in H'$. So, $H'E(S) \subseteq H'$. We now prove that

$H' \subseteq [s]_{\sigma_S}$. Let $x \in H'$. Then $x = k'$, for some $k \in H$. We have

$$\begin{aligned} k \in H \ &\Rightarrow \exists s \in S: \quad k \in [s]_{\sigma_S} \\ &\Rightarrow (k, s) \in \sigma_S \\ &\Rightarrow (k'k, k's) \in \sigma_S \\ &\Rightarrow (k'k)(k's)' \in E(S) && \text{(Proposition 1.37)} \\ &\Rightarrow (k's)' \in E(S) && \text{(S is E-unitary)} \\ &\Rightarrow k's \in E(S) \\ &\Rightarrow (k', s') \in \sigma_S && \text{(Proposition 1.37)} \\ &\Rightarrow k' \in [s']_{\sigma_S} \quad \text{and} \quad s' \in S. \end{aligned}$$

Therefore $H' \subseteq [s']_{\sigma_S}$ and $s' \in S$. Thus $H' \in C(S)$. We show now that $H = HH'H$ and $H' = H'HH'$. Let $h \in H$. Then $h = hh'h$ with $h' \in V(h)$. So $h \in HH'H$. Let $x \in HH'H$. Then $x = h_1h_2'h_3$ with $h_1, h_2, h_3 \in H$ and $h_2' \in V(h_2)$. We have

$$\begin{aligned} h_2, h_3 \in H \ &\Rightarrow h_2', h_3' \in H' \\ &\Rightarrow h_2', h_3' \in [s]_{\sigma_S}, \quad \text{for some } s \in S && (H' \in C(S)) \\ &\Rightarrow (h_3', h_2') \in \sigma_S && \text{(σ_S is an equivalence)} \\ &\Rightarrow h_3'(h_2')' \in E(S) && \text{(Proposition 1.37)} \\ &\Rightarrow h_3'h_2 \in E(S) \\ &\Rightarrow (h_3'h_2)' \in E(S) \\ &\Rightarrow h_2'h_3 \in E(S) \\ &\Rightarrow h_1h_2'h_3 \in HE(S) && (h_1 \in H) \\ &\Rightarrow h_1h_2'h_3 \in H && (H \in C(S)) \\ &\Rightarrow x \in H. \end{aligned}$$

So $HH'H \subseteq H$. Thus $HH'H = H$. Since $HH'H = H'$, for all $H \in C(S)$, and $H = (H')'$, it follows that $H'HH' = H'(H')'H' = H'$.

(iii) $E(C(S)) = \{H \subseteq E(S): \quad H \in C(S)\}$ is a left normal band. It is clear that all elements of $E(C(S))$ are idempotents, since they are subsets of the set of idempotents of S. Let $H, J, K \in E(C(S))$. We show that all idempotents of $C(S)$ belong to $\{H \subseteq E(S) : \quad H \in C(S)\}$. Let $A \in E(C(S))$. Then $A \in C(S)$ and $A^2 = A$. From $A \in C(S)$, it follows that there exists $s \in S$ such that

$$AE(S) \subseteq A \subseteq [s]_{\sigma_S}.$$

Let $a \in A$. Then $a = bc$, with $b, c \in A$. By Proposition 1.37 and since $c \in A$,

$$(c, s) \in \sigma_S \Rightarrow cs' \in E(S).$$

Since $a \in A$,

$$\begin{aligned}
(a, s) \in \sigma_S \quad &\Rightarrow as' \in E(S) && \text{(Proposition 1.37)} \\
&\Rightarrow (bc)s' \in E(S) \\
&\Rightarrow b(cs') \in E(S) \\
&\Rightarrow b \in E(S). && (cs' \in E(S) \text{ and } S \text{ is E-unitary})
\end{aligned}$$

We have

$$\begin{aligned}
b, c \in A \quad &\Rightarrow (b, c) \in \sigma_S \\
&\Rightarrow bc' \in E(S) && \text{(Proposition 1.37)} \\
&\Rightarrow (bc')' \in E(S) \\
&\Rightarrow cb' \in E(S) \\
&\Rightarrow cb \in E(S) \\
&\Rightarrow c \in E(S). && (b \in E(S) \text{ and } S \text{ is E-unitary})
\end{aligned}$$

Since S is an E-unitary semigroup, its band of idempotents is a subsemigroup of S and so $a = bc \in E(S)$. Thus $A \subseteq E(S)$. We show now that $HKJ = HJK$. Let $x \in HKJ$. Then $x = hkj$ with $h \in H$, $k \in K$ and $j \in J$. Since $H, J, K \subseteq E(S)$ and the band $E(S)$ is left normal, $h, j, k \in E(S)$ and so $x = hjk$. Hence $x \in HJK$. A similar argument proves that $HKJ \subseteq HJK$. Thus $HKJ = HJK$. Moreover, $C(S)$ is E-unitary since S is E-unitary.

By (i), (ii) and (iii), $C(S)$ is an E-unitary regular semigroup such that its idempotents constitute a left normal band.

(2) Let $s, t \in S$ be such that $s\varphi = t\varphi$. We have

$$\begin{aligned} s\varphi = t\varphi \ & \Leftrightarrow sE(S) = tE(S) \\ & \Rightarrow s = ss's = te \quad \text{and} \quad t = tt't = sf, \text{ for some } e, f \in E(S) \\ & \Rightarrow s = te = (sf)e = s(s's)fe = s(s's)ef = sef = te^2f = tef = sf = t. \end{aligned}$$

Then φ is injective. Now, we show that $(s\varphi)(t\varphi) = (st)\varphi$, that is, $(st)E(S) = sE(S)tE(S)$. Let $x \in (st)E(S)$. Then $x = (st)c$, for some $c \in E(S)$, and so

$$x = s(s's)tc \in sE(S)tE(S).$$

Thus $(st)E(S) \subseteq sE(S)tE(S)$. Let $y \in sE(S)tE(S)$. Then $y = sdtg$, for some $d, g \in E(S)$. We have

$$\begin{aligned} y \ & = s(s's)d(tt')tg \\ & = ss's(tt')dtg \qquad (E(S) \text{ is a left normal band}) \\ & = (st)(t'dtg). \end{aligned}$$

Also,

$$\begin{aligned} (t'dt)(t'dt) \ & = t'd(tt')dt \\ & = t'd^2(tt')t \\ & = t'dt \end{aligned}$$

and so $t'dt \in E(S)$. Since S is inverse, $t'dtg \in E(S)$. Then $y = (st)(t'dtg) \in (st)E(S)$ and therefore $sE(S)tE(S) \subseteq (st)E(S)$. Thus $(s\varphi)(t\varphi) = (st)\varphi$. Therefore φ is an embedding. □

Lemma 4.11. *Let S be an E-unitary regular semigroup in which the band $E(S)$ is a left normal band. Then $C(S)^1 = C(S)$ if and only if $E(S)$ is the identity element of $C(S)$.*

Proof: First, observe that $E(S) \in C(S)$. In fact, since $E(S)$ is a band and all idempotents of S are σ_S-related,

$$E(S)E(S) \subseteq E(S) \subseteq [e]_{\sigma_S}, \text{ for every } e \in E(S).$$

Now, if $C(S)^1 = C(S)$ then

$$E(S) = 1_{C(S)} E(S) \subseteq 1_{C(S)} \subseteq [x]_{\sigma_S}, \text{ for some } x \in S. \tag{4.4}$$

We have

$$\begin{aligned} a \in 1_{C(S)} &\Rightarrow \forall e \in E(S), \quad (a,e) \in \sigma_S \\ &\Rightarrow \forall e \in E(S), \quad ae = ae' \in E(S) \quad \text{(Lemma 1.37)} \\ &\Rightarrow a \in E(S) \qquad\qquad\qquad\quad (S \text{ is E-unitary}) \end{aligned}$$

and so $1_{C(S)} \subseteq E(S)$. Thus it follows from (4.4) that $1_{C(S)} = E(S)$.

The converse is obvious. □

Theorem 4.12. *Let S be an E-unitary regular semigroup such that $E(S)$ is a left normal band. Then S is embeddable into $E(C(S)) \operatorname{Wr} S/\sigma_S$.*

Proof: Clearly, $C(S)^1$ is an E-unitary semigroup. Let $\sigma_{C(S)^1}$ be the least group congruence on $C(S)^1$. By construction, each $\sigma_{C(S)^1}$-class has a unique σ_S-class as element. Thus $S/\sigma_S \simeq C(S)^1/\sigma_{C(S)^1}$.

For $[H]_{\sigma_{C(S)^1}} \neq E(C(S)^1)$, let H_0 be the unique σ_S-class contained in $[H]_{\sigma_{C(S)^1}}$ and $(E(C(S)^1))_0 = 1_{C(S)^1} = (E(C(S)^1))'_0$. By Theorem 4.9 and Proposition 4.10, it follows that the mapping $\varphi : S \to E(C(S)^1) \operatorname{Wr} S/\sigma_S$ defined by $s\varphi = (f_{sE(S)}, [s])$ is an embedding. If $E(C(S)) \neq E(C(S)^1)$ then, by Lemma 4.11, $[u]f_{sE(S)} \neq E_{C(S)^1}$, for every $[u] \in S/\sigma_S$. Since $E(C(S)) \subseteq E(C(S)^1)$, the map $s \mapsto (f_{sE(S)}, [s])$ is an embedding from S into $E(C(S)) \operatorname{Wr} S/\sigma_S$. □

5 | λ-semidirect Product

As shown in Example 3.11 of Chapter 3, the semigroup semidirect product of two inverse semigroups is not necessarily inverse. In order to overcome this difficulty, in [1] Billhardt modified the notion of semidirect product in the inverse case and obtained what he called a λ-semidirect product of inverse semigroups. This notion, which we now present, was later generalised for locally R-unipotent semigroups [3] (these are semigroups for which the semigroup eSe is R-unipotent, for all $e \in E(S)$).

5.1 Definitions and basic results

Let S and T be inverse semigroups such that S acts on T by endomorphisms on the left. Due to the axiom (SP1), we have

(LSP1) ${}^{s}(a^{-1}) = ({}^{s}a)^{-1}$, for all $s \in S$ and all $a \in T$;

(LSP2) ${}^{s}e$ is an idempotent of T, for all $s \in S$ and $e \in E(T)$.

Let

$$T *^{\lambda} S = \{(a, s) \in T \times S : \quad a = {}^{ss^{-1}}a\}$$

and let

$$(a, s)(b, r) = ({}^{(sr)(sr)^{-1}}a\, {}^{s}b, sr), \tag{5.1}$$

for all $(a,s),(b,r) \in T *^{\lambda} S$. For any $(a,s),(b,r) \in T *^{\lambda} S$, $({}^{(sr)(sr)^{-1}}a\,{}^{s}b, sr) \in T *^{\lambda} S$. In fact,

$$\begin{aligned}
{}^{(sr)(sr)^{-1}}({}^{(sr)(sr)^{-1}}a\,{}^{s}b) &= {}^{(sr)(sr)^{-1}(sr)(sr)^{-1}}a\,{}^{(sr)(sr)^{-1}s}b && \text{(SP1)}\\
&= {}^{(sr)(sr)^{-1}}a\,{}^{srr^{-1}s^{-1}s}b\\
&= {}^{(sr)(sr)^{-1}}a\,{}^{ss^{-1}srr^{-1}}b && \text{(idpts commute)}\\
&= {}^{(sr)(sr)^{-1}}a\,{}^{s}({}^{rr^{-1}}b) && \text{(SP2)}\\
&= {}^{(sr)(sr)^{-1}}a\,{}^{s}b. && ((b,r) \in T *^{\lambda} S)
\end{aligned}$$

So (5.1) defines a binary operation on $T *^{\lambda} S$. We have the following result:

Theorem 5.1. *Let S and T be inverse semigroups such that S acts on T by endomorphisms on the left. Then $T *^{\lambda} S$, as defined above, is an inverse semigroup with respect to the operation defined in* (5.1)*, with $(a,s)^{-1} = ({}^{s^{-1}}a^{-1}, s^{-1})$, for all $(a,s) \in T *^{\lambda} S$. If, in addition, S and T are monoids and axiom* (SP3) *holds then $T *^{\lambda} S$ is an inverse monoid with identity $(1_T, 1_S)$.*

Proof: Let $(a,s),(b,r),(c,u) \in T *^{\lambda} S$. Then

$$\begin{aligned}
(a,s)((b,r)(c,u)) &= (a,s)({}^{(ru)(ru)^{-1}}b\,{}^{r}c, ru)\\
&= ({}^{s(ru)(s(ru))^{-1}}a\,{}^{s}({}^{(ru)(ru)^{-1}}b\,{}^{r}c), s(ru))\\
&= ({}^{(sru)(sru)^{-1}}a\,{}^{s(ru)(ru)^{-1}}b\,{}^{sr}c, sru),
\end{aligned}$$

and so

$$(a,s)((b,r)(c,u)) = ({}^{(sru)(sru)^{-1}}a\,{}^{s(ru)(ru)^{-1}}b\,{}^{sr}c, sru). \tag{5.2}$$

Also,

$$\begin{aligned}
((a,s)(b,r))(c,u) &= ({}^{(sr)(sr)^{-1}}a\,{}^{s}b, sr)(c,u)\\
&= ({}^{(sr)u((sr)u)^{-1}}({}^{(sr)(sr)^{-1}}a\,{}^{s}b)\,{}^{sr}c, (sr)u)\\
&= ({}^{(sru)(sru)^{-1}(sr)(sr)^{-1}}a\,{}^{(sru)(sru)^{-1}s}b\,{}^{sr}c, sru),
\end{aligned}$$

and so

$$((a,s)(b,r))(c,u) = ({}^{(sru)(sru)^{-1}(sr)(sr)^{-1}}a\,{}^{(sru)(sru)^{-1}s}b\,{}^{sr}c, sru). \tag{5.3}$$

Since

$$\begin{aligned}(sru)(sru)^{-1}s &= sruu^{-1}r^{-1}s^{-1}s\\ &= s(ru)(ru)^{-1}(s^{-1}s)\\ &= s(s^{-1}s)(ru)(ru)^{-1}\\ &= s(ru)(ru)^{-1}\end{aligned}$$

and

$$\begin{aligned}(sru)(sru)^{-1}(sr)(sr)^{-1} &= sruu^{-1}r^{-1}s^{-1}(sr)(sr)^{-1}\\ &= (sru)u^{-1}(sr)^{-1}(sr)(sr)^{-1}\\ &= (sru)u^{-1}(sr)^{-1}\\ &= (sru)(sru)^{-1},\end{aligned}$$

and by (5.2) and (5.3), we obtain that

$$(a,s)((b,r)(c,u)) = ((a,s)(b,r))(c,u).$$

Hence the operation defined in (5.1) is associative and so $T *^{\lambda} S$ equipped with this operation is a semigroup.

Let $(a,s) \in T *^{\lambda} S$. Then $({}^{s^{-1}}a^{-1}, s^{-1}) \in T *^{\lambda} S$, since

$${}^{s^{-1}(s^{-1})^{-1}}({}^{s^{-1}}a^{-1}) = {}^{s^{-1}ss^{-1}}a^{-1} = {}^{s^{-1}}a^{-1}.$$

Note that

$$\begin{aligned}(a,s)({}^{s^{-1}}a^{-1}, s^{-1}) &= ({}^{ss^{-1}(ss^{-1})^{-1}}a\,{}^{s}({}^{s^{-1}}a^{-1}), ss^{-1})\\ &= ({}^{ss^{-1}ss^{-1}}a\,{}^{ss^{-1}}a^{-1}, ss^{-1})\\ &= ({}^{ss^{-1}}a({}^{ss^{-1}}a)^{-1}, ss^{-1}) && \text{(LSP1)}\\ &= (aa^{-1}, ss^{-1}). && ((a,s) \in T *^{\lambda} S)\end{aligned}$$

We have

$$\begin{aligned}(a,s)({}^{s^{-1}}a^{-1}, s^{-1})(a,s) &= (aa^{-1}, ss^{-1})(a,s)\\ &= ({}^{(ss^{-1})s((ss^{-1})s)^{-1}}(aa^{-1})\,{}^{ss^{-1}}a, ss^{-1}s)\\ &= ({}^{ss^{-1}}(aa^{-1})\,{}^{ss^{-1}}a, s)\\ &= ({}^{ss^{-1}}(aa^{-1}a), s) && \text{(SP1)}\\ &= ({}^{ss^{-1}}a, s)\\ &= (a,s) && ((a,s) \in T *^{\lambda} S)\end{aligned}$$

and

$$\begin{aligned}
({}^{s^{-1}}a^{-1}, s^{-1})(a,s)({}^{s^{-1}}a^{-1}, s^{-1}) &= ({}^{s^{-1}}a^{-1}, s^{-1})(aa^{-1}, ss^{-1}) \\
&= ({}^{s^{-1}(ss^{-1})(s^{-1}(ss^{-1}))^{-1}}({}^{s^{-1}}a^{-1})\,{}^{s^{-1}}(aa^{-1}), s^{-1}ss^{-1}) \\
&= ({}^{s^{-1}(s^{-1})^{-1}}({}^{s^{-1}}a^{-1})\,{}^{s^{-1}}(aa^{-1}), s^{-1}) \\
&= ({}^{s^{-1}ss^{-1}}(a^{-1})\,{}^{s^{-1}}(aa^{-1}), s^{-1}) && \text{(SP2)} \\
&= ({}^{s^{-1}}(a^{-1}aa^{-1}), s^{-1}) && \text{(SP1)} \\
&= ({}^{s^{-1}}a^{-1}, s^{-1}).
\end{aligned}$$

Hence $({}^{s^{-1}}a^{-1}, s^{-1}) \in V((a,s))$ and therefore $(a,s)^{-1} = ({}^{s^{-1}}a^{-1}, s^{-1})$.

We now determine the idempotents of $T *^{\lambda} S$ and show that they commute. If (e,x) is an idempotent of $T *^{\lambda} S$ then

$$\begin{aligned}
(e,x)(e,x) = (e,x) &\Leftrightarrow ({}^{xx(xx)^{-1}}e\,{}^{x}e, xx) = (e,x) \\
&\Leftrightarrow {}^{x^2(x^2)^{-1}}e\,{}^{x}e = e \quad \text{and} \quad x^2 = x \\
&\Leftrightarrow {}^{xx^{-1}}e\,{}^{x}e = e \quad \text{and} \quad x \in E(S) \\
&\Leftrightarrow {}^{xx^{-1}x}e = e \quad \text{and} \quad x \in E(S) \\
&\Leftrightarrow {}^{x}e = e \quad \text{and} \quad x \in E(S).
\end{aligned}$$

Then $e^2 = {}^{x}e\,{}^{x}e = {}^{x^2}e = {}^{x}e = e$ and so $e \in E(T)$. Conversely, suppose that $(e,x) \in T *^{\lambda} S$, $e \in E(T)$ and $x \in E(S)$. Then

$$e = {}^{xx^{-1}}e = {}^{xx}e = {}^{x}e$$

and so

$$\begin{aligned}
(e,x)(e,x) &= ({}^{x^2(x^2)^{-1}}e\,{}^{x}e, x^2) \\
&= ({}^{xx^{-1}}e\,{}^{x}e, x) \\
&= (ee, x) \\
&= (e,x).
\end{aligned}$$

Hence $E(T *^{\lambda} S) = \{(e,x) \in T *^{\lambda} S : \ e \in E(T), x \in E(S)\}$.

Let $(e,x),(f,y) \in E(T *^{\lambda} S)$. Since S and T are both inverse semigroups, the idempotents of S commute and the same happens with the idempotents of T and we have

$$\begin{aligned}
(e,x)(f,y) &= (\,{}^{(xy)(xy)^{-1}}e \,{}^{x}f, xy) \\
&= (\,{}^{x}(\,{}^{y(xy)^{-1}}ef), xy) \\
&= (\,{}^{x}(f \,{}^{y(xy)^{-1}}e), yx) \\
&= (\,{}^{x}f \,{}^{(xy)(xy)^{-1}}e, yx) \\
&= (\,{}^{xyy^{-1}}f \,{}^{(xy)(xy)^{-1}}e, yx) \\
&= (\,{}^{xy^{2}}f \,{}^{(xy)^{2}}e, yx) \\
&= (\,{}^{xy}f \,{}^{xy}e, yx) \\
&= (\,{}^{yx}f \,{}^{yx}e, yx)
\end{aligned}$$

and so

$$(e,x)(f,y) = (\,{}^{yx}f \,{}^{yx}e, yx). \tag{5.4}$$

Also, we have

$$\begin{aligned}
(f,y)(e,x) &= (\,{}^{(yx)(yx)^{-1}}f \,{}^{y}e, yx) \\
&= (\,{}^{yxx^{-1}y^{-1}}f \,{}^{yxx^{-1}}e, yx) \\
&= (\,{}^{y^{2}x^{2}}f \,{}^{yx^{2}}e, yx) \\
&= (\,{}^{yx}f \,{}^{yx}e, yx)
\end{aligned}$$

and so

$$(f,y)(e,x) = (\,{}^{yx}f \,{}^{yx}e, yx). \tag{5.5}$$

From (5.4) and (5.5), it follows that

$$(e,x)(f,y) = (f,y)(e,x),$$

for all $(e,x),(f,y) \in E(T *^{\lambda} S)$, that is, all idempotents of $T *^{\lambda} S$ commute. Hence $T *^{\lambda} S$ is a regular semigroup and its idempotents commute. Therefore it is an inverse semigroup.

Now, suppose that S and T are monoids and axiom (SP3) holds. Let $(a,s) \in T *^{\lambda} S$. Then

$$\begin{aligned}
(a,s)(1_T,1_S) &= ({}^{s1_S(s1_S)^{-1}}a\, {}^{s}1_T, s1_S) \\
&= ({}^{ss^{-1}}a\, {}^{s}1_T, s) \\
&= (a1_T, s) \qquad\qquad\qquad ((a,s) \in T *^{\lambda} S) \\
&= (a,s)
\end{aligned}$$

and

$$\begin{aligned}
(1_T,1_S)(a,s) &= ({}^{1_Ss(1_Ss)^{-1}}1_T\, {}^{1_S}a, 1_Ss) \\
&= ({}^{ss^{-1}}1_T\, {}^{1_S}a, s) \\
&= (1_Ta, s) \\
&= (a,s).
\end{aligned}$$

Therefore $(1_T,1_S)$ is the identity of $T *^{\lambda} S$. $\square$

The semigroup $T *^{\lambda} S$ is called a *λ-semidirect product of T by S*. A possible justification for this terminolgy is the notation used by Petrich in [13] for the idempotent ss^{-1}: he denoted this idempotent by $\lambda(s)$.

Proposition 5.2. *Let S and T be inverse semigroups such that S acts on T by endomorphisms on the left. If S and T are both groups and axiom* (SP3) *is satisfied then $T *^{\lambda} S$ is a group and is the classical semidirect product of the group T by the group S.*

Proof: Suppose that S and T are groups such that S acts on T by endomorphisms on the left and (SP3) holds. By Theorem 5.1 and its proof, the set of idempotents of $T *^{\lambda} S$ is

$$E(T *^{\lambda} S) = \{(a,s) \in T *^{\lambda} S : \quad a \in E(T), s \in E(S)\}.$$

Since S and T are groups, 1_S and 1_T are the unique idempotents of S and T, respectively. Then $T *^{\lambda} S$ has a unique idempotent $(1_T,1_S)$. Consequently, the inverse semigroup $T *^{\lambda} S$ is a group.

Let $(a,s),(b,r) \in T *^{\lambda} S$. Then

$$\begin{aligned}
(a,s)(b,r) &= ({}^{sr(sr)^{-1}}a\, {}^{s}b, sr) \\
&= ({}^{1_S}a\, {}^{s}b, sr) \\
&= (a\, {}^{s}b, sr)
\end{aligned}$$

and so the binary operation defined in (5.1) coincides with the one defined in (3.1). Thus $T *^{\lambda} S$ is the classical semidirect product of the group T by the group S. $\square$

The next result is a complement to Lemma 4.2 and its proof is similar. This result will be useful in the next section.

Lemma 5.3. *Let S be a semigroup and X be a non-empty set.*

(1) *S is a Clifford semigroup if and only if S^X is a Clifford semigroup;*

(2) *S is a group if and only if S^X is a group.*

Proof:

(1) Suppose that S is a Clifford semigroup. Then S is regular and $ex = xe$, for all $x \in S$ and all $e \in E(S)$. By Lemma 4.2, S^X is a regular semigroup. Let $f \in S^X$ and $\varepsilon \in E(S^X)$. We show that $\varepsilon f = f\varepsilon$. Notice that if $\varepsilon^2 = \varepsilon$ then $t\varepsilon \in E(S)$, for all $t \in X$:

$$(t\varepsilon)(t\varepsilon) = t\varepsilon^2 = t\varepsilon.$$

Let $x \in X$. Then

$$\begin{aligned} x(\varepsilon f) &= (x\varepsilon)(xf) \\ &= (xf)(x\varepsilon) \quad (x\varepsilon \in E(S) \text{ and } xf \in S) \\ &= x(f\varepsilon). \end{aligned}$$

Thus S^X is a Clifford semigroup.

Conversely, suppose that S^X is a Clifford semigroup. Then, by Lemma 4.2, S is regular. Let $s \in S$ and $e \in E(S)$. Define $f \in S^X$ by $xf = s$, for all $x \in X$, and $\varepsilon \in S^X$ by $x\varepsilon = e$, for all $x \in X$. Clearly, $\varepsilon \in E(S^X)$ since

$$x\varepsilon^2 = x\varepsilon\, x\varepsilon = ee = e = x\varepsilon.$$

Also, for any $x \in X$,

$$\begin{aligned} se &= (xf)(x\varepsilon) \\ &= x(f\varepsilon) \\ &= x(\varepsilon f) \qquad\qquad (S^X \text{ is a Clifford semigroup}) \\ &= (x\varepsilon)(xf) \\ &= es. \end{aligned}$$

Hence S is a Clifford semigroup.

(2) Suppose that S is a group and let 1_S be the identity of S. So S is an inverse semigroup with a unique idempotent. By Lemma 4.2, S^X is an inverse semigroup. Clearly, the constant map $i \in S^X$ defined by $xi = 1_S$, for all $x \in X$, is the identity of S^X. Let $f \in S^X$. Define $f' : X \to S$ by $xf' = (xf)^{-1}$, $x \in X$. Then

$$x(f'f) = (xf')(xf) = (xf)^{-1}(xf) = 1_S = x(ff')$$

and so

$$ff' = f'f = 1_{S^X}.$$

Thus S^X is a group.

Conversely, suppose that S^X is a group. Then S^X is an inverse semigroup with a single idempotent. By Lemma 4.2, the semigroup S is inverse and so it contains idempotents. We show that $E(S)$ has a unique element. Let $e, f \in E(S)$. Define $\varepsilon, \alpha \in S^X$ by $x\varepsilon = e$ and $x\alpha = f$, for all $x \in X$. Clearly, $\varepsilon, \alpha \in E(S^X)$ and so $\alpha = \varepsilon$. Thus $e = f$. Being an inverse semigroup with a single idempotent, S is a group. $\square$

Natural examples of λ-semidirect products are the so called λ-wreath products. Let S and T be inverse monoids. Let T^S be the set of all mappings from S to T. With respect to the multiplication defined by

$$\forall s \in S\ \forall f, g \in T^S, \quad s(fg) = (sf)(sg),$$

T^S is an inverse semigroup (Lemma 4.2), the inverse f^{-1} of $f \in T^S$ being defined by $sf^{-1} = (sf)^{-1}$. As shown in Chapter 4, S acts on T by endomorphisms on the left via

$$x({}^{s}f) = (xs)f,$$

for all $s \in S$, all $f \in T^S$ and all $x \in S$. The λ-semidirect product $T^S *^\lambda S$ is called the *λ-wreath product of T by S* and is denoted by $T \operatorname{Wr}^\lambda S$.

5.2 An application of λ-semidirect product

The main result of this section illustrates the importance of the λ-semidirect product on the theory of inverse semigroups by showing how to construct inverse semigroups from Clifford semigroups and fundamental semigroups. This construction is based on a certain class of congruences on inverse semigroups – the class of Billhard congruences.

A congruence ρ on an inverse semigroup S is called a *Billhardt congruence* if, for each $s \in S$, the set $\{t^{-1}t : \ t \in [s]_\rho\}$ contains a maximum element with respect to the natural partial order. An example of a class of Billhardt congruences is the class of idempotent-separating congruences on an inverse semigroup:

Proposition 5.4. *Every idempotent-separating congruence on an inverse semigroup is a Billhardt congruence.*

Proof: Let ρ be an idempotent-separating congruence on an inverse semigroup S. Let $s \in S$. Consider the set $\{t^{-1}t : \ t \in [s]_\rho\}$ and consider two elements of this set, $a^{-1}a$ and $b^{-1}b$. Then $a, b \in [s]_\rho$ and so $(a, b) \in \rho$. By Proposition 1.23, $(a^{-1}a, b^{-1}b) \in \rho$. Since ρ is an idempotent-separating congruence on S and $a^{-1}a, b^{-1}b \in E(S)$, $a^{-1}a = b^{-1}b$. Thus the set $\{t^{-1}t : \ t \in [s]_\rho\}$ contains a unique element and therefore ρ is a Billhardt congruence. $\square$

A *transversal of a congruence* ρ defined on S is a subset I of S such that $I \cap [s]_\rho$ has a unique element, for $s \in S$. We denote by s_0 this element. If, in addition, ρ is a Billhardt congruence and $s_0^{-1}s_0$ is the largest element of $\{t^{-1}t: \ t \in [s]_\rho\}$ then the tranversal I is called a *Billhardt transversal*.

Theorem 5.5. *Let ρ be a Billhardt congruence on an inverse semigroup S. Then S can be embedded in $\ker\rho \operatorname{Wr}^\lambda S/\rho$.*

Proof: Let ρ be a Billhardt congruence on an inverse semigroup S. Choose the Billhardt transversal for ρ. Let $[a], [b] \in S/\rho$. Suppose that $[a] = [b]$. Then $(a, b) \in \rho$. Since $(ss^{-1}, ss^{-1}) \in \rho$, for all $s \in S$, $(ass^{-1}, bss^{-1}) \in \rho$ and so $(ass^{-1})_0 = (bss^{-1})_0$. By Proposition 1.23, $(a^{-1}, b^{-1}) \in \rho$. Since $(s^{-1}, s^{-1}) \in \rho$, for all $s \in S$, $(s^{-1}a^{-1}, s^{-1}b^{-1}) \in \rho$, that is, $((as)^{-1}, (bs)^{-1}) \in \rho$ and so $(as)_0^{-1} = (bs)_0^{-1}$. Consequently,

$$(ass^{-1})_0 s(as)_0^{-1} = (bss^{-1})_0 s(bs)_0^{-1}.$$

Also, we have

$$\begin{aligned} s \in S \ &\Rightarrow \forall a \in S, \ \ ((ass^{-1})_0, ass^{-1}) \in \rho, \ ((as)_0^{-1}, (as)^{-1}) \in \rho \quad \text{and} \quad (s,s) \in \rho \\ &\Rightarrow \forall a \in S, \ \ ((ass^{-1})_0 s(as)_0^{-1}, ass^{-1}s(as)^{-1}) \in \rho \\ &\Rightarrow \forall a \in S, \ \ (ass^{-1})_0 s(as)_0^{-1} \in [(as)(as)^{-1}]. \end{aligned}$$

Since $(as)(as)^{-1}$ is an idempotent, we obtain that $(ass^{-1})_0 s(as)_0^{-1} \in \ker\rho$. Thus we can define a mapping $f_s : S/\rho \to \ker\rho$ by $[a]f_s = (ass^{-1})_0 s(as)_0^{-1}$.

By Proposition 1.24, $\ker\rho$ and S/ρ are both inverse semigroups and so we can define $\ker\rho \operatorname{Wr}^\lambda S/\rho$. Consider $\varphi : S \to \ker\rho \operatorname{Wr}^\lambda S/\rho$ defined by $s\varphi = (f_s, [s])$, with f_s defined as above. Clearly, if $s = w$, $s, w \in S$, then $s\varphi = w\varphi$. Let $s \in S$. Then, for any $x \in S$,

$$\begin{aligned} [x]^{\,[s][s]^{-1}} f_s &= [x]^{\,[s][s^{-1}]} f_s \\ &= [x]^{\,[ss^{-1}]} f_s \\ &= [xss^{-1}] f_s \\ &= ((xss^{-1})ss^{-1})_0 s((xss^{-1})s)_0^{-1} \\ &= (xss^{-1})_0 s(xs)_0^{-1} \\ &= [x] f_s \end{aligned}$$

and so ${}^{[s][s]^{-1}}f_s = f_s$. Hence $(f_s, [s]) \in \ker \rho \operatorname{Wr}^\lambda S/\rho$. Thus φ is well-defined. We show that φ is injective. Let $s, w \in S$ be such that $s\varphi = w\varphi$. We have

$$\begin{aligned} s\varphi = w\varphi \ &\Leftrightarrow (f_s, [s]) = (f_w, [w]) \\ &\Leftrightarrow \forall x \in S, \quad [x]f_s = [x]f_w \quad \text{and} \quad [s] = [w]. \end{aligned}$$

Since $(s, s_0) \in \rho$ and $s_0^{-1}s_0$ is the largest element of $\{t^{-1}t : \ t \in [s_0]\}$,

$$s^{-1}s \leq s_0^{-1}s_0. \tag{5.6}$$

From $(ss^{-1}, (ss^{-1})_0) \in \rho$, it follows that

$$(ss^{-1})^{-1}(ss^{-1}) \in \{t^{-1}t : \quad t \in [(ss^{-1})_0]\},$$

that is,

$$ss^{-1} \in \{t^{-1}t : \quad t \in [(ss^{-1})_0]\},$$

and so, by definition,

$$ss^{-1} \leq (ss^{-1})_0^{-1}(ss^{-1})_0. \tag{5.7}$$

From (5.6) and (5.7), it follows that

$$s = s(s^{-1}s) \leq ss_0^{-1}s_0$$

and

$$s = (ss^{-1})s \leq (ss^{-1})_0^{-1}(ss^{-1})_0 s,$$

whence, by (7) of Proposition 1.22,

$$s = s(s_0^{-1}s_0 s^{-1})s = ss^{-1}ss_0^{-1}s_0 = ss_0^{-1}s_0$$

and

$$s = ss^{-1}(ss^{-1})_0^{-1}(ss^{-1})_0 s = (ss^{-1})_0^{-1}(ss^{-1})_0 ss^{-1}s = (ss^{-1})_0^{-1}(ss^{-1})_0 s.$$

Then

$$s = (ss^{-1})_0^{-1}(ss^{-1})_0 s = (ss^{-1})_0^{-1}(ss^{-1})_0 ss_0^{-1}s_0.$$

Since

$$[ss^{-1}]f_s = ((ss^{-1})ss^{-1})_0 s((ss^{-1})s)_0^{-1} = (ss^{-1})_0 ss_0^{-1},$$

we have

$$s = (ss^{-1})_0^{-1}[ss^{-1}]f_s s_0.$$

Similar arguments show that

$$w = (ww^{-1})_0^{-1}\,[ww^{-1}]f_w\,w_0.$$

From $[s] = [w]$, it follows that $(ss^{-1})_0^{-1} = (ww^{-1})_0^{-1}$ and $s_0 = w_0$ and since $s\varphi = w\varphi$ and $[ss^{-1}] = [ww^{-1}]$, $[ss^{-1}]f_s = [ww^{-1}]f_w$. Then $s = w$ and so φ is injective. We now show that φ is a morphism, that is, $(sw)\varphi = s\varphi\,w\varphi$, for all $s, w \in S$. For every $s, w \in S$, we have

$$(sw)\varphi = (f_{sw}, [sw])$$

and

$$\begin{aligned} s\varphi\,w\varphi &= (f_s, [s])(f_w, [w]) \\ &= (\,{}^{[s][w]([s][w])^{-1}}f_s\,{}^{[s]}f_w, [s][w]) \\ &= (\,{}^{[(sw)(sw)^{-1}]}f_s\,{}^{[s]}f_w, [sw]). \end{aligned}$$

Let $[x] \in S/\rho$. Then

$$\begin{aligned} [x](\,{}^{[(sw)(sw)^{-1}]}f_s\,{}^{[s]}f_w) &= [x(sw)(sw)^{-1}]f_s\,[xs]f_w \\ &= (x(sw)(sw)^{-1}ss^{-1})_0 s(x(sw)(sw)^{-1}s)_0^{-1}(xsww^{-1})_0 w(xsw)_0^{-1} \\ &= (xsww^{-1}s^{-1}ss^{-1})_0 s(xsww^{-1}s^{-1}s)_0^{-1}(xsww^{-1})_0 w(xsw)_0^{-1} \\ &= (xsww^{-1}s^{-1})_0 s(xss^{-1}sww^{-1})_0^{-1}(xsww^{-1})_0 w(xsw)_0^{-1} \\ &= (x(sw)(sw)^{-1})_0 s(xsww^{-1})_0^{-1}(xsww^{-1})_0 w(xsw)_0^{-1}, \end{aligned}$$

that is,

$$[x](\,{}^{[(sw)(sw)^{-1}]}f_s\,{}^{[s]}f_w) = (x(sw)(sw)^{-1})_0 s(xsww^{-1})_0^{-1}(xsww^{-1})_0 w(xsw)_0^{-1}. \tag{5.8}$$

Also,

$$((x(sw)(sw)^{-1})_0 s, x(sw)(sw)^{-1}s) \in \rho,$$

that is,

$$((x(sw)(sw)^{-1})_0 s, xsww^{-1}s^{-1}s) \in \rho,$$

hence, since the idempotents of S commute,

$$((x(sw)(sw)^{-1})_0 s, xss^{-1}sww^{-1}) \in \rho,$$

and so

$$((x(sw)(sw)^{-1})_0 s, xsww^{-1}) \in \rho.$$

Therefore

$$((x(sw)(sw)^{-1})_0 s)^{-1}((x(sw)(sw)^{-1})_0 s) \leq (xsww^{-1})_0^{-1}(xsww^{-1})_0,$$

whence, by the definition of natural order for idempotents,

$$((x(sw)(sw)^{-1})_0 s)^{-1}((x(sw)(sw)^{-1})_0 s) =$$

$$= ((x(sw)(sw)^{-1})_0 s)^{-1}((x(sw)(sw)^{-1})_0 s)(xsww^{-1})_0^{-1}(xsww^{-1})_0,$$

hence, multiplying by $(x(sw)(sw)^{-1})_0 s$ on the left,

$$(x(sw)(sw)^{-1})_0 s = ((x(sw)(sw)^{-1})_0 s)(xsww^{-1})_0^{-1}(xsww^{-1})_0. \tag{5.9}$$

Using (5.9), the expression (5.8) is equal to $(x(sw)(sw)^{-1})_0(sw)(x(sw))_0^{-1}$ and so

$$[x]\left({}^{[sw(sw)^{-1}]}f_s\, {}^{[s]}f_w\right) = [x]f_{sw}.$$

Thus $(sw)\varphi = s\varphi\, w\varphi$. $\square$

Let μ be the maximum idempotent-separating congruence on an inverse semigroup S. By Proposition 5.4, μ is a Billhardt congruence and so it folllows from Theorem 5.5 that S can be embedded in $\ker\mu \operatorname{Wr}^\lambda S/\mu$. By Proposition 1.26, the semigroup S/μ is fundamental. By definition, the semigroup $\ker\mu \operatorname{Wr}^\lambda S/\mu$ is the λ-semidirect product $(\ker\mu)^{S/\mu} *^\lambda S/\mu$. By Corollary 1.30, $\ker\mu$ is a Clifford semigroup and so, since both semigroups $\ker\mu$ and S/μ are inverse, it follows from Lemma 5.3 that $(\ker\mu)^{S/\mu}$ is a Clifford semigroup. Thus we have the main result of this section.

Theorem 5.6. *Every inverse semigroup can be embedded in a λ-semidirect product of a Clifford semigroup by a fundamental semigroup.*

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