Blow-up and finite time extinction for $p(x,t)$-curl systems arising in electromagnetism

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Information

Abstract

We study a class of $p(x,t)$-curl systems arising in electromagnetism, with a nonlinear source term. Denoting by $\boldsymbol{h}$ the magnetic field, the source term considered is of the form $\lambda \boldsymbol{h} \left(\int_\Omega |\boldsymbol{h}|^2 \right)^{\frac{\sigma-2}{2}}$ where $\lambda \in \{-1,0,1\}$: when $\lambda \in \{-1,0\}$ we consider $0 < \sigma \le 2$ and for $\lambda = 1$ we have $\sigma \ge 1$.

We introduce a suitable functional framework and a convenient basis that allow us to apply the Galerkin's method and prove existence of local or global solutions, depending on the values of λ and σ.

We study the finite time extinction or the stabilization towards zero of the solutions when $\lambda \in \{-1,0\}$ and the blow-up of local solutions when $\lambda = 1$.

1 Introduction

The study of partial differential equations or systems with variable exponents is a recent research topic which had a very quick development that started when it was understood that variable exponents give better descriptions of the behavior of certain materials or phenomena. To the authors knowledge, this is one of the first works involving the $p(x,t)$-curl operator. In this work we intend to generalize the results in [1] to a similar problem but now with variable exponents.

Let Ω be a bounded simply connected domain of $\mathbb{R}^3$ with a $\mathscr{C}^{1,1}$ boundary denoted by Γ, $T \in \mathbb{R}^+$, $Q_T = \Omega \times (0,T)$ and $\Sigma_T = \Gamma \times (0,T)$.

In what follows, vector functions and spaces of vector functions will be denoted by boldface symbols. We will use ∂_x to denote the partial derivative of a function with respect to the variable x.

The divergence of a vector function $\boldsymbol{h} = (h_1, h_2, h_3)$ is denoted by

$$\nabla \cdot \boldsymbol{h} = \partial_{x_1} h_1 + \partial_{x_2} h_2 + \partial_{x_3} h_3$$

and the curl of $\boldsymbol{h}$ by

$$\nabla \times \boldsymbol{h} = (\partial_{x_2} h_3 - \partial_{x_3} h_2, \partial_{x_3} h_1 - \partial_{x_1} h_3, \partial_{x_1} h_2 - \partial_{x_2} h_1).$$

We recall the identity
$$-\Delta\boldsymbol{h} = \nabla\times(\nabla\times\boldsymbol{h}) - \nabla(\nabla\cdot\boldsymbol{h}), \tag{1}$$
where $\Delta\boldsymbol{h} = (\Delta h_1, \Delta h_2, \Delta h_3)$ and $\Delta h_i = \nabla\cdot(\nabla h_i)$, $i = 1, 2, 3$.

We wish to prove existence of solution for the system

$$\partial_t\boldsymbol{h} + \nabla\times\left(|\nabla\times\boldsymbol{h}|^{p(x,t)-2}\nabla\times\boldsymbol{h}\right) = \boldsymbol{f}(\boldsymbol{h}), \qquad \nabla\cdot\boldsymbol{h} = 0 \qquad \text{in } Q_T, \tag{2a}$$

$$|\nabla\times\boldsymbol{h}|^{p(x,t)-2}\nabla\times\boldsymbol{h}\times\boldsymbol{n} = \boldsymbol{0}, \qquad \boldsymbol{h}\cdot\boldsymbol{n} = 0 \qquad \text{on } \Sigma_T, \tag{2b}$$

$$\boldsymbol{h}(\,\cdot\,,0) = \boldsymbol{h}_0 \qquad \text{in } \Omega, \tag{2c}$$

where $\boldsymbol{f}(\boldsymbol{h}) = \lambda\,\boldsymbol{h}\left(\int_\Omega|\boldsymbol{h}|^2\right)^{\frac{\sigma-2}{2}}$ with $\lambda\in\{-1,0,1\}$ and σ a positive constant. Using the Galerkin's method, we prove existence of solution $\boldsymbol{h}\in\boldsymbol{X}(Q_T)\cap H^1(0,T;\boldsymbol{L}^2(\Omega))$ to the above problem. The space

$$\boldsymbol{X}(Q_T) = \left\{\boldsymbol{v}\in\boldsymbol{L}^2(Q_T) : \nabla\times\boldsymbol{v}\in\boldsymbol{L}^{p(\cdot,\cdot)}(Q_T),\ \nabla\cdot\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}_{|\Gamma} = 0\right\}$$

is the suitable functional framework to solve weakly the system (2). A first difficulty is the characterization of the space

$$\boldsymbol{W}^{p(\cdot)}(\Omega) = \left\{\boldsymbol{v}\in\boldsymbol{L}^{p(\cdot)}(\Omega) : \nabla\times\boldsymbol{v}\in\boldsymbol{L}^{p(\cdot)}(\Omega),\ \nabla\cdot\boldsymbol{v} = 0,\ \boldsymbol{v}\cdot\boldsymbol{n}_{|\Gamma} = 0\right\},$$

as a subspace of the Orlicz-Sobolev space $\boldsymbol{W}^{1,p(\cdot)}(\Omega)$ where the seminorm $\|\nabla\times\cdot\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$ is a norm equivalent to the one induced by the $\boldsymbol{W}^{1,p(\cdot)}$-norm (see Theorem 2.1). Others difficulties are the definitions of suitable countable topological bases of $\boldsymbol{W}^{p(\cdot)}(\Omega)$ and of $\boldsymbol{X}(Q_T)$ (see Proposition 2.3 and Proposition 3.2, respectively).

We prove existence of solutions to this system imposing that, besides natural assumptions, the function $p(\cdot,\cdot)$ is decreasing in time. We study the finite time extinction or blow-up of the solutions, depending on the values of the parameters λ and σ.

2   Functional framework − spatial variables

The main purposes of this section are the characterization of the space of divergence free vector functions belonging to $\boldsymbol{L}^{p(\cdot)}(\Omega)$ with curl in $\boldsymbol{L}^{p(\cdot)}(\Omega)$ and normal null trace as well as the construction of a suitable topological basis for this space.

2.1   The Orlicz–Sobolev spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$

We start by collecting a few well known facts from the theory of Sobolev spaces with variable exponent. For details about this theory and an exhaustive review of the existing bibliography, see the monograph [5].

Let $p : \bar\Omega \longrightarrow [1,\infty)$ be a continuous function. We will use the notation $p\in\mathscr{C}_{\log}(\bar\Omega)$ if p satisfies

$$\forall\,\zeta_1,\,\zeta_2\in\bar\Omega,\ |\zeta_1 - \zeta_2| < 1,\quad |p(\zeta_1) - p(\zeta_2)| \le \omega(|\zeta_1 - \zeta_2|), \qquad \limsup_{\tau\to 0^+}\omega(\tau)\log\frac{1}{\tau} = C. \tag{3}$$

Defining
$$A_{p(\cdot)}(f) = \int_\Omega|f|^{p(\cdot)},$$

we denote
$$L^{p(\cdot)}(\Omega) = \left\{f : \Omega\to\mathbb{R} \,:\, f \text{ is measurable and } A_{p(\cdot)}(f) < \infty\right\}.$$

The space $L^{p(\cdot)}(\Omega)$, equipped with the (Luxemburg) norm
$$\|f\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 :\ A_{p(\cdot)}\left(\tfrac{f}{\lambda}\right) \le 1\right\},$$

is a Banach space.

From now on we assume that

$$p^- = \min_{x \in \Omega} p(x) \quad \text{and} \quad p^+ = \max_{x \in \Omega} p(x), \quad 1 < p^-, \quad p^+ < \infty. \tag{4}$$

We define

$$W^{1,p(\cdot)}(\Omega) = \left\{ u \in L^{p(\cdot)}(\Omega) : |\nabla u|^{p(\cdot)} \in L^1(\Omega) \right\},$$

and we endow it with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \|u\|_{L^{p(\cdot)}(\Omega)} + \|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}. \tag{5}$$

We consider also

$$W_0^{1,p(\cdot)}(\Omega) = \left\{ u \in W^{1,p(\cdot)}(\Omega) : u_{|_\Gamma} = 0 \right\},$$

with the norm

$$\|u\|_{W_0^{1,p(\cdot)}(\Omega)} = \|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}.$$

Details about the trace of a function $u \in W^{1,p(\cdot)}(\Omega)$ can be found in [5].

Let us indicate the basic properties of the spaces $L^{p(\cdot)}(\Omega)$, $W^{1,p(\cdot)}(\Omega)$ and $W_0^{1,p(\cdot)}(\Omega)$ that we will need in the rest of this paper:

1. From the definition of the norm in $L^{p(\cdot)}(\Omega)$, we conclude that

$$\min\left(\|f\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|f\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right) \le A_{p(\cdot)}(f) \le \max\left(\|f\|_{L^{p(\cdot)}(\Omega)}^{p^-}, \|f\|_{L^{p(\cdot)}(\Omega)}^{p^+} \right); \tag{6}$$

2. Hölder's inequality is verified, i.e., for all $f \in L^{p(\cdot)}(\Omega)$, $g \in L^{p'(\cdot)}(\Omega)$ with

$$p(x) \in (1, \infty), \quad p'(x) = \frac{p(x)}{p(x) - 1},$$

the following inequality holds:

$$\int_\Omega |f\,g| \le \left(\frac{1}{p^-} + \frac{1}{p'^-} \right) \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)} \le 2 \|f\|_{L^{p(\cdot)}(\Omega)} \|g\|_{L^{p'(\cdot)}(\Omega)};$$

3. We also have that

$$\text{for} \quad q(\cdot) \le p(\cdot) \qquad L^{p(\cdot)}(\Omega) \subseteq L^{q(\cdot)}(\Omega) \quad \text{and} \quad \|f\|_{L^{q(\cdot)}(\Omega)} \le C \|f\|_{L^{p(\cdot)}(\Omega)};$$

4. The space $W^{1,p(\cdot)}(\Omega)$ is separable and reflexive, provided that $p \in \mathscr{C}(\bar{\Omega})$ and satisfies (4);

5. Assuming (3), we have $\mathscr{D}(\Omega)$ is dense in $W_0^{1,p(\cdot)}(\Omega)$ and this last space can be defined as the completion of $\mathscr{D}(\Omega)$ with respect to the norm (5). The density of smooth functions in the space $W_0^{1,p(\cdot)}(\Omega)$ is crucial for the understanding of these spaces. The condition of log-continuity of $p(\cdot)$ is the best known and the most frequently used sufficient condition for the density of $\mathscr{D}(\Omega)$ in $W_0^{1,p(\cdot)}(\Omega)$ (see [4, 5]). Although this condition is not necessary and can be substituted by other conditions (see [5, Chapter 9] for a discussion of this question) we keep it throughout the paper for the sake of simplicity of presentation;

6. Observing that $W_0^{1,p(\cdot)}(\Omega) \subseteq W_0^{1,p^-}(\Omega)$, the Sobolev inequality

$$\|f\|_{L^q(\Omega)} \le C \|f\|_{W^{1,p(\cdot)}(\Omega)}$$

holds, with $1 \le q < \frac{3p^-}{3-p^-}$ if $p^- < 3$, any q if $p^- = 3$ and $q = \infty$ if $p^- > 3$. Here $C = C(p^-, \Omega)$ is a positive constant.

2.2 The space $\boldsymbol{W}^{p(\cdot)}(\Omega)$

We define

$$\boldsymbol{W}^{p(\cdot)}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^{p(\cdot)}(\Omega) : \nabla \times \boldsymbol{v} \in \boldsymbol{L}^{p(\cdot)}(\Omega), \nabla \cdot \boldsymbol{v} = 0, \boldsymbol{v} \cdot \boldsymbol{n}_{|_\Gamma} = 0 \right\},$$

endowed with the norm

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{p(\cdot)}(\Omega)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} + \|\nabla \times \boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}.$$

Theorem 2.1. *Assume that $1 < p^- \le p(\cdot) \le p^+ < \infty$ and p satisfies (3). Then $\boldsymbol{W}^{p(\cdot)}(\Omega)$ is a closed subspace of $\boldsymbol{W}_{\boldsymbol{n}_0}^{1,p(\cdot)}(\Omega)$, where*

$$\boldsymbol{W}_{\boldsymbol{n}_0}^{1,p(\cdot)}(\Omega) = \left\{ \boldsymbol{v} \in \boldsymbol{W}^{1,p(\cdot)}(\Omega) : \boldsymbol{v} \cdot \boldsymbol{n}_{|_\Gamma} = 0 \right\}$$

Besides, if $p^- > \frac{6}{5}$, then $\|\nabla \times \cdot\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$ is a norm in $\boldsymbol{W}^{p(\cdot)}(\Omega)$ equivalent to the norm induced from $\boldsymbol{W}^{1,p(\cdot)}(\Omega)$. In particular,

$$\|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p(\cdot)}(\Omega)} \le C\|\nabla \times \boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)},$$

where $C = C(p^-, p^+, \Omega)$.

Proof. The inclusion $\{\boldsymbol{v} \in \boldsymbol{W}_{\boldsymbol{n}_0}^{1,p(\cdot)}(\Omega) : \nabla \cdot \boldsymbol{v} = 0\} \subseteq \boldsymbol{W}^{p(\cdot)}(\Omega)$ is immediate. We need to prove that $\boldsymbol{W}^{p(\cdot)}(\Omega) \subseteq \boldsymbol{W}_{\boldsymbol{n}_0}^{1,p(\cdot)}(\Omega)$

Let $\boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$ be such that $\nabla \times \boldsymbol{u} \in \boldsymbol{L}^{p(\cdot)}(\Omega)$, $\nabla \cdot \boldsymbol{u} = 0$ and $\boldsymbol{u} \cdot \boldsymbol{n}_{|_\Gamma} = 0$. Since Γ is compact, we can find $z_1, \ldots, z_k \in \Gamma$ and $r_1, \ldots, r_k \in \mathbb{R}^+$ such that, denoting $B_i = B(z_i, r_i)$ the ball centered in z_i with radius r_i and setting $\widehat{B}_i = B(z_i, \frac{r_i}{2})$, we have:

1. $\displaystyle\bigcup_{i=1}^{k} \widehat{B}_i \supseteq \Gamma$;

2. For $i = 1 \ldots k$, there exists a set $V_i^+ \subseteq \mathbb{R}^2 \times \mathbb{R}^+$ and a smooth function $\boldsymbol{\Phi}_i$ which is a bijection from $B_i \cap \Omega$ to V_i^+ and such that $\boldsymbol{\Phi}(\Gamma \cap B_i) = \overline{V_i^+} \cap (\mathbb{R}^2 \times \{0\})$;

3. Denote, for i fixed, $y = \boldsymbol{\Phi}_i(x)$ and $y_{0i} = \boldsymbol{\Phi}_i(z_i)$. Let $L = a_{jk}\partial_{y_j y_k} + b_j \partial_{y_j}$ be the expression of the Laplacian in the new variables (for details see [6, p. 97]). We choose the radius r_i of the ball B_i sufficiently small such that:

   (a)
   $$\max_{y \in \boldsymbol{\Phi}_i(B_i)} |a_{jk}(y) - a_{jk}(y_{0i})| < \frac{1}{36\,C}, \tag{7}$$

   where $C = C(p^-, p^+, \Omega)$ is the constant for the $\boldsymbol{W}_0^{1,p(\cdot)}(\Omega)$ estimate satisfied by the solutions of the equations

   $$\begin{cases} -a_{jk}(y_{0i})\partial_{y_j y_k}^2 \boldsymbol{v} = \boldsymbol{g} & \text{in } \Omega, \\ \boldsymbol{v} = \boldsymbol{0} & \text{on } \Gamma, \end{cases}$$

   where $\boldsymbol{g}$ is any element of $\boldsymbol{W}^{p(\cdot)}(\Omega)'$;

   (b)
   $$\operatorname{diam}(V_i^+)\,\|b_j\|_{W^{1,\infty}(\Omega)} \le \frac{1}{24\,C}, \tag{8}$$

   where $\operatorname{diam}(V_i^+)$ represents the diameter of V_i^+ defined in item 2, b_j are the coefficients of the first order terms of the operator defined above in this item and C is the constant mentioned in (a);

4. Since $\bar{\Omega}$ is compact, we can find $z_{k+1}, \ldots, z_m \in \Omega$ and $r_{k+1}, \ldots, r_m \in \mathbb{R}^+$ such that $\displaystyle\bigcup_{i=1}^{m} \widehat{B}_i \supseteq \bar{\Omega}$.

For $i \in \{k+1, \ldots, m\}$, let $\eta_i \in \mathscr{D}(B_i)$ be such that $0 \leq \eta_i \leq 1$ and $\eta_{i|_{\widehat{B}_i}} \equiv 1$. The function $\boldsymbol{v}_i = \eta_i \boldsymbol{u}$ satisfies the following problem

$$\begin{cases} -\Delta \boldsymbol{v}_i = -\eta_i \Delta \boldsymbol{u} - 2\nabla\eta_i \cdot \nabla \boldsymbol{u} - \boldsymbol{u}\Delta\eta_i & \text{in } B_i \\ \boldsymbol{v}_i = \boldsymbol{0} & \text{on } \partial B_i \end{cases}. \tag{9}$$

Observe that $\boldsymbol{F}_i(\boldsymbol{u}) = -\eta_i \Delta \boldsymbol{u} - 2\nabla\eta_i \cdot \nabla \boldsymbol{u} - \boldsymbol{u}\Delta\eta_i \in \boldsymbol{W}_0^{1,p(\cdot)}(B_i)'$ and so $\boldsymbol{v}_i \in \boldsymbol{W}_0^{1,p(\cdot)}(B_i)$ (see p. 439 of the proof of [5, Theorem 14.1.2]). Then $\boldsymbol{v}_i \in \boldsymbol{W}^{1,p(\cdot)}(B_i)$ and

$$\|\boldsymbol{u}\|_{\boldsymbol{W}^{1,p(\cdot)}(\widehat{B}_i)} \leq \|\boldsymbol{v}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(B_i)} \leq C\|\boldsymbol{F}_i(\boldsymbol{u})\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)'},$$

where $C = C(p^-, p^+, \Omega) > 0$.

In what follows C represents different positive constants. Using the identity (1) and recalling that $\nabla \cdot \boldsymbol{u} = 0$, we have

$$\begin{aligned} \|\eta_i \Delta \boldsymbol{u}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)'} &\leq \|\eta_i\|_{L^\infty(B_i)} \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'(\cdot)}(B_i) \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)} \leq 1}} \langle \Delta \boldsymbol{u}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)' \times \boldsymbol{W}^{1,p'(\cdot)}(B_i)} \\ &\leq \|\eta_i\|_{L^\infty(B_i)} \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'(\cdot)}(B_i) \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)} \leq 1}} \int_{B_i} |\nabla \times \boldsymbol{u} \cdot \nabla \times \boldsymbol{\varphi}| \\ &\leq 2\|\eta_i\|_{L^\infty(B_i)} \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'(\cdot)}(B_i) \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)} \leq 1}} \|\nabla \times \boldsymbol{u}\|_{L^{p(\cdot)}(B_i)} \|\nabla \times \boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'(\cdot)}(B_i)} \\ &\leq C\|\nabla \times \boldsymbol{u}\|_{L^{p(\cdot)}(B_i)}, \end{aligned}$$

$$\begin{aligned} \|\nabla\eta_i \cdot \nabla \boldsymbol{u}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)'} &\leq \|\eta_i\|_{L^\infty(B_i)} \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'(\cdot)}(B_i) \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)} \leq 1}} \langle \nabla \boldsymbol{u}, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)' \times \boldsymbol{W}^{1,p'(\cdot)}(B_i)} \\ &\leq \|\eta_i\|_{L^\infty(B_i)} \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'(\cdot)}(B_i) \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)} \leq 1}} \int_{B_i} |\boldsymbol{u} \cdot \nabla \boldsymbol{\varphi}| \\ &\leq 2\|\eta_i\|_{L^\infty(B_i)} \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'(\cdot)}(B_i) \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)} \leq 1}} \|\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(B_i)} \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)} \\ &\leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(B_i)} \end{aligned}$$

and

$$\begin{aligned} \|\boldsymbol{u}\Delta\eta_i\|_{\boldsymbol{W}^{1,p(\cdot)}(B_i)'} &\leq \|\Delta\eta_i\|_{L^\infty(B_i)} \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'(\cdot)}(B_i) \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)} \leq 1}} \int_{B_i} |\boldsymbol{u} \cdot \boldsymbol{\varphi}| \\ &\leq 2\|\Delta\eta_i\|_{L^\infty(B_i)} \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'(\cdot)}(B_i) \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)} \leq 1}} \|\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(B_i)} \|\boldsymbol{\varphi}\|_{\boldsymbol{L}^{p'(\cdot)}(B_i)} \\ &\leq C\|\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(B_i)}. \end{aligned}$$

So,

$$\|\boldsymbol{F}_i(\boldsymbol{u})\|_{\boldsymbol{W}^{1,p'(\cdot)}(B_i)'} \leq C\left(\|\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(B_i)} + \|\nabla \times \boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(B_i)}\right).$$

Let now $i \in \{1, \ldots, k\}$ and V_i^+ be the sets as defined in item 2. Define

$$V_i^- = \{(y_1, y_2, -y_3) : (y_1, y_2, y_3) \in V_i^+\} \quad \text{and} \quad V_i = V_i^+ \cup V_i^-.$$

Given a function $\varphi : V_i^+ \longrightarrow \mathbb{R}$, we define $\widetilde{\varphi} : V_i \longrightarrow \mathbb{R}$ by even reflection, i.e.,

$$\widetilde{\varphi}(y_1, y_2, y_3) = \begin{cases} \varphi(y_1, y_2, y_3) & \text{if } (y_1, y_2, y_3) \in V_i^+ \\ \varphi(y_1, y_2, -y_3) & \text{if } (y_1, y_2, y_3) \in V_i^- \end{cases}.$$

For a given function $\boldsymbol{v} : B_i \longrightarrow \mathbb{R}^3$ we call $\bar{\boldsymbol{v}} = \boldsymbol{v} \circ \boldsymbol{\Phi}_i^{-1}$, where $\boldsymbol{\Phi}_i$ is the change of variables defined in item 2. Recalling that $y = (y_1, y_2, y_3) = \boldsymbol{\Phi}_i(x)$, we have $\bar{\boldsymbol{v}}(y_1, y_2, y_3) = \boldsymbol{v}(x_1, x_2, x_3)$. Let $\eta_i \in \mathscr{D}(B_i)$, $0 \leq \eta_i \leq 1$, $\eta_{i|_{\widehat{B}_i}} \equiv 1$. For $\bar{\boldsymbol{v}}_i = \bar{\eta}_i \bar{\boldsymbol{u}}$ we have

$$\Delta \boldsymbol{v}_i(x) = a_{jk}(y) \partial^2_{y_i y_k} \bar{\boldsymbol{v}}_i(y) + b_j(y) \partial_{y_j} \bar{\boldsymbol{v}}_i(y),$$

where $(a_{jk})_{j,k}$ is strictly positively defined and $a_{jk}, b_j \in \mathscr{C}(\bar{V}_i)$ (for details see [6, p. 97]). Then the problem, in the variable x,

$$\begin{cases} -\Delta \boldsymbol{v}_i(x) = \boldsymbol{F}_i(\boldsymbol{u}(x)) & \text{in } B_i \cap \Omega \\ \boldsymbol{v}_i(x) = 0 & \text{on } \partial B_i \cap \Omega \\ \boldsymbol{v}_i(x) \cdot \boldsymbol{n} = 0 & \text{on } \Gamma \cap B_i \end{cases}$$

is transformed into the problem, in the variable y,

$$\begin{cases} -a_{jk}(y) \partial^2_{y_i y_k} \bar{\boldsymbol{v}}_i(y) = \boldsymbol{G}_i(\boldsymbol{u}(y)) + b_j(y) \partial_{y_j} \bar{\boldsymbol{v}}_i(y) & \text{in } V_i \\ \bar{\boldsymbol{v}}_i(y) = 0 & \text{on } \partial V_i \end{cases}, \tag{10}$$

where $\boldsymbol{G}_i(\boldsymbol{u})$ is the function that corresponds to $\boldsymbol{F}_i(\boldsymbol{u})$ defined in (9), but in the new variable y. Recalling the definition of $y_{i0} = \boldsymbol{\Phi}_i(z_i)$ given in item 3, we can rewrite the problem (10) as follows

$$\begin{cases} -a_{jk}(y_{0i}) \partial^2_{y_i y_k} \bar{\boldsymbol{v}}_i(y) = \boldsymbol{G}_i(\boldsymbol{u}(y)) + b_j(y) \partial_{y_j} \bar{\boldsymbol{v}}_i(y) - \big(a_{jk}(y_{0i}) - a_{jk}(y)\big) \partial^2_{y_i y_k} \bar{\boldsymbol{v}}_i(y) & \text{in } V_i \\ \bar{\boldsymbol{v}}_i(y) = 0 & \text{on } \partial V_i \end{cases},$$

and so

$$\|\bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(V_i)} \leq C \Big(\|\boldsymbol{G}_i(\boldsymbol{u})\|_{\boldsymbol{W}^{p(\cdot)}(V_i)'} + \|b_j(y) \partial_{y_j} \bar{\boldsymbol{v}}_i(y)\|_{\boldsymbol{W}^{p(\cdot)}(V_i)'}$$
$$+ \|\big(a_{jk}(y_{0i}) - a_{jk}(y)\big) \partial^2_{y_i y_k} \bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{p(\cdot)}(V_i)'} \Big). \tag{11}$$

We are going to estimate first the terms $\|b_j(y) \partial_{y_j} \bar{\boldsymbol{v}}_i(y)\|_{\boldsymbol{W}^{p(\cdot)}(V_i)'}$:

$$\langle b_j \partial_{y_j} \bar{\boldsymbol{v}}_i, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{1,p'(\cdot)}(V_i)' \times \boldsymbol{W}^{1,p'(\cdot)}(V_i)} = -\langle \bar{\boldsymbol{v}}_i, \partial_{y_j} b_j \boldsymbol{\varphi} + b_j \partial_{y_j} \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{1,p'(\cdot)}(V_i)' \times \boldsymbol{W}^{1,p'(\cdot)}(V_i)}$$
$$\leq \int_{V_i} |\bar{\boldsymbol{v}}_i| \left(|\boldsymbol{\varphi}| + |\partial_{y_j} \boldsymbol{\varphi}|\right) \|b_j\|_{W^{1,\infty}(V_i)}$$
$$\leq 2\|b_j\|_{W^{1,\infty}(V_i)} \|\bar{\boldsymbol{v}}_i\|_{\boldsymbol{L}^{p(\cdot)}(V_i)} \|\boldsymbol{\varphi}\|_{W^{1,p'(\cdot)}(V_i)}.$$

Applying the Poincaré inequality (see [5, Theorem 8.2.4]) we obtain

$$\langle b_j \partial \bar{\boldsymbol{v}}_i, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{1,p'(\cdot)}(V_i)' \times \boldsymbol{W}^{1,p'(\cdot)}(V_i)} \leq 2\, \mathrm{diam}(V_i) \|b_j\|_{W^{1,\infty}(V_i)} \|\bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(V_i)} \|\boldsymbol{\varphi}\|_{W^{i,p'(\cdot)}(V_i)}$$

and so, recalling that $\mathrm{diam}(V_i) \leq 2\, \mathrm{diam}(V_i^+)$ and (8),

$$\|b_j \partial_{y_j} \bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{1,p'(\cdot)}(V_i)} = \sup_{\substack{\boldsymbol{\varphi} \in \boldsymbol{W}^{1,p'(\cdot)}(V_i) \\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(V_i) \leq 1}}} \langle b_j \partial \bar{\boldsymbol{v}}_i, \boldsymbol{\varphi} \rangle_{\boldsymbol{W}^{1,p'(\cdot)}(V_i)' \times \boldsymbol{W}^{1,p'(\cdot)}(V_i)} \leq \frac{1}{12\,C} \|\bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(V_i)}. \tag{12}$$

Returning to (11), using assumption (7) and estimate (12),

$$\|\bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(V_i)} \leq C\Big(\|\boldsymbol{G}_i(\boldsymbol{u})\|_{\boldsymbol{W}^{p(\cdot)}(V_i)'} + \|b_j\partial_{y_j}\bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{1,p'(\cdot)}(V_i)}$$

$$+ \|a_{jk}(y_{0i}) - a_{jk}(y)\|_{L^\infty(V_i)} \sup_{\substack{\boldsymbol{\varphi}\in\boldsymbol{W}^{1,p'(\cdot)}(V_i)\\ \|\boldsymbol{\varphi}\|_{\boldsymbol{W}^{1,p'(\cdot)}(V_i)\leq 1}}} \int_{V_i} |\partial_{y_k}\bar{\boldsymbol{v}}_i \cdot \partial_{y_i}\boldsymbol{\varphi}|\Big)$$

$$\leq C\|\boldsymbol{G}_i(\boldsymbol{u})\|_{\boldsymbol{W}^{p(\cdot)}(V_i)'} + \frac{1}{2}\|\bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(V_i)}$$

and we conclude that

$$\|\bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(V_i)} \leq 2C\|\boldsymbol{G}_i(\boldsymbol{u})\|_{\boldsymbol{W}^{p(\cdot)}(V_i)'}.$$

So

$$\begin{aligned}\|\boldsymbol{u}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(\widehat{B}_i\cap\Omega)} &\leq \|\boldsymbol{v}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(B_i\cap\Omega)}\\ &\leq C_1\|\bar{\boldsymbol{v}}_i\|_{\boldsymbol{W}^{1,p(\cdot)}(V_i)}\\ &\leq C_2\|\boldsymbol{G}_i(\boldsymbol{u})\|_{\boldsymbol{W}^{p(\cdot)}(V_i)'}\\ &\leq C_3\|\boldsymbol{F}_i(\boldsymbol{u})\|_{\boldsymbol{W}^{p(\cdot)}(B_i)'}\\ &\leq C_4\big(\|\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(B_i)} + \|\nabla\times\boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(B_i)}\big).\end{aligned}$$

To conclude that $\boldsymbol{u} \in \boldsymbol{W}^{1,p(\cdot)}(\Omega)$ it is enough to use a partition of unity subordinated to the covering $\{\widehat{B}_i\}_{i=1,\dots,m}$.

Applying Peetre's Lemma, we conclude that there exists a positive $C = C(p^-, p^+, \Omega)$ such that

$$\forall \boldsymbol{v} \in \boldsymbol{W}^{p(\cdot)}(\Omega) \quad \|\boldsymbol{v}\|_{\boldsymbol{W}^{1,p(\cdot)}_{n_0}(\Omega)} \leq C\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}.$$

For details of the proof of this inequality, when p is constant, see [7, Theorem 2.1]. $\qquad\square$

Remark 2.2. *Let $v \in \boldsymbol{W}^{p(\cdot)}(\Omega)$. From item 6 of Subsection 2.1 and the previous theorem, the inequality*

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(\Omega)} \leq C\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \tag{13}$$

holds, with $1 \leq q < \frac{3p^-}{3-p^-}$ if $p^- < 3$, any q if $p^- = 3$ and $q = \infty$ if $p^- > 3$. Here $C = C(p^-, p^+, \Omega)$ is a positive constant.

2.3 A basis for $\boldsymbol{W}^{p(\cdot)}(\Omega)$

We wish to find out an appropriate countable topological basis of $\boldsymbol{W}^{p(\cdot)}(\Omega)$, to be able to define a family of approximating problems in finite dimensional subspaces.

Proposition 2.3. *There exists a countable topological basis $\{\boldsymbol{\psi}_n\}_n$ of $\boldsymbol{W}^{p(\cdot)}(\Omega)$ such that, for all $n \in \mathbb{N}$, $\boldsymbol{\psi}_n \in \boldsymbol{\mathcal{X}}$, where*

$$\boldsymbol{\mathcal{X}} = \big\{\boldsymbol{v} \in \boldsymbol{\mathscr{C}}^1(\bar{\Omega}) : \nabla\cdot\boldsymbol{v} = 0, \ \boldsymbol{v}\cdot\boldsymbol{n}_{|\Gamma} = 0\big\}.$$

Proof. A function $\boldsymbol{u} \in \boldsymbol{W}^{p(\cdot)}(\Omega)$ belongs to $\boldsymbol{W}^{1,p(\cdot)}(\Omega)$. So, there exists an extension, still denoted by $\boldsymbol{u}$, belonging to $\boldsymbol{W}^{1,p(\cdot)}(\mathbb{R}^3)$ (see [5, Theorem 8.5.2]). Given $\varepsilon > 0$, let $\rho_\varepsilon \in \mathscr{D}(\mathbb{R}^3)$ be a mollifier, define $\boldsymbol{u}_\varepsilon = \boldsymbol{u} * \rho_\varepsilon = (u_1 * \rho_\varepsilon, u_2 * \rho_\varepsilon, u_3 * \rho_\varepsilon)$ and recall that $\boldsymbol{u}_\varepsilon \in \mathscr{D}(\mathbb{R}^3)$ and $\boldsymbol{u}_\varepsilon \xrightarrow[\varepsilon\to 0]{} \boldsymbol{u}$ in $\boldsymbol{W}^{1,p(\cdot)}(\mathbb{R}^3)$.

Let v_ε be a solution of the problem

$$\begin{cases}-\Delta v_\varepsilon = -\nabla\cdot\boldsymbol{u}_\varepsilon & \text{in } \Omega\\ \frac{\partial v_\varepsilon}{\partial\boldsymbol{n}} = \boldsymbol{u}^\varepsilon\cdot\boldsymbol{n} & \text{on } \Gamma\end{cases}.$$

Then $v_\varepsilon \in \boldsymbol{W}^{2,q}(\Omega)$, for any $1 < q < \infty$ and, in particular, $v^\varepsilon \in \mathscr{C}^1(\bar{\Omega})$. Setting $\boldsymbol{z}_\varepsilon = \boldsymbol{u}_\varepsilon - \nabla v_\varepsilon$, we observe that $\nabla \cdot \boldsymbol{z}_\varepsilon = 0$, $\nabla \times \boldsymbol{z}_\varepsilon \in \mathscr{C}(\bar{\Omega}) \subseteq \boldsymbol{L}^{p(\cdot)}(\Omega)$ and $\boldsymbol{z}_\varepsilon \cdot \boldsymbol{n}_{|\Gamma} = 0$. So, $\boldsymbol{z}_\varepsilon \in \boldsymbol{\mathcal{X}} \subseteq \boldsymbol{W}^{p(\cdot)}(\Omega)$. Besides,

$$
\begin{aligned}
\|\boldsymbol{z}_\varepsilon - \boldsymbol{u}\|_{\boldsymbol{W}^{p(\cdot)}(\Omega)} &= \|\nabla \times \boldsymbol{z}_\varepsilon - \nabla \times \boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \\
&= \|\nabla \times \boldsymbol{u}_\varepsilon - \nabla \times \boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \\
&\le \|\nabla \times \boldsymbol{u}_\varepsilon - \nabla \times \boldsymbol{u}\|_{\boldsymbol{L}^{p(\cdot)}(\mathbb{R}^3)} \xrightarrow[\varepsilon \to 0]{} 0.
\end{aligned}
$$

Since $\boldsymbol{W}^{p(\cdot)}(\Omega)$ is a separable space, it admits a countable topological basis $\{\boldsymbol{\varphi}_n\}_n$. Given $n, m \in \mathbb{N}$, we construct a function $\boldsymbol{z}_{n,m}$ as above, with $\varepsilon = \frac{1}{m}$, such that $\boldsymbol{z}_{n,m} \in \boldsymbol{\mathcal{X}}$ and

$$
\|\boldsymbol{z}_{n,m} - \boldsymbol{\varphi}_n\|_{\boldsymbol{W}^{p(\cdot)}(\Omega)} \le \frac{1}{m}.
$$

But this implies that $\{\boldsymbol{z}_{n,m}\}_{n,m\in\mathbb{N}}$ is a countable subset of $\boldsymbol{\mathcal{X}}$ such that

$$
\left\{ \sum_{i=1}^{k} \sum_{j=1}^{l} \mu_{ij} z_{i,j} : k, l \in \mathbb{N}, \ \mu_{ij} \in \mathbb{R} \text{ for } i = 1, \dots, k \text{ and } j = 1, \dots, l \right\}
$$

is dense in $\boldsymbol{W}^{p(\cdot)}(\Omega)$. So, we can extract from $\{\boldsymbol{z}_{n,m}\}_{n,m}$ a topological countable vector basis of $\boldsymbol{W}^{p(\cdot)}(\Omega)$.

$\square$

3   Functional framework and weak formulation

In this section we present an adequate functional framework to define a weak formulation of problem (2) to which we will prove existence of solution.

Let $p : \bar{Q}_T \longrightarrow (1, \infty)$ be a given function. We will use the notation $p \in \mathscr{C}_{\log}(\bar{Q}_T)$ if p satisfies the log-continuity condition in the cylinder $\bar{Q}_T$:

$$
\forall \, \zeta_1, \zeta_2 \in \bar{Q}_T, \ \zeta_1 = (x, t), \ \zeta_2 = (y, \tau), \ |\zeta_1 - \zeta_2|^2 = |x - y|^2 + (t - \tau)^2 < 1,
$$

$$
|p(\zeta_1) - p(\zeta_2)| \le \omega(|\zeta_1 - \zeta_2|), \quad \limsup_{s \to 0^+} \omega(s) \log \frac{1}{s} = C. \quad (14)
$$

Throughout the rest of the text we use the notations

$$
p^- = \min_{\zeta \in \bar{Q}_T} p(\zeta), \qquad p^+ = \max_{\zeta \in \bar{Q}_T} p(\zeta).
$$

Let

$$
\boldsymbol{X}(Q_T) = \left\{ \boldsymbol{v} \in \boldsymbol{L}^2(Q_T) : \nabla \times \boldsymbol{v} \in \boldsymbol{L}^{p(\cdot,\cdot)}(Q_T), \ \nabla \cdot \boldsymbol{v} = 0, \ \boldsymbol{v} \cdot \boldsymbol{n}_{|\Gamma} = 0 \right\}
$$

endowed with the norm

$$
\|\boldsymbol{v}\|_{\boldsymbol{X}(Q_T)} = \|\boldsymbol{v}\|_{\boldsymbol{L}^2(Q_T)} + \|\nabla \times \boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot,\cdot)}(Q_T)}.
$$

Remark 3.1. *Recall that, by Theorem 2.1, given $\boldsymbol{v} \in \boldsymbol{X}(Q_T)$, we have, for a.e. $t \in (0, T)$,*

$$
\int_\Omega |\nabla \boldsymbol{v}(\cdot, t)|^{p(\cdot, t)} \le C_1 \int_\Omega |\nabla \times \boldsymbol{v}(\cdot, t)|^{p(\cdot, t)},
$$

where, setting $p^+(t) = \sup_{x \in \Omega} p(x, t)$ and $p^-(t) = \inf_{x \in \Omega} p(x, t)$, C_1 is a positive constant depending only on $p^+(t)$, $p^-(t)$ and Ω. Noticing that $p^- \le p^-(t) \le p(x, t) \le p^+(t) \le p^+$, the constant C_1 can be chosen independent of t.

Proposition 3.2. *Let $\{\boldsymbol{\psi}_n\}_n$ be a basis of $\boldsymbol{W}^{p(\cdot)}(\Omega)$ defined as in Proposition 2.3. Then the set*

$$
\boldsymbol{V} = \left\{ \sum_{k=1}^{m} \zeta_k(t) \boldsymbol{\psi}_k(x) : \zeta_k \in \mathscr{C}^1([0, T]), m \in \mathbb{N} \right\}
$$

is dense in $\boldsymbol{X}(Q_T)$.

Proof. We know that the set

$$\boldsymbol{Z} = \left\{ \sum_{k=1}^{m} \mu_k \boldsymbol{\psi}_k \, : \, \mu_k \in \mathbb{R}, \, m \in \mathbb{N} \right\}$$

is dense in $\boldsymbol{W}^{p(\cdot)}(\Omega)$. On the other hand, the set

$$\boldsymbol{Y}(Q_T) = \left\{ \alpha(t)\boldsymbol{\psi}(x) : \alpha \in \mathscr{C}^1([0,T]), \, \boldsymbol{\psi} \in \boldsymbol{W}^{p(\cdot)}(\Omega) \right\}$$

is dense in $L^q\big(0,T;\boldsymbol{W}^{p(\cdot)}(\Omega)\big)$, for any $1 < q < \infty$. But, as

$$\boldsymbol{X}(Q_T) \subseteq L^{\min\{2,p^-\}}\big(0,T;\boldsymbol{W}^{p(\cdot)}(\Omega)\big),$$

the conclusion follows. □

We can now present a weak formulation of problem (2): to find $\boldsymbol{h} \in \boldsymbol{X}(Q_T) \cap H^1\big(0,T;\boldsymbol{L}^2(\Omega)\big)$ such that, for a.e. $t \in (0,T)$,

$$\int_{\Omega} \partial_t \boldsymbol{h}(t) \cdot \boldsymbol{\psi} + \int_{\Omega} |\nabla \times \boldsymbol{h}(t)|^{p(\cdot,t)-2} \nabla \times \boldsymbol{h}(t) \cdot \nabla \times \boldsymbol{\psi} = \int_{\Omega} \boldsymbol{f}(\boldsymbol{h}(t)) \cdot \boldsymbol{\psi}, \, \forall \boldsymbol{\psi} \in \boldsymbol{W}^{p(\cdot)}(\Omega), \quad (15a)$$

$$\boldsymbol{h}(\cdot,0) = \boldsymbol{h}_0. \quad (15b)$$

Observe that, when $\boldsymbol{h} \in \boldsymbol{X}(Q_T) \cap H\big(0,T;\boldsymbol{L}^2(\Omega)\big)$ then $\boldsymbol{h} \in \mathscr{C}\big([0,T];\boldsymbol{L}^2(\Omega)\big)$ and so $\boldsymbol{h}(\cdot,0)$ has a meaning.

4 The case $\lambda \in \{-1,0\}$ and $0 < \sigma \le 2$

In this section we prove existence of global solutions of the problem (15), when $\boldsymbol{f}(\boldsymbol{h}) = \lambda \boldsymbol{h} \big(\int_{\Omega} |\boldsymbol{h}|^2 \big)^{\frac{\sigma-2}{2}}$ for $\lambda \in \{-1,0\}$. We also study the finite time extinction or asymptotic vanishing in time of the solutions.

4.1 Existence of solution

Let $\{\boldsymbol{\psi}_n\}_n$ be a basis of $\boldsymbol{W}^{p(\cdot)}(\Omega)$ defined as in Proposition 2.3. Assume that

$$\boldsymbol{h}_0 \in \boldsymbol{W}^{1,p(\cdot,0)}(\Omega), \quad (16)$$

and let $\boldsymbol{h}_{m0}$ be an approximation, in $\boldsymbol{W}^{1,p(\cdot,0)}(\Omega)$, of $\boldsymbol{h}_0$ such that $\boldsymbol{h}_{m0} \in \langle \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_m \rangle$. Setting

$$\boldsymbol{h}_m(t) = \sum_{i=1}^{m} \zeta_i^m(t)\boldsymbol{\psi}_i,$$

then the system of ODE's in the unknowns $\zeta_1^m, \ldots, \zeta_m^m$,

$$\int_{\Omega} \partial_t \boldsymbol{h}_m(t) \cdot \boldsymbol{\psi}_i + \int_{\Omega} |\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)-2} \nabla \times \boldsymbol{h}_m(t) \cdot \nabla \times \boldsymbol{\psi}_i$$

$$-\lambda \int_{\Omega} \boldsymbol{h}_m(t) \cdot \boldsymbol{\psi}_i \Big(\int_{\Omega} |\boldsymbol{h}_m(t)|^2 \Big)^{\frac{\sigma-2}{2}} = 0$$

$$\boldsymbol{h}_m(\cdot,0) = \boldsymbol{h}_{m0}$$

has a solution $(\zeta_1^m, \ldots, \zeta_m^m) \in \mathscr{C}^1\big([0,T]\big)^m$.

The above system is equivalent to

$$\int_{\Omega} \partial_t \boldsymbol{h}_m(t) \cdot \boldsymbol{\psi} + \int_{\Omega} |\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)-2} \nabla \times \boldsymbol{h}_m(t) \cdot \nabla \times \boldsymbol{\psi}$$

$$-\lambda \int_{\Omega} \boldsymbol{h}_m(t) \cdot \boldsymbol{\psi} \left(\int_{\Omega} |\boldsymbol{h}_m(t)|^2 \right)^{\frac{\sigma-2}{2}} = 0, \quad \forall \boldsymbol{\psi} \in \langle \boldsymbol{\psi}_1, \ldots \boldsymbol{\psi}_m \rangle \quad (17a)$$

$$\boldsymbol{h}_m(\cdot,0) = \boldsymbol{h}_{m0}. \quad (17b)$$

Proposition 4.1. *Assume that $1 < p^- \le p(\cdot,\cdot) \le p^+ < \infty$, p and $\boldsymbol{h}_0$ satisfy, respectively, (14) and (16). Let $\boldsymbol{h}_m$ be a solution of problem (17). Then there exists a positive constant, independent of m, such that*

$$\|\boldsymbol{h}_m\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} \le C, \qquad \|\nabla\times\boldsymbol{h}_m\|_{\boldsymbol{L}^{p(\cdot,\cdot)}(Q_T)} \le C. \tag{18}$$

Proof. Observe that we can use $\boldsymbol{h}_m(t)$ as a test function in (17), obtaining

$$\int_\Omega \partial_t \boldsymbol{h}_m(t) \cdot \boldsymbol{h}_m(t) + \int_\Omega |\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)} - \lambda \left(\int_\Omega |\boldsymbol{h}_m(t)|^2 \right)^{\frac{\sigma}{2}} = 0.$$

Integrating the above equality between 0 and t and denoting $Q_t = \Omega \times (0,t)$, we get

$$\frac{1}{2}\int_\Omega |\boldsymbol{h}_m(t)|^2 + \int_{Q_t} |\nabla\times\boldsymbol{h}_m(\tau)|^{p(\cdot,\tau)}d\tau - \lambda\int_0^t \left(\int_\Omega |\boldsymbol{h}_m(\tau)|^2 \right)^{\frac{\sigma}{2}} d\tau = \frac{1}{2}\int_\Omega |\boldsymbol{h}_{m0}|^2$$

and so

$$\sup_{0\le t\le T} \int_\Omega |\boldsymbol{h}_m(t)|^2 + 2\int_{Q_T} |\nabla\times\boldsymbol{h}_m|^{p(\cdot,\cdot)} \le \int_\Omega |\boldsymbol{h}_{m0}|^2,$$

which immediately yields the conclusion. $\qquad\square$

We will now prove that, under stronger assumptions on the function $p(\cdot,\cdot)$, the partial derivative of $\boldsymbol{h}$ with respect to t belongs to $\boldsymbol{L}^2(Q_T)$.

Proposition 4.2. *Assume that $1 < p^- \le p(\cdot,\cdot) \le p^+ < \infty$, p and $\boldsymbol{h}_0$ satisfy, respectively, (14) and (16), and also that there exists a positive constant c such that $-c \le \partial_t p \le 0$ a.e. in Q_T. Then*

$$\int_{Q_T} |\partial_t\boldsymbol{h}_m(t)|^2 + \sup_{0\le t\le T} \left(\int_\Omega \frac{|\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)}}{p(\cdot,t)} - \frac{\lambda}{\sigma}\left(\int_\Omega |\boldsymbol{h}_m(t)|^2 \right)^{\frac{\sigma}{2}} \right)$$
$$\le \int_\Omega \frac{|\nabla\times\boldsymbol{h}_{m0}|^{p(\cdot,0)}}{p(\cdot,0)} - \frac{\lambda}{\sigma}\left(\int_\Omega |\boldsymbol{h}_{m0}|^2 \right)^{\frac{\sigma}{2}} + \frac{c|\Omega|T}{(p^-)^2}. \tag{19}$$

In particular,

$$\|\partial_t\boldsymbol{h}_m\|_{\boldsymbol{L}^2(Q_T)} \le C. \tag{20}$$

Proof. We may use $\partial_t\boldsymbol{h}_m$ as test function in (17), obtaining

$$\int_\Omega |\partial_t\boldsymbol{h}_m(t)|^2 + \int_\Omega |\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)-2}\nabla\times\boldsymbol{h}_m(t)\cdot\nabla\times\partial_t\boldsymbol{h}_m(t) - \lambda\int_\Omega \boldsymbol{h}_m(t)\cdot\partial_t\boldsymbol{h}_m(t)\left(\int_\Omega |\boldsymbol{h}_m(t)|^2 \right)^{\frac{\sigma-2}{2}} = 0$$

and so

$$\int_\Omega |\partial_t\boldsymbol{h}_m(t)|^2 + \frac{d}{dt}\int_\Omega \frac{|\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)}}{p(\cdot,t)} - \frac{\lambda}{\sigma}\frac{d}{dt}\left(\int_\Omega |\boldsymbol{h}_m(t)|^2 \right)^{\frac{\sigma}{2}} = I, \tag{21}$$

where

$$I = \int_\Omega \frac{|\nabla\times\boldsymbol{h}_m|^{p(\cdot,t)}}{p(\cdot,t)^2}\left(-1 + p(\cdot,t)\log|\nabla\times\boldsymbol{h}_m(t)| \right)\partial_t p(\cdot,t). \tag{22}$$

We evaluate I by the following way:

$$I = \int_\Omega \frac{|\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)}}{p(\cdot,t)^2}\left(1 - p(\cdot,t)\log|\nabla\times\boldsymbol{h}_m(t)| \right)|\partial_t p(\cdot,t)|$$

$$= \int_{\Omega\cap\{1\ge p(\cdot,t)\log|\nabla\times\boldsymbol{h}_m(t)|\}} \frac{|\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)}}{p(\cdot,t)^2}\left(1 - p(\cdot,t)\log|\nabla\times\boldsymbol{h}_m(t)| \right)|\partial_t p(\cdot,t)|$$

$$+ \int_{\Omega\cap\{1< p(\cdot,t)\log|\nabla\times\boldsymbol{h}_m(t)|\}} \frac{|\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)}}{p(\cdot,t)^2}\left(1 - p(\cdot,t)\log|\nabla\times\boldsymbol{h}_m(t)| \right)|\partial_t p(\cdot,t)|$$

$$\le \int_{\Omega\cap\{1\ge p(\cdot,t)\log|\nabla\times\boldsymbol{h}_m(t)|\}} \frac{|\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)}}{p(\cdot,t)^2}\left(1 - p(\cdot,t)\log|\nabla\times\boldsymbol{h}_m(t)| \right)|\partial_t p(\cdot,t)|.$$

Next we use the following properties of the function $F(\eta) = \frac{\eta^{p(\cdot,\cdot)}}{p(\cdot,\cdot)^2}\big(1 - p(\cdot,\cdot)\log\eta\big)$, defined for $0 \leq \eta \leq e^{\frac{1}{p(\cdot,\cdot)}}$,

$$F(0) = F\left(e^{\frac{1}{p(\cdot,\cdot)}}\right) = 0, \quad F'(\eta) = -\eta^{p(\cdot,\cdot)-1}\log\eta, \quad \max_{0 \leq \eta \leq e^{\frac{1}{p(\cdot,\cdot)}}} F(\eta) = F(1) = \frac{1}{p(\cdot,\cdot)^2},$$

to conclude that

$$I \leq \frac{c|\Omega|}{(p^-)^2}. \tag{23}$$

Integrating equality (21) between 0 and t and using inequality (23), we prove (19). To prove (20) it is enough to notice that, from inequality (6),

$$\int_\Omega \frac{|\nabla \times \boldsymbol{h}_{m0}|^{p(\cdot,0)}}{p(\cdot,0)} \leq \frac{1}{p^-}\max\left\{\|\nabla \times \boldsymbol{h}_{m0}\|_{p(\cdot,0)}^{p^-}, \|\nabla \times \boldsymbol{h}_{m0}\|_{p(\cdot,0)}^{p^+}\right\}.$$

$\square$

Theorem 4.3. *Assume that $\frac{6}{5} < p^- \leq p(\cdot,\cdot) \leq p^+ < \infty$, p and $\boldsymbol{h}_0$ satisfy (14) and (16), respectively, and also that there exists a positive constant c such that $-c \leq \partial_t p \leq 0$ a.e. in Q_T. Then problem (15) has a solution $\boldsymbol{h} \in \boldsymbol{X}(Q_T) \cap H^1\big(0,T;\boldsymbol{L}^2(\Omega)\big)$.*
   Besides, if $\sigma \geq 1$, the solution is unique.

Proof. By the estimates (18) and (20), there exist $\boldsymbol{G}$ and $\boldsymbol{h}$ such that, at least for a subsequence, we have

$$\nabla \times \boldsymbol{h}_m \xrightarrow[m\to\infty]{} \boldsymbol{G} \quad \text{in } \boldsymbol{L}^{p(\cdot,\cdot)}(Q_T)\text{-weak},$$

$$\partial_t \boldsymbol{h}_m \xrightarrow[m\to\infty]{} \partial_t \boldsymbol{h} \quad \text{in } \boldsymbol{L}^2(Q_T)\text{-weak}.$$

Let $q = \min\{2, p^-\}$. Recall that, by Remark 3.1, given $\boldsymbol{v} \in \boldsymbol{X}(Q_T)$, we have

$$\int_\Omega |\nabla \boldsymbol{v}(\cdot,t)|^{p(\cdot,t)} \leq C_1 \int_\Omega |\nabla \times \boldsymbol{v}(\cdot,t)|^{p(\cdot,t)},$$

where C_1 is a constant that can be chosen independently of t. Then, there exists a positive constant C such that

$$\|\boldsymbol{v}\|_{\boldsymbol{L}^q(Q_T)} + \|\nabla \boldsymbol{v}\|_{\boldsymbol{L}^q(Q_T)} \leq C\left(\|\boldsymbol{v}\|_{\boldsymbol{L}^2(Q_T)} + \|\nabla \times \boldsymbol{v}\|_{\boldsymbol{L}^{p(\cdot,\cdot)}(Q_T)}\right).$$

Then $\{\boldsymbol{h}_m\}_m$ is bounded in $\boldsymbol{W}^{1,q}(Q_T)$ and this space is compactly included in $\boldsymbol{L}^{q^*}(Q_T)$. Here q^* is the critical Sobolev exponent and it is greater than 2 because $p^- > \frac{6}{5}$. So, at least for a subsequence, we have $\boldsymbol{h}_m \xrightarrow[m\to\infty]{} \boldsymbol{h}$ strongly in $\boldsymbol{L}^{q^*}(Q_T)$ and $\boldsymbol{h}_m(x,t) \xrightarrow[m\to\infty]{} \boldsymbol{h}(x,t)$ for a.e. $(x,t) \in Q_T$.
   Moreover, observing that

$$\frac{d}{dt}\int_\Omega |\boldsymbol{h}(\cdot,t) - \boldsymbol{h}_m(\cdot,t)|^2 = 2\int_\Omega \big(\boldsymbol{h}(\cdot,t) - \boldsymbol{h}_m(\cdot,t)\big) \cdot \big(\partial_t \boldsymbol{h}(\cdot,t) - \partial_t \boldsymbol{h}_m(\cdot,t)\big),$$

we conclude that

$$\int_\Omega |\boldsymbol{h}(\cdot,t) - \boldsymbol{h}_m(\cdot,t)|^2 \leq 2\left(\int_{Q_T}|\boldsymbol{h}-\boldsymbol{h}_m|^2\right)^{\frac{1}{2}}\left(\int_{Q_T}|\partial_t\boldsymbol{h}-\partial_t\boldsymbol{h}_m|^2\right)^{\frac{1}{2}} + \int_\Omega |\boldsymbol{h}_0 - \boldsymbol{h}_{0m}|^2 \xrightarrow[m\to\infty]{} 0,$$

which proves the strong convergence of $(\boldsymbol{h}_m)_m$ to $\boldsymbol{h}$ in $L^\infty\big(0,T;\boldsymbol{L}^2(\Omega)\big)$.
   For $N \in \mathbb{N}$, let $\boldsymbol{\varphi}(t) = \sum_{k=1}^N d_k(t)\boldsymbol{\psi}_k$. According to (17) we have

$$\int_\Omega \partial_t \boldsymbol{h}_m(t) \cdot \boldsymbol{\varphi}(t) + \int_\Omega |\nabla \times \boldsymbol{h}_m(\cdot,t)|^{p(\cdot,t)-2}\nabla \times \boldsymbol{h}_m(\cdot,t) \cdot \nabla \times \boldsymbol{\varphi}(t)$$

$$-\lambda\int_\Omega \boldsymbol{h}_m(\cdot,t) \cdot \boldsymbol{\varphi}(t)\left(\int_\Omega |\boldsymbol{h}_m(\cdot,t)|^2\right)^{\frac{\sigma-2}{2}} = 0. \tag{24}$$

Integrating the last equality with respect to t and passing to the limit as $m \to \infty$ for fixed N, we obtain

$$\int_{Q_T} \partial_t \boldsymbol{h} \cdot \boldsymbol{\varphi} + \int_{Q_T} \boldsymbol{G} \cdot \nabla \times \boldsymbol{\varphi} - \lambda \int_{Q_T} \boldsymbol{h} \cdot \boldsymbol{\varphi} \Big(\int_\Omega |\boldsymbol{h}|^2 \Big)^{\frac{\sigma-2}{2}} = 0, \tag{25}$$

first for $\boldsymbol{\varphi} = \sum_{k=1}^N d_k(t) \boldsymbol{\psi}_k$ and after, by density, for any $\boldsymbol{\varphi} \in \boldsymbol{X}(Q_T)$.

Let $A(\boldsymbol{v}) = |\nabla \times \boldsymbol{v}|^{p(\cdot,\cdot)-2} \nabla \times \boldsymbol{v}$ and recall that

$$\int_{Q_T} \big(A(\boldsymbol{v}) - A(\boldsymbol{w}) \big) \cdot \nabla \times (\boldsymbol{v} - \boldsymbol{w}) \geq 0. \tag{26}$$

We will now prove that

$$\int_{Q_T} \boldsymbol{G} \cdot \nabla \times \boldsymbol{\varphi} = \lim_{m \to \infty} \int_{Q_T} A(\boldsymbol{h}_m) \cdot \nabla \times \boldsymbol{\varphi} = \int_{Q_T} A(\boldsymbol{h}) \cdot \nabla \times \boldsymbol{\varphi}.$$

Subtracting (25) with $\boldsymbol{\varphi} = \boldsymbol{h}$ from (24), integrated in the interval $[0, T]$, with $\boldsymbol{\varphi} = \boldsymbol{h}_m$, we derive

$$J_m + \int_{Q_T} \Big(|\nabla \times \boldsymbol{h}_m|^{p(\cdot,\cdot)} - \boldsymbol{G} \cdot \nabla \times \boldsymbol{h} \Big) = 0, \tag{27}$$

where

$$J_m = \int_{Q_T} \big(\partial_t \boldsymbol{h}_m \cdot \boldsymbol{h}_m - \partial_t \boldsymbol{h} \cdot \boldsymbol{h} \big) - \lambda \left(\Big(\int_{Q_T} |\boldsymbol{h}_m|^2 \Big)^{\frac{\sigma}{2}} - \Big(\int_{Q_T} |\boldsymbol{h}|^2 \Big)^{\frac{\sigma}{2}} \right) \xrightarrow[m \to \infty]{} 0.$$

Subtracting (27) from (26) with $\boldsymbol{w} = \boldsymbol{h}_m$ and passing to the limit as $m \to \infty$, we arrive at inequality

$$0 \leq \int_{Q_T} \big(\boldsymbol{G} - A(\boldsymbol{v}) \big) \cdot \nabla \times (\boldsymbol{h} - \boldsymbol{v}). \tag{28}$$

Choosing $\boldsymbol{v} = \boldsymbol{h} - \nu \boldsymbol{w}$, where ν is a real positive number and $\boldsymbol{w}$ is any function in $\boldsymbol{X}(Q_T)$, and substituting it into (28), we have

$$0 \leq \int_{Q_T} \big(\boldsymbol{G} - A(\boldsymbol{h} - \nu \boldsymbol{w}) \big) \cdot \nabla \times \boldsymbol{w}.$$

Letting $\nu \to 0$ we obtain the inequality

$$0 \leq \int_{Q_T} \big(\boldsymbol{G} - A(\boldsymbol{h}) \big) \cdot \nabla \times \boldsymbol{w}$$

and so $\boldsymbol{G} = A(\boldsymbol{h})$. Recalling that $\boldsymbol{h} \in H^1(0,T; \boldsymbol{L}^2(\Omega))$ then $\boldsymbol{h} \in \mathscr{C}([0,T]; \boldsymbol{L}^2(\Omega))$.

Setting $q = \min\{p^-, 2\}$, the set $\{\boldsymbol{h}_m\}_m$ is bounded in

$$\boldsymbol{Z} = \big\{ \boldsymbol{v} \in L^q\big(0,T; \boldsymbol{W}^{p^-}(\Omega)\big), \partial_t \boldsymbol{v} \in \boldsymbol{L}^2(Q_T) \big\}$$

and we will prove below that $\boldsymbol{Z}$ is compactly included in $\mathscr{C}([0,T]; \boldsymbol{L}^2(\Omega))$. Observe that

$$\int_\Omega |\boldsymbol{h}_m(t+\delta) - \boldsymbol{h}_m(t)|^q = \int_\Omega \left| \int_t^{t+\delta} \partial_t \boldsymbol{h}_m(\tau) \, d\tau \right|^q$$

$$\leq \int_\Omega \delta^{q-1} \int_t^{t+\delta} |\partial_t \boldsymbol{h}_m(\tau)|^q \, d\tau$$

$$\leq \delta^{q-1} \|\partial_t \boldsymbol{h}_m\|_{\boldsymbol{L}^q(Q_T)}^q \leq \delta^q C.$$

We have $\boldsymbol{W}^{p^-}(\Omega) \subseteq \boldsymbol{L}^2(\Omega) \subseteq \boldsymbol{L}^q(\Omega)$, being the first inclusion compact because $p^- > \frac{6}{5}$. On the other hand $\{\boldsymbol{h}_m\}_m$ is a bounded subset of $L^q\big(0,T; \boldsymbol{W}^{p^-}(\Omega)\big)$ and, denoting $\tau_\delta(\boldsymbol{f}(t)) = f(t+\delta)$, we have

$$\|\tau_\delta(\boldsymbol{h}_m) - \boldsymbol{h}_m\|_{L^\infty(0,T-\delta;\boldsymbol{L}^q(\Omega))} = \sup_{t \in [0,T-\delta]} \int_\Omega |\boldsymbol{h}_m(t+\delta) - \boldsymbol{h}_m(t)|^q \xrightarrow[\delta \to 0]{} 0.$$

Then, by [8, Theorem 5], $\{\boldsymbol{h}_m\}_m$ is compactly included in $\mathscr{C}\big([0,T]; \boldsymbol{L}^2(\Omega)\big)$. So, at least for a subsequence, we have $\boldsymbol{h}_m \underset{m\to\infty}{\to} \boldsymbol{h}$ in $\mathscr{C}\big([0,T]; \boldsymbol{L}^2(\Omega)\big)$ and, in particular, $\boldsymbol{h}(0) = \lim_m \boldsymbol{h}_m(0) = \lim_m \boldsymbol{h}_{m0} = \boldsymbol{h}_0$.

This concludes the proof that $\boldsymbol{h}$ solves the problem (15).

To prove the uniqueness of solution in the case $\sigma \geq 1$, we follow the steps in [1]. The proof in the case $\lambda = 0$ is immediate. Let $\lambda = -1$ and $\boldsymbol{h}_1$ and $\boldsymbol{h}_2$ be two solutions of (15). Use $\boldsymbol{h}_1 - \boldsymbol{h}_2$ as test function in the problem solved by $\boldsymbol{h}_1$ and by $\boldsymbol{h}_2$. Then we get, after subtraction,

$$\frac{1}{2}\int_\Omega |\boldsymbol{h}_1(t) - \boldsymbol{h}_2(t)|^2 + \int_{Q_t} \Big(|\nabla\times\boldsymbol{h}_1|^{p(\cdot,\cdot)-2}\nabla\times\boldsymbol{h}_1 - |\nabla\times\boldsymbol{h}_2|^{p(\cdot,\cdot)-2}\nabla\times\boldsymbol{h}_2 \Big) \cdot \nabla\times(\boldsymbol{h}_1 - \boldsymbol{h}_2)$$
$$+ \int_0^t \Big(\int_\Omega |\boldsymbol{h}_1|^2\Big)^{\frac{\sigma-2}{2}} \Big(\int_\Omega \boldsymbol{h}_1 \cdot (\boldsymbol{h}_1 - \boldsymbol{h}_2)\Big) - \int_0^t \Big(\int_\Omega |\boldsymbol{h}_2|^2\Big)^{\frac{\sigma-2}{2}} \Big(\int_\Omega \boldsymbol{h}_2 \cdot (\boldsymbol{h}_1 - \boldsymbol{h}_2)\Big) = 0.$$

Recalling (26), we get

$$\frac{1}{2}\int_\Omega |\boldsymbol{h}_1(t) - \boldsymbol{h}_2(t)|^2 + \int_0^t \Big(\int_\Omega |\boldsymbol{h}_1|^2\Big)^{\frac{\sigma-2}{2}} \Big(\int_\Omega \boldsymbol{h}_1 \cdot (\boldsymbol{h}_1 - \boldsymbol{h}_2)\Big) - \int_0^t \Big(\int_\Omega |\boldsymbol{h}_2|^2\Big)^{\frac{\sigma-2}{2}} \Big(\int_\Omega \boldsymbol{h}_2 \cdot (\boldsymbol{h}_1 - \boldsymbol{h}_2)\Big) \leq 0.$$

Calling $y_1(t) = \int_\Omega |\boldsymbol{h}_1(t)|^2$ and $y_2(t) = \int_\Omega |\boldsymbol{h}_2(t)|^2$, we have, for $\sigma \geq 1$,

$$0 \geq \int_0^t \Big(\int_\Omega |\boldsymbol{h}_1|^2\Big)^{\frac{\sigma-2}{2}} \Big(\int_\Omega \boldsymbol{h}_1 \cdot (\boldsymbol{h}_1 - \boldsymbol{h}_2)\Big) - \int_0^t \Big(\int_\Omega |\boldsymbol{h}_2|^2\Big)^{\frac{\sigma-2}{2}} \Big(\int_\Omega \boldsymbol{h}_2 \cdot (\boldsymbol{h}_1 - \boldsymbol{h}_2)\Big)$$
$$\geq \int_0^t \Big(y_1^{\frac{\sigma}{2}} + y_2^{\frac{\sigma}{2}} - y_1^{\frac{\sigma-1}{2}} y_2^{\frac{1}{2}} - y_2^{\frac{\sigma-1}{2}} y_1^{\frac{1}{2}}\Big) = \int_0^t \Big(y_1^{\frac{\sigma-1}{2}} - y_2^{\frac{\sigma-1}{2}}\Big)\Big(y_1^{\frac{1}{2}} - y_2^{\frac{1}{2}}\Big) \geq 0,$$

and the above inequality implies that $y_1(t) = y_2(t) = 0$ for a.e. $t \in (0,T)$ and

$$\int_0^t \Big(\int_\Omega |\boldsymbol{h}_1|^2\Big)^{\frac{\sigma-2}{2}} \Big(\int_\Omega \boldsymbol{h}_1 \cdot (\boldsymbol{h}_1 - \boldsymbol{h}_2)\Big) - \int_0^t \Big(\int_\Omega |\boldsymbol{h}_2|^2\Big)^{\frac{\sigma-2}{2}} \Big(\int_\Omega \boldsymbol{h}_2 \cdot (\boldsymbol{h}_1 - \boldsymbol{h}_2)\Big) = 0.$$

Consequently,

$$\frac{1}{2}\int_\Omega |\boldsymbol{h}_1(t) - \boldsymbol{h}_2(t)|^2 \leq 0,$$

which implies that $\boldsymbol{h}_1 = \boldsymbol{h}_2$ a.e. in Q_T. $\qquad\square$

4.2 Finite time extinction and asymptotic behavior

We are going to study now the finite time extinction or stabilization in time, towards zero, of the solutions of problem (15), depending on the choice of the parameters λ and σ.

Theorem 4.4. *Let $\boldsymbol{h}$ be a solution of problem* (15) *with $1 < p(\cdot,\cdot) < \infty$, $\lambda = -1$, $0 < \sigma < 2$ and $\boldsymbol{h}_0 \in \boldsymbol{L}^2(\Omega)$. Then there exists a positive $t_* = \frac{1}{2-\sigma}\|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}^{2-\sigma}$ such that, for $t \geq t_*$, we have $\|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)} = 0$.*

Proof. Observing that a solution $\boldsymbol{h}$ of problem (15) satisfies

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{h}(t)|^2 + \int_\Omega |\nabla\times\boldsymbol{h}(t)|^{p(\cdot,t)} + \Big(\int_\Omega |\boldsymbol{h}(t)|^2\Big)^{\frac{\sigma}{2}} = 0, \tag{29}$$

denoting

$$Y(t) = \int_\Omega |\boldsymbol{h}(t)|^2 = \|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)}^2$$

we obtain that Y satisfies the differential inequality

$$Y'(t) + 2Y(t)^{\frac{\sigma}{2}} \leq 0$$

and so

$$\|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall t \geq \frac{1}{2-\sigma}\Big(\int_\Omega |\boldsymbol{h}_0|^2\Big)^{\frac{2-\sigma}{2}}.$$

$\qquad\square$

Remark 4.5. *Notice that, according to* (29)*, we derive, for the limit case $\sigma = 2$ and also for $\sigma > 2$, the asymptotic extinction of the solution when $t \to \infty$. In fact, using the above notation,*

- *if $\sigma = 2$,*

$$Y(t) \le Y(0)e^{-2t};$$

- *if $\sigma > 2$, we have*

$$Y(t) \le \frac{Y(0)}{\left(1 + t(\sigma - 2)Y(0)^{\frac{\sigma-2}{2}}\right)^{\frac{2}{\sigma-2}}}.$$

Now we are going to investigate the asymptotic behavior of $\boldsymbol{h}(t)$ with respect to t, where $\boldsymbol{h}$ solves problem (15) when the term $\boldsymbol{f}(\boldsymbol{h})$ is absent, i.e., $\lambda = 0$.

Theorem 4.6. *Let $\boldsymbol{h}$ be a solution of problem* (15) *with $1 < p(\cdot,\cdot) \le p^+ < 2$, $\lambda = 0$ and $\boldsymbol{h}_0 \in \boldsymbol{L}^2(\Omega)$. Then there exists a positive $t_* < \infty$ such that $\|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)} = 0$ for $t \ge t_*$.*

Proof. Taking into account the first estimate in (18) we can assume, without loss of generality, that $\|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)} \le 1$. Using the energy relation

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{h}(t)|^2 + \int_\Omega |\nabla \times \boldsymbol{h}(t)|^{p(\cdot,t)} = 0,$$

(6) and (13), we can write, for any fixed t,

$$\|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)} \le C \max\left(\left(\int_\Omega |\nabla \times \boldsymbol{h}(t)|^{p(\cdot,t)}\right)^{\frac{1}{p^-}}, \left(\int_\Omega |\nabla \times \boldsymbol{h}(t)|^{p(\cdot,t)}\right)^{\frac{1}{p^+}}\right)$$

and so

$$\|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)}^{p^+} = \min\left(\|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)}^{p^-}, \|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)}^{p^+}\right) \le C \int_\Omega |\nabla \times \boldsymbol{h}(t)|^{p(\cdot,t)}.$$

Setting $Y(t) = \|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)}^2$, the last inequality and the energy relation lead us to the following ordinary differential inequality

$$Y'(t) + CY(t)^{\frac{p^+}{2}} \le 0, \tag{30}$$

which gives

$$Y(t)^{\frac{2-p^+}{2}} \le Y(0)^{\frac{2-p^+}{2}} - \frac{2-p^+}{2}Ct.$$

This completes the proof with $t_* = \frac{2}{C(2-p^+)}\|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}^{2-p^+}$. $\qquad\square$

Now we consider a limit situation when

$$1 < p(\cdot,\cdot) \le \sup_{x\in\Omega} p(x,t) = p^+(t) \le 2 \text{ and } p^+(t) \nearrow 2 \text{ as } t \to \infty. \tag{31}$$

Theorem 4.7. *Let $\boldsymbol{h}$ be a solution of problem* (15) *with $p(\cdot,\cdot)$ satisfying* (31)*, $\lambda = 0$ and $\boldsymbol{h}_0 \in \boldsymbol{L}^2(\Omega)$. Assume that the exponent $p^+(t)$ is monotone increasing and*

$$\int_0^\infty \frac{dt}{e^{\frac{t}{2}(2-p^+(t))}} < \infty. \tag{32}$$

Then there exists a positive $t_* < \infty$ such that $\|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)} = 0$ for $t \ge t_*$.

The proof of this theorem will use the following lemma.

Lemma 4.8 ([4, Lemma 6.7] and [2, Lemma 9.1]). *Let a nonnegative function $\Theta(t)$ satisfy the conditions*

$$\begin{cases} \Theta'(t) + C\Theta^{\mu(t)}(t) \le 0 & \text{for a.e. } t \ge 0 \text{ with } \mu(t) \in (0,1) \text{ and } C \text{ a positive constant} \\ \Theta(t) \le \Theta(0) < \infty, \quad \Theta(0) > 0 \end{cases}.$$

If the exponent $\mu(t)$ is monotone increasing, then $\Theta(t) \equiv 0$ for all $t \ge t_*$ with t_* defined from the equality

$$C\int_0^{t_*} \Theta^{\mu(s)-1}(0)\, ds = \int_0^\infty \frac{dz}{e^{z(1-\mu(z))}}.$$

We present now the proof of Theorem 4.7.

Proof of Theorem 4.7. Applying Lemma 4.8 to the inequality (30) we derive

$$C\int_0^{t_*} \|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}^{p^+(s)-2}\, ds = \int_0^\infty \frac{dt}{e^{\frac{t}{2}(2-p^+(t))}} < \infty.$$

It follows that there exists a positive $t_* < \infty$ such that $\|\boldsymbol{h}(t)\|_{\boldsymbol{L}^2(\Omega)} = 0$ for $t \geq t_*$. □

Remark 4.9. *A simple example of an exponent $p^+(t) = \sup\limits_{x\in\Omega} p(x,t)$ satisfying the conditions of the Theorem 4.7 is*

$$p^+(t) = 2\Big(1 - \alpha\frac{\log t}{t}\Big), \quad 1 < \alpha, \quad e \leq t.$$

5   The case $\lambda = 1$ and $\sigma \geq 1$

In this section we prove existence of global or local solutions of the problem (15), for $\boldsymbol{f}(\boldsymbol{h}) = \boldsymbol{h}\Big(\int_\Omega |\boldsymbol{h}|^2\Big)^{\frac{\sigma-2}{2}}$ and $\sigma \geq 1$. We also study the blow-up of local solutions.

5.1   Existence of solution

We consider, as in Subsection 4.1, a basis $\{\boldsymbol{\psi}_n\}_n$ of $\boldsymbol{W}^{p(\cdot)}(\Omega)$, defined as in Proposition 2.3. Assuming that $\boldsymbol{h}_0$ satisfies (16) let $\boldsymbol{h}_{m0}$ be an approximation, in $\boldsymbol{W}^{1,p(\cdot,0)}(\Omega)$, of $\boldsymbol{h}_0$ such that $\boldsymbol{h}_{m0} \in \langle\boldsymbol{\psi}_1,\ldots,\boldsymbol{\psi}_m\rangle$.
Denoting

$$\boldsymbol{h}_m(t) = \sum_{i=1}^m \zeta_i^m(t)\boldsymbol{\psi}_i,$$

then the system of ODE's in the unknowns $\zeta_1^m,\ldots,\zeta_m^m$,

$$\int_\Omega \partial_t\boldsymbol{h}_m(t)\cdot\boldsymbol{\psi_i} + \int_\Omega |\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)-2}\nabla\times\boldsymbol{h}_m(t)\cdot\nabla\times\boldsymbol{\psi_i} \quad = \int_\Omega \boldsymbol{h}_m(t)\cdot\boldsymbol{\psi_i}\Big(\int_\Omega |\boldsymbol{h}_m(t)|^2\Big)^{\frac{\sigma-2}{2}}$$

$$\boldsymbol{h}_m(\,\cdot\,,0) = \boldsymbol{h}_{m0}$$

has a solution $(\zeta_1^m,\ldots,\zeta_m^m) \in \mathscr{C}^1\big([0,T]\big)^m$.
This problem is equivalent to the following problem

$$\int_\Omega \partial_t\boldsymbol{h}_m(t)\cdot\boldsymbol{\psi} + \int_\Omega |\nabla\times\boldsymbol{h}_m(t)|^{p(\cdot,t)-2}\nabla\times\boldsymbol{h}_m(t)\cdot\nabla\times\boldsymbol{\psi}$$

$$= \int_\Omega \boldsymbol{h}_m(t)\cdot\boldsymbol{\psi_i}\Big(\int_\Omega |\boldsymbol{h}_m(t)|^2\Big)^{\frac{\sigma-2}{2}}, \quad \forall\boldsymbol{\psi} \in \langle\boldsymbol{\psi}_1,\ldots\boldsymbol{\psi}_m\rangle \tag{33a}$$

$$\boldsymbol{h}_m(\,\cdot\,,0) = \boldsymbol{h}_{m0}. \tag{33b}$$

Theorem 5.1. *Assume that $\frac{6}{5} < p^- \leq p(\cdot,\cdot) \leq p^+ < \infty$, p and $\boldsymbol{h}_0$ satisfy (14) and (16), respectively. Assume, in addition, that there exists a positive constant c such that $-c \leq \partial_t p \leq 0$ a.e. in Q_T.*

1. *If*

$$1 \leq \sigma \leq \max\{2, p^-\} \tag{34}$$

   then problem (15) has a solution $\boldsymbol{h} \in \boldsymbol{X}(Q_T)\cap H^1\big(0,T;\boldsymbol{L}^2(\Omega)\big)$ for any positive T.

2. *If*

$$\sigma > \max\{2, p^+\} \tag{35}$$

   then problem (15) has a solution $\boldsymbol{h} \in \boldsymbol{X}(Q_T)\cap H^1\big(0,T;\boldsymbol{L}^2(\Omega)\big)$ for a small $T_{\max} > 0$.

Proof. First of all we derive a priori estimates, independent of m, for the approximations $\boldsymbol{h}_m$. These estimates will be global, for any finite T, in (34) and local, for a small $T_{\max} > 0$, in (35).

We start by proving that, in item 1, for any finite T there exists a positive constant C such that

$$\|\boldsymbol{h}_m\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))} + \int_{Q_T} |\nabla \times \boldsymbol{h}_m|^{p(\cdot,\cdot)} \leq C, \tag{36}$$

$$\int_{Q_T} |\partial_t \boldsymbol{h}_m|^2 + \sup_{0 \leq t \leq T} \int_\Omega |\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)} \leq C. \tag{37}$$

Using $\boldsymbol{h}_m(t)$ as test function in (33), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\boldsymbol{h}_m(t)|^2 + \int_\Omega |\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)} = \left(\int_\Omega |\boldsymbol{h}_m(t)|^2\right)^{\frac{\sigma}{2}}. \tag{38}$$

We split now the proof of the estimate (36) in two cases: $1 \leq \sigma \leq 2$ and $2 < \sigma < p^-$.

- The case $1 \leq \sigma \leq 2$

  The function $Y(t) = \int_\Omega |\boldsymbol{h}_m(t)|^2$ satisfies the inequality $Y'(t) \leq 2Y(t)^{\frac{\sigma}{2}}$. So

  $$\left(\int_\Omega |\boldsymbol{h}_m(t)|^2\right)^{\frac{2-\sigma}{2}} \leq (2-\sigma)t + \left(\int_\Omega |\boldsymbol{h}_{m0}(t)|^2\right)^{\frac{2-\sigma}{2}}, \text{ if } \sigma < 2,$$

  and

  $$\int_\Omega |\boldsymbol{h}_m(t)|^2 \leq e^{2T}\int_\Omega |\boldsymbol{h}_{m0}(t)|^2, \text{ if } \sigma = 2,$$

  and the estimate (36) is immediately obtained.

- The case $2 < \sigma < p^-$

  According to the formula (13) of Remark 2.2 and the inequalities (6), for any fixed $t \in [0,T]$, we have that

  $$\left(\int_\Omega |\boldsymbol{h}_m(t)|^2\right)^{\frac{\sigma}{2}} \leq C \|\nabla \times \boldsymbol{h}_m(t)\|^\sigma_{\boldsymbol{L}^{p(\cdot,t)}(\Omega)}$$

  $$\leq \begin{cases} C\left(\int_\Omega |\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)}\right)^{\frac{\sigma}{p^-}}, & \text{if } \int_\Omega |\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)} > 1 \\ C\left(\int_\Omega |\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)}\right)^{\frac{\sigma}{p^+}}, & \text{if } \int_\Omega |\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)} \leq 1 \end{cases}.$$

  Taking into account that $\sigma < p^- \leq p^+$ and applying Young inequality, we obtain

  $$\left(\int_\Omega |\boldsymbol{h}_m(t)|^2\right)^{\frac{\sigma}{2}} \leq \frac{1}{2}\int_\Omega |\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)} + C'.$$

  Substituting the last estimate in (38) and integrating with respect to t, we conclude the proof of the estimate (36).

Now we derive the estimate (37). Using $\partial_t \boldsymbol{h}_m$ as test function in (33) we obtain

$$\int_\Omega |\partial_t \boldsymbol{h}_m|^2 + \frac{d}{dt}\int_\Omega \frac{|\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)}}{p(\cdot,t)} = I + \frac{d}{dt}\frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}_m(t)|^2\right)^{\frac{\sigma}{2}},$$

where I is defined in (22) and can be estimated, as in (23), by $|I| \leq \frac{c}{(p^-)^2}|\Omega|$. Integrating between 0 and t, we have

$$\int_{Q_t} |\partial_t \boldsymbol{h}_m|^2 + \int_\Omega \frac{|\nabla \times \boldsymbol{h}_m(t)|^{p(\cdot,t)}}{p(\cdot,t)} \leq \int_\Omega \frac{|\nabla \times \boldsymbol{h}_{m0}|^{p(\cdot,0)}}{p(\cdot,0)} + \frac{c}{(p^-)^2}|\Omega|\,T$$

$$+ \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}_m(t)|^2\right)^{\frac{\sigma}{2}} - \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}_{m0}|^2\right)^{\frac{\sigma}{2}}. \tag{39}$$

Applying the estimate in (36) for $\|\boldsymbol{h}_m\|_{L^\infty(0,T;\boldsymbol{L}^2(\Omega))}$, we obtain estimate (37).

Now we derive local estimates (36) and (37), for $T > 0$ small enough, in the case of item 2. Assuming that

$$t < T_{\max} = \frac{1}{(\sigma - 2)\|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}^{\sigma-2}},$$

it follows from (38) that

$$\int_\Omega |\boldsymbol{h}_m(t)|^2 \leq \left(1 - t(\sigma - 2)\left(\int_\Omega |\boldsymbol{h}_{m0}|^2\right)^{\frac{\sigma-2}{2}}\right)^{\frac{2}{2-\sigma}} \int_\Omega |\boldsymbol{h}_{m0}|^2$$

and we obtain the estimate (36).

Substituting the last estimate in (39) we obtain (37) for $t < T_{\max}$. To conclude the proof of the existence of global solution, in the case (34), and local solution, in the case (35), we only need to argue as in the last part of the proof of Theorem 4.3. □

5.2 Blow-up

In this subsection, following some ideas of paper [3], we study the blow-up of local solutions of problem (15). We consider first the case where p depends only on x and afterwards on (x,t).

Theorem 5.2. *Let $\boldsymbol{h}$ be a solution of problem (15) with $1 < p(\cdot) < \infty$, $\lambda = 1$ and $\boldsymbol{h}_0 \in \boldsymbol{L}^2(\Omega)$. Suppose that*

$$E(0) = \int_\Omega \frac{|\nabla \times \boldsymbol{h}_0|^{p(\cdot)}}{p(\cdot)} - \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}_0|^2\right)^{\frac{\sigma}{2}} \leq 0$$

and $\sigma > \max\{2, p^+\}$.

Then, if $\mu \in (\frac{1}{\sigma}, \frac{1}{p^+})$, the solutions of problem (15) blow up on the interval $(0, t_{\max})$, where

$$t_{\max} = \frac{\mu\sigma}{(\sigma - 2)(\mu\sigma - 1)\|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}^{\sigma-2}}.$$

Proof. Let

$$E(t) = \int_\Omega \frac{|\nabla \times \boldsymbol{h}(t)|^{p(\cdot)}}{p(\cdot)} - \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}(t)|^2\right)^{\frac{\sigma}{2}}.$$

Using $\partial_t \boldsymbol{h}(t)$ as test function in (15), we obtain

$$\int_\Omega |\partial_t \boldsymbol{h}(t)|^2 + \partial_t \int_\Omega \frac{|\nabla \times \boldsymbol{h}(t)|^{p(\cdot)}}{p(\cdot)} = \frac{1}{\sigma}\partial_t\left(\int_\Omega |\boldsymbol{h}(t)|^2\right)^{\frac{\sigma}{2}}$$

and so

$$\int_{Q_t} |\partial_t \boldsymbol{h}|^2 + \int_\Omega \frac{|\nabla \times \boldsymbol{h}(t)|^{p(\cdot)}}{p(\cdot)} = \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}(t)|^2\right)^{\frac{\sigma}{2}} - \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}_0|^2\right)^{\frac{\sigma}{2}} + \int_\Omega \frac{|\nabla \times \boldsymbol{h}_0|^{p(\cdot)}}{p(\cdot)}$$

i.e.

$$E(t) + \int_{Q_t} |\partial_t \boldsymbol{h}|^2 = E(0). \tag{40}$$

Setting

$$F(t) = \frac{1}{2}\int_{Q_t} |\boldsymbol{h}|^2,$$

we have

$$F'(t) = \frac{1}{2}\int_\Omega |\boldsymbol{h}(t)|^2 \quad \text{and} \quad F''(t) = \int_\Omega \partial_t \boldsymbol{h}(t) \cdot \boldsymbol{h}(t).$$

Using now $\boldsymbol{h}(t)$ as test function in (15), we get

$$F''(t) + \int_\Omega |\nabla \times \boldsymbol{h}(t)|^{p(\cdot)} = \left(\int_\Omega |\boldsymbol{h}(t)|^2\right)^{\frac{\sigma}{2}}.$$

Recalling the definition of $E(t)$ and (40) we have, for any μ,

$$\mu F''(t) \geq E(t) + \mu\left(\left(\int_\Omega |\boldsymbol{h}(t)|^2\right)^{\frac{\sigma}{2}} - \int_\Omega |\nabla\times\boldsymbol{h}(t)|^{p(\cdot)}\right) \geq \left(\frac{1}{p^+} - \mu\right)\int_\Omega |\nabla\times\boldsymbol{h}(t)|^{p(\cdot)} + \left(\mu - \frac{1}{\sigma}\right)\left(\int_\Omega |\boldsymbol{h}(t)|^2\right)^{\frac{\sigma}{2}}.$$

So, for $\frac{1}{\sigma} < \mu < \frac{1}{p^+}$,

$$\mu F''(t) \geq \left(\mu - \tfrac{1}{\sigma}\right) 2^{\frac{\sigma}{2}} F'(t)^{\frac{\sigma}{2}}$$

and consequently

$$\int_\Omega |\boldsymbol{h}(t)|^2 \geq \frac{1}{\left(\frac{\mu\sigma-1}{\mu\sigma}(2-\sigma)t + \|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}^{2-\sigma}\right)^{\frac{2}{\sigma-2}}}$$

for

$$t < t_{\max} = \frac{\mu\sigma}{(\sigma-2)(\mu\sigma-1)\|\boldsymbol{h}_0\|_{\boldsymbol{L}^2(\Omega)}^{\sigma-2}}, \tag{41}$$

concluding the proof. $\qquad\square$

Theorem 5.3. *Let $\boldsymbol{h}$ be a solution of problem* (15) *with $1 < p(\cdot,\cdot) < \infty$, $-c \leq \partial_t p \leq 0$ a.e. in Q_T, where c is a positive constant, $\lambda = 1$ and $\boldsymbol{h}_0 \in \boldsymbol{L}^2(\Omega)$. Suppose that*

$$E(0) = \int_\Omega \frac{|\nabla\times\boldsymbol{h}_0|^{p(\cdot,0)}}{p(\cdot,0)} - \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}_0|^2\right)^{\frac{\sigma}{2}} < 0,$$

$\sigma > \max\{2, p^+\}$ and c small enough such that $t_{\max} < \frac{|E(0)|(p^-)^2}{|\Omega|c}$, for $t_{\max}$ defined in (41). *Then any solution of problem* (15) *blows up when $t \nearrow t_{\max}$.*

Proof. Using $\partial_t \boldsymbol{h}(t)$ as test function in (15) and recalling that p depends now on (x,t), following the calculations done in the proof of Proposition 4.2, we obtain

$$\int_\Omega |\partial_t \boldsymbol{h}(t)|^2 + \partial_t \int_\Omega \frac{|\nabla\times\boldsymbol{h}(t)|^{p(\cdot,t)}}{p(\cdot,t)} = I + \frac{1}{\sigma}\partial_t\left(\int_\Omega |\boldsymbol{h}(t)|^2\right)^{\frac{\sigma}{2}},$$

where

$$I = \int_\Omega \frac{|\nabla\times\boldsymbol{h}(t)|^{p(\cdot,t)}}{p(\cdot,t)^2}\left(-1 + p(\cdot,t)\log|\nabla\times\boldsymbol{h}(t)|\right)\partial_t p(\cdot,t).$$

Integrating between 0 and t, we have

$$\int_{Q_t} |\partial_t \boldsymbol{h}|^2 + \int_\Omega \frac{|\nabla\times\boldsymbol{h}(t)|^{p(\cdot,t)}}{p(\cdot,t)} = \int_0^t I + \int_\Omega \frac{|\nabla\times\boldsymbol{h}_0|^{p(\cdot,0)}}{p(\cdot,0)} + \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}(t)|^2\right)^{\frac{\sigma}{2}} - \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}_0|^2\right)^{\frac{\sigma}{2}}.$$

Setting

$$E(t) = \int_\Omega \frac{|\nabla\times\boldsymbol{h}(t)|^{p(\cdot,t)}}{p(\cdot,t)} - \frac{1}{\sigma}\left(\int_\Omega |\boldsymbol{h}(t)|^2\right)^{\frac{\sigma}{2}},$$

we conclude that

$$E(t) + \int_{Q_t} |\partial_t \boldsymbol{h}|^2 = E(0) + \int_0^t I(\tau)d\tau.$$

But, from (23), we know that $I \leq \frac{c\,|\Omega|}{(p^-)^2}$. So, choosing t_0 small enough, we have

$$\forall\, t \leq t_0 \quad E(0) + \int_0^t I \leq 0$$

and we conclude the proof as in the previous theorem. $\qquad\square$

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